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Walks on ordinals, Asplund spaces, and norming Markuševič bases

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An Asplund space with norming Markuševič basis that is not weakly compactly generated,
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Markuševič bases

Let $\mathcal{X}$ be a Banach space. A system $\{u_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a **Markuševič basis** (**M-basis**, for short) for $\mathcal{X}$ if

- $\langle \varphi_\beta, u_\alpha \rangle = \delta_{\alpha,\beta}$,
- $\mathrm{span}\{u_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$,
- $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is w^*-dense in $\mathcal{X}^*$.

$\{\langle \varphi_\alpha, x\rangle \colon \alpha \in \Gamma\}$ are the **coordinates** of $x \in \mathcal{X}$

$\{\langle \psi, x_\alpha\rangle \colon \alpha \in \Gamma\}$ are the **coordinates** of $\psi \in \mathcal{X}^*$.

- **Markuševič, 1943.** Every separable Banach space has an M-basis.
- **Amir–Lindenstrauss, 1968.** Every WCG Banach space has an M-basis;

  Def : $\mathcal{X}$ is **WCG** if it contains a linearly dense weakly compact subset.
- **Johnson, 1970.** ℓ_∞ has no M-basis.

Norming M-bases

- Several classes of Banach spaces can be characterised by the existence of M-bases with additional properties.
- So it is tempting to ask if $\operatorname{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ exhausts $\mathcal{X}^*$ in a stronger sense.
- $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is **shrinking** if $\operatorname{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}^*$.
- An M-basis $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is λ-**norming** $(0<\lambda\leqslant 1)$ if

$$\lambda\|x\|\leqslant \sup\{|\langle\varphi,x\rangle|\colon \varphi\in\operatorname{span}\{\varphi_\alpha\}_{\alpha\in\Gamma},\ \|\varphi\|\leqslant 1\}.$$

- Separable Banach spaces have a 1-norming M-basis (Markuševič).
- Every reflexive Banach space has a shrinking M-basis.
- **Alexandrov–Plichko, 2006.** $\mathcal{C}([0,\omega_1])$ has no norming M-basis.

Norming M-bases and WCG spaces

- Which class of Banach spaces is characterised by admitting a norming M-basis?
- **John–Zizler, 1974.** Do WCG spaces have a norming M-basis?

Theorem (Hájek, Advances 2019)

There exists a WCG $\mathcal{C}(\mathcal{K})$ space with no norming M-basis.

Def : $\mathcal{X}$ is **Asplund** if every its separable subspace has separable dual.

- $\mathcal{C}(\mathcal{K})$ is Asplund iff $\mathcal{K}$ is scattered.

- **Godefroy, $\sim$1990.** Let $\mathcal{X}$ be an Asplund space with a norming M-basis. Is $\mathcal{X}$ WCG?

Theorem A (Hájek, R., Somaglia, Todorčević, Advances 2021)

There exists an Asplund space $\mathcal{X}$ with a 1-norming M-basis such that $\mathcal{X}$ is not WCG.

The core of the construction

Our example is a subspace of an Asplund $\mathcal{C}(\mathcal{K})$ (that is not WCG).

- **Problem.** Is there a $\mathcal{C}(\mathcal{K})$ example?

We now explain how to build $\mathcal{K}$.

- $\mathcal{P}(\Gamma) \equiv \{0,1\}^\Gamma$ by $A \leftrightarrow 1_A$;
- This gives a compact 'product' topology on $\mathcal{P}(\Gamma)$.

Theorem B (HRST)

There exists a family $\mathcal{F}_\varrho \subseteq [\omega_1]^{<\omega}$ of finite subsets of ω_1 such that $\mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}$ has the following properties:

(i) $\{\alpha\} \in \mathcal{K}_\varrho$ for every $\alpha < \omega_1$,
(ii) $[0,\alpha) \in \mathcal{K}_\varrho$ for every $\alpha \leqslant \omega_1$,
(iii) if $A \in \mathcal{K}_\varrho$ is an infinite set, then $A = [0,\alpha)$ for some $\alpha \leqslant \omega_1$,
(iv) $\mathcal{K}_\varrho$ is scattered.

Todorčević's ϱ-functions

S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), 261–294.

S. Todorčević, *Walks on ordinals and their characteristics*. Birkhäuser Verlag, Basel, 2007.

- We consider functions $\varrho\colon [\omega_1]^2 \to \omega$.
- We identify $[\omega_1]^2 = \{(\alpha, \beta) \in {\omega_1}^2 : \alpha < \beta\}$.
  - Thus, we write $\varrho(\alpha, \beta)$, with $\alpha < \beta$, for $\varrho(\{\alpha, \beta\})$.
- We also add the 'boundary condition' $\varrho(\alpha, \alpha) = 0$.

Definition (Todorčević)

A ϱ-**function** on ω_1 is a function $\varrho\colon [\omega_1]^2 \to \omega$ such that:

(ϱ1) $\{\xi \leqslant \alpha \colon \varrho(\xi, \alpha) \leqslant n\}$ is finite, for every $\alpha < \omega_1$ and $n < \omega$,

(ϱ2) $\varrho(\alpha, \gamma) \leqslant \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$,

(ϱ3) $\varrho(\alpha, \beta) \leqslant \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$.

Definition of the compact $\mathcal{K}_\varrho$

Proposition (Todorčević)

There exists a function $\varrho\colon [\omega_1]^2 \to \omega$ such that ($\alpha < \beta < \gamma < \omega_1$):

- $\varrho(\alpha, \beta) > 0$;
- $\varrho(\alpha, \gamma) \leqslant \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$;
- $\varrho(\alpha, \gamma) \neq \varrho(\beta, \gamma)$.

$$F_n(\alpha) := \{\xi \leqslant \alpha \colon \varrho(\xi, \alpha) \leqslant n\}$$

$$\mathcal{F}_\varrho := \{F_n(\alpha) \colon n < \omega,\ \alpha < \omega_1\} \qquad \text{and} \qquad \mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}.$$

Fact

- $|F_n(\alpha)| \leqslant n + 1$;
- $(F_n(\alpha))_{n<\omega}$ converges to $[0, \alpha]$.

The compact $\mathcal{K}_\varrho$ satisfies Theorem B.

Semi-Eberlein compacta

Definition (Kubiś and Leiderman, 2004)

A compact space is **semi-Eberlein** if it is homeomorphic to a compact $\mathcal{K} \subseteq [0,1]^\Gamma$ such that $c_0(\Gamma) \cap \mathcal{K}$ is dense in $\mathcal{K}$.

Kubiś and Leiderman (2004). No semi-Eberlein compact space has a P-point.

- Used to find a Corson, not semi-Eberlein space.
- A point $p \in \mathcal{K}$ is a **P-point** if it is not isolated and for every choice of $(U_j)_{j<\omega}$ nhoods of p, $\cap U_j$ is a nhood of p.

Question (Kubiś and Leiderman, 2004)

Can a semi-Eberlein compact space have weak P-points?

- A point $p \in \mathcal{K}$ is a **weak P-point** if it is not isolated and no countable set in $\mathcal{K} \setminus \{p\}$ accumulates at p.
- The compact space $\mathcal{K}_\varrho$ in Theorem B is semi-Eberlein and it has a weak P-point.

The end

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Thank you for your attention!

Th B $\implies$ Th A (in 1 slide)

Not even a sketch of a proof

- We define a biorthogonal system $\{f_\gamma; \mu_\gamma\}_{\gamma<\omega_1}$ in $\mathcal{C}(\mathcal{K}_\varrho)$:

$$f_\gamma \in \mathcal{C}(\mathcal{K}_\varrho) \qquad f_\gamma(A) = \begin{cases} 1 & \gamma \in A \\ 0 & \gamma \notin A \end{cases} \qquad (A \in \mathcal{K}_\varrho)$$

$$\mu_\gamma := \delta_{\{\gamma\}} \in \mathcal{M}(\mathcal{K}_\varrho) \qquad \mu_\gamma(S) = \begin{cases} 1 & \{\gamma\} \in S \\ 0 & \{\gamma\} \notin S \end{cases} \qquad (S \subseteq \mathcal{K}_\varrho).$$

- $\langle \mu_\alpha, f_\gamma \rangle = f_\gamma(\{\alpha\}) = \delta_{\alpha,\gamma}$, so *it is* biorthogonal.
- The space that we are looking for is

$$\mathcal{X}_\varrho := \overline{\operatorname{span}}\{f_\gamma\}_{\gamma<\omega_1} \subseteq \mathcal{C}(\mathcal{K}_\varrho).$$