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Symmetrically separated sequences in the unit sphere of a Banach space

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Proem

Hereinafter, X is an **infinite-dimensional** Banach space.

The Riesz lemma (1916). There exists a sequence $(x_n)_{n=1}^{\infty}$ in the unit sphere S_X with $\|x_n - x_k\| \geqslant 1$ for $n \neq k$ (*i.e.* a 1-*separated* sequence).

Kottman's theorem (1975). The unit sphere S_X contains a $(1+)$-*separated* sequence $(x_n)_{n=1}^{\infty}$, *i.e.* $\|x_n - x_k\| > 1$ for $n \neq k$.

A very short proof: J. Diestel *Sequences and series in Banach spaces*, pp. 7–8.

The Elton–Odell theorem (1981). S_X contains a $(1+\varepsilon)$-separated sequence $(x_n)_{n=1}^{\infty}$ (for some $\varepsilon > 0$).

Kryczka–Prus (2000). If X is non-reflexive, the unit sphere of X contains a $\sqrt[5]{4}$-separated sequence.

Symmetric separation

Actually, in Riesz' lemma $\|x_n \pm x_k\| \geqslant 1$ for $n \neq k \in \mathbb{N}$.

Definition

A sequence $(x_n)_{n=1}^{\infty}$ in a normed space X is *symmetrically* $(\delta+)$*-separated* (respectively, *symmetrically* δ*-separated*) if $\|x_n \pm x_k\| > \delta$ (respectively, $\|x_n \pm x_k\| \geqslant \delta$) for distinct n, k.

The symmetric Kottman's constant

$$K^s(X) := \sup \left\{ \sigma > 0 \colon \exists (x_n)_{n=1}^{\infty} \subset B_X : \|x_n \pm x_k\| \geqslant \sigma \ \forall n \neq k \right\}.$$

Problem (J. M. F. Castillo and P. L. Papini, 2011):
Is $K^s(X) > 1$ for every infinite-dimensional Banach space?

Some quantitative results

- $K^s(\ell_p) = 2^{1/p}$ for $p \in [1, \infty)$;
- $K^s(X) = 2$ if X contains c_0 or ℓ_1 (James' non-distortion theorem);
- $K^s(X) = 2$ if X has a c_0 (or ℓ_1) quotient;
- $K^s(X) = 2$ if X has an ℓ_1 spreading model;
- **Castillo–Papini (2011).** If X is a $\mathscr{L}_\infty$-space, then $K^s(X) = 2$;
- **Delpech (2010).** $K^s(X) \geqslant 1 + \overline{\delta}_X(1)$.

Symmetry and cotype

Prus (2010). If X has cotype $q < \infty$, then $K(X) \geqslant 2^{1/q}$.

Problem: What about $K^s(X)$?

Theorem (P. Hájek, T. Kania, and R.)

Let X be an infinite-dimensional Banach space. If X contains a normalized basic sequence with a *lower q-estimate*, then $K^s(X) \geqslant 2^{1/q}$.
In particular, if X has non-trivial cotype q, then $K^s(X) \geqslant 2^{1/q}$.

A sequence $(x_n)_{n=1}^{\infty}$ is a normed space X has a *lower q-estimate* if

$$c \cdot \left(\sum_{i=1}^{N} |a_i|^q \right)^{1/q} \leqslant \left\| \sum_{i=1}^{N} a_i x_i \right\| .$$

Symmetric Kottman's theorem

Theorem (P. Hájek, T. Kania, and R.)

Let X be an infinite-dimensional Banach space. Then the unit sphere of X contains a symmetrically $(1+)$-separated sequence $(x_n)_{n=1}^{\infty}$, *i.e.* $\|x_n \pm x_k\| > 1$ for $n \neq k$.

Time flies

Towards a symmetric Elton–Odell theorem

Theorem (P. Hájek, T. Kania, and R.)

Let X be a Banach space that contains a boundedly complete basic sequence. Then, for some $\varepsilon > 0$, the unit sphere of X contains a symmetrically $(1+\varepsilon)$-separated sequence.

Consequences. Let X be infinite-dimensional. Then $K^s(X) > 1$ if:

- X is reflexive;
- X contains a (subspace isomorphic to a) separable dual;
- in particular, X has RNP (or, more generally, PCP);
- X contains an unconditional basic sequence (or, more generally, an infinite-dimensional subspace isomorphic to a Banach lattice).

A non-separable interlude

Theorem (P. Hájek, T. Kania, and R.)

Let X be a reflexive Banach space. Then:

- the unit sphere of X contains a symmetrically $(1+)$-separated family with cardinality $\mathrm{dens}(X)$;
- for every cardinal $\lambda \leqslant \mathrm{dens}(X)$ with $\mathrm{cf}(\lambda)$ uncountable there exists ($\varepsilon > 0$ and) a symmetrically $(1+\varepsilon)$-separated family of unit vectors with cardinality λ.

The same circle of ideas also provides us with (weaker) assertions concerning Radon–Nikodym spaces, duals to weak-Asplund spaces, strictly convex spaces, and something else.

Proof of the symmetric Kottman

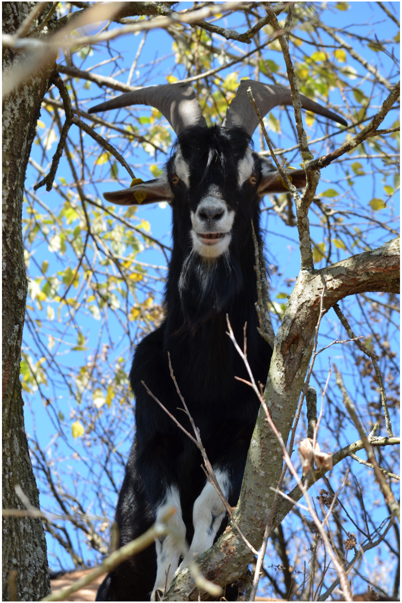
For $\tilde{X} \subseteq X$, $\dim(\tilde{X}) = \infty$, we say that

$\tilde{X}$ has (🪫) if: $\exists x \in S_{\tilde{X}}, \exists Y \subseteq \tilde{X}, \dim(Y) = \infty : \forall y \in S_Y \; \|x + y\| > 1.$

Case 1: Every $\tilde{X} \subseteq X$, $\dim(\tilde{X}) = \infty$, has (🪫). This is very easy.
Case 2: WLOG, X has ($\neg$🪫). This is easily equivalent to:

$$(🔋) \quad \forall x \in B_X, \forall Y \subseteq X, \dim(Y) = \infty, \; \exists y \in S_Y : \|x + y\| \leqslant 1.$$

Thank you for your attention!



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