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On some open problems in Banach space theory

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Milano
February 1, 2019

References

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[HKR2] P. Hájek, T. Kania, and T. Russo, Separated sets and Auerbach systems in Banach spaces, preprint (2018), `arXiv:1711.05149`.

[HR1] P. Hájek and T. Russo, Some remarks on smooth renormings of Banach spaces, *J. Math. Anal. Appl.* **455** (2017), 1272–1284.

[HR2] P. Hájek and T. Russo, An uncountable version of Pták's combinatorial lemma, *J. Math. Anal. Appl.* **470** (2019), 1070–1080.

Overview

[HR1] Is dedicated to the problem of smooth approximation of norms in separable Banach spaces.

- **Hájek–Talponen, 2014:** If a separable Banach space admits a C^k smooth norm, then every equivalent norm can be approximated by C^k smooth ones.
- We sharpen this result by obtaining an 'asymptotically optimal' approximation, which, in particular, preserves every asymptotic property of the underlying Banach space.

[HR2] Is dedicated to a well known lemma due to Vlastimil Pták.

- Pták's lemma asserts the validity of a combinatorial property for the cardinal number ω.
- We show that the analogous property for the cardinal ω_1 is undecidable in ZFC.
- We also give sufficient conditions on a cardinal number κ for the validity of Pták's lemma for κ.

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Separated unit vectors

Hereinafter, X is an **infinite-dimensional** Banach space.

A subset A of X is *δ-separated* (resp. *$(\delta+)$-separated*) if $\|x-y\| \geqslant \delta$ (resp. $\|x-y\| > \delta$) for distinct $x, y \in A$.

The Riesz lemma (1916). The unit sphere of X contains a 1-separated sequence. (Consequently, B_X is not compact.)

Kottman's theorem (1975). The unit sphere of X contains a $(1+)$-separated sequence.

The Elton–Odell theorem (1981). S_X contains a $(1+\varepsilon)$-separated sequence (for some $\varepsilon > 0$, that depends on X).

Symmetric separation

Actually, in Riesz' lemma $\|x_n \pm x_k\| \geqslant 1$ for $n \neq k \in \mathbb{N}$.

Definition

A sequence $(x_n)_{n=1}^{\infty}$ in a normed space X is *symmetrically* $(\delta+)$*-separated* (respectively, *symmetrically* δ*-separated*) if $\|x_n \pm x_k\| > \delta$ (respectively, $\|x_n \pm x_k\| \geqslant \delta$) for distinct n, k.

The symmetric Kottman's constant

$$K^s(X) := \sup\left\{\sigma > 0\colon \exists (x_n)_{n=1}^{\infty} \subset B_X : \|x_n \pm x_k\| \geqslant \sigma \;\; \forall n \neq k\right\}.$$

Problem (J.M.F. Castillo and P.L. Papini, 2011):
Is $K^s(X) > 1$ for every infinite-dimensional Banach space?

Towards a symmetric Elton–Odell theorem

Theorem, [HKR1]

Let X be an infinite-dimensional Banach space. Then the unit sphere of X contains a symmetrically $(1+)$-separated sequence.

Theorem, [HKR1]

Let X be a Banach space that contains a boundedly complete basic sequence. Then, for some $\varepsilon > 0$, the unit sphere of X contains a symmetrically $(1+\varepsilon)$-separated sequence.

Consequences. Let X be infinite-dimensional. Then $K^s(X) > 1$ if:

- X is reflexive;
- X contains a (subspace isomorphic to a) separable dual;
- in particular, X has RNP (or, more generally, PCP);
- X contains an unconditional basic sequence.

Some quantitative results

- $K^s(\ell_p) = 2^{1/p}$, for $p \in [1, \infty)$;
- $K^s(X) = 2$, if X contains c_0 or ℓ_1 (James' non-distortion theorem);
- $K^s(X) = 2$, if X has a c_0 (or ℓ_1) quotient;
- **Castillo–Papini (2011).** If X is a $\mathscr{L}_\infty$-space, then $K^s(X) = 2$;
- **Delpech (2010).** $K^s(X) \geqslant 1 + \delta_X(1)$;
- **[HKR1].** $K^s(X) = 2$, if X has an ℓ_1 spreading model;
- **[HKR1].** In particular, $K^s(X) = 2$ for every renorming X of Tsirelson's space T;
- **[HKR1].** If X has non-trivial cotype q, then $K^s(X) \geqslant 2^{1/q}$.

Uncountable separated sets

Henceforth, X is a **non-separable** Banach space. Therefore B_X contains an uncountable ε-separated subset, for some $\varepsilon > 0$.

- Does S_X contain an uncountable $(1+)$-separated subset?
- Can we find a $(1+)$-separated subset with cardinality $\mathrm{dens}(X)$?
- What about $(1+\varepsilon)$-separated?

Elton–Odell (1981). Let $\mathcal{F} \subseteq S_{c_0(\Gamma)}$ be $(1+\varepsilon)$-separated, for some $\varepsilon > 0$. Then $\mathcal{F}$ is countable.

Theorem, [HKR2]

Let $\mathcal{F} \subseteq S_{c_0(\Gamma)}$ *be* $(1+)$*-separated. Then* $|\mathcal{F}| \leqslant \omega_1$.

A main question: Let X be non-separable. Does the unit sphere of X contain an uncountable $(1+)$-separated subset?

A glimpse at the literature

Non-separable $C(K)$ spaces:

- The unit sphere of $C(K)$ contains an uncountable $(1+)$-separated set (Kania–Kochanek; significantly improved by Cúth–Kurka–Vejnar);
- The existence of an uncountable $(1+\varepsilon)$-separated subset of the unit sphere of $C(K)$ is independent of ZFC (Koszmider).

Theorem (Kania–Kochanek, 2016)

- *Let X be a non-separable, reflexive Banach space. Then there is an uncountable $(1+)$-separated subset $\mathcal{F} \subseteq S_X$;*
- *Let X be super-reflexive and $\lambda \leqslant \operatorname{dens} X$ have uncountable cofinality. Then, for some $\varepsilon > 0$, S_X contains a $(1+\varepsilon)$-separated subset with cardinality λ.*

Reflexivity

Theorem, [HKR2]

Let X be a reflexive Banach space. Then:

- *The unit sphere of X contains a $(1+)$-separated subset of cardinality $\operatorname{dens} X$;*
- *For every $\lambda \leqslant \operatorname{dens} X$ with $\operatorname{cf} \lambda$ uncountable, S_X contains a $(1+\varepsilon)$-separated subset with cardinality λ.*

- We have better and optimal results, with simpler proofs;
- We obtain both clauses by means of the same argument;
- The same circle of ideas covers further classes of Banach spaces, most notably, Banach spaces with the Radon–Nikodym property.

Super-reflexivity

Example (Kania–Kochanek): the unit sphere of

$$X := \left(\bigoplus_{n \in \mathbb{N}} \ell_{p_n}(\omega_n) \right)_{\ell_2} \qquad (p_n)_{n=1}^{\infty} \subseteq (1, \infty),\ p_n \nearrow \infty$$

does not contain $(1+\varepsilon)$-separated subsets of cardinality $\omega_\omega = \operatorname{dens} X$.
However, X is not super-reflexive.

Theorem, [HKR2]

Let X be a super-reflexive Banach space. Then there exist $\varepsilon > 0$ and a $(1+\varepsilon)$-separated subset of S_X of cardinality $\operatorname{dens} X$.

Large Banach spaces

Theorem, [HKR2]

- *Assume that X is a 'large' Banach space (more precisely, assume $w^*\text{-}\operatorname{dens} X^* > \exp_2 \mathfrak{c}$). Then both S_X and S_{X^*} contain an uncountable $(1+)$-separated subset.*

  In particular, the assumption is satisfied whenever $\operatorname{dens} X > \exp_3 \mathfrak{c}$.

- *Let X be a WLD space with $\operatorname{dens} X \geqslant \mathfrak{c}^+$. Then S_X and S_{X^*} contain uncountable $(1+)$-separated subsets.*

Open problem: What about X WLD with $\operatorname{dens} X = \omega_1$?

An important ingredient in these proofs is the use of Auerbach systems; in particular, the existence of a 'large' Auerbach system allows us to construct such separated sets.

Auerbach systems

An *Auerbach system* is a collection $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma} \subseteq X \times X^*$ such that $\langle x^*_\alpha, x_\beta\rangle = \delta_{\alpha,\beta}$ and $\|x_\gamma\| = \|x^*_\gamma\| = 1$.

Theorem, [HKR2]

- *Let $\kappa \geqslant \mathfrak{c}$ be a cardinal number and let X be a Banach space with $w^*\text{-}\mathrm{dens}\, X^* > \exp_2 \kappa$. Then X admits a subspace Y with Auerbach basis and such that $\mathrm{dens}\, Y = \kappa^+$.*
- *Every WLD Banach space X with $\mathrm{dens}\, X > \omega_1$ contains a subspace Y with Auerbach basis and such that $\mathrm{dens}\, Y = \mathrm{dens}\, X$.*

For X separable, the result is a well-know theorem of M. Day.

Problem (1): What happens if X is WLD and $\mathrm{dens}\, X = \omega_1$?

Renorming $c_0(\omega_1)$

Problem (2), [GMZ]. Does there exist a non-separable Banach space X with unconditional basis such that no non-separable subspace of X has an Auerbach basis?

Theorem, [HKR2]

(CH) *There exists a renorming* $|||\cdot|||$ *of the space* $c_0(\omega_1)$ *such that the space* $(c_0(\omega_1), |||\cdot|||)$ *contains no uncountable Auerbach systems.*

Consequently, assuming the Continuum Hypothesis:

- There exists a non-separable Banach space with unconditional basis whose no non-separable subspace admits an Auerbach basis.
- There exists a WLD Banach space X with $\operatorname{dens} X = \omega_1$ every whose non-separable subspace fails to have an Auerbach basis.

Therefore, under CH, we can answer both Problems (1) and (2).

Thank you for your attention!