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Separated families of unit vectors

An interplay of geometry and combinatorics

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Workshop in Abstract Analysis

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Proem

Hereinafter, X is an **infinite-dimensional** Banach space.

The Riesz lemma (1916). There exists a sequence $(x_n)_{n=1}^{\infty}$ in the unit sphere S_X with $\|x_n - x_k\| \geqslant 1$ for $n \neq k$ (*i.e.* a 1-*separated* sequence).

Kottman's theorem (1975). The unit sphere S_X contains a $(1+)$-*separated* sequence $(x_n)_{n=1}^{\infty}$, *i.e.* $\|x_n - x_k\| > 1$ for $n \neq k$.

The Elton–Odell theorem (1981). The unit sphere S_X contains a $(1+\varepsilon)$-separated sequence $(x_n)_{n=1}^{\infty}$ (for some $\varepsilon > 0$).

Kottman's constant

Definition

$$K(X) := \sup \left\{ \sigma > 0 \colon \exists (x_n)_{n=1}^{\infty} \subset B_X : \|x_n - x_k\| \geqslant \sigma \ \forall n \neq k \right\}.$$

- **Elton–Odell.** $K(X) > 1$ for every X;
- **Kryczka–Prus (2000).** $K(X) \geqslant \sqrt[5]{4}$ for non-reflexive X;
- **Delpech (2010).** $K(X) \geqslant 1 + \overline{\delta}_X(1)$ ($\geqslant 1 + \delta_X(1)$);
- **Maluta–Papini (2009).** $K(X) \leqslant 2 - 2\delta_X(1)$;
- $K(\ell_p) = 2^{1/p}$ for $p \in [1, \infty)$ (note the equality!);
- $K(X) = 2$ if X contains c_0 or ℓ_1 (James' non-distortion theorem).

Problem (E. Maluta and P. L. Papini, 2009)

Assume that $K(X, |||\cdot|||) = 2$ for every renorming $|||\cdot|||$ of X. Does it follow that X is non-reflexive?

- **Mathematical folklore(?)** If X admits a spreading model isomorphic to ℓ_1, then $K(X) = 2$;
- Every spreading model of the Tsirelson's space T is isomorphic to ℓ_1 (and, of course, the same for every renorming of T).

Consequently, $K(T, |||\cdot|||) = 2$ for every renorming $|||\cdot|||$ of T. Still, T is well known to be reflexive.

Definition

Let $\mathcal{B}$ denote the family of bounded, closed subsets of X.
A map $\phi : \mathcal{B} \to [0, \infty)$ is a *measure of non-compactness* if:

(i) $\phi(B) = 0$ iff B is relatively compact;
(ii) $\phi(B) = \phi(\overline{B})$;
(iii) $\phi(B_1 \cup B_2) = \max\{\phi(B_1), \phi(B_2)\}$.

Examples:

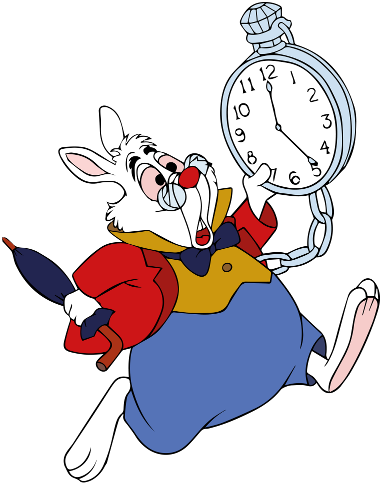
- $\phi(B) = \begin{cases} 0 & B \text{ is relatively compact} \\ 1 & \text{otherwise;} \end{cases}$
- $\chi(B) = \inf\{\varepsilon > 0 \colon B \text{ is covered by finitely many balls of radius } \varepsilon\}$;
- $\beta(B) = \sup\{\sigma > 0 \colon B \text{ contains an infinite } \sigma\text{-separated set}\}$.

Kirszbraun's theorem (1934). Let A be a subset of a Hilbert space H_1 and $f\colon A \to H_2$ be a Lipschitz map, where H_2 is a Hilbert space too. Then there exists an extension $\tilde{f}\colon H_1 \to H_2$ with $Lip(\tilde{f}) = Lip(f)$.

Theorem (N. J. Kalton, 2007)

Let X be an infinite-dimensional Banach space, A a subset of X and $f\colon A \to c_0$ a Lipschitz map. Then f admits an extension $\tilde{f}\colon X \to c_0$ such that $Lip(\tilde{f}) \leqslant K(X) \cdot Lip(f)$.
Moreover, $K(X)$ is the optimal constant in the above bound.

Time flies



Uncountable families

Henceforth, X is a **non-separable** Banach space.

B_X contains an uncountable ε-separated family, for some $\varepsilon > 0$.

- Does S_X contain an uncountable $(1+)$-separated subset?
- What about uncountable $(1+\varepsilon)$-separated subsets?
- Can we find a $(1+)$-separated subset with cardinality $\mathrm{dens}(X)$?
- What about $(1+\varepsilon)$-separated?

Learning from the Tomasz

Theorem (T. Kania and T. Kochanek, 2016)

- *Let X be a reflexive Banach space. Then there is an uncountable $(1+)$-separated family $\mathcal{F} \subseteq S_X$;*
- *Let X be super-reflexive and $\lambda \leqslant \mathrm{dens}(X)$ have uncountable cofinality. Then, for some $\varepsilon > 0$, S_X contains a $(1+\varepsilon)$-separated family with cardinality λ;*
- *The unit sphere of a non-separable $C(K)$-space contains an uncountable $(1+)$-separated subset.*

M. Cúth, O. Kurka, and B. Vejnar, 2017: for quite many compact spaces, there is a $(1+)$-separated subset of the unit sphere of $C(K)$, whose cardinality equals $\mathrm{dens}(C(K))$.

The combinatorial side of the moon

Theorem (J. Elton and E. Odell, 1981)

Let $\mathcal{F} \subseteq S_{c_0(\Gamma)}$ be $(1+\varepsilon)$-separated, for some $\varepsilon > 0$. Then $\mathcal{F}$ is countable.

The Δ-system lemma. Let $\mathcal{G}$ be an uncountable family of finite subsets of a set S. Then there are an uncountable subfamily $\mathcal{G}_0$ of $\mathcal{G}$ and a finite set $\Delta \subseteq S$ such that

$$G \cap H = \Delta \quad \text{for} \quad G \neq H \in \mathcal{G}_0.$$

Theorem (P. Hájek, T. Kania, and R., 201?)

Let $\mathcal{F} \subseteq S_{c_0(\Gamma)}$ be $(1+)$-separated. Then $|\mathcal{F}| \leqslant \omega_1$.

And its geometric face

Theorem (P. Hájek, T. Kania, and R., 201?)

Let X be a reflexive Banach space. Then:

- the unit sphere of X contains a $(1+)$-separated family with cardinality $\mathrm{dens}(X)$;
- for every cardinal $\lambda \leqslant \mathrm{dens}(X)$ with uncountable cofinality there exists ($\varepsilon > 0$ and) a $(1+\varepsilon)$-separated family of unit vectors with cardinality λ.

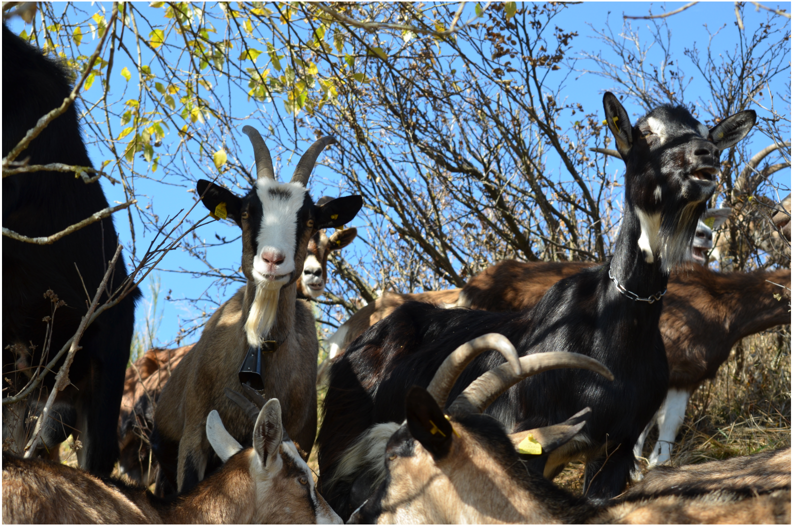
The same circle of ideas also provides us with (weaker) assertions concerning Radon–Nikodym spaces, duals to weak-Asplund spaces, strictly convex spaces (and perhaps something else).

And its geometric face

1 picture = 1 proof

(credits to Marco Russo)

Thank you for your attention!



(Ph: Marco Russo)