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Projections onto spaces of polynomials

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The starting point

- A Banach space X has the *AP* if for every compact set $K \subseteq X$ and $\varepsilon > 0$ there exists a finite-rank, bounded linear operator $T \colon X \to X$ such that $\|Tx - x\| < \varepsilon$ $(x \in K)$;
- X has λ-*BAP* if additionally $\|T\| \leqslant \lambda$.

Theorem (Godefroy and Kalton, 2003)

A Banach space X has the λ-BAP if and only if $\mathcal{F}(X)$ has the λ-BAP.

- In particular, $\mathcal{F}(\ell_2)$ has the MAP ($\equiv 1$-BAP).

Problem (Godefroy)

Does $Lip_0(\ell_2)$ have the AP?

- **Grothendieck (1955).** The AP passes from X^* to X;
- Hence, this would be a stronger result.

Our approach

For a Banach space X, $\mathcal{P}(^2X)$ is a subspace of $Lip_0(X)$.

- $\mathcal{P}(^2X)$ is the Banach space of bounded 2-homogeneous polynomials on X.
  - $P \in \mathcal{P}(^2X)$ if there is a bounded bilinear map $M\colon X \times X \to \mathbb{R}$ such that $P(x) = M(x, x)$;
  - $\|P\|_{\mathcal{P}} = \sup_{x \in B_X} |P(x)|$.
- But polynomials are **not** Lipschitz functions!
- However, they are Lipschitz on the unit ball.
  - Therefore, $\mathcal{P}(^2X)$ is a natural subspace of $Lip_0(B_X)$;
  - Moreover, $\|\cdot\|_{\mathcal{P}}$ is equivalent to $\|\cdot\|_{Lip}$.
- Consequently, $\mathcal{P}(^2X)$ is naturally isomorphic to a subspace of $Lip_0(B_X)$, via the restriction map

$$P \mapsto P\restriction_{B_X}.$$

Why such approach?

- The AP passes to complemented subspaces;
- **Dineed and Mujica (2015).** $\mathcal{P}(^2\ell_2)$ does not have the AP;
- So, if $\mathcal{P}(^2\ell_2) \subseteq \mathit{Lip}_0(B_{\ell_2})$ is complemented, then $\mathit{Lip}_0(B_{\ell_2})$ fails to have the AP.

Question

Is $\mathcal{P}(^2\ell_2) \subseteq \mathit{Lip}_0(B_{\ell_2})$ a complemented subspace?

- **Kaufmann (2015).** $\mathit{Lip}_0(X)$ is isomorphic to $\mathit{Lip}_0(B_X)$;
- Thus, a positive answer to this question would yield that $\mathit{Lip}_0(\ell_2)$ fails to have the AP.

Some more motivation

Question (*Repetita iuvant*)

Is $\mathcal{P}(^2\ell_2) \subseteq Lip_0(B_{\ell_2})$ a complemented subspace?

Theorem (Lindenstrauss, 1964)

X^* is a 1-complemented subspace of $Lip_0(X)$.

- Evidently, $X^* = \mathcal{P}(^1X)$;
- The above question is also about the possibility to extend Lindenstrauss' result to polynomials;
- If 'yes', we can answer in the negative Godefroy's question;
- If 'no', Lindenstrauss' result admits no polynomial version.

Sooo..., Yes or no?

Theorem (Hájek and R.)

NO. $\mathcal{P}(^2\ell_2) \subseteq Lip_0(B_{\ell_2})$ is not complemented.

- We didn't solve the problem we started with;
- But, at least, we can tell that this is not the correct approach.
- **Aron and Schottenloher (1976).** $\mathcal{P}(^n\ell_1)$ is isomorphic to ℓ_∞.

The result follows from a finite-dimensional, quantitative counterpart.

Theorem (Hájek and R.)

Let E_n be $\mathbb{R}^n$ with euclidean norm. If Q is any projection from $Lip_0(B_{E_n})$ onto $\mathcal{P}(^2E_n)$, then

$$\|Q\| \geqslant c \cdot \left(n - 2\sqrt{2}\right)^{1/5}.$$

A few consequences

- If a Banach space X contains $(\ell_2^n)_{n=1}^\infty$ uniformly complemented, then $\mathcal{P}(^2X)$ is not complemented in $Lip_0(B_X)$;
- **Tzafriri (1974).** If a Banach space admits an unconditional basis, then there is $p \in \{1, 2, \infty\}$ such that $(\ell_p^n)_{n=1}^\infty$ is uniformly complemented in X;
- If, additionally, X has non-trivial type, it must be $p = 2$.

Corollary/Theorem (Hájek and R.)

If a Banach space X an unconditional basis and non-trivial type, then $\mathcal{P}(^2X)$ is not complemented in $Lip_0(B_X)$.

- ℓ_p $(1 < p < \infty)$;
- L_p $(1 < p < \infty)$;
- Recall: $\mathcal{P}(^2\ell_1)$ is complemented in $Lip_0(B_{\ell_1})$ (Aron–Schottenloher).