Faculty of Electrical Engineering
Czech Technical University in Prague

Overcomplete sets in Banach spaces

Tommaso Russo
(Joint work in progress with J. Somaglia)

Workshop on Abstract Analysis and Convex Analysis
Università Cattolica del Sacro Cuore
November 8, 2019

International Mobility of Researchers in CTU
Project number: CZ.02.2.69/0.0/0.0/16_027/0008465

EVROPSKÁ UNIE
Evropské strukturální a investiční fondy
Operační program Výzkum, vývoj a vzdělávání

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

Table of contents

Geometric vectors

Given $\lambda \in (0,1)$, consider the vector

$$g_\lambda := (1, \lambda, \lambda^2, \lambda^3, \lambda^4, \dots) \in c_0.$$

Klee (1958). If $J \subseteq [0, 1/2]$ is infinite, then

$$\overline{\operatorname{span}}\{g_\lambda \colon \lambda \in J\} = c_0.$$

Therefore, if $J = (\lambda_j)_{j=0}^\infty$ is any injective sequence, then every subsequence of $(g_{\lambda_j})_{j=0}^\infty$ is linearly dense.

Definition

A sequence $(x_j)_{j=0}^\infty$ in a normed space X is **overcomplete** if every its subsequence is complete (*i.e.*, linearly dense).

Existence of overcomplete sequences

Theorem (Klee, 1958)

Every separable Banach space contains an overcomplete sequence.

Proof.

- Let $(x_j)_{j=0}^{\infty}$ be a linearly dense sequence for X, $\|x_j\| = 1$.
- Consider the geometric vectors

$$g_\lambda := \sum_{j=0}^{\infty} \lambda^j x_j.$$

- If $(\lambda_j)_{j=0}^{\infty}$ is an injective sequence in $[0, 1/2]$, $(g_{\lambda_j})_{j=0}^{\infty}$ is overcomplete (same argument as before). □

A dichotomy

Definition (Terenzi)

A sequence $(x_j)_{j=0}^{\infty}$ in a normed space X is **overfilling** if it is overcomplete for $\overline{\mathrm{span}}\{x_j\}_{j=0}^{\infty}$.
In other words, every its subsequence is complete in $\overline{\mathrm{span}}\{x_j\}_{j=0}^{\infty}$.

Terenzi (1978). Let $(x_j)_{j=0}^{\infty}$ be a sequence in a Banach space X. Then $(x_j)_{j=0}^{\infty}$ admits a subsequence $(y_j)_{j=0}^{\infty}$ that satisfies one of the following alternatives:

(i) $(y_j)_{j=0}^{\infty}$ is a basic sequence;
(ii) $(y_j)_{j=0}^{\infty}$ is overfilling;
(iii) $y_j = u_j + v_j$, where $(u_j)_{j=0}^{\infty}$ is basic and $(v_j)_{j=0}^{\infty}$ is overfilling.

Compactness

Theorem (Fonf and Zanco, 2014)

Every bounded overcomplete sequence in a Banach space is relatively compact.

In the same paper:

- The notion of **overtotal** sequence is introduced;
- Both notions are weakened ($\rightsquigarrow$ **almost overcomplete** and **almost overtotal** sequences);
- Almost overtotal sequences are used to obtain a simple proof of the following result:
  Cariello and Seoane-Sepúlveda (2014). Let Y be a closed, infinite-dimensional subspace of ℓ_∞. Then there is a non-zero vector $y \in Y$ with infinitely many null coordinates.

Our motivation

- The proof of the result by Fonf and Zanco still bears some elements of mystery to the speaker;
- Some recent constructions use ‘geometric’ vectors to build some non-separable Banach spaces:

[HKR] P. Hájek, T. Kania, and T. Russo, Separated sets and Auerbach systems in Banach spaces, `arXiv:1711.05149`.

[H] P. Hájek, Hilbert generated Banach spaces need not have a norming Markushevich basis, *Adv. Math.* **351** (2019), 702–717.

[HR] P. Hájek and T. Russo, On densely isomorphic normed spaces, `arXiv:1910.01527`.

Overcomplete sequences have been studied in separable, complete normed spaces.

Overcomplete sequences in normed spaces

The argument by Klee needs completeness, for the convergence of geometric series.
There exists a 'finitely supported' construction (see, *e.g.*, Gurariy–Lusky, *Geometry of Muntz Spaces...*, p. 24).

Brass (1963). Every separable normed space admits an overcomplete sequence.

Proof.

Let $(x_j)_{j=1}^{\infty}$ be normalised and complete in X. Set, for $n \in \mathbb{N}$,

$$y_n := \sum_{j=1}^{n} \frac{1}{j^n} x_j$$

Then $(y_j)_{j=1}^{\infty}$ is overcomplete. □

Terenzi (1982). Overcompleteness is not stable.

Do we still have compactness?

Proposition R&S

Let X be an incomplete, separable normed space. Then there exists an overcomplete sequence in X that is not relatively compact.

Proof.

- Let $(y_j)_{j=1}^{\infty}$ converge 'fast' to a vector of $\hat{X} \setminus X$;
- Let $(x_j)_{j=1}^{\infty}$ be normalised and complete in X;
- The desired sequence is

$$g_n = y_n + \sum_{j=1}^{n} \frac{1}{(j+1)^n} x_j.$$

□

Towards ω_1 and beyond

Definition

Let X be a Banach space. A subset S of X, with $|S| = \operatorname{dens} X$, is **overcomplete** if every subset Λ of S, with $|\Lambda| = |S|$, is complete in X.

Particular case. If $\operatorname{dens} X = \omega_1$, A is overcomplete if $|A| = \omega_1$ and every its uncountable subset is complete.

- Do non-separable Banach spaces have overcomplete sets?
- Which properties can these overcomplete sets have?
- Relatively compact sets are separable. So, no overcomplete set can be relatively compact if X is non-separable.
- What about weak compactness? $\rightsquigarrow$ WCG spaces.

The basic existence result

'Avoid countably many hyperplanes'

Theorem R&S

(CH) Let X be a Banach space with $\operatorname{dens} X = \operatorname{dens} X^* = \omega_1$. Then X contains an overcomplete set.

Proof.

- $\mathfrak{c} \leqslant |X^*| \leqslant (\omega_1)^\omega = \mathfrak{c}$.
- Let $(H_\alpha)_{\alpha<\omega_1}$ be an enumeration of all hyperplanes of X.
- Find an injective sequence $(x_\beta)_{\beta<\omega_1}$ with

$$x_\beta \notin H_\alpha \qquad (\alpha < \beta).$$

- Every hyperplane of X contains at most countably many x_β's. □

Negative results

Every Banach space is union of $\mathfrak{c}$ hyperplanes. Therefore,

Proposition R&S

Let X be a Banach space such that $\mathrm{cf}(\mathrm{dens}\, X) \geqslant \mathfrak{c}^+$. Then X contains no overcomplete set.

By the same argument, we also obtain:

- Let X be a Banach space with M-basis and such that $\mathrm{cf}(\mathrm{dens}\, X) \geqslant \omega_2$. Then X contains no overcomplete set;
- Indeed, such X is union of ω_1 hyperplanes;
- We can improve the above result (using Hajnal Theorem).

Theorem R&S

Let X be a Banach space with M-basis. If $\mathrm{dens}\, X \geqslant \omega_2$, X contains no overcomplete set.

Recap

WLD Banach spaces $\equiv$ A huge class of Banach spaces, with a weird def.

Theorem R&S

Let X be a WLD Banach space.

(i) (CH) If $\operatorname{dens} X = \omega_1$, X contains an overcomplete set;

(ii) If $\operatorname{dens} X \geqslant \omega_2$, X contains no overcomplete set.

Problem. What about (i) under, say, MA_{ω_1}?

Theorem R&S

$\ell_1(\omega_1)$ does not contain overcomplete sets.

Density

- Every overcomplete sequence in an infinite-dimensional Banach space is nowhere dense.
  [It is relatively compact, after all.]
- A piece of folklore: every finite-dimensional Banach space contains a dense overcomplete sequence.

Proposition R&S

(CH) Let X be a Banach space with $\operatorname{dens} X = \operatorname{dens} X^* = \omega_1$. Then:

- X contains a dense overcomplete set;
- B_X contains a $(1-\varepsilon)$-separated overcomplete set.

Every uncountable subset of a WLD Banach space has a weak cluster point.

Compactness

- No overcomplete set in a non-separable Banach space can be relatively compact;
- If an overcomplete set for X is relatively weakly compact, then X is WCG;
- If X is reflexive, any (bounded) overcomplete set is relatively weakly compact.

Theorem $^{\mathrm{R\&S}}$

(CH) Let X be a WCG Banach space with $\operatorname{dens} X = \omega_1$. Then X contains a relatively weakly compact overcomplete set.

Every WCG Banach space admits a (bounded) weakly compact M-basis.

Thank you for your attention!