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# On densely isomorphic normed spaces

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Banach spaces and optimization
Métabief, France
June 16–21, 2019

International Mobility of Researchers in CTU
Project number: CZ.02.2.69/0.0/0.0/16_027/0008465

EVROPSKÁ UNIE
Evropské strukturální a investiční fondy
Operační program Výzkum, vývoj a vzdělávání

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

# The starting question



## Problem

*How different can two dense subspaces of a Banach space be?*

In this talk, subspaces are **NOT** assumed to be closed.

**Example 1:** $c_{00}$ and $c_0$ (as dense subspaces of $c_0$).

- $c_{00}$ is meager, has a countable Hamel basis, is 'very small';
- $c_0$ is a Baire space.

**Example 2:** An incomplete normed space $X$ and its completion $\widetilde{X}$.

**Example 3:** $c_{00}$ and $(\ell_1, \|\cdot\|_\infty)$ (as dense subspaces of $c_0$).

- There is a (non-equivalent) complete norm on $(\ell_1, \|\cdot\|_\infty)$;
- There is no complete norm on $c_{00}$.

# Setting the scene



## Definition

Two normed spaces $X$ and $Y$ are *densely isomorphic* if there exist dense subspaces $X_0$ of $X$ and $Y_0$ of $Y$ such that $X_0$ and $Y_0$ are isomorphic.

- Isomorphic normed spaces are densely isomorphic, *a fortiori*;
- Every normed space $X$ is densely isomorphic to its completion $\widetilde{X}$;
- If $X$ and $Y$ are densely isomorphic, then $\widetilde{X}$ and $\widetilde{Y}$ are isomorphic;
- Two densely isomorphic Banach spaces are isomorphic.

In particular, given densely isomorphic normed spaces, we can assume that they are dense subspaces of the same Banach space.

## Problem

*Let $Y$ and $Z$ be dense subspaces of a Banach space $X$. Must $Y$ and $Z$ be densely isomorphic?*

# The separable case



## Theorem A

*Let $Y$ and $Z$ be dense subspaces of a separable Banach space $X$. Then $Y$ and $Z$ are densely isomorphic.*

*Proof.* We apply a perturbation argument to an M-basis of $X$.

- Let $\{e_j; e_j^*\}_{j=1}^{\infty}$ be a bounded M-basis for $X$ and $(\varepsilon_j)_{j=1}^{\infty} \searrow 0$;
- Find $(y_j)_{j=1}^{\infty} \subseteq Y$ and $(z_j)_{j=1}^{\infty} \subseteq Z$ with $\|y_j - e_j\|, \|z_j - e_j\| < \varepsilon_j$;
- As in the proof of the small perturbation lemma, we prove that $Y_0 := \mathrm{span}(y_j)_{j=1}^{\infty}$ and $Z_0 := \mathrm{span}(z_j)_{j=1}^{\infty}$ are isomorphic;
- Finally, $Y_0$ and $Z_0$ are dense in $X$.

## Remark

Any two dense subspaces $Y$ and $Z$ of $\ell_1(\Gamma)$ are densely isomorphic.

# The non-separable nature of the problem



A Banach space $X$ is *weakly Lindelöf determined* (hereinafter, *WLD*) if the dual ball $B_{X^*}$ is a Corson compact in the relative $w^*$-topology.

For our purposes, $X$ is WLD if it admits an M-basis $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ that *countably supports* $X^*$, *i.e.*,

$$\operatorname{supp} x^* := \{\gamma \in \Gamma \colon \langle x^*, x_\gamma \rangle \neq 0\}$$

is a countable subset of $\Gamma$, for every $x^* \in X^*$.

## Theorem B

*Let $X$ be a WLD Banach space such that $\omega_1 \leqslant \operatorname{dens} X \leqslant \mathfrak{c}$. Then there exists a closed subspace $X_0$ of $X$, with $\operatorname{dens} X_0 = \operatorname{dens} X$, that contains two dense subspaces $Y$ and $Z$ which are not densely isomorphic.*

**Corollary.** There exist two dense subspaces $Y$ and $Z$ of the Hilbert space $\ell_2(\omega_1)$ that are not densely isomorphic.

# Finding the subspaces



- Let $\{e_\gamma; e_\gamma^*\}_{\gamma<\Gamma}$ be a normalised M-basis for $X$ ($\Gamma := \operatorname{dens} X$);
- Pick an injective long sequence $(q_\gamma)_{\omega\leqslant\gamma<\Gamma}$ in $(0,1)$;
- Set

$$\tilde{e}_\gamma := e_\gamma + \sum_{j=1}^{\infty} (q_\gamma)^j \cdot e_j \qquad (\omega \leqslant \gamma < \Gamma);$$

- $X_0 := \overline{\operatorname{span}}\{\tilde{e}_\gamma\}_{\omega\leqslant\gamma<\Gamma}$ (a WLD Banach space);
- $Y := \operatorname{span}\{\tilde{e}_\gamma\}_{\omega\leqslant\gamma<\Gamma}$;
- $Z := \operatorname{span}\{v_\alpha\}_{\alpha<\Gamma}$, where $\{v_\alpha; \varphi_\alpha\}_{\alpha<\Gamma}$ is an M-basis for $X_0$;
- **Fact 1.** $(e_n^*)_{n=1}^{\infty}$ separates points on $Y$ (Vandermonde matrices);
- **Fact 2.** No dense subspace of $Z$ admits a separating sequence of functionals.

Therefore, $Y$ and $Z$ are not densely isomorphic.

# A stronger result, under CH



## Theorem C

(CH) *Let $X$ be a WLD Banach space with* $\operatorname{dens} X = \omega_1$. *Then there exists a dense subspace $Y$ of $X$ that contains no uncountable biorthogonal system.*

**Particular case.** (CH) There exists a dense subspace of the Hilbert space $\ell_2(\omega_1)$ that contains no uncountable biorthogonal system.

## Lemma

Let $\{e_\alpha; e_\alpha^*\}_{\alpha\in\Gamma}$ be an M-basis for a Banach space $X$. Then every non-separable subspace of $Z := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$ contains an uncountable biorthogonal system.

Therefore, no non-separable subspace of $Y$ is isomorphic to a subspace of $Z$ (and, in particular, $Y$ and $Z$ are not densely isomorphic).

*Thank you for your attention!*