Czech Technical University in Prague

Smooth renormings on dense subspaces of Banach spaces

Tommaso Russo
russotom@fel.cvut.cz

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The existence of a smooth norm on a Banach space bears several geometric consequences. Just to name a few:

- If a separable Banach space X has a C^1-smooth norm, X^* is separable;
- **Meshkov (1978).** If X and X^* admit a C^2-smooth norm, then X is isomorphic to a Hilbert space;
- **Fabian, Whitfield, Zizler (1983).** If X admits a C^2-smooth norm, either it contains c_0, or it is super-reflexive with type 2;
- **Deville (1989).** If X has a C^∞-smooth norm, either it contains c_0, or it is super-reflexive, with exact cotype $2k$, and it contains ℓ_{2k}.

A couple of references:

[DGZ] Deville–Godefroy–Zizler, *Smoothness and Renorming...*;

[HJ] Hájek–Johanis, *Smooth Analysis in Banach spaces*.

Let X be a normed space with a countable Hamel basis.

- **Vanderwerff (1992).** X has a Fréchet smooth norm;
- **Hájek (1995).** X has a C^∞-smooth norm;
- **Deville, Fonf, Hájek (1998).** Are analytic norms dense?

It was asked several times if the existence of a smooth norm on some 'large' subset of a Banach space X has similar consequences for X:

- Benyamini–Lindenstrauss, *Geometric Nonlinear Functional Analysis*;
- Guirao–Montesinos–Zizler, *Open problems...*, Problem 149: Does the space of finitely supported vectors in $\ell_1(\Gamma)$ have a C^1-smooth norm (when Γ is uncountable)?
- **(Rather bold?) Problem.** Given a Banach space X, is there a dense subspace of X that admits a C^k-smooth norm?
    - Yes, if X is separable (Hájek, 1995).

Analytic norms

Let ℓ_∞^F be the dense subspace of ℓ_∞ comprising sequences that assume finitely many values, *i.e.*,

$$\ell_\infty^F = \operatorname{span}\{\chi_A : A \subseteq \mathbb{N}\}.$$

Theorem (Dantas, Hájek, and R.)

The normed space ℓ_∞^F admits an analytic norm.

Corollaries

(i) Every normed space with a countable Hamel basis admits an analytic norm (and analytic norms are dense);

(ii) The space of finitely supported vectors in $\ell_1(\mathfrak{c})$ admits an analytic norm.

- The spaces considered in the corollary are subspaces of ℓ_∞;
- Approximate them suitably with elements from ℓ_∞^F;
- Transfer the smooth norm from ℓ_∞^F to them.

Let $\ell_\infty^{F,c}(\Gamma)$ be the dense subspace of $\ell_\infty^c(\Gamma)$ comprising functions with countable support and that assume finitely many values, *i.e.*,

$$\ell_\infty^{F,c}(\Gamma) = \operatorname{span}\{\chi_A : A \subseteq \Gamma, |A| \leqslant \omega\}.$$

Theorem (Dantas, Hájek, and R.)

Every renorming of $\ell_\infty^{F,c}(\omega_1)$ contains an isometric copy of $\ell_\infty^{F,c}(\omega_1)$.

- $\ell_\infty^{F,c}(\omega_1)$ has no Gâteaux-smooth norm;
- **Partington (1980).** Every renorming of $\ell_\infty^c(\omega_1)$ contains $\ell_\infty^c(\omega_1)$ isometrically.

Unconditional bases and C^∞-smoothness

Theorem (Dantas, Hájek, and R.)

Let X be a Banach space with long unconditional Schauder basis and let Y be the linear span of such basis. Then, Y admits a C^∞-smooth norm.

In particular, let Y be the space of finitely supported vectors in $\ell_1(\Gamma)$.

- Y admits a C^∞-smooth norm;
- if $|\Gamma| \leqslant \mathfrak{c}$, Y admits an analytic norm;
- Recall the question by Guirao, Montesinos, and Zizler.

Proposition (Dantas, Hájek, and R.)

No dense subspace of $c_0(\omega_1)$ admits an analytic norm.

- We cannot get analytic norms in the theorem above;
- Compare $c_0(\omega_1)$ and $\ell_1(\omega_1)$.

What's next?

- In the above results, we obtained a smooth norm on *one specific* dense subspace;
- What about other dense subspaces?
- The linear span of an M-basis is a 'very small' subspace:
  - Every its separable subspace has a countable Hamel basis;
  - In particular, it contains no infinite-dimensional Banach space.
- Can we build smooth norms on 'large' dense subspaces?
  - Operator ranges (*i.e.*, linear images of Banach spaces);
  - Normed spaces every whose dense subspace contains an infinite-dimensional Banach space.
- Is there a Banach space no whose dense subspace has a C^k-smooth norm?
- Let X be a Banach space such that every dense subspace contains a further dense subspace with a C^k-smooth norm. What can we say about X?

But not today

Thank you for your attention!