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Smoothness in normed spaces

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Workshop on Banach spaces and Banach lattices II
ICMAT (Madrid), 9-13 May 2022

- Renorming theory = Find an equivalent norm on $\mathcal{X}$ with the strongest possible form of a certain property.
- Let $\mathcal{U} \subseteq \mathcal{X}$ be open. $f : \mathcal{U} \to \mathcal{Y}$ is differentiable at $x \in \mathcal{U}$ if there is $f'(x) \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - \langle f'(x), h \rangle}{\|h\|} = 0.$$

- C^k-smoothness, rules of calculus, Implicit Function theorem, ...
- If $p \notin \mathbb{N}$, the ℓ_p norm is $C^{\lfloor p \rfloor}$-smooth, but not $C^{\lfloor p \rfloor + 1}$.
  And ℓ_p has no $C^{\lfloor p \rfloor + 1}$-smooth norm!
- Smooth references:

Deville, Godefroy, Zizler, *Smoothness and Renormings in Banach Spaces*.

Hájek, Johanis, *Smooth analysis in Banach spaces*.

- If a separable Banach space $\mathcal{X}$ has a C^1-smooth norm, then $\mathcal{X}$ is Asplund (*i.e.*, $\mathcal{X}^*$ is separable).
  - No closed, inf-dim subspace of ℓ_1 has a C^1-smooth norm.
- **Meshkov (1978).** If $\mathcal{X}$ and $\mathcal{X}^*$ admit a C^2-smooth norm, then $\mathcal{X}$ is isomorphic to a Hilbert space.
- **Fabian, Whitfield, Zizler (1983).** If $\mathcal{X}$ admits a C^2-smooth norm, either it contains c_0, or it is super-reflexive with type 2.
- **Deville (1989).** If $\mathcal{X}$ has a C^∞-smooth norm, either it contains c_0, or it is super-reflexive, with exact cotype $2k$, and it contains ℓ_{2k}.
  - Can the first case actually happen? The c_0 norm doesn't seem smooth...
- **Pechanec, Whitfield, Zizler (1981) – Fabian, Zizler (1997).** If $\mathcal{X}$ has a LFC norm, then it is c_0-saturated and Asplund.

All these proofs require $\mathcal{X}$ to be complete (variational principles).

Normed spaces

- Let $\mathcal{X}$ be a normed space with a countable algebraic basis.
  - **Vanderwerff (1992).** $\mathcal{X}$ has a C^1-smooth norm.
  - **Hájek (1995).** $\mathcal{X}$ has a C^∞-smooth norm.
  - **Deville, Fonf, Hájek (1998).** $\mathcal{X}$ has an analytic norm.
- **Guirao, Montesinos, Zizler**, *Open problems...*, Problem 149: Does the space of finitely supported vectors in $\ell_1(\Gamma)$ have a C^1-smooth norm (when Γ is uncountable)?
- **Dantas, Hájek, R. (JMAA'20).** Given a Banach space $\mathcal{X}$, is there a dense subspace of $\mathcal{X}$ that admits a C^k-smooth norm?
- **Benyamini, Lindenstrauss**, *Geometric Nonlinear Functional Analysis*
  - Does the existence of a smooth norm on some 'large' subset of a Banach space $\mathcal{X}$ imply that $\mathcal{X}$ is Asplund?
  - Is there a norm on ℓ_1 that is differentiable outside a countable union of closed hyperplanes?

- $c_0(\Gamma)$ has a C^∞-smooth norm.
- Normed spaces of countable dim have a C^∞-smooth norm.

Theorem (Dantas, Hájek, R., JMAA'20)

$\ell_\infty^F := span\{1_A : A \subseteq \mathbb{N}\}$ has a C^∞-smooth norm.

- Take a sequence $\varepsilon_j \searrow 0$ and define $T\colon \ell_\infty^F \to \ell_\infty$ by

$$(x(j))_{j=1}^\infty \mapsto ((1+\varepsilon_j)\cdot x(j))_{j=1}^\infty.$$

- Look at the picture. ■
- Take $\mathcal{X} \subseteq \ell_\infty$ of countable dimension, $\mathcal{X} = \mathrm{span}\{e_j\}_{j=1}^\infty$.
- Take $\{v_j\}_{j=1}^\infty \subseteq \ell_\infty^F$ 'very close' to e_js.
- $\mathcal{X}$ is isomorphic to $\mathrm{span}\{v_j\}_{j=1}^\infty$ (small perturbation lemma).
  - Well, I'm cheating a bit, $\{e_j\}_{j=1}^\infty$ has to be an M-basis.

Fundamental biorthogonal systems

Theorem (Dantas, Hájek, R., JMAA'20)

Let $\mathcal{X}$ be a Banach space with long unconditional basis and let $\mathcal{Y}$ be the linear span of such basis. Then, $\mathcal{Y}$ has a C^∞-smooth norm.

Theorem (Dantas, Hájek, R., `arXiv:2201.03379`)

Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system $\{e_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$. Consider $\mathcal{Y} := \mathrm{span}\{e_\alpha\}_{\alpha\in\Gamma}$. Then:

- (i) *$\mathcal{Y}$ admits a polyhedral and LFC norm.*
- (ii) *$\mathcal{Y}$ admits a C^∞-smooth and LFC norm.*
- (iii) *$\mathcal{Y}$ admits a C^1-smooth LUR norm.*

Moreover, such norms are dense.

The norm $\|\cdot\|$ is *LFC* on $\mathcal{X}$ if for each $x \in \mathcal{S}_\mathcal{X}$ there exist an open nhood $\mathcal{U}$ of x, functionals $\varphi_1, \ldots, \varphi_k \in \mathcal{X}^*$, and $G\colon \mathbb{R}^k \to \mathbb{R}$ such that

$$\|y\| = G(\langle\varphi_1, y\rangle, \ldots, \langle\varphi_k, y\rangle) \qquad \text{for every } y \in \mathcal{U}.$$

Can we aim for more?

- **Dantas, Hájek, R. (JMAA'20).** No dense subspace of $c_0(\omega_1)$ admits an analytic norm.
- **Fabian, Whitfield, Zizler (1983).** Let $\mathcal{Y}$ be a normed space with a $C^{1,+}_{\text{loc}}$-smooth (*e.g.*, C^2-smooth) LUR norm $\|\cdot\|$. Then the completion of $\mathcal{Y}$ is super-reflexive.
- What about dense subspaces that are not the span of a fundamental biorthogonal system?
  - **Hájek, R., JFA'20.** Different dense subspaces of a Banach space can be extremely different.
  - See Slide 8 for more about this.
- **Main problem.** Is there a Banach space $\mathcal{X}$ such that no dense subspace of $\mathcal{X}$ has a C^k-smooth norm?

$\{e_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}\subseteq\mathcal{X}\times\mathcal{X}^*$ is a *fundamental biorthogonal system* for $\mathcal{X}$ if

- $\langle\varphi_\beta,e_\alpha\rangle=\delta_{\alpha,\beta}$,
- $\mathrm{span}\{e_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$.

Which Banach spaces admit a fundamental biorthogonal system?

- Plichko spaces (*e.g.*, WCG, reflexive, $c_0(\Gamma)$, $L_1(\mu)$ for a finite measure, $C(\mathcal{K})$ for $\mathcal{K}$ Valdivia),
- **Kalenda (2020).** Every space with projectional skeleton (duals of Asplund spaces, preduals of Von Neumann algebras, preduals of JBW*-triples),
- $\ell_\infty(\Gamma)$, $\ell_\infty^c(\Lambda)$ when $|\Lambda|\leqslant\mathfrak{c}$,
- $C(\mathcal{T})$, when $\mathcal{T}$ is a tree,
- **Davis, Johnson (1973).** $\mathcal{X}$ with $\mathrm{dens}\,\mathcal{X}=\kappa$ that has a WCG quotient of density κ,
- **Todorčević (2006).** All Banach spaces of density ω_1, under MM.

Other dense subspaces

Theorem (Dantas, Hájek, R., *in preparation*)

Let $1 \leqslant p < \infty$ and $r \in (0,p)$. The dense subspace $\ell_r(\Gamma)$ of $\ell_p(\Gamma)$ has a C^∞-smooth norm.

- ℓ_p has a dense subspace of dimension continuum with a C^∞-smooth norm.
- If $p > 1$ such subspace is an operator range.
- **Rosenthal (1970).** Every non-separable operator range in $\ell_1(\Gamma)$ contains $\ell_1(\omega_1)$.
- Let $\mathcal{X}$ be a separable Banach space. Does it have a dense subspace of dimension continuum with a C^∞-smooth norm?
- Can a dense hyperplane in ℓ_1 have a smooth norm?
- Does the space of simple functions in L_1 have a smooth norm?

- **S. Dantas, P. Hájek, and T. Russo, Smooth norms in dense subspaces of Banach spaces, *J. Math. Anal. Appl.* 487 (2020), 123963.**
- **P. Hájek and T. Russo, On densely isomorphic normed spaces, *J. Funct. Anal.* 279 (2020), 108667.**
- **S. Dantas, P. Hájek, and T. Russo, Smooth and polyhedral norms via fundamental biorthogonal systems, `arXiv:2201.03379`.**
- **S. Dantas, P. Hájek, and T. Russo, Smooth norms in dense subspaces of $\ell_p(\Gamma)$ and operator ranges, *in preparation*.**

Thank you for your attention!