

Institute of Mathematics
Czech Academy of Sciences

An Asplund space with norming M-basis that is not WCG

Tommaso Russo
tommaso.russo.math@gmail.com

P. Hájek, T. Russo, J. Somaglia, and S. Todorčević,
*An Asplund space with norming Markuševič basis
that is not weakly compactly generated,*
Adv. Math. **392**, 108041 (2021).

TOPOSYM 2022
Prague, Czech Republic
July 25–29, 2022



Let \mathcal{X} be a Banach space. A system $\{u_\alpha; \varphi_\alpha\}_{\alpha \in \Gamma} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a **Markuševič basis (M-basis, for short)** for \mathcal{X} if

- ▶ $\langle \varphi_\beta, u_\alpha \rangle = \delta_{\alpha, \beta}$,
- ▶ $\text{span}\{u_\alpha\}_{\alpha \in \Gamma}$ is dense in \mathcal{X} ,
- ▶ $\text{span}\{\varphi_\alpha\}_{\alpha \in \Gamma}$ is w^* -dense in \mathcal{X}^* .

$\{\langle \varphi_\alpha, x \rangle : \alpha \in \Gamma\}$ are the **coordinates** of $x \in \mathcal{X}$
 $\{\langle \psi, \varphi_\alpha \rangle : \alpha \in \Gamma\}$ are the **coordinates** of $\psi \in \mathcal{X}^*$.

- ▶ **Markuševič, 1943.** Every separable Banach space has an M-basis.
- ▶ **Amir–Lindenstrauss, 1968.** Every WCG Banach space has an M-basis;
Def: \mathcal{X} is **WCG** if it contains a linearly dense weakly compact subset.
- ▶ **Johnson, 1970.** ℓ_∞ has no M-basis.



- ▶ Several classes of Banach spaces can be characterised by the existence of M-bases with additional properties.
- ▶ So it is tempting to ask if $\text{span}\{\varphi_\alpha\}_{\alpha \in \Gamma}$ exhausts \mathcal{X}^* in a stronger sense.
- ▶ $\{u_\alpha; \varphi_\alpha\}_{\alpha \in \Gamma}$ is **shrinking** if $\text{span}\{\varphi_\alpha\}_{\alpha \in \Gamma}$ is dense in \mathcal{X}^* .
- ▶ An M-basis $\{u_\alpha; \varphi_\alpha\}_{\alpha \in \Gamma}$ is **λ -norming** ($0 < \lambda \leq 1$) if

$$\lambda \|x\| \leq \sup\{|\langle \varphi, x \rangle| : \varphi \in \text{span}\{\varphi_\alpha\}_{\alpha \in \Gamma}, \|\varphi\| \leq 1\}.$$

- ▶ Separable Banach spaces have a 1-norming M-basis (Markuševič).
- ▶ Every reflexive Banach space has a shrinking M-basis.
- ▶ **Alexandrov–Plichko, 2006.** $\mathcal{C}([0, \omega_1])$ has no norming M-basis.



- ▶ Which class of Banach spaces is characterised by admitting a norming M-basis?
- ▶ **John–Zizler, 1974.** Do WCG spaces have a norming M-basis?

Theorem (Hájek, Advances 2019)

There exists a WCG $\mathcal{C}(\mathcal{K})$ space with no norming M-basis.

Def: \mathcal{X} is **Asplund** if every its separable subspace has separable dual.

- ▶ $\mathcal{C}(\mathcal{K})$ is Asplund iff \mathcal{K} is scattered.
- ▶ **Godefroy, ~1990.** Let \mathcal{X} be an Asplund space with a norming M-basis. Is \mathcal{X} WCG?

Theorem A (Hájek, R., Somaglia, Todorčević, Advances 2021)

There exists an Asplund space \mathcal{X} with a 1-norming M-basis such that \mathcal{X} is not WCG.



Our example is a subspace of an Asplund $\mathcal{C}(\mathcal{K})$ (that is not WCG).

- **Problem.** Is there a $\mathcal{C}(\mathcal{K})$ example?

We now explain how to build \mathcal{K} .



- $\mathcal{P}(\Gamma) \equiv \{0, 1\}^\Gamma$ by $A \leftrightarrow 1_A$;
- This gives a compact 'product' topology on $\mathcal{P}(\Gamma)$.

Theorem B (HRST)

There exists a family $\mathcal{F}_\varrho \subseteq [\omega_1]^{<\omega}$ of finite subsets of ω_1 such that $\mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}$ has the following properties:

- (i) $\{\alpha\} \in \mathcal{K}_\varrho$ for every $\alpha < \omega_1$,
- (ii) $[0, \alpha) \in \mathcal{K}_\varrho$ for every $\alpha \leq \omega_1$,
- (iii) if $A \in \mathcal{K}_\varrho$ is an infinite set, then $A = [0, \alpha)$ for some $\alpha \leq \omega_1$,
- (iv) \mathcal{K}_ϱ is scattered.



-  S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), 261–294.
-  S. Todorčević, *Walks on ordinals and their characteristics*. Birkhäuser Verlag, Basel, 2007.
- ▶ We consider functions $\varrho: [\omega_1]^2 \rightarrow \omega$.
 - ▶ We identify $[\omega_1]^2 = \{(\alpha, \beta) \in \omega_1^2 : \alpha < \beta\}$.
 - ▶ Thus, we write $\varrho(\alpha, \beta)$, with $\alpha < \beta$, for $\varrho(\{\alpha, \beta\})$.
 - ▶ We also add the 'boundary condition' $\varrho(\alpha, \alpha) = 0$.

Definition (Todorčević)

A **ϱ -function** on ω_1 is a function $\varrho: [\omega_1]^2 \rightarrow \omega$ such that:

- ($\varrho 1$) $\{\xi \leq \alpha : \varrho(\xi, \alpha) \leq n\}$ is finite, for every $\alpha < \omega_1$ and $n < \omega$,
- ($\varrho 2$) $\varrho(\alpha, \gamma) \leq \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$,
- ($\varrho 3$) $\varrho(\alpha, \beta) \leq \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$.



Proposition (Todorčević)

There exists a function $\varrho: [\omega_1]^2 \rightarrow \omega$ such that $(\alpha < \beta < \gamma < \omega_1)$:

- ▶ $\varrho(\alpha, \beta) > 0$;
- ▶ $\varrho(\alpha, \gamma) \leq \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$;
- ▶ $\varrho(\alpha, \gamma) \neq \varrho(\beta, \gamma)$.

$$F_n(\alpha) := \{\xi \leq \alpha : \varrho(\xi, \alpha) \leq n\}$$

$$\mathcal{F}_\varrho := \{F_n(\alpha) : n < \omega, \alpha < \omega_1\} \quad \text{and} \quad \mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}.$$

Fact

- ▶ $|F_n(\alpha)| \leq n + 1$;
- ▶ $(F_n(\alpha))_{n < \omega}$ converges to $[0, \alpha]$.

The compact \mathcal{K}_ϱ satisfies Theorem B.



Definition (Kubiś and Leiderman, 2004)

A compact space is **semi-Eberlein** if it is homeomorphic to a compact $\mathcal{K} \subseteq [0, 1]^{\Gamma}$ such that $c_0(\Gamma) \cap \mathcal{K}$ is dense in \mathcal{K} .

Kubiś and Leiderman (2004). No semi-Eberlein compact space has a P-point.


- ▶ Used to find a Corson, not semi-Eberlein space.
- ▶ A point $p \in \mathcal{K}$ is a **P-point** if it is not isolated and for every choice of $(U_j)_{j < \omega}$ nhoods of p , $\cap U_j$ is a nhod of p .

Question (Kubiś and Leiderman, 2004)

Can a semi-Eberlein compact space have weak P-points?

- ▶ A point $p \in \mathcal{K}$ is a **weak P-point** if it is not isolated and no countable set in $\mathcal{K} \setminus \{p\}$ accumulates at p .
- ▶ The compact space \mathcal{K}_q in Theorem B is semi-Eberlein and it has a weak P-point.



 P. Hájek, T. Russo, J. Somaglia, and S. Todorčević,
*An Asplund space with norming Markuševič basis
that is not weakly compactly generated,*
Adv. Math. **392**, 108041 (2021).

Thank you for your attention!

Th B \implies Th A (in 1 slide)

Not even a sketch of a proof



- ▶ We define a biorthogonal system $\{f_\gamma; \mu_\gamma\}_{\gamma < \omega_1}$ in $\mathcal{C}(\mathcal{K}_\varrho)$:

$$f_\gamma \in \mathcal{C}(\mathcal{K}_\varrho) \qquad f_\gamma(A) = \begin{cases} 1 & \gamma \in A \\ 0 & \gamma \notin A \end{cases} \quad (A \in \mathcal{K}_\varrho)$$

$$\mu_\gamma := \delta_{\{\gamma\}} \in \mathcal{M}(\mathcal{K}_\varrho) \qquad \mu_\gamma(S) = \begin{cases} 1 & \{\gamma\} \in S \\ 0 & \{\gamma\} \notin S \end{cases} \quad (S \subseteq \mathcal{K}_\varrho).$$

- ▶ $\langle \mu_\alpha, f_\gamma \rangle = f_\gamma(\{\alpha\}) = \delta_{\alpha, \gamma}$, so it is biorthogonal.
- ▶ The space that we are looking for is

$$\mathcal{X}_\varrho := \overline{\text{span}}\{f_\gamma\}_{\gamma < \omega_1} \subseteq \mathcal{C}(\mathcal{K}_\varrho).$$