An uncountable version of Pták's lemma

(Joint work with P. Hájek)

Tommaso Russo
Faculty of Electrical Engineering
Czech Technical University in Prague

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Convex means

Given a set S, a *convex mean* λ on S is a function $\lambda\colon S \to [0,\infty)$ such that:

(i) the set $\operatorname{supp}(\lambda) := \{s \in S\colon \lambda(s) \neq 0\}$ is finite

(ii)
$$\sum_{s\in S} \lambda(s) = 1.$$

A family $\mathcal{F} \subseteq 2^S$ is *hereditary* if $G \subseteq F \in \mathcal{F}$ yields $G \in \mathcal{F}$.

Pták's combinatorial lemma, 1959

Let S be an infinite set and let $\mathcal{F} \subseteq [S]^{<\omega}$ be an hereditary family. If

$$(\dagger) \qquad \delta := \inf\left\{\sup_{F\in\mathcal{F}} \lambda(F)\colon \lambda \text{ is a convex mean on } S\right\} > 0,$$

there exists an infinite subset M of S such that every finite subset of M is in $\mathcal{F}$.

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Applications of the lemma: Krein Theorem

A subset A of a Banach space X *interchanges limits* if for every pair of sequences $(x_n) \subseteq A$ and $(x_k^*) \subseteq B_{X^*}$ the existence of

$$\lim_{n\to\infty} \lim_{k\to\infty} \langle x_k^*, x_n \rangle \qquad \lim_{k\to\infty} \lim_{n\to\infty} \langle x_k^*, x_n \rangle$$

implies their equality.

Grothendieck, 1952: A bounded subset A of a Banach space X is relatively weakly compact if and only if it interchanges limits.

Theorem (Pták, 1963)

If a bounded subset A of a Banach space interchanges limits, then so does $\operatorname{conv} A$.

Krein's theorem then follows immediately.

Applications of the lemma: Mazur Theorem

Mazur Theorem

Let $(f_n)_{n=1}^{\infty} \subseteq C(K)$ be a bounded sequence of continuous functions that converge pointwise to a continuous function f. Then f can be uniformly approximated by convex combinations of the f_n.

The classical proof depends on the Hahn–Banach and Riesz Representation Theorems. Pták's lemma allows for a self-contained simple proof, which, in particular, involves no measure theory whatsoever.

- Assume that $f = 0$ and $\|f_n\| \leqslant 1$. Consider, for $x \in K$, the finite set
$$F_x := \{n \in \mathbb{N} \colon |f_n(x)| \geqslant \varepsilon/2\}.$$
- Let $\mathcal{F}$ comprise all subsets of F_x, $x \in K$.
- Pták's lemma yields a convex mean λ on $\mathbb{N}$ such that $\lambda(F_x) < \varepsilon/2$ whenever $x \in K$.
- The function $\sum_{n=1}^{\infty} \lambda(n) f_n$ is then as desired.

A couple of ideas from the proof
(S.F. Bellenot, R. Haydon, and E. Odell)

- Condition (†) implies that

$$\|x\| := \sup_{F\in\mathcal{F}} \left|\sum_{s\in F} x(s)\right| \qquad x \in c_{00}(S)$$

  is equivalent to the $\ell_1(S)$-norm. Therefore, the completion X of $(c_{00}(S), \|\cdot\|)$ is isomorphic to $\ell_1(S)$.
- $F \in \mathcal{F}$ naturally defines a functional $F^* \in B_{X^*}$ by $x \mapsto \sum_{s\in F} x(s)$; since $\mathcal{F}^* := \{F^* : F \in \mathcal{F}\}$ is 1-norming for X, X embeds in $C\left(\overline{\mathcal{F}^*}^{w^*}\right)$.
- $\overline{\mathcal{F}^*}^{w^*}$ can be identified with the closure $\overline{\mathcal{F}}$ of $\mathcal{F}$ in the pointwise topology of 2^S.

Consequently, $C(\overline{\mathcal{F}})$ contains a copy of $\ell_1(S)$.

Uncountable extensions

Let κ be an infinite cardinal number.We say that *Pták's lemma holds true for* κ if for every set S with $|S| \geqslant \kappa$ and every hereditary family $\mathcal{F} \subseteq [S]^{<\omega}$ such that

$$(\dagger) \qquad \delta := \inf \left\{ \sup_{F \in \mathcal{F}} \lambda(F) \colon \lambda \text{ is a convex mean on } S \right\} > 0,$$

there is a subset M of S, with $|M| = \kappa$, such that every finite subset of M belongs to $\mathcal{F}$.

Theorem (Hájek and R., JMAA 2019)

The validity of Pták's lemma for ω_1 is independent of ZFC.

(i) (MA_{ω_1}) Pták's lemma holds true for ω_1;

(ii) (CH) Pták's lemma fails to hold for ω_1.

If κ is regular and $\lambda^\omega < \kappa$ whenever $\lambda < \kappa$, then Pták's lemma is true for κ.

A closely related problem

Let K be a Corson compact and τ be an uncountable cardinal.

Problem: Does $\ell_1(\tau)$ embed in $C(K)$?

- MA_{ω_1} implies that $C(K)$ is WLD, whence the answer is no;
- Under CH, $\ell_1(\omega_1)$ may embed in $C(K)$ (Erdős' space);
- $\ell_1(\mathfrak{c}^+)$ does not embed in $C(K)$ (Haydon);
- Is it consistent that $\ell_1(\omega_2)$ embeds in $C(K)$?

Thank you for your attention!