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Separated families of unit vectors

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A separable overture

Hereinafter, X is an **infinite-dimensional** Banach space.

The Riesz lemma (1916). There exists a sequence $(x_n)_{n=1}^{\infty}$ in the unit sphere S_X with $\|x_n - x_k\| \geqslant 1$ for $n \neq k$.

In plain words: S_X contains a 1-*separated* sequence.

Kottman's theorem (1975). There exists a sequence $(x_n)_{n=1}^{\infty}$ in the unit sphere S_X such that $\|x_n - x_k\| > 1$ for $n \neq k$.

In plain words: S_X contains a $(1+)$-*separated* sequence.

The Elton–Odell theorem (1981). S_X contains a $(1+\varepsilon)$-separated sequence $(x_n)_{n=1}^{\infty}$ (for some $\varepsilon > 0$, that depends on X).

And its non-separable counterpart

General problem: How large can separated subsets of S_X be?

- If X is separable, they are at most countable!
- If X is non-separable, S_X contains (for some $\delta > 0$) a δ-separated subset of cardinality $\operatorname{dens} X$.
- Does S_X contain a $(1+)$-separated subset of cardinality $\operatorname{dens} X$?
- What about $(1+\varepsilon)$-separation?

A few reassuring examples:

- In $S_{\ell_\infty(\Gamma)}$ we have a 2-separated set of cardinality $2^{|\Gamma|}$; ✓
- In $\ell_p(\Gamma)$, the canonical basis suffices; ✓
- $S_{c_0(\omega_1)}$ contains an uncountable $(1+)$-separated set. ✗

Limitations: $c_0(\Gamma)$ spaces

Remark (J. Elton and E. Odell, 1981)

Let $\mathcal{F} \subseteq S_{c_0(\Gamma)}$ be $(1+\varepsilon)$-separated, for some $\varepsilon > 0$. Then $\mathcal{F}$ is countable.

The proof is a simple exercise for students. They may want to have the hint that the Δ-*system lemma* is the key.

Theorem A

Let $\mathcal{F} \subseteq S_{c_0(\Gamma)}$ *be* $(1+)$*-separated. Then* $|\mathcal{F}| \leqslant \omega_1$*.*

A main question: Let X be non-separable. Does the unit sphere of X contain an uncountable $(1+)$-separated subset?

The rôle of Auerbach systems

Theorem B

- *Assume that X is a 'large' Banach space (more precisely, assume $w^*\text{-}\mathrm{dens}\, X^* > \exp_2 \mathfrak{c}$). Then both S_X and S_{X^*} contain an uncountable $(1+)$-separated family.*

  In particular, the assumption is satisfied whenever $\mathrm{dens}\, X > \exp_3 \mathfrak{c}$.
- *Let X be a WLD space with $\mathrm{dens}\, X \geqslant \mathfrak{c}^+$. Then S_X and S_{X^*} contain uncountable $(1+)$-separated families.*

For this, we need to prove general results concerning the existence of Auerbach systems in Banach spaces. But this is another story (*i.e.*, talk).

(Sub-)Question: What about X WLD with $\mathrm{dens}\, X = \omega_1$?

Time flies

(credits to Marco Russo)

A glimpse at the literature

Non-separable $C(K)$ spaces:

- The unit sphere of $C(K)$ contains an uncountable $(1+)$-separated set (Kania–Kochanek; significantly improved by Cúth–Kurka–Vejnar);
- The existence of an uncountable $(1+\varepsilon)$-separated family in the unit sphere of $C(K)$ is independent of ZFC (Koszmider).

Theorem (T. Kania and T. Kochanek, 2016)

- *Let X be a non-separable, reflexive Banach space. Then there is an uncountable $(1+)$-separated family $\mathcal{F} \subseteq S_X$;*
- *Let X be super-reflexive and $\lambda \leqslant \operatorname{dens} X$ have uncountable cofinality. Then, for some $\varepsilon > 0$, S_X contains a $(1+\varepsilon)$-separated family with cardinality λ.*

Reflexive spaces

Theorem C

- *Let X be a reflexive Banach space. Then there is a $(1+)$-separated family $\mathcal{F} \subseteq S_X$ such that $|\mathcal{F}| = \mathrm{dens}\, X$;*
- *Let X be reflexive and $\lambda \leqslant \mathrm{dens}\, X$ have uncountable cofinality. Then, for some $\varepsilon > 0$, S_X contains a $(1+\varepsilon)$-separated family with cardinality λ.*

- We obtain both clauses by means of the same argument;
- The same circle of ideas covers, *e.g.*, RNP spaces: if X has RNP,
  - there is a $(1+)$-separated family $\mathcal{F} \subseteq S_X$ with $|\mathcal{F}| = w^*\text{-}\mathrm{dens}\, X^*$;
  - for every $\lambda \leqslant w^*\text{-}\mathrm{dens}\, X^*$ with $\mathrm{cf}(\lambda) > \omega$, S_X contains a $(1+\varepsilon)$-separated family with cardinality λ.

Super-reflexivity

Example (Kania–Kochanek): the unit sphere of

$$X := \left(\bigoplus_{n\in\mathbb{N}} \ell_{p_n}(\omega_n) \right)_{\ell_2} \qquad (p_n)_{n=1}^{\infty} \subseteq (1,\infty),\ p_n \nearrow \infty$$

does not contain $(1+\varepsilon)$-separated subsets of cardinality $\omega_\omega = \operatorname{dens} X$. However, X is not super-reflexive.

Theorem D

Let X be a super-reflexive Banach space. Then there exist $\varepsilon > 0$ and a $(1+\varepsilon)$-separated subset of S_X of cardinality $\operatorname{dens} X$.

Thank you for your attention!