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How different can two dense subspaces of a Banach space be?

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P. Hájek and T. Russo, *On densely isomorphic normed spaces*.
Preprint available at arXiv:1910.01527.

48^{th} Winter School in Abstract Analysis
Svratka, Czech Republic

January 11–18, 2020

International Mobility of Researchers in CTU
Project number: CZ.02.2.69/0.0/0.0/16_027/0008465

EVROPSKÁ UNIE
Evropské strukturální a investiční fondy
Operační program Výzkum, vývoj a vzdělávání

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

Smoothness and structure

The existence of a smooth norm on a Banach space bears several geometric consequences. Just to name a few:

- If a separable Banach space X has a C^1-smooth norm, X^* is separable;
- If X^* admits a dual C^1-smooth norm, X is reflexive;
- **Meshkov (1978).** If X and X^* admit a C^2-smooth norm, then X is isomorphic to a Hilbert space;
- **Fabian, Whitfield, Zizler (1983).** If X admits a C^2-smooth norm, either it contains c_0, or it is super-reflexive with type 2;
- **Deville (1989).** If X has a C^∞-smooth norm, either it contains c_0, or it is super-reflexive, with exact cotype $2k$, and it contains ℓ_{2k}.

Smoothness on subsets

It was asked several times if the existence of a smooth norm on some 'large' subset has similar consequences:

- Benyamini–Lindenstrauss, *Geometric Nonlinear Functional Analysis*, p. 96: Is there an equivalent norm on ℓ_1 that is Fréchet differentiable outside a countable union of hyperplanes?
- Guirao–Montesinos–Zizler, *Open problems...*, Problem 149: Does the space of finitely supported vectors in $\ell_1(\Gamma)$ have a C^1-smooth norm (when Γ is uncountable)?

Some recent results, joint with Sheldon Dantas and Petr Hájek:

- There exists a dense subspace of ℓ_∞ with a C^∞-smooth norm;
- If $(e_\gamma)_{\gamma\in\Gamma}$ is a long unconditional basis for a Banach space, its linear span admits a C^∞-smooth norm.

What about other dense subspaces?

How different can two dense subspaces of a Banach space be?

Separable Banach spaces

In this talk, subspaces are **NOT** assumed to be closed.

In every separable Banach space there is a canonical (smallest) dense subspace, that is densely contained in every other dense subspace. More precisely:

Folklore

Let $\{e_j; e_j^*\}_{j=1}^{\infty}$ be an M-basis for a separable Banach space X. Then every dense subspace of X contains a dense subspace isomorphic to $\mathrm{span}\{e_j\}_{j=1}^{\infty}$.

- $\mathrm{span}\{e_j\}_{j=1}^{\infty}$ is this 'minimal' dense subspace;
- **Grivaux (2003).** Such a minimal subspace is additionally unique up to isomorphisms;
- This feature breaks down completely in many non-separable Banach spaces; most notably, in (non-separable) Hilbert spaces.

Different dense subspaces

Theorem A

Let X be a non-separable WLD Banach space. Then there are two dense subspaces Y and Z of X whose every dense subspaces are non-isomorphic.

- If $\mathrm{dens}\, X \leqslant \mathfrak{c}$, there exist two dense subspaces Y and Z such that no non-separable subspace of Y is isomorphic to a subspace of Z;
- A striking particular case is given by $\ell_2(\mathfrak{c})$;
- A 'minimal' subspace as before has to be separable.

Definition

Two normed spaces X and Y are *densely isomorphic* if there exist dense subspaces X_0 of X and Y_0 of Y such that X_0 and Y_0 are isomorphic.

Th A (restated). Every non-separable WLD Banach space contains two dense subspaces that are not densely isomorphic.

Towards a study of dense subspaces

- Every two dense subspaces of $\ell_1(\Gamma)$ are densely isomorphic.
  - In particular, every dense subspace of $\ell_1(\Gamma)$ contains a further dense subspace with a C^∞-smooth norm.
- What about ℓ_∞?
  - There exists a dense subspace with a C^∞-smooth norm;
  - Is there a dense subspace whose every dense subspace fails to have smooth norms?
- We can look at different properties.
  - Let X be a Banach space and P be a property;
  - Find Y dense in X whose **every** dense subspace has P;
  - Find Z dense in X **no** whose dense subspace has P;
  - P witnesses that Y and Z are not densely isomorphic.
- We can say that X *densely has* P if every dense subspace of X has a dense subspace with P.
- What about uncountable biorthogonal systems?

Biorthogonal systems

Theorem B

(CH) *Let X be a WLD Banach space with $\operatorname{dens} X = \omega_1$. Then there exists a dense subspace Y of X that contains no uncountable biorthogonal system.*

Particular case. (CH) There exists a dense subspace of the Hilbert space $\ell_2(\omega_1)$ that contains no uncountable biorthogonal system.

Lemma

Let $\{e_\alpha; e^*_\alpha\}_{\alpha\in\Gamma}$ be an M-basis for a Banach space X. Then every non-separable subspace of $Z := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$ contains an uncountable biorthogonal system.

- Again, a common subspace is separable;
- Y and Z are not densely isomorphic.

Inner product spaces

For an orthonormal system $\{e_\gamma\}_{\gamma\in\Gamma}$ in an inner produce space H, TFAE:

(i) $\{e_\gamma\}_{\gamma\in\Gamma}$ is complete (*i.e.*, linearly dense);

(ii) $\{e_\gamma\}_{\gamma\in\Gamma}$ is a Schauder basis;

(iii) Parseval's equality $\|x\|^2 = \sum_{\gamma\in\Gamma} |\langle e_\gamma, x\rangle|^2$ holds for every $x \in H$.

$$\Downarrow$$

(iv) $\{e_\gamma\}_{\gamma\in\Gamma}$ is maximal.

Gudder (1974). There exists a non-separable inner product space that contains no uncountable orthonormal system.
[See Halmos, *A Hilbert space problem book*, Problem 54.]

Buhagiar, Chetcuti, and Weber (2008). If $\operatorname{dens} H \geqslant \mathfrak{c}^+$, H contains an uncountable orthonormal system (actually, of cardinality $\mathfrak{c}^+$).

Thank you for your attention!