Lipschitz functions on Euclidean spaces

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Dramatis personae

- Let $(\mathcal{M}, d, 0_{\mathcal{M}})$ be a pointed metric space, with distinguished point $0_{\mathcal{M}}$.
- $\mathrm{Lip}_0(\mathcal{M}) := \{f\colon \mathcal{M} \to \mathbb{R}\colon f \text{ is Lipschitz, } f(0_{\mathcal{M}}) = 0\}$.
- $$\|f\|_{\mathrm{Lip}} = \mathrm{Lip}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} \colon x \neq y \in \mathcal{M}\right\}.$$
- $(\mathrm{Lip}_0(\mathcal{M}), \|\cdot\|_{\mathrm{Lip}})$ is a Banach space.
- Define $\delta_p \in \mathrm{Lip}_0(\mathcal{M})^*$ by $\langle \delta_p, f\rangle = f(p)$. Then $\|\delta_p\| = d(0_{\mathcal{M}}, p)$.
- $\mathcal{F}(\mathcal{M}) := \overline{\mathrm{span}}\{\delta_p\colon p \in \mathcal{M}\} \subseteq \mathrm{Lip}_0(\mathcal{M})^*$ is the **Lipschitz-free space** over $\mathcal{M}$.
- $\mathcal{F}(\mathcal{M})^* = \mathrm{Lip}_0(\mathcal{M})$ and $\mathcal{F}(\mathcal{M})$ linearises Lipschitz functions on $\mathcal{M}$.
- Henceforth, $\mathcal{M}$ is a Banach space and $0_{\mathcal{M}}$ is the origin.

Are spaces of Lipschitz functions different?

Problem (Candido–Cúth–Doucha)

Are there two separable, infinite-dimensional Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathrm{Lip}_0(\mathcal{X})$ and $\mathrm{Lip}_0(\mathcal{Y})$ are not isomorphic?

- **Candido–Kaufmann ('21).** w^*-dens$(\mathrm{Lip}_0(\mathcal{X})^*) = \mathrm{dens}(\mathcal{X})$.
- $\mathrm{Lip}_0(\mathbb{R}) \simeq \ell_\infty$.
- **Kisljakov ('75), Cúth–Doucha–Wojtaszczyk ('16).** If $\dim(\mathcal{X}) \geqslant 2$, there is no onto map from a $\mathcal{C}(\mathcal{K})$ space to $\mathrm{Lip}_0(\mathcal{X})$.
- Is $\mathrm{Lip}_0(\mathbb{R}^k) \simeq \mathrm{Lip}_0(\mathbb{R}^n)$, for $k, n \geqslant 2$ distinct?
- Is there an infinite-dimensional $\mathcal{X}$ with $\mathrm{Lip}_0(\mathcal{X}) \simeq \mathrm{Lip}_0(\mathbb{R}^2)$?
- **Candido–Cúth–Doucha ('19).** If $\mathcal{X}$ is a separable, inf.-dim. $\mathscr{L}_p$-space, then $\mathrm{Lip}_0(\mathcal{X}) \simeq \mathrm{Lip}_0(\ell_p)$ when $p < \infty$ and $\mathrm{Lip}_0(\mathcal{X}) \simeq \mathrm{Lip}_0(c_0)$ when $p = \infty$.
  - **Dutrieux–Ferenczi ('05).** The case $\mathcal{X} = \mathcal{C}(\mathcal{K})$.
- **Candido–Cúth–Doucha ('19).** Is $\mathrm{Lip}_0(L_p) \simeq \mathrm{Lip}_0(L_q)$ for $p \neq q$?

Approximation properties at our service

- A Banach space $\mathcal{X}$ has the **approximation property** (**AP**) if for every compact set $\mathcal{K} \subseteq \mathcal{X}$ and $\varepsilon > 0$ there exists a finite-rank operator $T \colon \mathcal{X} \to \mathcal{X}$ such that $\|Tx - x\| < \varepsilon$ $(x \in \mathcal{K})$.
- $\mathcal{X}$ has the λ-**BAP** if additionally $\|T\| \leqslant \lambda$. **MAP** $\equiv 1$-**BAP**.
- AP and BAP pass to complemented subspaces and from $\mathcal{X}^*$ to $\mathcal{X}$.
- **Johnson ('75).** If $\mathrm{Lip}_0(\mathcal{X})$ has AP/BAP, then $\mathcal{X}$ has it too.
  - **Lindenstrauss ('64)** $\mathcal{X}^*$ is 1-complemented in $\mathrm{Lip}_0(\mathcal{X})$.
- However, the converse fails.
  - There exists $\mathcal{X}$ with MAP such that $\mathcal{X}^*$ fails AP.
- So, just take $\mathcal{X}$ infinite-dimensional such that $\mathrm{Lip}_0(\mathcal{X})$ has AP.
- Well, "just"...
- **Godefroy.** Does $\mathrm{Lip}_0(\ell_2)$ have the AP?
- Actually, does $\mathrm{Lip}_0(\mathbb{R}^n)$ have the AP/BAP, for $n \geqslant 2$?
- **Godefroy–Kalton ('03).** $\mathcal{F}(\mathcal{X})$ has λ-BAP if and only if $\mathcal{X}$ has it.

What is a polynomial, and why?

Problem (Godefroy)

Does $\mathrm{Lip}_0(\ell_2)$ have the AP?

- Let's try to disprove it, judiciously.
- Candidates complemented subspaces of $\mathrm{Lip}_0(\ell_2)$ that fail the AP?
  - **Szankowski ('81).** $\mathcal{L}(\ell_2)$ fails the AP.
  - **Floret ('97), Dineed–Mujica ('15).** $\mathcal{P}(^2\ell_2)$ fails the AP.
    - In fact, $\mathcal{P}(^2\ell_2) \simeq \mathcal{B}(\ell_2)$.
- $\mathcal{P}(^2\mathcal{X})$ is the space of bounded 2-homogeneous polynomials on $\mathcal{X}$.
  - $P \in \mathcal{P}(^2\mathcal{X})$ if there is a bounded bilinear map $M \colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that $P(x) = M(x,x)$;
  - There is a unique symmetric such bilinear map;
  - $\|P\|_{\mathcal{P}} = \sup_{x \in B_{\mathcal{X}}} |P(x)|$.
- But polynomials are **not** Lipschitz functions!

Polynomial vs. Lipschitz

- However, they are Lipschitz on the unit ball.
  - Therefore, $\mathcal{P}(^2\mathcal{X})$ is a natural subspace of $\mathrm{Lip}_0(B_{\mathcal{X}})$.
  - Moreover, $\|\cdot\|_{\mathcal{P}}$ is equivalent to $\|\cdot\|_{\mathrm{Lip}}$.
- Consequently, $\mathcal{P}(^2\mathcal{X})$ is naturally isomorphic to a subspace of $\mathrm{Lip}_0(B_{\mathcal{X}})$, via the restriction map

$$P \mapsto P\restriction_{B_{\mathcal{X}}}.$$

(Another) Problem

Is $\mathcal{P}(^2\ell_2) \subseteq \mathrm{Lip}_0(B_{\ell_2})$ a complemented subspace?

- **Kaufmann ('15).** $\mathrm{Lip}_0(\mathcal{X}) \simeq \mathrm{Lip}_0(B_{\mathcal{X}})$.

Summary:

If $\mathcal{P}(^2\ell_2) \subseteq \mathrm{Lip}_0(B_{\ell_2})$ is complemented, then $\mathrm{Lip}_0(\ell_2)$ fails to have the AP.

Polynomial vs. Linear

Problem (*Repetita iuvant*)

Is $\mathcal{P}(^2\ell_2) \subseteq Lip_0(B_{\ell_2})$ a complemented subspace?

- **Lindenstrauss ('64).** $\mathcal{X}^* = \mathcal{P}(^1\mathcal{X})$ is 1-complemented in $Lip_0(\mathcal{X})$.
- No wonder, $\mathcal{P}(^d\mathcal{X}) \equiv$ bdd, d-homogeneous poly on $\mathcal{X}$.
- Is there a version of Lindenstrauss' result for polynomials?
- **Pełczyński ('57).** Every d-homogeneous polynomial from ℓ_p to ℓ_q is compact if $dq < p$.
- Polynomial Schur and Dunford-Pettis properties.
- Norm-attaining polynomials.
- Hahn-Banach extensions of polynomials.
- Can you, **please**, **state** something?

So, Yes or No?

Theorem (Hájek, R., '22)

$\mathcal{P}(^2\ell_2) \subseteq \mathrm{Lip}_0(B_{\ell_2})$ is not complemented.

- We still don't know if $\mathrm{Lip}_0(\ell_2)$ has the AP.
  - Maybe, that's an indication that it does.
  - At least, we **know** that this is not the correct approach.

Theorem (Hájek, R., '22)

If $\mathcal{X}$ contains uniformly complemented $(\ell_2^n)_{n=1}^{\infty}$, then $\mathcal{P}(^2\mathcal{X})$ is not complemented in $\mathrm{Lip}_0(B_{\mathcal{X}})$.

- In particular, if $\mathcal{X}$ has non-trivial type...
- Moreover, for every $d \geqslant 2$, $\mathcal{P}(^d\mathcal{X})$ and $\mathcal{P}_0^d(\mathcal{X})$ are not complemented in $\mathrm{Lip}_0(B_{\mathcal{X}})$.

- $\mathcal{P}^d(\mathcal{X})$ is the space of polynomials of degree at most d on $\mathcal{X}$.

Forbidden area

Enter at your own risk

- **Problem.** Is $\mathcal{P}(^2 c_0)$ is complemented in $\mathrm{Lip}_0(B_{c_0})$?
  - **Alencar ('75).** $\mathcal{P}(^2 c_0)$ has a Schauder basis (so BAP).
- **Aron–Schottenloher ('76).** $\mathcal{P}(^d \ell_1) \simeq \ell_\infty$.
  - $\mathcal{L}(^d \ell_1) \simeq \ell_\infty$ (a computation).
  - $\mathcal{P}(^d \ell_1)$ is complemented in $\mathcal{L}(^d \ell_1)$ (Polarisation formula).
  - **Lindenstrauss ('67).** ℓ_∞ is prime.
- **Arias–Farmer ('96).** If $\mathcal{X}$ is a separable $\mathscr{L}_1$-space, $\mathcal{P}(^d \mathcal{X}) \simeq \ell_\infty$.

(Main) Proposition

Let E be $\mathbb{R}^n$ with Euclidean norm. If Q is any projection from $\mathrm{Lip}_0(B_E)$ onto $\mathcal{P}(^2 E)$, then

$$\|Q\| \geqslant c \cdot n^{1/5}.$$

Thank you for your attention!