Lipschitz functions in Banach spaces I

SAA - Seminario di Analisi Astratta

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Geometric Nonlinear Functional Analysis

A main topic in the nonlinear geometry of Banach spaces is the investigation of the metric structure of a Banach space, thus forgetting the underling vector space.

In particular the goal is to understand to what extent the metric structure already characterizes the linear structure of the space, namely which linear properties of a given Banach space can be deduced from its metric structure.

The isometric structure implies the linear one

Theorem (Mazur - Ulam 1932)

Let $f : X \twoheadrightarrow Y$ be an onto isometry of two real Banach spaces. Then f is affine; equivalently, if $f(0) = 0$ f is linear.

The result tells that the isometric structure of (real) Banach spaces completely encodes the linear structure.

Lipschitz embeddings

Theorem (Aronszajn, Christensen, Mankiewicz 1976)

If a separable Banach space X Lipschitz embeds in a space Y with the RNP, then X linearly embeds in Y.

Corollary

RNP, reflexivity, superreflexivity, being isomorphic to a Hilbert space are preserved under Lipschitz embedding.
Namely, if X Lipschitz embeds in Y and Y is RN, reflexive, superreflexive or isomorphic to a Hilbert space, then the same holds for X.

Uniqueness of the Lipschitz structure

Theorem (Godefroy - Kalton - Lancien 2000)

The Lipschitz structure of c_0 implies the linear one: i.e. if X is Lipschitz equivalent to c_0, then it is in fact isomorphic to c_0.

Theorem (Johnson - Lindenstrauss - Schechtman 1996)

The uniform structure of ℓ_p, $1 < p < \infty$, implies the linear one.

The uniform structure implies the local one

Theorem (Ribe 1976)

Let X and Y be uniformly homeomorphic; then they are crudely finitely representable in each other (i.e. there is $C > 0$ such that every finite-dimensional subspace of X is C-isomorphic to a finite-dimensional subspace of Y and vice versa).

In particular every local property of Banach spaces (e.g. superreflexivity, type and cotype) is preserved under uniform homeomorphism.

Uniform vs linear

Theorem (Johnson - Lindenstrauss - Schechtman 1996)

Let $\mathcal{T}^{(p)}$ be the p-convexification of (a modification of) the Tsirelson space ($1 < p < \infty$). Then X is uniformly homeomorphic to $\mathcal{T}^{(p)}$ iff it is isomorphic to $\mathcal{T}^{(p)}$ or to $\mathcal{T}^{(p)} \oplus \ell_p$.

Theorem (Ribe, Aharoni - Lindenstrauss 1985)

Let $1 \leq p, q, p_n < \infty$ be such that $p_n \to p$ and $p \notin \{p_n, q\}$. The spaces

$$\left(\sum \ell_{p_n}\right)_q \quad \text{and} \quad \left(\sum \ell_{p_n}\right)_q \oplus \ell_p$$

are uniformly homeomorphic but not isomorphic.

Lipschitz vs linear

Lemma

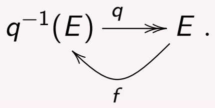
ℓ_∞/c_0 contains an isometric copy of $c_0(\mathbb{R})$.

Proof: in fact, let $\{N_\gamma\}_{\gamma\in\mathbb{R}}$ be a family as in Sierpinski's lemma and consider $\xi_\gamma := \chi_{N_\gamma} \in \ell_\infty$. Then $\{q(\xi_\gamma)\}_{\gamma\in\mathbb{R}}$ is 1-equivalent to the $c_0(\mathbb{R})$ canonical basis.

Note in passing that $c_0(\mathbb{R})$ does not embed in ℓ_∞, so ℓ_∞/c_0 does not embed in ℓ_∞ and we deduce Phillips' theorem.

Aharoni-Lindenstrauss example

Consider $E := \overline{\text{span}}\,\{q(\xi_\gamma)\}_{\gamma\in\mathbb{R}} \cong c_0(\mathbb{R})$ and $q^{-1}(E) = \overline{\text{span}}\,\{c_0, \xi_\gamma\}_{\gamma\in\mathbb{R}}$. The main point is that there is a Lipschitz lifting $f : E \to q^{-1}(E)$ of the quotient map:



It easily follows that

$$\begin{array}{ccc} E \oplus c_0 & \longrightarrow & q^{-1}(E) \\ (x, y) & \to & f(x) + y \\ (q(z), z - f(q(z))) & \leftarrow & z \end{array}$$

are Lipschitz equivalent, but not isomorphic.

Quotient maps

Let $q : X \twoheadrightarrow Y$ be a quotient map.
If it admits a Lipschitz lifting, then the spaces X and $Y \oplus \ker q$ are Lipschitz equivalent.
Of course if there is a linear lifting, the two spaces are linearly isomorphic too.

Theorem (Godefroy - Kalton 2003)

Let $q : X \twoheadrightarrow Y$ be a quotient map and let Y be separable. If q admits a Lipschitz lifting, then it admits a linear lifting too.

Lipschitz analogue to $\langle X, X^* \rangle$

Given a Banach space X, consider the Banach space

$$Lip_0(X) := \{f : X \to \mathbb{R} : f(0) = 0 \text{ and } f \text{ is Lipschitz}\},$$

$$\|f\|_{Lip_0(X)} := Lip(f).$$

Define $\delta_x \in Lip_0(X)^*$ by $\delta_x(f) := f(x)$ and let δ_X be the map $x \mapsto \delta_x$; $\delta_X : X \to Lip_0(X)^*$ is a (nonlinear!) isometry.

Definition

The *Lipschitz-free space over* X, or *Arens-Eells space over* X, is

$$\mathcal{F}(X) := \overline{\operatorname{span}} \{\delta_x : x \in X\}.$$

Example

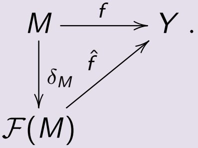
$\mathcal{F}(\mathbb{R}) \cong L_1(\mathbb{R})$ and $\mathcal{F}(\mathbb{N}) \cong \ell_1$.

Universal property of $\mathcal{F}(M)$

More in general, let (M, d) be a metric space with a distinguished point 0_M; we can analogously define $\mathcal{F}(M)$.

Theorem

Let Y be a Banach space and $f : M \to Y$ be Lipschitz with $f(0_M) = 0$. Then there exists a unique linear map $\hat{f} : \mathcal{F}(M) \to Y$ with $\left\|\hat{f}\right\| = Lip(f)$ that extends f, in the sense that



Proof

Proof of the universal property

We define $\hat{f}(\delta_x) := f(x)$; the set $\{\delta_x : x \in M, x \neq 0_M\}$ is l.i., so we extend linearly to span $\{\delta_x : x \in M\}$. This extension is a bounded operator with norm at most $Lip(f)$.

Corollary

$\mathcal{F}(M)^* \cong Lip_0(M)$.

In fact, use $X = \mathbb{R}$; the map $f \mapsto \hat{f}$ is the required one.

Linearization

Corollary

Let $f : M \to N$ be a Lipschitz map with $f(0_M) = 0_N$. Then there is a unique $\hat{f} : \mathcal{F}(M) \to \mathcal{F}(N)$ with $\left\|\hat{f}\right\| = Lip(f)$ and

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ \downarrow{\scriptstyle \delta_M} & & \downarrow{\scriptstyle \delta_N} \\ \mathcal{F}(M) & \xrightarrow{\ \hat{f}\ } & \mathcal{F}(N) \end{array}$$

- f Lipschitz (isometric) embedding $\implies$ $\hat{f}$ isomorphic (isometric) embedding;
- f onto $\implies$ $\hat{f}$ onto;
- f Lipschitz retraction $\implies$ $\hat{f}$ linear projection.

The map β_X

We now focus on the case that $M = X$, a Banach space. Consider

$$\begin{array}{rcc} \beta_X : \mathcal{F}(X) & \longrightarrow & X \\ \sum_{i=1}^n \alpha^i \delta_{x_i} & \mapsto & \sum_{i=1}^n \alpha^i x_i. \end{array}$$

Then β_X is an (isometric) quotient map and of course $\beta_X \circ \delta_X = id_X$; hence β_X always admits a Lipschitz lifting! We deduce

$$\mathcal{F}(X) \text{ is Lipschitz equivalent to } X \oplus \ker \beta_X.$$

Of course if β_X admits a linear lifting, the spaces are isomorphic too.

The converse

Definition

A Banach space X has the *(isometric) Lipschitz-lifting property* if $\beta_X : \mathcal{F}(X) \to X$ admits a (non-expansive) linear lifting.

Theorem (Godefroy - Kalton 2003)

X has the Lipschitz-lifting property iff $\mathcal{F}(X)$ is isomorphic to $X \oplus \ker \beta_X$.

Proof

We use two (easy) facts:

- Complemented subspaces of spaces with the Lipschitz-lifting property also have it;
- $\mathcal{F}(X)$ has the isometric Lipschitz-lifting property.

Other counterexamples

Hence $\mathcal{F}(X)$ is always Lipschitz equivalent to $X \oplus \ker \beta_X$ and they are isomorphic iff X has the Lipschitz-lifting property.

Theorem (Godefroy - Kalton 2003)

- Every separable Banach space has the isometric Lipschitz-lifting property;
- A WCG Banach space has the Lipschitz-lifting property iff it is separable.

The second assertion follows easily from the following: if X is WCG, then every w-compact subset of $\mathcal{F}(X)$ is separable.

A.P. and B.A.P.

- X has λ-B.A.P. iff $\mathcal{F}(X)$ has it; in particular the B.A.P. is invariant under Lipschitz equivalence;
- If X is a finite-dimensional Banach space, $\mathcal{F}(X)$ has M.A.P.;
- $\mathcal{F}(\mathbb{U})$ has the M.A.P., where $\mathbb{U}$ is the Urysohn space;
- if K is a convex subset of a finite-dimensional normed space, or a countable proper metric space, then $\mathcal{F}(K)$ has M.A.P.;
- There are compact spaces K for which $\mathcal{F}(K)$ fails even the A.P..

Schauder bases

- $\mathcal{F}(\ell_1)$ and $\mathcal{F}(\mathbb{R}^n)$ admit a Schauder basis;
- if $\mathcal{N}$ is a net in c_0 or in $\mathbb{R}^n$, then $\mathcal{F}(\mathcal{N})$ admits a Schauder basis;
- if M is the integer grid in c_0, then $\mathcal{F}(M)$ admits a monotone Schauder basis.