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Hunting vector spaces inside non-linear objects

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Some lineability results

- **Gurariy (1966).** There is an infinite-dimensional subspace Y of $C([0,1])$ such that every $f \in Y, f \neq 0$ is nowhere differentiable.
  - **Fonf–Gurariy–Kadets (1999).** Such Y can be a closed subspace.
  - **Gurariy (1966).** If Y is a closed subspace of $C([0,1])$ and every element of Y is differentiable, then Y is finite-dimensional.
- **Grothendieck (1954).** If μ is a finite measure, Y is a closed subspace of $L_p(\mu)$, $0 < p < \infty$, and $Y \subseteq L_\infty(\mu)$, then Y is finite-dimensional.
- **Plichko–Zagorodnyuk (1998).** If X is an infinite-dimensional complex Banach space and $P\colon X \to \mathbb{C}$ is a polynomial with $P(0)=0$, $P^{-1}(0)$ contains an infinite-dimensional vector space.
  - **Avilés–Todorčević (2007).** There is a polynomial $P\colon \ell_1(\omega_1) \to \mathbb{C}$ with $P(0)=0$ such that every subspace $Y \subseteq P^{-1}(0)$ is separable.
  - **Aron–Hajék (2006).** If X is a real infinite-dimensional, separable Banach space, there is an odd polynomial $P\colon X \to \mathbb{R}$ such that $P^{-1}(0)$ contains no infinite-dimensional vector space.

Aron–Bernal–Pellegrino–Seoane (2016), *Lineability*.

Dense lineability in spaces of sequences

- A subset M of a Banach space X is **lineable** if $M \cup \{0\}$ contains an infinite-dimensional vector space.
- M is **densely lineable** if it contains a vector subspace dense in X.
- If $1 \leqslant q < p < \infty$, $\ell_p \setminus \ell_q$ is densely lineable.
  - **Kitson–Timoney (2011).** Let X be a separable Banach space and $T_n \colon Z_n \to X$ be bounded linear maps. If $Y := \operatorname{span}\left(\bigcup T_n[Z_n]\right)$ is not closed in X, then $X \setminus Y$ is densely lineable.
  - So, $\ell_p \setminus \bigcup_{q<p} \ell_q$ is densely lineable.
  - **Nestoridis (2020).** An explicit construction.
- **Nestoridis (2020).** Is $\ell_\infty \setminus c_0$ densely lineable?
  - **Papathanasiou (2022).** Yes, it is.
  - **Leonetti, R., Somaglia.** An alternative simpler proof.
    With a technique that can be adapted to prove much more.

- Let X be a Banach space with a projectional skeleton and such that $\operatorname{dens} X \leqslant \mathfrak{c}$ and let Y be a closed subspace of X such that the quotient X/Y is infinite-dimensional. Then $X \setminus Y$ is densely lineable.
  - Reflexive, WCG, WLD, Plichko, $L_1(\mu)$, $C(K)$ where K is Valdivia, or a compact Abelian group, ...
- If Y is a subspace of ℓ_∞ and $\operatorname{dens} Y < \mathfrak{c}$, then $\ell_\infty \setminus Y$ is densely lineable.
- If $\mathcal{I}$ is a meager ideal, $\ell_\infty \setminus c_0(\mathcal{I})$ is densely lineable.
- $\ell_\infty^c(\Gamma) \setminus c_0(\Gamma)$ is densely lineable.
- $\ell_\infty(\Gamma) \setminus \ell_\infty^{<|\Gamma|}(\Gamma)$ is densely lineable: there is a dense vector subspace of $\ell_\infty(\Gamma)$ whose each non-zero vector has $|\Gamma|$-many non-zero coordinates.

Lineability of $\ell_\infty \setminus c_0$

- Let $(B_j)_{j\in\omega}$ be a partition of ω, such that $|B_j| = \omega$.
- Take a bijection between $2^\omega \times \omega$ and $(0,1)$. Hence $(A,k) \leftrightarrow q_{A,k}$.
- For $A \subseteq \omega$ and $k \in \omega$ define

$$f_{A,k} := \mathbb{1}_A + 2^{-k} \sum_{j\in\omega} (q_{A,k})^j \, \mathbb{1}_{B_j}.$$

- $f_{A,k} \to \mathbb{1}_A$ as $k \to \infty$. So $\operatorname{span}\{f_{A,k}\}$ is dense in ℓ_∞.

Lemma (Strong linear independence of geometric sequences)

Let $\lambda_1, \ldots, \lambda_n \in (0,1)$ be mutually distinct scalars and let $\beta_1 \ldots, \beta_n \in \mathbb{R}$ not all equal to 0. Then the sequence

$$\left(\beta_1 \lambda_1^j + \cdots + \beta_n \lambda_n^j\right)_{j\in\omega}$$

attains infinitely many distinct values.

Question. Let $\lambda \leqslant \kappa$ be two cardinals and Γ be a set with $|\Gamma| = \kappa$. Is there a family $\mathcal{A} \subseteq [\Gamma]^\lambda$ with $|\mathcal{A}| = \kappa^\lambda$ and such that for every $A_0, A_1, \ldots, A_n \in \mathcal{A}$

$$A_0 \setminus (A_1 \cup \cdots \cup A_n) \text{ has cardinality } \lambda?$$

Yes, assuming one of the following:

1. $\kappa = \kappa^\lambda$ (disjoint sets)
2. λ is the least cardinal with $\kappa < \kappa^\lambda$ (almost-disjoint family)
3. $\lambda = \kappa$ (independent family).

Prize. Free beer until the end of the conference.

Thank you for your attention!