49^{th} Winter School in Abstract Analysis, Sněžné, January 8–15, 2022

Smooth and polyhedral norms via fundamental biorthogonal systems

Tommaso Russo
(Joint with Sheldon Dantas and Petr Hájek)

Institute of Mathematics, Czech Academy of Sciences
Faculty of Electrical Engineering, Czech Technical University

Smoothness and structure

The existence of a smooth norm on a Banach space $\mathcal{X}$ bears several geometric consequences. Just to name a few:

- If a separable Banach space $\mathcal{X}$ has a C^1-smooth norm, then $\mathcal{X}$ is Asplund (*i.e.*, $\mathcal{X}^*$ is separable);
- **Meshkov (1978).** If $\mathcal{X}$ and $\mathcal{X}^*$ admit a C^2-smooth norm, then $\mathcal{X}$ is isomorphic to a Hilbert space;
- **Fabian, Whitfield, Zizler (1983).** If $\mathcal{X}$ admits a C^2-smooth norm, either it contains c_0, or it is super-reflexive with type 2;
- **Deville (1989).** If $\mathcal{X}$ has a C^∞-smooth norm, either it contains c_0, or it is super-reflexive, with exact cotype $2k$, and it contains ℓ_{2k};
- **Pechanec, Whitfield, Zizler (1981) – Fabian, Zizler (1997).** If $\mathcal{X}$ has a LFC norm, then it is c_0-saturated and Asplund.

All these results require $\mathcal{X}$ to be complete (typically, via variational principles).

Normed spaces

It was asked several times if the existence of a smooth norm on some 'large' subset of a Banach space $\mathcal{X}$ has similar consequences for $\mathcal{X}$:

- **Benyamini, Lindenstrauss**, *Geometric Nonlinear Functional Analysis*
    - Must $\mathcal{X}$ be Asplund?
    - Is there a norm on ℓ_1 that is differentiable outside a countable union of closed hyperplanes?
- **Guirao, Montesinos, Zizler**, *Open problems...*, Problem 149: Does the space of finitely supported vectors in $\ell_1(\Gamma)$ have a C^1-smooth norm (when Γ is uncountable)?
- **Dantas, Hájek, R. (JMAA'20).** Given a Banach space $\mathcal{X}$, is there a dense subspace of $\mathcal{X}$ that admits a C^k-smooth norm?
    - **Hájek (1995).** Yes, if $\mathcal{X}$ is separable.
    - **Deville, Fonf, Hájek (1998).** Even an analytic norm.

Normed spaces

Let $\mathcal{X}$ be a normed space with a countable algebraic basis.

- **Hájek (1995).** $\mathcal{X}$ has a C^∞-smooth norm (such norms are dense);
- **Deville, Fonf, Hájek (1998).** Are analytic norms dense?

Theorem (Dantas, Hájek, R., JMAA'20)

(i) *ℓ_∞ admits a dense subspace with an analytic norm;*
(ii) *Every normed space with a countable algebraic basis admits an analytic norm (and analytic norms are dense);*
(iii) *The space of finitely supported vectors in $\ell_1(\mathfrak{c})$ admits an analytic norm.*

Theorem (Dantas, Hájek, R., JMAA'20)

Let $\mathcal{X}$ be a Banach space with long unconditional Schauder basis and let $\mathcal{Y}$ be the linear span of such basis. Then, $\mathcal{Y}$ admits a C^∞-smooth norm.

Fundamental biorthogonal systems

Theorem (Dantas, Hájek, R., *in preparation*)

Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$. Consider $\mathcal{Y} := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$. Then:

(i) *$\mathcal{Y}$ admits a polyhedral and LFC norm,*

(ii) *$\mathcal{Y}$ admits a C^∞-smooth and LFC norm,*

(iii) *$\mathcal{Y}$ admits a C^∞-smooth and LFC bump,*

(iv) *$\mathcal{Y}$ admits locally finite, σ-uniformly discrete C^∞-smooth and LFC partitions of unity,*

(v) *$\mathcal{Y}$ admits a C^1-smooth LUR norm.*

Moreover, norms as in (i), (ii), and (v) are dense.

The norm $\|\cdot\|$ is *LFC* on $\mathcal{X}$ if for each $x \in \mathcal{S}_\mathcal{X}$ there exist an open nhood $\mathcal{U}$ of x and functionals $\varphi_1, \dots, \varphi_k \in \mathcal{X}^*$ such that, for $y, z \in \mathcal{U}$,

$$\langle \varphi_j, y\rangle = \langle \varphi_j, z\rangle \ (j = 1, \dots, k) \implies \|y\| = \|z\|.$$

Can we aim for more?

- **Dantas, Hájek, R. (JMAA'20).** No dense subspace of $c_0(\omega_1)$ admits an analytic norm.
- **Fabian, Whitfield, Zizler (1983).** Let $\mathcal{Y}$ be a normed space with a $C^{1,+}_{\text{loc}}$-smooth (*e.g.*, C^2-smooth) LUR norm $\|\cdot\|$. Then the completion of $\mathcal{Y}$ is super-reflexive.
- Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system. Then densely many norms on $\mathcal{X}$ are differentiable on a dense G_δ (actually, open) set.
  Compare:
  - $\mathcal{X}$ is Asplund iff every norm on $\mathcal{X}$ is differentiable on a dense G_δ.
  - **Moreno (1997).** Every Banach space admits a norm that is differentiable on a dense open set.
- What about dense subspaces that are not the span of a fundamental biorthogonal system?
  - **Hájek, R., JFA'20.** Different dense subspaces of a Banach space can be extremely different.

How general is the result?

$\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma} \subseteq \mathcal{X}\times\mathcal{X}^*$ is a *fundamental biorthogonal system* for $\mathcal{X}$ if

- $\langle \varphi_\beta, e_\alpha\rangle = \delta_{\alpha,\beta}$,
- $\operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$.

Which Banach spaces admit a fundamental biorthogonal system?

- Plichko spaces (*e.g.*, WCG, reflexive, $c_0(\Gamma)$, $L_1(\mu)$ for a finite measure, $C(\mathcal{K})$ for $\mathcal{K}$ Valdivia),
- **Kalenda (2020).** Every space with projectional skeleton (duals of Asplund spaces, preduals of Von Neumann algebras, preduals of JBW*-triples),
- $\ell_\infty(\Gamma)$, $\ell_\infty^c(\Lambda)$ when $|\Lambda| \leqslant \mathfrak{c}$,
- $C(\mathcal{T})$, when $\mathcal{T}$ is a tree,
- **Todorčević (2006).** All Banach spaces of density ω_1, under MM,
- **Davis, Johnson (1973).** $\mathcal{X}$ with $\operatorname{dens}\mathcal{X} = \kappa$ that has a WCG quotient of density κ.

References

- **S. Dantas, P. Hájek, and T. Russo, Smooth norms in dense subspaces of Banach spaces, *J. Math. Anal. Appl.* 487 (2020), 123963.**
- **P. Hájek and T. Russo, On densely isomorphic normed spaces, *J. Funct. Anal.* 279 (2020), 108667.**
- **S. Dantas, P. Hájek, and T. Russo, Smooth and polyhedral norms via fundamental biorthogonal systems, *to appear (on* `arXiv` *before the end of the WS)*.**

Thank you for your attention!