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Asplund Banach spaces with norming Markuševič bases

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P. Hájek, T. Russo, J. Somaglia, and S. Todorčević,
An Asplund space with norming Markuševič basis that is not weakly compactly generated.
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Banach spaces webinars
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Markušević bases

A system $\{u_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a **Markušević basis** (**M-basis**, for short) for $\mathcal{X}$ if

- $\langle \varphi_\beta, u_\alpha \rangle = \delta_{\alpha,\beta}$,
- $\operatorname{span}\{u_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$,
- $\operatorname{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is w^*-dense in $\mathcal{X}^*$.

$\{\langle \varphi_\alpha, x\rangle : \alpha \in \Gamma\}$ are the **coordinates** of $x \in \mathcal{X}$

$\{\langle \psi, x_\alpha\rangle : \alpha \in \Gamma\}$ are the **coordinates** of $\psi \in \mathcal{X}^*$.

- **Markušević, 1943.** Every separable Banach space has an M-basis.
- **Amir–Lindenstrauss, 1968.** Every WCG Banach space has an M-basis;

  Def : $\mathcal{X}$ is **WCG** if it contains a linearly dense weakly compact subset.
- **Johnson, 1970.** ℓ_∞ has no M-basis.

Existence of Norming M-bases

- It is tempting to ask if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ exhausts $\mathcal{X}^*$ in a stronger sense.
- $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is **shrinking** if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}^*$.
- A subspace $\mathcal{Z}$ of $\mathcal{X}^*$ is λ**-norming** ($0<\lambda\leqslant 1$) if

$$\lambda\|x\| \leqslant \sup\{|\langle\varphi, x\rangle| \colon \varphi\in\mathcal{Z},\ \|\varphi\|\leqslant 1\}.$$

Plainly, $\mathcal{X}^*$ is 1-norming, by the Hahn–Banach theorem.

Definition

An M-basis $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is λ**-norming** ($0<\lambda\leqslant 1$) if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is a λ-norming subspace, namely if

$$\lambda\|x\| \leqslant \sup\{|\langle\varphi, x\rangle| \colon \varphi\in\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma},\ \|\varphi\|\leqslant 1\}.$$

- Separable Banach spaces have a 1-norming M-basis (Markuševič).
- Every reflexive Banach space has a shrinking M-basis.

[HRST] $\mathcal{C}(\mathcal{K})$ has a 1-norming M-basis, if $\mathcal{K}$ is adequate.

How strong are them?

- **Alexandrov–Plichko, 2006.** $\mathcal{C}([0, \omega_1])$ has no norming M-basis;
  - but it has a countably 1-norming, strong M-basis
  - and a monotone long Schauder basis.
- Schauder bases and (long) unconditional bases are norming M-bases.
- **(Folklore) Fact.** Let $\{f_\gamma; \mu_\gamma\}_{\gamma<\omega_1}$ be a λ-norming M-basis for a Banach space $\mathcal{X}$. Then there exists an uncountable subset Λ of ω_1 such that $(f_\gamma)_{\gamma\in\Lambda}$ is a long basic sequence in $\mathcal{X}$.
  - If $\mathcal{Y} \subseteq \mathcal{X}$ is separable, there is a countable $S \subseteq \omega_1$ such that $\mathrm{span}\{\mu_\gamma\}_{\gamma\in S}$ is λ-norming for $\mathcal{Y}$; insert in Mazur's technique. ✓
  - So, norming M-bases are 'stronger than' long basic sequences.
- **John–Zizler, 1974.** Do WCG spaces have a norming M-basis?

Theorem (Hájek, 2019)

There exists a WCG $\mathcal{C}(\mathcal{K})$ space with no norming M-basis.
Actually, $\mathcal{K}$ is uniform Eberlein, so $\mathcal{C}(\mathcal{K})$ is also Hilbert-generated.

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Towards the main result

- Properties of M-bases can characterise classes of Banach spaces: weakly compact, shrinking, countably norming, ...
- What about norming M-bases?

Def : $\mathcal{X}$ is **Asplund** if every its separable subspace has separable dual.

  - $\mathcal{C}(\mathcal{K})$ is Asplund iff $\mathcal{K}$ is scattered.

- **Troyanski, John–Zizler, Orihuela–Valdivia, Fabian, ...** TFAE:

  (i) $\mathcal{X}$ has a shrinking M-basis;
  (ii) $\mathcal{X}$ is WCG and Asplund;
  (iii) $\mathcal{X}$ is WLD and Asplund;
  (iv) $\mathcal{X}$ is WLD and $\mathcal{X}^*$ admits a dual LUR norm.

Ex : $\ell_1(\Gamma)$ admits a 1-norming M-basis.

  - Norming M-basis $\not\Longrightarrow$ WCG, or Asplund ($\ell_1(\omega_1)$);
  - Norming M-basis and WCG $\not\Longrightarrow$ Asplund (ℓ_1).

- **Godefroy ∼1990.** Let $\mathcal{X}$ be an Asplund space with a norming M-basis. Is $\mathcal{X}$ WCG?

Our answer

Theorem A

There exists an Asplund space $\mathcal{X}$ with a 1-norming M-basis $\{f_\gamma; \mu_\gamma\}_{\gamma<\omega_1}$ such that $\mathcal{X}$ is not WCG.

- $\{f_\gamma; \mu_\gamma\}_{\gamma<\omega_1}$ is additionally Auerbach, *i.e.*, $\|f_\gamma\| = \|\mu_\gamma\| = 1$.
- We can also assume that $(f_\gamma)_{\gamma<\omega_1}$ is a monotone long Schauder basis.

Proof : Recall the folklore fact: there is $\Lambda \subseteq \omega_1$ uncountable s.t. $(f_\gamma)_{\gamma\in\Lambda}$ is a monotone long basic sequence.

- The space $\mathcal{X}_\Lambda := \overline{\text{span}}\{f_\gamma\}_{\gamma\in\Lambda}$ also satisfies Th A. ✓

$\mathcal{X}$ is constructed as a subspace of a $\mathcal{C}(\mathcal{K})$ space, $\mathcal{K}$ scattered.

- We shall explain how to build $\mathcal{K}$, *cf.* Th B in the next slide.
- Our example is a subspace of an Asplund $\mathcal{C}(\mathcal{K})$ (that is not WCG).
- **Problem.** Is there a $\mathcal{C}(\mathcal{K})$ example?

A peculiar compact space

- $\mathcal{P}(\Gamma) \equiv \{0,1\}^\Gamma$ by $A \leftrightarrow 1_A$;
- This gives a compact 'product' topology on $\mathcal{P}(\Gamma)$;
- If $A \in \mathcal{P}(\Gamma)$, $a_0, \dots, a_n \in A$, $b_0, \dots, b_n \notin A$

$$\mathcal{U} := \{B \in \mathcal{P}(\Gamma) \colon a_0, \dots, a_n \in B,\ b_0, \dots, b_n \notin B\}$$

is a nghd of A in $\mathcal{P}(\Gamma)$.

Theorem B

There exists a family $\mathcal{F}_\varrho \subseteq [\omega_1]^{<\omega}$ of finite subsets of ω_1 such that $\mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}$ has the following properties:

(i) $\{\alpha\} \in \mathcal{K}_\varrho$ for every $\alpha < \omega_1$,
(ii) $[0, \alpha) \in \mathcal{K}_\varrho$ for every $\alpha \leqslant \omega_1$,
(iii) if $A \in \mathcal{K}_\varrho$ is an infinite set, then $A = [0, \alpha)$ for some $\alpha \leqslant \omega_1$,
(iv) $\mathcal{K}_\varrho$ is scattered.

- We define a biorthogonal system $\{f_\gamma; \mu_\gamma\}_{\gamma<\omega_1}$ in $\mathcal{C}(\mathcal{K}_\varrho)$:

$$f_\gamma \in \mathcal{C}(\mathcal{K}_\varrho) \qquad f_\gamma(A) = \begin{cases} 1 & \gamma \in A \\ 0 & \gamma \notin A \end{cases} \qquad (A \in \mathcal{K}_\varrho)$$

$$\mu_\gamma := \delta_{\{\gamma\}} \in \mathcal{M}(\mathcal{K}_\varrho) \qquad \mu_\gamma(S) = \begin{cases} 1 & \{\gamma\} \in S \\ 0 & \{\gamma\} \notin S \end{cases} \qquad (S \subseteq \mathcal{K}_\varrho).$$

- $\langle \mu_\alpha, f_\gamma \rangle = f_\gamma(\{\alpha\}) = \delta_{\alpha,\gamma}$, so *it is* biorthogonal.
- The space that we are looking for is

$$\mathcal{X}_\varrho := \overline{\mathrm{span}}\{f_\gamma\}_{\gamma<\omega_1} \subseteq \mathcal{C}(\mathcal{K}_\varrho).$$

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Todorčević's ϱ-functions

S. Todorčević, Partitioning pairs of countable ordinals, *Acta Math.* **159** (1987), 261–294.

S. Todorčević, *Walks on ordinals and their characteristics*. Birkhäuser Verlag, Basel, 2007.

- We consider functions $\varrho\colon [\omega_1]^2 \to \omega$.
- We identify $[\omega_1]^2 = \{(\alpha,\beta) \in {\omega_1}^2 : \alpha < \beta\}$.
  - Thus, we write $\varrho(\alpha,\beta)$, with $\alpha < \beta$, for $\varrho(\{\alpha,\beta\})$.
- We also add the 'boundary condition' $\varrho(\alpha,\alpha) = 0$.

Definition (Todorčević)

A ϱ**-function** on ω_1 is a function $\varrho\colon [\omega_1]^2 \to \omega$ such that:

(ϱ1) $\{\xi \leqslant \alpha\colon \varrho(\xi,\alpha) \leqslant n\}$ is finite, for every $\alpha < \omega_1$ and $n < \omega$,

(ϱ2) $\varrho(\alpha,\gamma) \leqslant \max\{\varrho(\alpha,\beta), \varrho(\beta,\gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$,

(ϱ3) $\varrho(\alpha,\beta) \leqslant \max\{\varrho(\alpha,\gamma), \varrho(\beta,\gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$.

Definition of the compact $\mathcal{K}_\varrho$

Proposition (Todorčević)

There exists a function $\varrho\colon [\omega_1]^2 \to \omega$ such that ($\alpha < \beta < \gamma < \omega_1$):

- $\varrho(\alpha, \beta) > 0$;
- $\varrho(\alpha, \gamma) \leqslant \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$;
- $\varrho(\alpha, \gamma) \neq \rho(\beta, \gamma)$.

$$F_n(\alpha) := \{\xi \leqslant \alpha \colon \varrho(\xi, \alpha) \leqslant n\}$$

$$\mathcal{F}_\varrho := \{F_n(\alpha) \colon n < \omega,\ \alpha < \omega_1\} \qquad \text{and} \qquad \mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}.$$

Fact

- $F_0(\alpha) = \{\alpha\}$;
- $|F_n(\alpha)| \leqslant n + 1$;
- $(F_n(\alpha))_{n<\omega}$ converges to $[0, \alpha]$.

Th B again, aka $\mathcal{K}_\varrho$ verifies Th B

Th B (again, but smaller).

The compact space $\mathcal{K}_\varrho$ has the following properties:

(i) $\{\alpha\} \in \mathcal{K}_\varrho$ for every $\alpha < \omega_1$,

(ii) $[0, \alpha) \in \mathcal{K}_\varrho$ for every $\alpha \leqslant \omega_1$,

(iii) if $A \in \mathcal{K}_\varrho$ is an infinite set, then $A = [0, \alpha)$ for some $\alpha \leqslant \omega_1$,

(iv) $\mathcal{K}_\varrho$ is scattered.

Proof. (i) $\{\alpha\} = F_0(\alpha) \in \mathcal{K}_\varrho$. ✓

(ii) If $\alpha = \alpha' + 1$, $[0, \alpha) = [0, \alpha'] = \lim_{n<\omega} F_n(\alpha') \in \mathcal{K}_\varrho$.

If α is limit, $([0, \beta + 1))_{\beta<\alpha}$ converges to $[0, \alpha)$. ✓

(iii) $\implies$ (iv)

- Let $\mathcal{D} \subseteq \mathcal{K}_\varrho$ be closed. Pick $D_0 \in \mathcal{D}$ s.t. $\exists\, \alpha < \beta$, $\alpha \notin D_0$, $\beta \in D_0$.
- Pick a maximal element $M \in \mathcal{D}$ with $\alpha \notin M$, $D_0 \subseteq M$.
- M is a finite set, by (iii).
- $\mathcal{U} := \{D \in \mathcal{D} \colon \alpha \notin D, M \subseteq D\} = \{M\}$, so M is isolated in $\mathcal{D}$. ✓

Proof of (iii), $|A| = \omega$

(iii) if $A \in \mathcal{K}_\varrho$ is infinite, then $A = [0, \alpha)$ for some $\alpha \leqslant \omega_1$;

Def : $F_n(\alpha) := \{\xi \leqslant \alpha \colon \varrho(\xi, \alpha) \leqslant n\}$.

Assume first $|A| = \omega$.

- Pick $\alpha \in A$ and $\tilde{\alpha} < \alpha$; we need $\tilde{\alpha} \in A$.
- Pick a sequence $(F_{n_k}(\alpha_k))_{k<\omega} \to A$ (Fréchet–Urysohn property).
- If $(n_k)_{k<\omega}$ is bounded, then $|F_{n_k}(\alpha_k)| \leqslant M$; so, A is finite.
- WLOG, $\varrho(\tilde{\alpha}, \alpha) \leqslant n_k$.
- Also, assume $\alpha \in F_{n_k}(\alpha_k)$, namely $\alpha \leqslant \alpha_k$ and $\varrho(\alpha, \alpha_k) \leqslant n_k$.
- By triangle inequality,

$$\varrho(\tilde{\alpha}, \alpha_k) \leqslant \max\{\varrho(\tilde{\alpha}, \alpha), \varrho(\alpha, \alpha_k)\} \leqslant n_k,$$

so $\tilde{\alpha} \in F_{n_k}(\alpha_k)$. Passing to the limit, $\tilde{\alpha} \in A$. ✓

Proof of (iii), $|A| = \omega_1$

$\Sigma(\Gamma) := \{x \in [0,1]^\Gamma : \mathrm{supp}(x) \text{ is countable}\}$.

Theorem (Deville–Godefroy, Kalenda)

Let $\mathcal{K} \subseteq [0,1]^{\omega_1}$ be a compact set such that $\mathcal{K} \cap \Sigma(\omega_1)$ is dense in $\mathcal{K}$. Let $x \in \mathcal{K} \setminus \Sigma(\omega_1)$. Then there exists an embedding $\varphi\colon [0,\omega_1] \to \mathcal{K}$:

(i) $\varphi(\alpha) \in \mathcal{K} \cap \Sigma(\omega_1)$, for $\alpha < \omega_1$,

(ii) $\mathrm{supp}(\varphi(\alpha)) \subseteq \mathrm{supp}(\varphi(\beta))$, for $\alpha < \beta \leqslant \omega_1$,

(iii) $\varphi(\omega_1) = x$.

- In our case, there is $\varphi\colon [0,\omega_1] \to \mathcal{K}_\varrho$ with
  - (i) $|\varphi(\alpha)| \leqslant \omega$, for $\alpha < \omega_1$,
  - (ii) $\varphi(\alpha) \subseteq \varphi(\beta)$, for $\alpha < \beta \leqslant \omega_1$,
  - (iii) $\varphi(\omega_1) = A$, $\implies \cup_{\alpha<\omega_1}\varphi(\alpha) = A$.
- Hence, the sets $\varphi(\alpha)$ are infinite, for α large.
- By the previous case, such $\varphi(\alpha)$ are initial intervals. ✓ ■

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Definition (Kubiś and Leiderman, 2004)

A compact space is **semi-Eberlein** if it is homeomorphic to a compact $\mathcal{K} \subseteq [0,1]^\Gamma$ such that $c_0(\Gamma) \cap \mathcal{K}$ is dense in $\mathcal{K}$.

Recall that $\Sigma(\Gamma) := \{x \in [0,1]^\Gamma : \mathrm{supp}(x) \text{ is countable}\}$.

A compact space is ...

$$\begin{array}{ccc} \text{Valdivia} & \hookleftarrow & \text{semi-Eberlein} \\ \uparrow & & \uparrow \\ \text{Corson} & \hookleftarrow & \text{Eberlein} \end{array}$$

if it is homeomorphic to $\mathcal{K} \subseteq [0,1]^\Gamma$ such that ...

$$\begin{array}{ccc} \mathcal{K} \cap \Sigma(\Gamma) \text{ is dense in } \mathcal{K} & \hookleftarrow & \mathcal{K} \cap c_0(\Gamma) \text{ is dense in } \mathcal{K} \\ \uparrow & & \uparrow \\ \mathcal{K} \subseteq \Sigma(\Gamma) & \hookleftarrow & \mathcal{K} \subseteq c_0(\Gamma) \end{array}$$

Theorem (Kubiś and Leiderman, 2004)

No semi-Eberlein compact space has a P-point.

- Used to show that there is $\mathcal{K}$ Corson, not semi-Eberlein.
- A point $p \in \mathcal{K}$ is a **P-point** if it is not isolated and for every choice of $(U_j)_{j<\omega}$ nghds of p, $\cap U_j$ is a nghd of p.

Question (Kubiś and Leiderman, 2004)

Can a semi-Eberlein compact space have weak P-points?

- A point $p \in \mathcal{K}$ is a **weak P-point** if it is not isolated and no sequence in $\mathcal{K} \setminus \{p\}$ converges to p.
- The compact space $\mathcal{K}_\varrho$ in Theorem B is semi-Eberlein and it has a weak P-point.

Recap

Theorem A

There exists an Asplund space $\mathcal{X}$ with a 1-norming M-basis such that $\mathcal{X}$ is not WLD.

Theorem B

There exists a family $\mathcal{F}_\varrho \subseteq [\omega_1]^{<\omega}$ of finite subsets of ω_1 such that $\mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}$ has the following properties:

(i) $\{\alpha\} \in \mathcal{K}_\varrho$ for every $\alpha < \omega_1$,

(ii) $[0, \alpha) \in \mathcal{K}_\varrho$ for every $\alpha \leqslant \omega_1$,

(iii) if $A \in \mathcal{K}_\varrho$ is an infinite set, then $A = [0, \alpha)$ for some $\alpha \leqslant \omega_1$,

(iv) $\mathcal{K}_\varrho$ is scattered.

Thank you for your attention!

The Banach space $\mathcal{X}_\varrho$

The space that we are looking for is

$$\mathcal{X}_\varrho := \overline{\mathrm{span}}\{f_\gamma\}_{\gamma<\omega_1} \subseteq \mathcal{C}(\mathcal{K}_\varrho).$$

What do we know already?

- $\mathcal{X}_\varrho$ is Asplund (as $\mathcal{K}_\varrho$ is scattered);
- $\{f_\gamma; \mu_\gamma\restriction_{\mathcal{X}_\varrho}\}_{\gamma<\omega_1}$ is a biorthogonal system in $\mathcal{X}_\varrho$.

What do we still need?

- $\mathcal{X}_\varrho$ is not WLD;
- $\mathrm{span}\{\mu_\gamma\restriction_{\mathcal{X}_\varrho}\}_{\gamma<\omega_1}$ is a 1-norming subspace for $\mathcal{X}_\varrho$;
- In particular, $\mathrm{span}\{\mu_\gamma\restriction_{\mathcal{X}_\varrho}\}_{\gamma<\omega_1}$ is w^* dense, so $\{f_\gamma; \mu_\gamma\restriction_{\mathcal{X}_\varrho}\}_{\gamma<\omega_1}$ is an M-basis for $\mathcal{X}_\varrho$.

$\mathcal{X}_\varrho$ is not WLD

- We shall show that $[0,\omega_1]$ embeds in $(B_{\mathcal{X}_\varrho{}^*}, w^*)$.
- Define $\iota\colon [0,\omega_1] \to \mathcal{K}_\varrho$ by $\alpha \mapsto [0,\alpha)$ (recall that $[0,\alpha) \in \mathcal{K}_\varrho$).

$$\begin{array}{ccccc} [0,\omega_1] & \overset{\iota}{\hookrightarrow} & \mathcal{K}_\varrho & \overset{\delta}{\hookrightarrow} & (B_{\mathcal{M}(\mathcal{K}_\varrho)}, w^*) \\ & & & \overset{e}{\searrow} & \Big\downarrow q \\ & & & & (B_{\mathcal{X}_\varrho{}^*}, w^*) \end{array}$$

- **Claim.** e is injective.
  - Note that $e(A) := \delta_A \restriction_{\mathcal{X}_\varrho}$;
  - Let $A \neq B \in \mathcal{K}_\varrho$ and pick $\gamma \in A \setminus B$;
  - $\langle \delta_A \restriction_{\mathcal{X}_\varrho}, f_\gamma \rangle = \langle \delta_A, f_\gamma \rangle = f_\gamma(A) = 1$;
  - $\langle \delta_B \restriction_{\mathcal{X}_\varrho}, f_\gamma \rangle = f_\gamma(B) = 0$.

■

$\mathrm{span}\{\mu_\gamma\restriction_{\mathcal{X}_\varrho}\}_{\gamma<\omega_1}$ is 1-norming

Claim. Let $A \in \mathcal{F}_\varrho$. Then

$$\delta_A\restriction_{\mathcal{X}_\varrho} = \sum_{\alpha\in A} \delta_{\{\alpha\}}\restriction_{\mathcal{X}_\varrho}.$$

Proof. Just check that $\langle \delta_A, f_\gamma\rangle = \left\langle \sum_{\alpha\in A} \delta_{\{\alpha\}}, f_\gamma \right\rangle$. ✓

In particular,

$$(\dagger) \qquad \{\delta_A\restriction_{\mathcal{X}_\varrho} \colon A \in \mathcal{F}_\varrho\} \subseteq \mathrm{span}\{\mu_\gamma\restriction_{\mathcal{X}_\varrho}\}_{\gamma<\omega_1}.$$

Finally, for every $f \in \mathcal{X}_\varrho$ we have

$$\|f\| = \max_{A\in\mathcal{K}_\varrho} |f(A)| = \sup_{A\in\mathcal{F}_\varrho} |f(A)| = \sup_{A\in\mathcal{F}_\varrho} |\langle \delta_A, f\rangle|$$

$$\overset{(\dagger)}{\leqslant} \sup\left\{ |\langle \mu, f\rangle| \colon \mu \in \mathrm{span}\{\mu_\gamma\restriction_{\mathcal{X}_\varrho}\}_{\gamma<\omega_1}, \|\mu\| \leqslant 1 \right\}.$$