Weak compactness, double limits, and a combinatorial lemma

Workshop in
Analisi Astratta
e Analisi Convessa

Milano

November 9, 2018

Tommaso Russo

russotom@fel.cvut.cz

Faculty of Electrical Engineering
Czech Technical University in Prague

A weakly compact overview

Let A be a subset of a Banach space X. We say that:

- A is *relatively weakly sequentially compact* if every sequence in A admits a weakly convergent subsequence in X.
- A is *relatively weakly countably compact* if every sequence in A admits a weak cluster point in X.

The Eberlein–Šmulian Theorem. A subset A of a Banach space is relatively weakly compact iff it is relatively weakly sequentially compact (iff it is relatively weakly countably compact).

Proof of Šmulian's implication

Given a sequence (x_n) in A, the set $\overline{A}^w \cap \overline{\mathrm{span}}(x_n)$ is a weakly compact subset of a separable Banach space. Therefore, it is metrisable. ■

Grothendieck's insight

A subset A of a Banach space X *interchanges limits* if for every pair of sequences $(x_n) \subseteq A$ and $(x_k^*) \subseteq B_{X^*}$ the existence of

$$\lim_{n\to\infty} \lim_{k\to\infty} \langle x_k^*, x_n \rangle \qquad \lim_{k\to\infty} \lim_{n\to\infty} \langle x_k^*, x_n \rangle$$

implies their equality.

Theorem (Grothendieck, 1952)

A bounded subset A of a Banach space X is relatively weakly compact if and only if it interchanges limits.

Note: If A is relatively weakly countably compact and $(x_n) \subseteq A$ and $(x_k^*) \subseteq B_{X^*}$ are as above, then let x be a weak cluster point of (x_n), x^* be a w^* cluster point of (x_n^*). Clearly,

$$\lim_n \lim_k \langle x_k^*, x_n \rangle = \langle x^*, x \rangle = \lim_k \lim_n \langle x_k^*, x_n \rangle.$$

Proof of Grothendieck's theorem

- **Strategy:** Prove that $\overline{A}^{w^*} \subseteq X^{**}$ is actually contained in X.
  So, assume that $\psi \in \overline{A}^{w^*}$ satisfies $d := \mathrm{dist}(\psi, X) > 0$.
- **Fact:** If F is a finite subset of X and $\varepsilon > 0$, (by Goldstine thm) there is $x^* \in B_{X^*}$ such that

$$|\langle f, x^* \rangle| < \varepsilon \ \ (f \in F) \qquad \text{and} \qquad |\langle \psi, x^* \rangle - d| < \varepsilon.$$

- **Step1:** Fix arbitrarily $a_1 \in A$, use the Fact to find $x_1^* \in B_{X^*}$ with

$$|\langle a_1, x_1^* \rangle| < 1 \qquad \text{and} \qquad |\langle \psi, x_1^* \rangle - d| < 1.$$

- **Step2:** As $\psi \in \overline{A}^{w^*}$, we may find $a_2 \in A$ such that

$$|\langle \psi - a_2, x_1^* \rangle| < 1/2.$$

  Then, using the Fact to $\{a_1, a_2\}$, find $x_2^* \in B_{X^*}$ such that

$$|\langle a_1, x_2^* \rangle| < 1/2, \ |\langle a_2, x_2^* \rangle| < 1/2, \qquad \text{and} \qquad |\langle \psi, x_2^* \rangle - d| < 1/2.$$

- **Step3:** Then use the functionals x_1^*, x_2^* and find $a_3 \in A$ such that

$$|\langle \psi - a_3, x_1^* \rangle| < 1/3,\ |\langle \psi - a_3, x_2^* \rangle| < 1/3.$$

The Fact applied to $\{a_1, a_2, a_3\}$ yields $x_3^* \in B_{X^*}$ such that

$$|\langle a_1, x_3^* \rangle| < 1/3,\ |\langle a_2, x_3^* \rangle| < 1/3,\ |\langle a_3, x_3^* \rangle| < 1/3,\ |\langle \psi, x_3^* \rangle - d| < 1/3.$$

- **Induction:** Yields sequences $(a_n)_{n=1}^{\infty} \subseteq A$ and $(x_k^*)_{k=1}^{\infty} \subseteq B_{X^*}$ with

$$|\langle \psi - a_n, x_k^* \rangle| < 1/n \qquad (k \leqslant n-1)$$

$$|\langle a_n, x_k^* \rangle| < 1/k \qquad (n \leqslant k)$$

$$|\langle \psi, x_k^* \rangle - d| < 1/k.$$

- **The end:** $\lim_k \langle a_n, x_k^* \rangle = 0$, whence $\lim_n \lim_k \langle a_n, x_k^* \rangle = 0$.
On the other hand,

$$\lim_n \langle a_n, x_k^* \rangle = \langle \psi, x_k^* \rangle$$

$$\lim_k \lim_n \langle a_n, x_k^* \rangle = \lim_k \langle \psi, x_k^* \rangle = d \neq 0.$$

∎

Reflexivity

Notice that, by discarding finitely many terms, in the above proof we obtain two sequences $(a_n)_{n=1}^\infty \subseteq A$ and $(x_k^*)_{k=1}^\infty \subseteq B_{X^*}$ such that

$$\langle a_n, x_k^* \rangle \approx \begin{cases} d & k < n \\ 0 & k \geqslant n. \end{cases}$$

In case X is not reflexive, we can apply the result with $A = B_X$, in which case we obtain the above with $d = 1$.

Theorem (James, 1964)

A Banach space X fails to be reflexive if and only if for every $\theta \in (0, 1)$ there are sequences $(x_n)_{n=1}^\infty \subseteq B_X$ and $(x_k^*)_{k=1}^\infty \subseteq B_{X^*}$ such that

$$\langle x_n, x_k^* \rangle = \begin{cases} \theta & k < n \\ 0 & k \geqslant n. \end{cases}$$

Convex means

Given a set S, a *convex mean* λ on S is a function $\lambda\colon S \to [0,\infty)$ such that:

(i) the set $\operatorname{supp}(\lambda) := \{s \in S\colon \lambda(s) \neq 0\}$ is finite

(ii)

$$\sum_{s\in S} \lambda(s) = 1.$$

A family $\mathcal{F} \subseteq 2^S$ is *hereditary* if $G \subseteq F \in \mathcal{F}$ yields $G \in \mathcal{F}$.

Pták's combinatorial lemma, 1959

Let S be an infinite set and let $\mathcal{F} \subseteq [S]^{<\omega}$ be an hereditary family. If

$$(\dagger) \qquad \delta := \inf\left\{\sup_{F\in\mathcal{F}} \lambda(F)\colon \lambda \text{ is a convex mean on } S\right\} > 0,$$

there exists an infinite subset M of S such that every finite subset of M is in $\mathcal{F}$.

Two uses of Pták's lemma

Krein theorem

The closed convex hull of a weakly compact set is weakly compact.

It suffices to prove that $\operatorname{conv} C$ interchanges limits, whenever C does. Pták's lemma allows to give an elementary proof of this implication.

Mazur Theorem

Let $(f_n)_{n=1}^{\infty} \subseteq C(K)$ be a bounded sequence of continuous functions that converge pointwise to a continuous function f. Then f can be uniformly approximated by convex combinations of the f_n.

The classical proof depends on the Hahn–Banach and Riesz Representation Theorems. Pták's lemma allows for a self-contained simple proof, which, in particular, involves no measure theory whatsoever.

Proof of Mazur theorem

Assume that $f = 0$ and $\|f_n\| \leqslant 1$. Fix $\varepsilon > 0$ and consider, for $x \in K$, the finite set

$$F_x := \{n \in \mathbb{N} \colon |f_n(x)| \geqslant \varepsilon/2\}.$$

We shall apply Pták's lemma to the hereditary family $\mathcal{F}$ comprising all subsets of F_x, $x \in K$.

We can easily see that the conclusion of Pták's lemma fails to hold for $\mathcal{F}$, so there exists a convex mean λ on $\mathbb{N}$ such that $\lambda(F_x) < \varepsilon/2$ whenever $x \in K$.

The function $\sum_{n=1}^{\infty} \lambda(n) f_n$ is then as desired. In fact, if $x \in K$, we have

$$\left| \sum_{i=1}^{\infty} \lambda(i) f_i(x) \right| \leqslant \sum_{i \in F_x} \lambda(i) |f_i(x)| + \sum_{i \notin F_x} \lambda(i) |f_i(x)|$$

$$\leqslant \sum_{i \in F_x} \lambda(i) + \varepsilon/2 \sum_{i \notin F_x} \lambda(i) \leqslant \lambda(F_x) + \varepsilon/2 < \varepsilon.$$

■

Uncountable extensions

Let κ be an infinite cardinal number.We say that *Pták's lemma holds true for* κ if for every set S with $|S| \geqslant \kappa$ and every hereditary family $\mathcal{F} \subseteq [S]^{<\omega}$ such that

$$(\dagger) \qquad \delta := \inf\left\{\sup_{F\in\mathcal{F}} \lambda(F) \colon \lambda \text{ is a convex mean on } S\right\} > 0,$$

there is a subset M of S, with $|M| = \kappa$, such that every finite subset of M belongs to $\mathcal{F}$.

Theorem (Hájek and R., JMAA 2019)

The validity of Pták's lemma for ω_1 is independent of ZFC.

(i) (MA_{ω_1}) Pták's lemma holds true for ω_1;

(ii) (CH) Pták's lemma fails to hold for ω_1;

(iii) If κ is regular and $\lambda^\omega < \kappa$ whenever $\lambda < \kappa$, then Pták's lemma is true for κ.

A closely related problem

Let K be a Corson compact and τ be an uncountable cardinal.

Problem: Does $\ell_1(\tau)$ embed in $C(K)$?

- MA_{ω_1} implies that $C(K)$ is WLD, whence the answer is no;
- Under CH, $\ell_1(\omega_1)$ may embed in $C(K)$ (Erdős' space);
- $\ell_1(\mathfrak{c}^+)$ does not embed in $C(K)$ (Haydon);
- Is it consistent that $\ell_1(\omega_2)$ embeds in $C(K)$?

Thank you for your attention!