Smooth approximations of norms with asymptotic improvement

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Smooth bumps and structure

- **Meshkov (1978).** If X and X^* admit a C^2-smooth bump, then X is isomorphic to a Hilbert space.
- **Fabian, Whitfield, Zizler (1983).** If X admits a bump with locally uniformly continuous derivative, then either X contains a copy of c_0 or it is super-reflexive.
  If X admits a bump with locally Lipschitz derivative and it contains no copy of c_0, then X is (super-reflexive) with type 2.
- **Deville (1989).** Assume that X admits a C^∞-smooth bump and it contains no copy of c_0. Then X is of exact cotype $2k$, for some integer k, and it contains a copy of ℓ_{2k}.

Density of smooth norms

- If X admits a C^1-smooth norm and X^* admits a dual LUR norm (e.g. if X is WCG), then every equivalent norm on X can be approximated by a C^1-smooth one.
- **Hájek, Talponen (2014).** If X is separable and it admits a C^k-smooth norm, then every equivalent norm on X can be approximated by a C^k-smooth one.
- **Bible, Smith (2016).** Every equivalent norm on $c_0(\Gamma)$ can be approximated by a C^∞-smooth one.

Here, "approximated" stands for approximated uniformly on bounded sets and with arbitrary precision.

Asymptotic behavior

Assume that the Banach space X admits a Schauder basis $\{e_i\}_{i\geq 1}$. Let X^N be

$$X^N := \overline{\text{span}}\,\{e_i\}_{i\geq N+1} = \ker P_N.$$

Here, P_N is the natural projection onto $\text{span}\,\{e_i\}_{i=1}^N$. We also denote by $P^N := I - P_N$ the complementary projection onto X^N.

Problem (Guirao, Montesinos, Zizler)

Can an approximating norm be chosen so that the approximation improves on X^N?

The main result

Theorem (Hájek, R.)

Assume that X admits a C^k-smooth renorming. Then for every equivalent norm $\|\cdot\|$ on X and every sequence $\{\varepsilon_N\}_{N\geq 0}$ of positive numbers, there is a C^k-smooth renorming $|||\cdot|||$ of X such that

$$\Big| |||\cdot||| - \|\cdot\| \Big| \leq \varepsilon_N \|\cdot\| \qquad \text{on } X^N.$$

In other words, we can approximate every equivalent norm with a C^k-smooth one in a way that on the "tail vectors" the approximation improves as fast as we wish.

Sketch of the proof 1/3: a geometric lemma

Lemma

Let $(X, \|\cdot\|)$ be a Banach space with Schauder basis $\{e_i\}_{i\geq 1}$ with basis constant K. Denote the unit ball of X by B, fix $k \in \mathbb{N}$, a parameter $\lambda > 0$, and consider the sets

$$D := \left\{x \in X : \left\|P^k x\right\| \leq 1/2\right\} \cap (1+\lambda) \cdot B,$$

$$C := \overline{\text{conv}}\,\{D, B\}\,.$$

Then

$$C \cap X^k \subseteq \left(1 + \lambda \frac{K}{K+1/2}\right) \cdot B.$$

The picture doesn't fit in here. ☹

Sketch of the proof 2/3: iteration

Applying iteratively the lemma (and doing something else, in fact), we find a sequence of norms $\{|||\cdot|||_n\}_{n\geq 0}$ (all close to $\|\cdot\|$) such that, for some $\gamma_n \in (0,1)$:

- for every $x \in X$ there is n_0 such that for $n \geq n_0$

$$|||x|||_n = \frac{1+\lambda_n \frac{1+\gamma_n}{2}}{1+\lambda_n} |||x|||_{n-1};$$

- if $x \in X^N$, then for $n = 1, \ldots, N$ we have

$$|||x|||_n = \frac{1+\lambda_n \frac{1+\gamma_n}{2}}{1+\lambda_n \gamma_n} |||x|||_{n-1}.$$

Sketch of the proof 3/3: gluing together

Let $|||\cdot|||_{(s),n}$ be a C^k-smooth approximation of $|||\cdot|||_n$, with

$$|||\cdot|||_n \leq |||\cdot|||_{(s),n} \leq (1+\delta_n)\,|||\cdot|||_n\,.$$

Now find $\varphi_n : [0,\infty) \to [0,\infty)$ to be C^∞-smooth, convex and such that $\varphi_n(1) = 1$ and $\varphi_n = 0$ on $[0, 1-\delta_n]$. Define $\Phi : X \to [0,\infty]$ by

$$\Phi(x) := \sum_{n\geq 0} \varphi_n\left(|||x|||_{(s),n}\right).$$

Then the Minkowski functional $|||\cdot|||$ of $\{\Phi \leq 1\}$ is the desired norm.

Two polyhedral remarks

Theorem (Deville, Fonf, Hájek; 1998)

Let X be a separable polyhedral Banach space. Then every equivalent norm on X can be approximated by:

1. a polyhedral norm.
2. a C^∞-smooth LFC norm.

"Proposition" (Hájek, R.)

In the above, the approximations can be chosen to improve on the tail vectors.