Smooth norms and biorthogonal systems

(j/w Sheldon Dantas and Petr Hájek)

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Thank you, Gilles

- *Septièmes Journées Besançon–Neuchâtel*, Besançon, Jun ’17.

[DGZ] Deville, Godefroy, Zizler, *Smoothness and Renorming in Banach spaces*.

- Conferences: Marseille, Birmingham, Prague, Svratka, Métabief.
- Many online talks.
- Three papers acknowledge Gilles for suggestions.
- Two papers (**Hájek, R. (MJOM’22)** and **Hájek, R., Somaglia, Todorčević (JAIM’21)**) originate from his questions.
- Recommendation letters for my PostDoc applications.

Thanks Gilles for your passion for Mathematics, for sharing your knowledge with young researchers, suggesting us new problems, helping with applications, ...

One-two combo

- **Main problem.** Given a Banach space $\mathcal{X}$, is there a dense subspace of $\mathcal{X}$ that admits a C^k-smooth norm?
- **The main result.** Yes, a C^∞-smooth norm, if $\mathcal{X}$ has a fundamental biorthogonal system.

Who cares? / Motivation / History / ...

- If a separable Banach space $\mathcal{X}$ has a C^1-smooth norm, then $\mathcal{X}$ is Asplund (*i.e.*, $\mathcal{X}^*$ is separable);
- **Meshkov (1978).** If $\mathcal{X}$ and $\mathcal{X}^*$ admit a C^2-smooth norm, then $\mathcal{X}$ is isomorphic to a Hilbert space;
- **Deville (1989).** If $\mathcal{X}$ has a C^∞-smooth norm, either it contains c_0, or it is super-reflexive, with type 2, exact cotype $2k$, and it contains ℓ_{2k}.

All these results require $\mathcal{X}$ to be complete (variational principles).

Normed spaces

- Let $\mathcal{X}$ be a normed space with a countable algebraic basis.
  - **Vanderwerff (1992).** $\mathcal{X}$ has a C^1-smooth norm.
  - **Hájek (1995).** $\mathcal{X}$ has a C^∞-smooth norm.
  - **Dantas, Hájek, R. (JMAA'20).** $\mathcal{X}$ has an analytic norm.
- **Guirao, Montesinos, Zizler**, *Open problems...*, Problem 149: Does the space of finitely supported vectors in $\ell_1(\Gamma)$ have a C^1-smooth norm (when Γ is uncountable)?
- **Dantas, Hájek, R. (JMAA'20).** Given a Banach space $\mathcal{X}$, is there a dense subspace of $\mathcal{X}$ that admits a C^k-smooth norm?
- **Benyamini, Lindenstrauss**, *Geometric Nonlinear Functional Analysis*
  - Does the existence of a smooth norm on some 'large' subset of a Banach space $\mathcal{X}$ imply that $\mathcal{X}$ is Asplund?
  - Is there a norm on ℓ_1 that is differentiable outside a countable union of closed hyperplanes?

A couple of previous results

Theorem (Dantas, Hájek, R., JMAA'20)

$\ell_\infty^F := \mathrm{span}\{1_A \colon A \subseteq \mathbb{N}\}$ *has an analytic norm.*
Consequences:

- ℓ_∞ *admits a dense subspace with an analytic norm;*
- *Every normed space with a countable algebraic basis admits an analytic norm (and analytic norms are dense);*
- *The space of finitely supported vectors in* $\ell_1(\mathfrak{c})$ *admits an analytic norm.*

Theorem (Dantas, Hájek, R., JMAA'20)

Let $\mathcal{X}$ *be a Banach space with long unconditional Schauder basis and let* $\mathcal{Y}$ *be the linear span of such basis. Then,* $\mathcal{Y}$ *admits a* C^∞*-smooth norm.*

The main result, now for real

Theorem (Dantas, Hájek, R., `arXiv:2201.03379`)

Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$. Consider $\mathcal{Y} \coloneqq \mathrm{span}\{e_\alpha\}_{\alpha\in\Gamma}$. Then:

(i) *$\mathcal{Y}$ admits a polyhedral and LFC norm,*

(ii) *$\mathcal{Y}$ admits a C^∞-smooth and LFC norm,*

(iii) *$\mathcal{Y}$ admits a C^∞-smooth and LFC bump,*

(iv) *$\mathcal{Y}$ admits locally finite, σ-uniformly discrete C^∞-smooth and LFC partitions of unity,*

(v) *$\mathcal{Y}$ admits a C^1-smooth LUR norm.*

Moreover, norms as in (i), (ii), and (v) are dense.

The norm $\|\cdot\|$ is *LFC* on $\mathcal{X}$ if for each $x \in \mathcal{S}_\mathcal{X}$ there exist an open nhood $\mathcal{U}$ of x, functionals $\varphi_1, \dots, \varphi_k \in \mathcal{X}^*$, and $G\colon \mathbb{R}^k \to \mathbb{R}$ such that

$$\|y\| = G\big(\langle\varphi_1, y\rangle, \dots, \langle\varphi_k, y\rangle\big) \qquad \text{for every } y \in \mathcal{U}.$$

Can we aim for more?

- **Pełczyński (1957) – Dantas, Hájek, R. (JMAA'20).** No dense subspace of $c_0(\omega_1)$ admits an analytic norm.
- **Fabian, Whitfield, Zizler (1983).** Let $\mathcal{Y}$ be a normed space with a $C^{1,+}_{\rm loc}$-smooth (*e.g.*, C^2-smooth) LUR norm $\|\cdot\|$. Then the completion of $\mathcal{Y}$ is super-reflexive.
- What about dense subspaces that are not the span of a fundamental biorthogonal system?
  - **Hájek, R., JFA'20.** Distinct dense subspaces of a Banach space can be extremely different.
- **Main problem.** Is there a Banach space $\mathcal{X}$ such that no dense subspace of $\mathcal{X}$ has a C^k-smooth norm?

How general is the result?

$\{e_\alpha; \varphi_\alpha\}_{\alpha \in \Gamma} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a *fundamental biorthogonal system* for $\mathcal{X}$ if

- $\langle \varphi_\beta, e_\alpha \rangle = \delta_{\alpha,\beta}$,
- $\operatorname{span}\{e_\alpha\}_{\alpha \in \Gamma}$ is dense in $\mathcal{X}$.

Which Banach spaces admit a fundamental biorthogonal system?

- Plichko spaces (*e.g.*, WCG, reflexive, $c_0(\Gamma)$, $L_1(\mu)$ for any measure, $\mathcal{C}(\mathcal{K})$ for $\mathcal{K}$ Valdivia or an abelian group),
- **Kalenda (2020).** Every space with projectional skeleton (duals of Asplund spaces, preduals of Von Neumann algebras, preduals of JBW*-triples),
- $\ell_\infty(\Gamma)$, $\ell_\infty^c(\Lambda)$ when $|\Lambda| \leqslant \mathfrak{c}$,
- $\mathcal{C}(\mathcal{T})$, when $\mathcal{T}$ is a tree,
- **Todorčević (2006).** All Banach spaces of density ω_1, under MM,
- **Davis, Johnson (1973).** $\mathcal{X}$ with $\operatorname{dens} \mathcal{X} = \kappa$ that has a WCG quotient of density κ.

References

- S. Dantas, P. Hájek, and T. Russo, Smooth norms in dense subspaces of Banach spaces, *J. Math. Anal. Appl.* **487** (2020), 123963.
- P. Hájek and T. Russo, On densely isomorphic normed spaces, *J. Funct. Anal.* **279** (2020), 108667.
- S. Dantas, P. Hájek, and T. Russo, Smooth and polyhedral norms via fundamental biorthogonal systems, `arXiv:2201.03379`.
- S. Dantas, P. Hájek, and T. Russo, Smooth norms in dense subspaces of $\ell_p(\Gamma)$ and operator ranges, `arXiv:2205.11282`.

The end

And in the end
The love you take
Is equal to the love you make
(The End, Lennon – McCartney, 1969)

4. Just for fun: Gilles Godefroy is really good, but the shortening [God] for him looks to be an exaggeration ☺

(From Vladimir Kadets' report on Rihhard Nadel's PhD Thesis, August 2020)

Thank you for your attention!