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(1+)-meters apart: Separated sets in Covid times

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Riesz' lemma and separated sets

Hereinafter, $\mathcal{X}$ is an **infinite-dimensional** Banach space.

Riesz' lemma (1916). There exists a sequence $(x_n)_{n=1}^{\infty}$ in the unit sphere $S_{\mathcal{X}}$ of $\mathcal{X}$ with $\|x_n - x_k\| \geqslant 1$ for $n \neq k$.

- Actually, one has $\|x_n \pm x_k\| \geqslant 1$ for $n \neq k$.

A set $\mathcal{A} \subseteq \mathcal{X}$ and $\delta > 0$, is:

- δ**-separated** if $\|a - b\| \geqslant \delta$ for $a \neq b \in \mathcal{A}$
- $(\delta+)$**-separated** if $\|a - b\| > \delta$ ...
- **symmetrically** δ**-separated** if $\|a \pm b\| \geqslant \delta$ ...
- **symmetrically** $(\delta+)$**-separated** if $\|a \pm b\| > \delta$ ...

We are interested in (symmetrically) $(1+)$ or $(1+\varepsilon)$-separated subsets of $S_{\mathcal{X}}$.

- $\mathcal{A} \subseteq S_{\mathcal{X}}$ is symmetrically $(1+)$- (resp. $(1+\varepsilon)$)-separated if $\mathcal{A} \cup -\mathcal{A}$ is $(1+)$- (resp. $(1+\varepsilon)$)-separated.

Kottman and Elton–Odell

Kottman's theorem (1975). The unit sphere $S_{\mathcal{X}}$ contains a (1+)-separated sequence $(x_n)_{n=1}^{\infty}$, *i.e.*, $\|x_n - x_k\| > 1$ for $n \neq k$.

The Elton–Odell theorem (1981). The unit sphere $S_{\mathcal{X}}$ contains a $(1+\varepsilon)$-separated sequence $(x_n)_{n=1}^{\infty}$ (for some $\varepsilon > 0$).

The (symmetric) Kottman constant

$$K(\mathcal{X}) := \sup\left\{\sigma > 0 \colon B_{\mathcal{X}} \text{ contains a } \sigma\text{-separated sequence}\right\}$$
$$K^s(\mathcal{X}) := \sup\left\{\sigma > 0 \colon B_{\mathcal{X}} \text{ contains a symmetrically } \sigma\text{-sep...}\right\}$$

- By Elton–Odell $K(\mathcal{X}) > 1$, for every $\mathcal{X}$.
- **Castillo–Papini (2011).** Is $K^s(\mathcal{X}) > 1$?
- Is there a symmetric version of Kottman's theorem?
- Writing $B_{\mathcal{X}}$ or $S_{\mathcal{X}}$ is equivalent.

Examples and estimates for K^s

- $K^s(c_0) = 2$;
- $K^s(\ell_p) = 2^{1/p}$ for $p \in [1, \infty)$ (note the equality!);
- $K^s(\mathcal{X}) = 2$ if $\mathcal{X}$ contains c_0 or ℓ_1 (James' non-distortion);
- **Kottman (1975).** $K^s(\mathcal{X}) \geqslant 2^{1/p}$ if $\mathcal{X}$ contains ℓ_p;
- $K^s(\mathcal{X}) = 2$ if $\mathcal{X}$ has a c_0 (or ℓ_1) quotient;
- **Castillo–Papini (2011).** If $\mathcal{X}$ is an $\mathscr{L}_\infty$-space, then $K^s(\mathcal{X}) = 2$;
- **Delpech (2010).** $K^s(\mathcal{X}) \geqslant 1 + \delta_{\mathcal{X}}(1)$;
- **Maluta–Papini (2009).** $K^s(\mathcal{X}) \leqslant K(\mathcal{X}) \leqslant 2 - 2\delta_{\mathcal{X}}(1)$;
- **Castillo–González–Kania–Papini (2020).** $K^s(\mathcal{X}) \cdot K^s(\mathcal{X}^*) \geqslant 2$;
- **Kryczka–Prus (2000).** $K(\mathcal{X}) \geqslant \sqrt[5]{4}$ for non-reflexive $\mathcal{X}$.

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A symmetric version of Kottman's theorem

Theorem (P. Hájek, T. Kania, and R., JFA'18)

The unit sphere of every $\mathcal{X}$ contains a symmetrically $(1+)$-separated sequence $(x_n)_{n=1}^{\infty}$, i.e. $\|x_n \pm x_k\| > 1$ for $n \neq k$.

For $\mathcal{X}_0 \subseteq \mathcal{X}$, $\dim(\mathcal{X}_0) = \infty$, we say that $\mathcal{X}_0$ has ($\Box$) if:

$$\exists x \in S_{\mathcal{X}_0}, \exists \mathcal{Y} \subseteq \mathcal{X}_0, \dim(\mathcal{Y}) = \infty \colon \forall y \in S_{\mathcal{Y}} \; \|x + y\| > 1.$$

Case 1: Every $\mathcal{X}_0 \subseteq \mathcal{X}$, $\dim(\mathcal{X}_0) = \infty$, has ($\Box$).

Case 2: Pick $\mathcal{X}_0$ that has ($\neg\Box$). WLOG $\mathcal{X}_0 = \mathcal{X}$. Equivalently:

$$(\blacksquare) \qquad \forall x \in B_{\mathcal{X}}, \forall \mathcal{Y} \subseteq \mathcal{X}, \dim(\mathcal{Y}) = \infty, \; \exists y \in S_{\mathcal{Y}} \colon \|x + y\| \leqslant 1.$$

Contd, Case 2

$$(\text{battery})\qquad \forall x \in B_{\mathcal{X}}, \forall \mathcal{Y} \subseteq \mathcal{X}, \dim(\mathcal{Y}) = \infty,\ \exists y \in S_{\mathcal{Y}}\colon\ \|x+y\| \leqslant 1.$$

Now that we have the symmetric Kottman, is also $K^s(\mathcal{X}) > 1$?

Hájek–Kania–R., JFA'18: $K^s(\mathcal{X}) > 1$ if:

- $\mathcal{X}$ contains a boundedly complete basic sequence,
  - $\mathcal{X}$ is reflexive,
  - $\mathcal{X}$ contains a separable dual,
  - $\mathcal{X}$ has the Radon–Nikodym property,
- $\mathcal{X}$ contains an unconditional basic sequence,
- $\mathcal{X}$ has cotype $q < \infty$.

Is $K^s(\mathcal{X}) > 1$?

Theorem (R., RACSAM'19)

For every $\mathcal{X}$, $K^s(\mathcal{X}) > 1$, namely, the unit sphere of $\mathcal{X}$ contains a symmetrically $(1+\varepsilon)$-separated sequence.

Check the proof of Elton–Odell: if $\mathcal{X}$ doesn't contain c_0 and $(x_j)_{j=1}^{\infty}$ is normalised and weakly null, it admits a $(1+\varepsilon)$-separated normalised block sequence.

Is $K^s(\mathcal{X}) > 1$?

Theorem (R., RACSAM'19)

For every $\mathcal{X}$, $K^s(\mathcal{X}) > 1$, namely, the unit sphere of $\mathcal{X}$ contains a symmetrically $(1+\varepsilon)$-separated sequence.

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- Tuning the argument: If $S_{\mathcal{X}}$ contains a $(1+\varepsilon)$-separated weakly null sequence, it contains a symmetrically $(\sqrt{1+\varepsilon})$-septd one.
- Hence, $K^s(\mathcal{X}) \geqslant \sqrt{K(\mathcal{X})}$ if
  - $\mathcal{X}$ is reflexive with the non-strict Opial property, or
  - $\mathcal{X}$ has a suppression 1-unconditional Schauder basis.
- Does $K^s(\mathcal{X}) \geqslant \sqrt{K(\mathcal{X})}$ hold for every reflexive $\mathcal{X}$?
- How large can $K^s(\mathcal{X}) - K(\mathcal{X})$ be in general?

Open problems

1. In a **complex** Banach space $\mathcal{X}$ consider **toroidal separation**:
   1. Is there $(x_n)_{n=1}^{\infty}$ with $\|x_n - \theta x_k\| > 1$ ($\theta \in \mathbb{C}$, $|\theta| = 1$, $n \neq k$)?
   2. Do we have a 'toroidal' version of the Elton–Odell theorem?
2. **Kottman (1970).** The **isomorphic** Kottman constant:

   $$\widetilde{K}(\mathcal{X}) := \inf\{K(\mathcal{Y}) \colon \mathcal{Y} \text{ isomorphic to } \mathcal{X}\}$$

   1. Is $\widetilde{K} > 1$? (Of course, also $\widetilde{K}^s$ could be defined.)
   2. Can we choose the ε in the Elton–Odell theorem to be renorming invariant?
   3. $\widetilde{K}(\mathcal{X}) > 1$ if $\mathcal{X}$ contains c_0, or ℓ_p.
3. **Diestel (1984).** The 'subspace' Kottman constant:

   $$K^D(\mathcal{X}) := \inf\{K(\mathcal{Y}) \colon \mathcal{Y} \text{ subspace of } \mathcal{X}\}$$

   1. $K^D(\ell_p) = 2^{1/p}$, $K^D(c_0) = 2$.
   2. $K^D(\mathcal{X}) = 1$ if $\mathcal{X}$ contains ℓ_{p_n}, $p_n \to \infty$.
   3. For which spaces is $K^D(\mathcal{X}) > 1$?

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What can we hope for?

Henceforth, $\mathcal{X}$ is a **non-separable** Banach space.
General problem: How large can separated subsets of $S_{\mathcal{X}}$ be?

$B_{\mathcal{X}}$ contains an uncountable ε-separated set, for some $\varepsilon > 0$.

- Does $S_{\mathcal{X}}$ contain an uncountable (1+)-separated subset?
- What about uncountable $(1+\varepsilon)$-separated subsets?
- Can we find a such subsets with cardinality $\text{dens}(\mathcal{X})$?

A few reassuring examples:

- $c_0(\Gamma)$: the sphere contains an uncountable (1+)-separated set;
- In $\ell_p(\Gamma)$, the canonical basis is $2^{1/p}$-separated;
- In the ball of $\ell_\infty(\Gamma)$ we have a 2-separated set of cardinality $2^{|\Gamma|}$.

Did we hope for too much? More on $c_0(\Gamma)$

Elton–Odell (1981). In $c_0(\Gamma)$, $(1+\varepsilon)$-separated subsets of the ball are at most countable.

Δ-system lemma

Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be uncountably many finite subsets of S. Then there are $\Gamma_0 \subseteq \Gamma$ uncountable and $\Delta \subseteq S$ finite such that

$$A_\alpha \cap A_\beta = \Delta \quad \textit{for} \quad \alpha \neq \beta \in \Gamma_0.$$

Theorem (P. Hájek, T. Kania, and R., TAMS'20)

Let $\mathcal{F} \subseteq S_{c_0(\Gamma)}$ be $(1+)$-separated. Then $|\mathcal{F}| \leqslant \omega_1$.

- Does S_X contain an uncountable $(1+)$-separated subset?
- For which X also an uncountable $(1+\varepsilon)$-separated one?

Theorem (P. Hájek, T. Kania, and R., TAMS'20)

$S_{\mathcal{X}}$ and $S_{\mathcal{X}^*}$ contain uncountable $(1+)$-separated sets if:

- *$\mathcal{X}$ is WLD, $\operatorname{dens}\mathcal{X} > \mathfrak{c}$, or*
- *$\mathcal{X}$ is 'large' (more precisely, $w^*\text{-}\operatorname{dens}\mathcal{X}^* > 2^{2^{\mathfrak{c}}}$).*

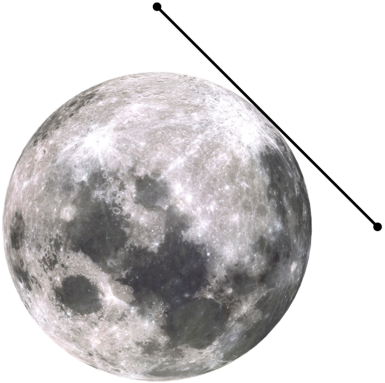
- WLD spaces of density ω_1? Renormings of $c_0(\omega_1)$?

$C(\mathcal{K})$ spaces:

- **Kania–Kochanek (2016).** The ball contains an uncountable $(1+)$-separated set.
- **Koszmider (2018).** It is undecidable if the ball contains an uncountable $(1+\varepsilon)$-separated set.
- Does the ball contain a $(1+)$-separated set of cardinality $\operatorname{dens} C(\mathcal{K})$?
  - **Cúth–Kurka–Vejnar (2019).** Yes, if $\operatorname{dens} C(\mathcal{K}) \leqslant \mathfrak{c}$.

Theorem (P. Hájek, T. Kania, and R., TAMS'20)

- The sphere of a reflexive $\mathcal{X}$ contains a $(1+)$-separated set of cardinality $\mathrm{dens}(\mathcal{X})$;



Theorem (P. Hájek, T. Kania, and R., TAMS'20)

- The sphere of a reflexive $\mathcal{X}$ contains a (1+)-separated set of cardinality $\mathrm{dens}(\mathcal{X})$;
- if $\mathcal{X}$ is reflexive and $\lambda \leqslant \mathrm{dens}(\mathcal{X})$ has uncountable cofinality, the sphere of $\mathcal{X}$ contains a $(1+\varepsilon)$-separated set of cardinality λ;
- if $\mathcal{X}$ is super-reflexive, the sphere contains a $(1+\varepsilon)$-separated set of cardinality $\mathrm{dens}(\mathcal{X})$.

Example (Kania–Kochanek, 2016). the unit sphere of

$$\mathcal{X} := \left(\bigoplus_{n\in\mathbb{N}} \ell_{p_n}(\omega_n) \right)_{\ell_2} \qquad (p_n)_{n=1}^{\infty} \subseteq (1,\infty),\ p_n \nearrow \infty$$

does not contain $(1+\varepsilon)$-separated subsets of cardinality $\omega_\omega = \mathrm{dens}\,\mathcal{X}$.

References

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- T. Russo, A note on symmetric separation in Banach spaces, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **113** (2019), 3649–3658.
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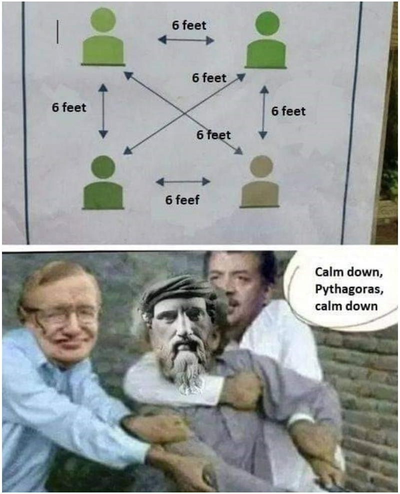

Thank you for your attention!