Separated Families of Unit Vectors

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Proem

Hereinafter, X is an **infinite-dimensional** Banach space.

The Riesz lemma (1916). There exists a sequence $(x_n)_{n=1}^\infty$ in the unit sphere S_X with $\|x_n - x_k\| \geqslant 1$ for $n \neq k$ (*i.e.* a 1-*separated* sequence).

Kottman's theorem (1975). The unit sphere S_X contains a $(1+)$-*separated* sequence $(x_n)_{n=1}^\infty$, *i.e.* $\|x_n - x_k\| > 1$ for $n \neq k$.

The Elton–Odell theorem (1981). The unit sphere S_X contains a $(1+\varepsilon)$-separated sequence $(x_n)_{n=1}^\infty$ (for some $\varepsilon > 0$).

Kottman's constant

Definition

$$K(X) := \sup\left\{\begin{array}{l}\sigma > 0\colon \exists (x_n)_{n=1}^\infty \subset B_X\colon \\ \|x_n - x_k\| \geqslant \sigma \;\; \forall n \neq k\end{array}\right\}$$

- **Elton–Odell.** $K(X) > 1$ for every X;
- **Kryczka–Prus (2000).** $K(X) \geqslant \sqrt[5]{4}$ for non-reflexive X;
- **Maluta–Papini (2009).** $K(X) \leqslant 2 - 2\delta_X(1)$;

Main goals of the project

- Obtain symmetric analogues to the above results, *i.e.* with $\|x_n \pm x_k\| \geqslant 1 + \varepsilon$;
- Investigate the symmetric analogue to Kottman's constant:
$$K^s(X) := \sup\left\{\begin{array}{l}\sigma > 0\colon \exists (x_n)_{n=1}^\infty \subset B_X\colon \\ \|x_n \pm x_k\| \geqslant \sigma \;\; \forall n \neq k\end{array}\right\};$$
- Is it possible to obtain uncountable $(1+)$ (resp. $(1+\varepsilon)$)-separated families if the space X is non-separable?

Symmetric separation

Theorem (Hájek, Kania, and R.)

Let X be an infinite-dimensional Banach space. Then the unit sphere of X contains a symmetrically $(1+)$-separated sequence $(x_n)_{n=1}^\infty$, *i.e.* $\|x_n \pm x_k\| > 1$ for $n \neq k$.

- $K^s(\ell_p) = 2^{1/p}$ for $p \in [1,\infty)$;
- $K^s(X) = 2$ if X contains c_0 or ℓ_1 (James' non-distortion theorem);
- $K^s(X) = 2$ if X has a c_0 (or ℓ_1) quotient;
- $K^s(X) = 2$ if X has an ℓ_1 spreading model;
- **Castillo–Papini (2011).** If X is a $\mathscr{L}_\infty$-space, then $K^s(X) = 2$;
- **Delpech (2010).** $K^s(X) \geqslant 1 + \delta_X(1)$.
- **Hájek, Kania, and R.** $K^s(X) \geqslant 2^{1/q}$ if X has nontrivial cotype q.

$(1+\varepsilon)$-separation

Theorem (Hájek, Kania, and R.)

Let X be a Banach space that contains a boundedly complete basic sequence. Then, for some $\varepsilon > 0$, the unit sphere of X contains a symmetrically $(1+\varepsilon)$-separated sequence.

Consequences. Let X be infinite-dimensional. Then $K^s(X) > 1$ if:

- X is reflexive;
- X contains a (subspace isomorphic to a) separable dual;
- in particular, X has RNP (or, more generally, PCP);
- X contains an unconditional basic sequence (or, more generally, an infinite-dimensional subspace isomorphic to a Banach lattice).

Around reflexivity

Kania–Kochanek (2016)

- *Let X be a reflexive Banach space. Then there is an uncountable $(1+)$-separated family $\mathcal{F} \subseteq S_X$;*
- *Let X be super-reflexive and $\lambda \leqslant \operatorname{dens}(X)$ have uncountable cofinality. Then, for some $\varepsilon > 0$, S_X contains a $(1+\varepsilon)$-separated family with cardinality λ;*

Theorem (Hájek, Kania, and R.)

Let X be a reflexive Banach space. Then:

(i) the unit sphere of X contains a $(1+)$-separated family with cardinality $\operatorname{dens}(X)$;

(ii) for every cardinal $\lambda \leqslant \operatorname{dens}(X)$ with uncountable cofinality there exists ($\varepsilon > 0$ and) a $(1+\varepsilon)$-separated family of unit vectors with cardinality λ;

(iii) if X is super-reflexive, there exist $\varepsilon > 0$ and a symmetrically $(1+\varepsilon)$-separated subset of S_X of cardinality $\operatorname{dens}(X)$.

Strategy of the proof

- If $\varphi \in S_{X^*}$ exposes $x \in S_X$, then for every unit vector $y \in \ker\varphi$ one has $\|x + y\| > 1$;
- The unit ball of a reflexive space contains plenty of exposed points (Lindenstrauss–Troyanski);
- Proceed by transfinite induction to reach (i).
- If $\varphi \in S_{X^*}$ strongly exposes $x \in S_X$, then for every unit vector $y \in \ker\varphi$ one has $\|x + y\| \geqslant 1 + \varepsilon$ (for some $\varepsilon > 0$);
- A cofinality argument then proves (ii).

The same circle of ideas also applies to other classes of Banach spaces including Radon–Nikodym spaces, duals to weak-Asplund spaces, strictly convex spaces, LUR spaces.

The rôle of Auerbach systems

A simple lemma

If X contains an Auerbach system with cardinality $\mathfrak{c}^+$, then the unit sphere of X (and therefore that of X^*) contains an uncountable $(1+)$-separated subset.

- Let $\{e_\alpha; e^*_\alpha\}$ be one such system;
- Find an uncountable homogeneous subset for the colouring
$$\{\alpha,\beta\} \mapsto \begin{cases}(>) & \|e_\alpha - e_\beta\| > 1\\ (\leqslant) & \|e_\alpha - e_\beta\| \leqslant 1;\end{cases}$$
- In case $(>)$ we are fine, otherwise we replace e_α with some $\tilde{e}_\alpha$ adding some small 'front tails'.

Corollary (*cf.* Petr's talk)

- Let X be a WLD space with $\operatorname{dens}(X) \geqslant \mathfrak{c}^+$. Then S_X and S_{X^*} contain uncountable $(1+)$-separated families.
- Let X be a 'large' Banach space. Then S_X contains an uncountable $(1+)$-separated family.

The above can not be improved

Theorem (Hájek, Kania, and R.)

Let $A \subseteq S_{c_0(\Gamma)}$ be a $(1+)$-separated set. Then $|A| \leqslant \omega_1$.

References

[1] P. Hájek, T. Kania, and T. Russo, Symmetrically separated sequences in the unit sphere of a Banach space, to appear in *J. Funct. Anal.*; *arXiv:1711.05149v2*.

[2] P. Hájek, T. Kania, and T. Russo, *in preparation (the title too!)*.