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What do dense subspaces of Hilbert spaces look like?

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P. Hájek and T. Russo, *On densely isomorphic normed spaces*.
Preprint available at `arXiv:1910.01527`.

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Smoothness and structure

- The existence of a smooth norm on a Banach space bears several geometric consequences. Just to name a few:
  - If a separable Banach space X has a C^1 norm, X^* is separable;
  - **Deville (1989).** If X has a C^∞-smooth norm, either it contains c_0, or it is super-reflexive, with exact cotype $2k$, and it contains ℓ_{2k}.
- It was asked several times if the existence of a smooth norm on some dense subspaces has similar consequences:
  - Benyamini–Lindenstrauss, *Geometric Nonlinear Functional Analysis*;
  - Guirao–Montesinos–Zizler, *Open problems in the geometry and analysis of Banach spaces*.
- There exists a dense subspace of ℓ_∞ with a C^∞-smooth norm;
  - Is there a dense subspace of ℓ_∞ no whose dense subspace has a smooth norm?
- How different can two dense subspaces of a Banach space be?

Separable Banach spaces

In every separable Banach space there is a canonical (smallest) dense subspace, that is densely contained in every other dense subspace. More precisely:

(Folklore?) Proposition

Let $\{e_j; e_j^*\}_{j=1}^{\infty}$ be an M-basis for a separable Banach space X. Then every dense subspace of X contains a dense subspace isomorphic to $\mathrm{span}\{e_j\}_{j=1}^{\infty}$.

- $\mathrm{span}\{e_j\}_{j=1}^{\infty}$ is this 'minimal' dense subspace;
- This feature breaks down completely in many non-separable Banach spaces;
- In the talk, we will see this in Hilbert spaces;
- Closed subspaces of Hilbert spaces are what they are, but dense subspaces can be quite diverse.

Inner product spaces

For an orthonormal system $\{e_\gamma\}_{\gamma\in\Gamma}$ in an inner produce space H, TFAE:

(i) $\{e_\gamma\}_{\gamma\in\Gamma}$ is complete (*i.e.*, linearly dense);
(ii) $\{e_\gamma\}_{\gamma\in\Gamma}$ is a Schauder basis;
(iii) Parseval's equality $\|x\|^2 = \sum_{\gamma\in\Gamma} |\langle e_\gamma, x\rangle|^2$ holds for every $x \in H$.

$$\Downarrow$$

(iv) $\{e_\gamma\}_{\gamma\in\Gamma}$ is maximal.

Gudder (1974). There exists a non-separable inner product space that contains no uncountable orthonormal system.
[See Halmos, *A Hilbert space problem book*, Problem 54.]

Buhagiar, Chetcuti, and Weber (2008). If $\operatorname{dens} H \geqslant \mathfrak{c}^+$, H contains an uncountable orthonormal system (actually, of cardinality $\mathfrak{c}^+$).

Different dense subspaces

Theorem A

Let H be a non-separable Hilbert space. Then there are two dense subspaces Y and Z of H whose every dense subspaces are non-isomorphic.

- In $\ell_2(\mathfrak{c})$ there exist two dense subspaces Y and Z such that no non-separable subspace of Y is isomorphic to a subspace of Z;
- A 'minimal' subspace as before has to be separable.

Definition

Two normed spaces X and Y are *densely isomorphic* if there exist dense subspaces X_0 of X and Y_0 of Y such that X_0 and Y_0 are isomorphic.

Th A (restated). Every non-separable Hilbert space contains two dense subspaces that are not densely isomorphic.

Biorthogonal systems

Theorem B

(CH) *There exists a dense subspace of the Hilbert space $\ell_2(\omega_1)$ that contains no uncountable biorthogonal system.*

Recall: Every inner product space with density at least $\mathfrak{c}^+$ contains an uncountable orthonormal system (Buhagiar–Chetcuti–Weber).

Lemma

Let $\{e_\alpha\}_{\alpha<\mathfrak{c}}$ be the canonical basis for $\ell_2(\mathfrak{c})$. Then every non-separable subspace of $Z := \mathrm{span}\{e_\alpha\}_{\alpha<\mathfrak{c}}$ contains an uncountable biorthogonal system.

Again, a common subspace is separable.

☢ Thm$^{\mathrm{s}}$ A and B are valid, more generally, for WLD Banach spaces. ☢

Thank you for your attention!