Linear subspaces of ℓ_∞ with fixed number of accumulation points

j/w Paolo Leonetti and Jacopo Somaglia
(LRS $\equiv$ Leonetti, R., and Somaglia)

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Monsters, Inc.

- **Bolzano (1834) — Weierstraß (1872).** There is a continuous nowhere differentiable function $f : \mathbb{R} \to \mathbb{R}$.
  - For $a \in (0, 1)$ and b an odd integer with $ab > 1 + 3\pi/2$,

$$f(x) := \sum_{k=0}^{\infty} a^k \cos(b^k \pi x).$$

- **Banach (1931).** The set of continuous nowhere differentiable functions is residual in $C([0, 1])$.
- **Gurariy (1966).** There is an infinite-dim linear subspace Y of $C([0, 1])$ such that every $f \in Y$, $f \neq 0$ is nowhere differentiable.
  - **Fonf–Gurariy–Kadets (1999).** Such Y can be a closed subspace.
  - **Bayart–Quarta (2007).** Y can be dense in $C([0, 1])$.
- **Gurariy (1966).** If Y is a closed subspace of $C([0, 1])$ and every element of Y is differentiable, then Y is finite-dimensional.

Zeros of polynomials

- **Plichko–Zagorodnyuk (1998).** If X is an infinite-dimensional complex Banach space and $P\colon X \to \mathbb{C}$ is a polynomial with $P(0) = 0$, $P^{-1}(0)$ contains an infinite-dimensional vector space.
  - **Avilés–Todorčević (2007).** There is a polynomial $P\colon \ell_1(\omega_1) \to \mathbb{C}$ with $P(0) = 0$ such that every subspace $Y \subseteq P^{-1}(0)$ is separable.
  - **Aron–Hájek (2006).** If X is a real, separable Banach space, there is an odd polynomial $P\colon X \to \mathbb{R}$ such that $P^{-1}(0)$ contains no infinite-dimensional vector space.

A subset M of a normed space X is

- **lineable**, if $M \cup \{0\}$ contains an infinite-dim linear subspace.
- **spaceable**, if $M \cup \{0\}$ contains an infinite-dim closed subspace.
- **densely lineable in** X, if $M \cup \{0\}$ contains a linear subspace dense in X.

Aron–Bernal–Pellegrino–Seoane (2016), *Lineability*.

The complement of a subspace

Let Y be a linear subspace of X. Is $X \setminus Y$ lineable, spaceable, or densely lineable?

- $X \setminus Y$ is lineable iff X/Y is infinite-dim.
- **Bernal–Ordóñez (2014).** For separable X, iff $X \setminus Y$ is densely lineable in X.
- **Papathanasiou (2022).** $\ell_\infty \setminus c_0$ is densely lineable in ℓ_∞.

Lemma (Bernal–Ordóñez, 2014 — LRS)

Let X be a normed space with $\kappa = \mathrm{dens}(X)$ and Y be a linear subspace. Then the following are equivalent:

(i) $X \setminus Y$ is densely lineable in X,
(ii) $X \setminus Y$ is κ-lineable,
(iii) $\kappa \leqslant \dim(X/Y)$.

Proof of (iii) $\Rightarrow$ (i)

- Let $\{B_\alpha\}_{\alpha\in\kappa}$, $B_\alpha \neq \emptyset$ be a basis for the topology of X.
- By transfinite induction take vectors $\{x_\alpha\}_{\alpha\in\kappa}$ such that

$$x_\alpha \in B_\alpha \setminus \operatorname{span}(Y \cup \{x_\gamma\}_{\gamma\in\alpha}) \text{ for all } \alpha < \kappa.$$

- By (iii), $\operatorname{span}(Y \cup \{x_\gamma\}_{\gamma\in\alpha}) \neq X$ for all $\alpha < \kappa$.
- $V \coloneqq \operatorname{span}\{x_\alpha\}_{\alpha\in\kappa}$ is dense in X and $V \cap Y = \{0\}$. ∎

Notation. For $x \in \ell_\infty$ and a cardinal number κ,

$$\mathrm{L}_x \coloneqq \{\eta \in \mathbb{R} : \eta \text{ is an accumulation point of } x\}$$
$$\mathrm{L}(\kappa) \coloneqq \{x \in \ell_\infty : |\mathrm{L}_x| = \kappa\}.$$

We will study (dense) lineability and spaceability of sets $\mathrm{L}(\kappa)$ in ℓ_∞.

- **Papathanasiou (2022).** $\mathrm{L}(\mathfrak{c})$ is densely lineable in ℓ_∞.
- **LRS.** $\mathrm{L}(\omega)$ and $\bigcup_{2\leqslant n<\omega} \mathrm{L}(n)$ are densely lineable in ℓ_∞.
- Notice that $\mathrm{L}(\kappa) = \emptyset$ for uncountable $\kappa < \mathfrak{c}$.

Spaceability

Wilansky (1975). If Y is a closed subspace of X, $X \setminus Y$ is spaceable iff X/Y is infinite-dim.

- Let (y_n) be a basic sequence in Y.
- Let $q\colon X \to X/Y$ be the quotient map and take (x_n) in X such that $(q(x_n))$ is a basic sequence in X/Y.
- For $\varepsilon_n > 0$ small, $(y_n + \varepsilon_n x_n)$ is a basic sequence equiv to (y_n).
- Take $V \coloneqq \overline{\mathrm{span}}\{y_n + \varepsilon_n x_n\}$. Then $V \cap Y = \{0\}$ (easy). ■

However, we can't apply this to study spaceability of $\mathrm{L}(\mathfrak{c})$, $\mathrm{L}(\omega)$, and $\bigcup_{2 \leqslant n < \omega} \mathrm{L}(n)$.

Theorem (LRS)

$\mathrm{L}(\mathfrak{c})$ and $\mathrm{L}(\omega)$ are spaceable.
More precisely, $\mathrm{L}(\omega) \cup \{0\}$ contains c_0 isometrically and $\mathrm{L}(\mathfrak{c}) \cup \{0\}$ contains ℓ_∞ isometrically.

Finitely many accumulation points

Theorem (LRS)

$\bigcup_{2 \leqslant n < \omega} \mathrm{L}(n)$ is not spaceable.

Among others, in the proof we use the following:

- Let $n \geqslant 2$. Then $\mathrm{L}(n) \cup \cdots \cup \mathrm{L}(n+d)$ is $(d+1)$-lineable, but not $(d+2)$-lineable.
- More generally, we could study the lineability of $\bigcup_{n \in A} \mathrm{L}(n)$, where $A \subseteq \omega$ with $\min A \geqslant 2$.
- If A is an interval, we have a complete result, but for 'sparse' A?

Theorem (LRS)

- $\bigcup_{2 \leqslant n < \omega} \mathrm{L}(n!)$ and $\bigcup_{1 \leqslant n < \omega} \mathrm{L}(3^n)$ are not 2-lineable.
- $\bigcup_{1 \leqslant n < \omega} \mathrm{L}(2n+1)$ is $\mathfrak{c}$-lineable.

Further results?

- Similar results hold if we replace accumulation points with $\mathcal{I}$-cluster points.
- We also study the same problem in $\mathbb{R}^\omega$ (with pointwise topology).
  - $\mathrm{L}(\omega)$ is not spaceable in $\mathbb{R}^\omega$.
- We collect several problems for further research.
  - Does $\mathrm{L}(\omega)$ contain a closed non-separable subspace?
  - Is $\bigcup_{1\leqslant n<\omega} \mathrm{L}(2n)$ lineable?
  - Is $\bigcup_{1\leqslant n<\omega} \mathrm{L}(2n+1)$ densely lineable in ℓ_∞?

P. Leonetti, T. Russo, and J. Somaglia, *Dense lineability and spaceability in certain subsets of ℓ_∞*, arXiv:2203.08662.

Thank you for your attention!