Smoothness in long sequence spaces

j/w Sheldon Dantas and Petr Hájek

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What and Why

Prerequisites:

- Banach space & norm;
- Dense subspace;
- Derivative (in the Fréchet sense).

So far so good.

Question. Let $\mathcal{X}$ be a Banach space. Is there a dense subspace $\mathcal{Y}$ of $\mathcal{X}$ with a C^k-smooth norm?
Note. We always mean C^k-smooth on $\mathcal{Y} \setminus \{0\}$ (equiv. on $S_{\mathcal{Y}}$).

Separable Banach spaces

- $\mathcal{X}$ has a C^1-smooth norm $\iff$ $\mathcal{X}^*$ is separable.
  - No closed, inf-dim subspace of ℓ_1 has a C^1-smooth norm.
- **Vanderwerff (1992).** $\mathcal{X}$ has a dense subspace with a C^1-smooth norm.
- If $p \notin \mathbb{N}$, the ℓ_p norm is $C^{\lfloor p \rfloor}$-smooth, but not $C^{\lfloor p \rfloor+1}$.
  And ℓ_p has no $C^{\lfloor p \rfloor+1}$-smooth norm!
- **Deville (1989).** If $\mathcal{X}$ has a C^∞-smooth norm, either it contains c_0, or it is super-reflexive, with exact cotype $2k$, and it contains ℓ_{2k}.
- **Hájek (1995).** $\mathcal{X}$ has a dense subspace with a C^∞-smooth norm.
- **Dantas, Hájek, R. (JMAA'20).** $\mathcal{X}$ has a dense subspace with an analytic norm.

Important. All the above dense subspaces are the linear span of an M-basis.

We don't like separable spaces anymore

- **Benyamini, Lindenstrauss**, *Geometric Nonlinear Functional Analysis*
  - Does the existence of a smooth norm on some 'large' subset of a separable Banach space $\mathcal{X}$ imply that $\mathcal{X}^*$ is separable?
  - Is there a norm on ℓ_1 that is differentiable outside a countable union of closed hyperplanes?
- **Guirao, Montesinos, Zizler**, *Open problems...*, Problem 149: Does the space of finitely supported vectors in $\ell_1(\Gamma)$ have a C^1-smooth norm (when Γ is uncountable)?

Some more recent results.

- **Dantas, Hájek, R. (JMAA'20).** ℓ_∞ and $\ell_1(\mathfrak{c})$ have a dense subspace with an analytic norm.
- **Dantas, Hájek, R. (JMAA'20).** Every $\mathcal{X}$ with a (long) uncond. basis has a dense subspace with a C^∞-smooth norm.
- **Dantas, Hájek, R. (IMRN'22+).** Every $\mathcal{X}$ with a fundamental biorthogonal system ... dense ... C^∞-smooth.

Different subspaces?

(Still) Important. All the above dense subspaces are the linear span of some biorthogonal system.

- With the exception of ℓ_∞.

Main open problem. Is there a Banach space $\mathcal{X}$ such that no dense subspace has a C^k-smooth norm?

- **Pełczyński (1957) – Dantas, Hájek, R. (JMAA'20).** No dense subspace of $c_0(\omega_1)$ has an analytic norm.
- Natural strategy: "diagonalise" different dense subspaces.
- Every dense subspace of $\ell_1(\Gamma)$ contains an isomorphic copy of the linear span of the canonical basis.

Hájek, R. (JFA'20). Let $\mathcal{X}$ be a WLD Banach space with $\operatorname{dens}\mathcal{X} \leqslant \mathfrak{c}$. Then there are two dense subspaces $\mathcal{Y}$ and $\mathcal{Z}$ of $\mathcal{X}$ such that no non-separable subspace of $\mathcal{Y}$ is isomorphic to a subspace of $\mathcal{Z}$.

- **Question.** Uncountably many such subspaces?

Where are the long sequence spaces?

Theorem (Dantas, Hájek, R., arXiv:2205.11282)

Let $1 \leqslant p < \infty$ *and* $r \in (0, p)$. *The dense subspace*

$$\mathcal{Y}_p \coloneqq \Big\{ y \in \ell_p(\Gamma) \colon \|y\|_q < \infty \text{ for some } q \in (0,p) \Big\} = \bigcup_{0<q<p} \ell_q(\Gamma)$$

of $\ell_p(\Gamma)$ *has a* C^∞*-smooth norm.*

- ℓ_p has a dense subspace of dimension continuum with a C^∞-smooth norm.
- If $p > 1$ such subspace contains an operator range.
  - Hence, it is not the linear span of an M-basis.
- **Rosenthal (1970).** Every non-separable operator range in $\ell_1(\Gamma)$ contains $\ell_1(\omega_1)$.
- Can a dense hyperplane in ℓ_1 have a smooth norm?
- Does the space of simple functions in L_1 have a smooth norm?

References

- S. Dantas, P. Hájek, and T. Russo, Smooth norms in dense subspaces of Banach spaces, *J. Math. Anal. Appl.* **487** (2020), 123963.
- P. Hájek and T. Russo, On densely isomorphic normed spaces, *J. Funct. Anal.* **279** (2020), 108667.
- S. Dantas, P. Hájek, and T. Russo, Smooth and polyhedral norms via fundamental biorthogonal systems, *Int. Math. Res. Not. IMRN* (online first), rnac211.
- S. Dantas, P. Hájek, and T. Russo, Smooth norms in dense subspaces of $\ell_p(\Gamma)$ and operator ranges, `arXiv:2205.11282`.

Thank you for your attention!