On Auerbach systems in Banach spaces

(Joint work with P. Hájek and T. Kania)

Geometrical Aspects of Banach Spaces

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Systems of coordinates

- A collection $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma} \subseteq X \times X^*$ is a *biorthogonal system* if

$$\langle x_\alpha^*, x_\beta\rangle = \delta_{\alpha,\beta} \qquad (\alpha, \beta \in \Gamma).$$

- If $\|x_\gamma\| = \|x_\gamma^*\| = 1$, the system is said to be an *Auerbach system*.
- An *M-basis* is a biorthogonal system such that:

$$\overline{\mathrm{span}}\{x_\gamma\}_{\gamma\in\Gamma} = X \qquad \& \qquad \overline{\mathrm{span}}^{w^*}\{x_\gamma^*\}_{\gamma\in\Gamma} = X^*.$$

- An *Auerbach basis* is an M-basis with $\|x_\gamma\| = \|x_\gamma^*\| = 1$.

Auerbach (1934). Every finite-dimensional normed space contains an Auerbach basis.

Day (1962). Every infinite-dimensional Banach space contains am infinite-dimensional subspace with an Auerbach basis, in particular it contains an infinite Auerbach system.

And spaces with few of them

- **Kunen (1975).** (CH) There exists a non-separable Banach space that contains no uncountable biorthogonal system.
  - Other examples: ($\clubsuit$) Ostaszewski (1975), ($\diamondsuit$) Shelah (1985).
  - **Todorčević (2006).** (MM) Every non-separable Banach space contains uncountable biorthogonal systems.
- **Johnson (1970).** ℓ_∞ has no M-basis.
- **Plichko (1986).** $c_0[0,1] + C[0,1]$ has no Auerbach basis.
- **Godun–Lin–Troyanski (1993).** Every non-separable Banach space X with B_{X^*} w^*-separable, admits an equivalent norm with no Auerbach basis.
  - **Godun (1990).** The particular case $X = \ell_1(\mathfrak{c})$.

Problem (1): Guirao, Montesinos, and Zizler (2016), *Open problems...*

Does there exist a non-separable Banach space X with unconditional basis such that no non-separable subspace of X has an Auerbach basis?

‘Large’ Banach spaces

Theorem A

Let $\kappa \geqslant \mathfrak{c}$ be a cardinal number and let X be a Banach space with $w^*\text{-dens}\, X^* > \exp_2 \kappa$. Then X admits a subspace Y with Auerbach basis and such that $\operatorname{dens} Y = \kappa^+$.

Proof. Set $\lambda := w^*\text{-dens}\, X^*$. By transfinite induction, find unit vectors $(e_\alpha)_{\alpha<\lambda}$ and unit functionals $(\varphi_{\alpha,\beta})_{\alpha<\beta<\lambda}$ such that:

(i) $\varphi_{\alpha,\beta}$ is a norming functional for $e_\alpha - e_\beta \neq 0$ $(\alpha < \beta < \lambda)$;

(ii) $e_\gamma \in \ker \varphi_{\alpha,\beta}$ $(\alpha < \beta < \gamma < \lambda)$.

We invoke the Erdős–Rado theorem, for the colouring

$$\{\alpha, \beta, \gamma\} \mapsto \langle \varphi_{\beta,\gamma}, e_\alpha \rangle \qquad (\alpha < \beta < \gamma < \lambda).$$

We can build the desired Auerbach system out of a monochromatic set of cardinality κ^+.

WLD spaces

A Banach space X is *weakly Lindelöf determined* (hereinafter, *WLD*) if the dual ball B_{X^*} is a Corson compact in the relative w^*-topology.

For our purposes, X is WLD if it admits an M-basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ that *countably supports* X^*, *i.e.*,

$$\operatorname{supp} x^* := \{\gamma \in \Gamma : \langle x^*, x_\gamma\rangle \neq 0\}$$

is a countable subset of Γ, for every $x^* \in X^*$.

Theorem B

Every WLD Banach space X with $\operatorname{dens} X > \omega_1$ contains a subspace Y with Auerbach basis and such that $\operatorname{dens} Y = \operatorname{dens} X$.

Problem (2): What happens if $\operatorname{dens} X = \omega_1$?

The main result

Theorem C

(CH) *There exists a renorming* $\lvert\lvert\lvert\cdot\rvert\rvert\rvert$ *of the space* $c_0(\omega_1)$ *such that the space* $(c_0(\omega_1), \lvert\lvert\lvert\cdot\rvert\rvert\rvert)$ *contains no uncountable Auerbach systems.*

Consequently, assuming the Continuum Hypothesis:

- There exists a non-separable Banach space with unconditional basis whose no non-separable subspace admits an Auerbach basis.
- There exists a WLD Banach space X with $\operatorname{dens} X = \omega_1$ every whose non-separable subspace fails to have an Auerbach basis.

Therefore, under CH, we can answer both Problems (1) and (2).

- For $\alpha < \omega_1$, let σ_α be an enumeration of the set $[0, \alpha)$.
- Select $\lambda \in (0, 1/6)$ and define $\varphi_\alpha \in \ell_1(\omega_1)$ by

$$\varphi_\alpha(\eta) = \begin{cases} 1 & \text{if } \eta = \alpha \\ 0 & \text{if } \eta > \alpha \\ \lambda^k & \text{if } \eta < \alpha,\ \eta = \sigma_\alpha(k). \end{cases}$$

- Define a new (equivalent) norm on $c_0(\omega_1)$ to be

$$|||x||| := \sup_{\alpha<\omega_1} |\langle \varphi_\alpha, x\rangle| \qquad (x \in c_0(\omega_1)).$$

The main property.
If $u \in c_0(\omega_1) \setminus \{0\}$ and $\operatorname{supp} u < \alpha$, then

$$\langle \varphi_\alpha, u\rangle := \sum_{\beta<\alpha} u(\beta)\langle \varphi_\alpha, e_\beta\rangle = \sum_{k=1}^{\infty} u(\sigma_\alpha(k))\lambda^k$$

is a (non-trivial) real-analytic function of λ and therefore it has countably many zeros.
Consequently, for every countable subset $(u_n)_{n=1}^{\infty}$ of $c_0(\omega_1) \setminus \{0\}$, we may select λ such that $\langle \varphi_\alpha, u_n\rangle \neq 0$ for every $n \in \mathbb{N}$.

Application to (1+)-separated sets

Theorem D

If X contains an Auerbach system with cardinality $\mathfrak{c}^+$, then the unit sphere of X (and therefore that of X^*) contains an uncountable (1+)-separated subset.

Consequences:

- Assume that $w^*\text{-}\mathrm{dens}\, X^* > \exp_2 \mathfrak{c}$. Then both S_X and S_{X^*} contain an uncountable (1+)-separated subset.

  In particular, the assumption is satisfied whenever $\mathrm{dens}\, X > \exp_3 \mathfrak{c}$.
- Let X be a WLD space with $\mathrm{dens}\, X > \mathfrak{c}$. Then S_X and S_{X^*} contain uncountable (1+)-separated subsets.

Recapitulation

For every WLD Banach space X with $\operatorname{dens} X > \mathfrak{c}$, S_X and S_{X^*} contain uncountable (1+)-separated subsets.

Note: This is somewhat sharp, the conclusion can not be improved.

Theorem E

Let $\mathcal{F} \subseteq S_{c_0(\kappa)}$ be (1+)-separated. Then $|\mathcal{F}| \leqslant \omega_1$.

Open problem: What about WLD spaces with density character ω_1?

In other words, if X is a WLD Banach space and $\operatorname{dens} X = \omega_1$, does S_X contain an uncountable (1+)-separated subset?

Thank you for your attention!