Morning tales

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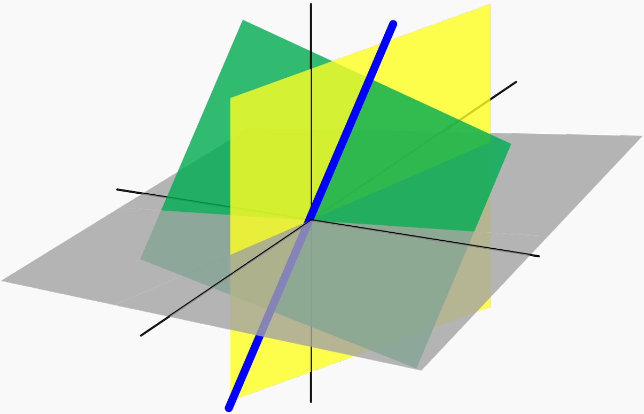
Joke fail

Yes, I did ask Sheldon to move my talk to the morning

Once upon a time in Linear Algebra



Some basis stuff

- Every Banach space $\mathcal{X}$ has a linear basis $\{v_\alpha\}_{\alpha\in\Gamma}$.
- Even when $\mathcal{X}$ is separable, the index set Γ is uncountable.
  - So, linear bases don't generalise complete orthonormal systems.
- Moreover, the linear functionals $\sum c_\alpha v_\alpha \mapsto c_\alpha$ are never continuous.
- A sequence $(e_n)_{n=1}^{\infty}$ is a **Schauder basis** if for all $x \in \mathcal{X}$ there are unique scalars $(x_n)_{n=1}^{\infty}$ with

$$x = \sum_{n=1}^{\infty} x_n e_n \qquad \text{(the series converges in } \mathcal{X}\text{).}$$

- The **coordinate functionals** $\varphi_n \colon \sum x_n e_n \mapsto x_n$ are continuous.
- Two drawbacks:
  - Schauder bases can only exist in separable spaces.
  - **Enflo ('73).** Not every separable Banach space has a Schauder basis.

M for Markushevich

A system $\{e_n; \varphi_n\}_{n=1}^{\infty} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a **Markushevich basis** (**M-basis**) if:

(i) $\langle \varphi_k, e_n \rangle = \delta_{k,n}$,

(ii) $\overline{\mathrm{span}}\{e_n\} = \mathcal{X}$,

(iii) $\overline{\mathrm{span}}^{w^*}\{\varphi_n\} = \mathcal{X}^*$.

- Drawback: $\sum_{n=1}^{\infty} \langle \varphi_n, x \rangle e_n$ might not converge!
- Advantages:
  - **Markushevich ('43).** Every separable Banach space has an M-basis.
  - The definition extends to all Banach spaces (just change label!).

Example: The trigonometric system $\{t \mapsto e^{ikt}\}_{k \in \mathbb{Z}}$ is not a Schauder basis of $\mathcal{C}(\mathbb{T})$ (or $L^1(\mathbb{T})$), but it is an M-basis.

- **Johnson ('70).** ℓ_∞ has no M-basis.

Let us welcome the main character

We actually have more:

- If $\mathcal{X}^*$ is separable, $\mathcal{X}$ admits an M-basis with $\overline{\mathrm{span}}\{\varphi_\alpha\} = \mathcal{X}^*$.
- Every separable Banach space, for every $\varepsilon > 0$, admits an M-basis $\{e_n; \varphi_n\}_{n=1}^\infty$ with $\|e_n\| \cdot \|\varphi_n\| \leqslant 1 + \varepsilon$.
- Every separable Banach space admits a 1-norming M-basis.

A subspace $\mathcal{Z}$ of $\mathcal{X}^*$ is **λ-norming** $(0 < \lambda \leqslant 1)$ if

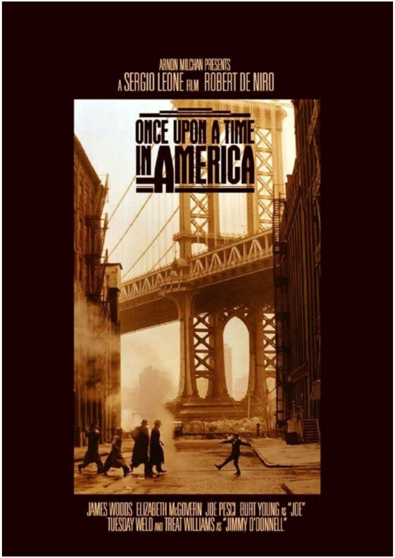
$$\lambda\|x\| \leqslant \sup\{|\langle\varphi, x\rangle| \colon \varphi \in \mathcal{Z},\ \|\varphi\| \leqslant 1\}.$$

Plainly, $\mathcal{X}^*$ is 1-norming, by the Hahn–Banach theorem.

Definition. An M-basis $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$ is **λ-norming** $(0 < \lambda \leqslant 1)$ if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is a λ-norming subspace, namely if

$$\lambda\|x\| \leqslant \sup\{|\langle\varphi, x\rangle| \colon \varphi \in \mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma},\ \|\varphi\| \leqslant 1\}.$$

Once upon a time in America



"Do WCG spaces!"
Viktor Klee $\to$ Vaclav Zizler, '69

- **John–Zizler ('74).** Every Banach space with norming M-basis has a PRI and a LUR norm.
- **Amir–Lindenstrauss ('68).** A Banach space $\mathcal{X}$ is **WCG** if it admits a weakly compact subset with dense linear span.
  - **Amir–Lindenstrauss ('68).** WCG spaces have a PRI.
  - **Troyanski ('71).** WCG spaces have a LUR norm.
- Perhaps **WCG** $\longleftrightarrow$ **norming M-basis**?
  - The canonical basis of $\ell_1(\Gamma)$ is 1-norming.
  - **John–Zizler ('74).** Does every WCG space have a norming M-basis?
- More recent results: WCG spaces, or spaces with norming M-basis, are **Plichko**. And Plichko spaces have a PRI and a LUR norm.

Theorem (Hájek, Advances '19)

There exists a WCG $\mathcal{C}(\mathcal{K})$ space with no norming M-basis.

Problems everywhere

- **John–Zizler ('74).** If a Banach space $(\mathcal{X}, \|\cdot\|)$ has a 1-norming M-basis and $\|\cdot\|$ is Fréchet differentiable, then $\mathcal{X}$ is WCG.
- $\mathcal{X}$ has an M-basis $\{e_\alpha, \varphi_\alpha\}_{\alpha\in\Gamma}$ with $\overline{\mathrm{span}}\{\varphi_\alpha\} = \mathcal{X}^*$, iff $\mathcal{X}$ is Asplund and WCG.
  - If a Banach space has a Fréchet norm, then it is Asplund.
- **Godefroy ($\sim$'90).** Let $\mathcal{X}$ be an Asplund space with norming M-basis. Is $\mathcal{X}$ WCG?

Theorem (Hájek, R., Somaglia, Todorčević, Advances '21)

There exists an Asplund space $\mathcal{X}$ with a 1-norming M-basis such that $\mathcal{X}$ is not WCG.

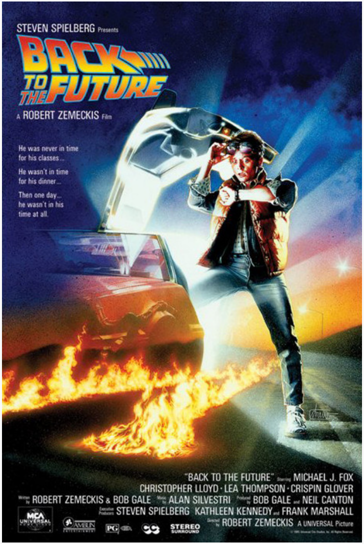
- $\mathcal{X}$ is a subspace of an Asplund $\mathcal{C}(\mathcal{K})$ space, which is not WCG.
- **Problem.** Is there a $\mathcal{C}(\mathcal{K})$ counterexample?
  - **(The same) Problem.** Let $\mathcal{K}$ be a scattered compact space such that $\mathcal{C}(\mathcal{K})$ space has a norming M-basis. Is $\mathcal{K}$ Eberlein?

Forbidden area

Enter at your own risk

- **Problem.** Assume that a $\mathcal{C}(\mathcal{K})$ space has a norming M-basis. Must $\mathcal{K}$ be Valdivia?
- **Deville–Godefroy ('93).** A Valdivia compact space $\mathcal{K}$ is Corson iff it does not contain $[0, \omega_1]$.
- **Alster ('79).** A scattered Corson compact is Eberlein.
- **Problem.** Let $\mathcal{K}$ be a Valdivia compact such that $[0, \omega_1] \subseteq \mathcal{K}$. Does it follow that $\mathcal{C}(\mathcal{K})$ has no norming M-basis?
- If both answers are YES, there is no $\mathcal{C}(\mathcal{K})$ counterexample to Godefroy's problem.
  - $\mathcal{K}$ must be Valdivia. Distinguish two cases, if $[0, \omega_1] \subseteq \mathcal{K}$, or not.
- Well, maybe you should consider the case $\mathcal{K} = [0, \omega_1]$...
- **READ BELOW (AND FORGET THE ABOVE).**
- Well, maybe you should consider the case $\mathcal{K} = [0, \omega_1]$...

Back to the future



How could I give a talk with no ω_1?

- Well, maybe you should consider the case $\mathcal{K} = [0, \omega_1]$...
- **Alexandrov–Plichko ('06).** $\mathcal{C}[0, \omega_1]$ admits no norming M-basis.

Theorem (R. and Somaglia, '23+)

$C[0, \omega_1]$ embeds in no Banach space with a norming M-basis.

- So if $[0, \omega_1]$ is continuous image of $\mathcal{K}$, $\mathcal{C}(\mathcal{K})$ has no norming M-basis.
  - This does <u>not</u> solve the second problem from the previous slide!
- If $\mathcal{K} = \mathcal{T}$ is a tree (with the coarse wedge topology), then: $\mathcal{T}$ scattered and $\mathcal{C}(\mathcal{T})$ with norming M-basis implies $\mathcal{T}$ Eberlein.
  - Some topological results on (scattered) trees with the coarse wedge topology.

Hold on, did we do something?

- **Alexandrov–Plichko ('06).** $\mathcal{C}[0,\omega_1]$ admits no norming M-basis.
- **R.–Somaglia ('23+).** $\mathcal{C}[0,\omega_1]$ does not embed in a Banach space with norming M-basis.
- Are they actually different results?

Problem

Let $\mathcal{X}$ be a Banach space with norming M-basis and $\mathcal{Y}$ be a subspace of $\mathcal{X}$. Must $\mathcal{Y}$ have a norming M-basis?

- **Vanderwerff–Whitfield–Zizler ('94).** Yes, if $\mathcal{Y}$ is WCG (WLD).
- ℓ_∞ does not embed in a space with norming M-basis (no LUR).
- **Kubiś ('07).** The analogue for Plichko spaces has negative answer.
- **Problem**, **Kalenda ('00).** Do all subspaces of $\ell_1(\Gamma)$ have a norming M-bases? Are they Plichko?

(A few, recent) References

P. Hájek
Hilbert generated Banach spaces need not have a norming Markushevich basis
Adv. Math. **351** (2019), 702–717.

P. Hájek, T. Russo, J. Somaglia, and S. Todorčević
An Asplund space with norming Markuševič basis that is not weakly compactly generated
Adv. Math. **392** (2021), 108041.

T. Russo and J. Somaglia
Banach spaces of continuous functions without norming Markushevich bases
Mathematika (in press), `arXiv:2305.11737`.

So, in the end, norming or morning?

Thamk you for your attemtiom!