La geometria nonlineare degli spazi di Banach

PhD ^2 - PhD Days

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13 Giugno 2016

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Some linear questions

Given a Banach space X, we want to understand its structure.

- Can we characterize its subspaces?
- Does it embed in some "simpler" space?
- Can we find "nice" embeddings, i.e. complemented ones?
- Which are the "nice" subspaces?

And some examples of answers

- Every separable Banach space admits an isometric copy in $\mathbb{G}$. Moreover, many spaces can be embedded nicely: every separable Lindenstrauss space admits a 1-complemented copy in $\mathbb{G}$.
- Every separable Banach space embeds isometrically in $C([0,1])$ too, but nice embeddings are difficult to find. It is conjectured that if X has an isomorphic and complemented copy in $C([0,1])$, then $X \sim C(K)$ for some K.
- If X embeds with complementation in c_0 (or ℓ_p), then $X \sim c_0$ (resp. $X \sim \ell_p$).

What is the nonlinear theory?

The nonlinear geometry of Banach spaces considers a Banach space from the point of view of the metric structure, thus forgetting the underling vector space.

- Given X and Y are they uniformly homeomorphic? Lipschitz homeomorphic?
- Can we uniformly (or Lipschitz) embed Y in X?
- If $Y \subseteq X$ is there a retraction from X onto Y?
- Can we (at least partially) recover the linear structure?

And why?

- "Why not?"
- We can ask the same for Y a metric space and use the results to study a certain metric space.
- The nonlinear classification is coarser that the linear one, we can use it to understand if two spaces are really very different or not.

Recovering all the structure

Theorem (Mazur-Ulam 1932)

Let X and Y be real Banach spaces and $f : X \twoheadrightarrow Y$ be isometric. Then f is affine; equivalently, if $f(0) = 0$ f is linear.

- From a mathematical point of view, the theorem is not that interesting.
- From a "meta-mathematical" one it is incredibly powerful: if a map preserves the metric structure of a Banach space, then it is forced to preserve the whole linear structure. In other words the linear structure is entirely contained in the metric one.

The topology gives no information

Teorema (Kadec 1967 - Torunczyk 1981)

All separable Banach spaces are mutually homeomorphic.
More in general, two Banach spaces are homeomorphic iff they have the same density character.

The nonlinear classification lies between these two extreme situations.

Midpoints

Proof of the Mazur-Ulam theorem

For $x, y \in X$ define the metric midpoint set

$$K_0(x,y) := Mid(x,y) := \left\{ u \in X : d(u,x) = d(u,y) = \frac{1}{2} d(x,y) \right\},$$

$$K_{n+1}(x,y) := \left\{ u \in K_n(x,y) : K_n(x,y) \subseteq B\left(u, \frac{1}{2}\mathrm{diam}(K_n(x,y))\right) \right\}.$$

- $f(K_n(x,y)) = K_n(f(x), f(y))$;
- $\cap_n K_n(x,y) = \left\{\frac{x+y}{2}\right\}$.

Heuristic use of approximate midpoints

More in general we can set for $\delta > 0$

$$Mid(x,y,\delta) := \left\{ u \in X : d(u,x), d(u,y) \leq \frac{1+\delta}{2} d(x,y) \right\}$$

and we have

$$f\left(Mid(x,y,\delta)\right) \subseteq Mid\left(f(x), f(y), (1+\delta) \cdot Lip(f) \cdot Lip(f^{-1}) - 1\right).$$

This inclusion shows that if the approximate midpoints of Y are small and the ones of X are large, then no bi-Lipschitz $f : X \to Y$ can exist since such a map should preserve large sets, if properly defined (the most obvious choice is the diameter).

A rigorous and quantitative statement

Proposition

Let $f : X \to Y$ be a uniform homeomorphism and $0 < \delta < 1/2$. Then there are $x, y \in X$, with $\|x - y\|$ arbitrarily large, such that

$$f\left(Mid(x, y, \delta)\right) \subseteq Mid(f(x), f(y), 5\delta).$$

Uniformly continuous implies Lipschitz at large

Let $f : X \to Y$ be uniformly continuous; then for every $\delta > 0$ there is K_δ such that for every $\|x - y\| \geq \delta$ one has

$$\|f(x) - f(y)\| \leq K_\delta \|x - y\|.$$

Implementation of the idea

Theorem (Enflo - Bourgain - Gorelik 1994)

If $1 \leq p < \infty$ and $p \neq 2$ the spaces ℓ_p and $L^p([0,1])$ are not uniformly homeomorphic.

Example of large midpoint sets

Let $x \in S_{L^1([0,1])}$, then there is a sequence $\{x_j\}_{j=1}^{\infty} \subseteq Mid(x, -x)$ such that $\|x_i - x_j\| = 1$ for $i \neq j$.
In fact, let $\{A_1, A_2\}$ be a partition of $[0,1]$ such that $\int_{A_i} |x| = 1/2$ and define $x_1 := -x \cdot \chi_{A_1} + x \cdot \chi_{A_2}$; then split each A_i in two parts and iterate.

The Gorelik principle

The case $p > 2$ is the most delicate and requires a more careful concept of small and large which is based on the so called Gorelik principle: a uniform homeomorphism between Banach spaces can not map a large radius ball of a finite-codimensional subspace in a small neighborhood of an infinite-codimensional subspace. Explicitly:

Theorem (Gorelik)

Let $f : X \to Y$ be a uniform homeomorphism, $X_1 \subseteq X$ be finite-codimensional and $Y_1 \subseteq Y$ be infinite-codimensional. If for $\alpha, \beta > 0$ one has

$$f\left(\alpha \cdot B_{X_1}\right) \subseteq Y_1 + \beta \cdot B_Y = \{y \in Y : dist(y, Y_1) \leq \beta\},$$

then $\alpha \leq 4\omega_{f^{-1}}(2\beta)$.

Two examples of partially recovered linearity

Theorem (Johnson - Lindenstrauss - Schechtman 1996)

If X is uniformly homeomorphic to ℓ_p for $1 < p < \infty$, then actually $X \sim \ell_p$.

Theorem (Ribe 1976 "uniform homeomorphism implies same local structure")

If X and Y are uniformly homeomorphic, then they are crudely finitely representable in each other, i.e. there is $C > 0$ such that every finite-dimensional subspace of X is C-isomorphic to a finite-dimensional subspace of Y and vice versa.
In particular L_p and L_r are not uniformly homeomorphic for $p \neq r$.

Different classifications

- There exists a separable Banach space E such that for some $p \in [1, \infty)$ E and $E \oplus \ell_p$, are uniformly homeomorphic, but not isomorphic; actually they are not even Lipschitz homeomorphic.
- This shows that the uniform classification of Banach spaces is different from the linear one and from the Lipschitz one.
- Since in some cases uniformly homeomorphic implies isomorphic, it is natural to hope that in "many" cases Lipschitz homeomorphic implies isomorphic.

The most obvious idea

Given a Lipschitz embedding $f : X \to Y$ we know how to associate to it a linear map, it suffices to take the derivative at some point $df_x : X \to Y$.
It is also clear that

$$\frac{1}{Lip(f^{-1})}\|u\| \leq \|df_x(u)\| \leq Lip(f)\|u\| :$$

just let $t \to 0$ in

$$\frac{1}{Lip(f^{-1})}\|u\| \leq \left\|\frac{f(x+tu)-f(x)}{t}\right\| \leq Lip(f)\|u\| .$$

and its trivial consequence

Corollary

If the Lipschitz embedding $f : X \to Y$ is Gâteaux-differentiable at some point, X linearly embeds in Y.

Warning!!!

- Is f necessarily Gâteaux-differentiable at some point?
- If f is onto, can we assure df_x to be surjective too?

Surjective derivatives

- It is unknown at present.
- If a Lipschitz homeomorphism f is Fréchet-differentiable at x, then df_x is onto. However:

In general there is no hope to have Fréchet-derivatives

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be $\varphi(t) := \begin{cases} 2t & t \geq 0 \\ t & t < 0 \end{cases}$ and define $f : \ell_2 \to \ell_2$ by $f\left((x^i)_{i=1}^{\infty}\right) := \left(\varphi(x^i)\right)_{i=1}^{\infty}$. Then f is a nowhere Fréchet-differentiable Lipschitz homeomorphism.

Existence of Gâteaux-derivatives

Theorem (Aharoni 1974)

Every separable Banach space can be Lipschitz embedded in c_0.

However very few Banach spaces can be linearly embedded in it (for example only non-reflexive ones). Hence:

- c_0 is not small in the Lipschitz setting;
- A Lipschitz embedding $f : X \to c_0$ is in general nowhere Gâteaux-differentiable;
- We need conditions on Y for every Lipschitz map $f : X \to Y$ to be Gâteaux-differentiable at some point.

The correct condition

Theorem (Aronszajn - Christensen - Mankiewicz 1976)

Let X be separable, Y have the Radon-Nikodým property and $f : X \to Y$ be Lipschitz. Then f is Gâteaux-differentiable at some point.

The RNP is actually a necessary condition since a Banach space F has RNP iff every Lipschitz function $f : \mathbb{R} \to F$ is differentiable at some point (equivalently a.e.).

Combining with the trivial corollary

The previous corollary now yields that if a separable Banach space Lipschitz embeds in a Radon-Nikodým one, then it linearly embeds too.

Corollary

Let a separable Banach space X be Lipschitz embedded in Y.

- If Y has RNP, X has RNP too;
- if Y is reflexive, then so is X;
- if Y is isomorphic to a Hilbert space, then so is X.

We notice that the above fails for uniform embeddings since one can show that $L_1([0,1])$ can be uniformly embedded in a Hilbert space (but it does not have the RNP).

The main open problem

- Lipschitz homeomorphism is a rather strong assumption, is it equivalent to isomorphism?
- The answer is NO, the first counterexample is due to Aharoni and Lindenstrauss (1978);
- more recently (2003) Godefroy and Kalton introduced a general construction, the so-called Lipschitz-free spaces, that allows, among other things, to obtain further examples;
- all these examples are non-separable and non-reflexive. The main open problem in this area is to investigate the reflexive and separable cases.

"almost-everywhere" differentiability

- In the above theorem one actually shows that the set of points where f is not Gáteaux-differentiable is negligible in a suitable sense;
- in infinite-dimensional Banach spaces there is no analogue to Lebesgue measure, so one needs other concepts;
- we just mention here one such concept:

Haar null sets

A borel subset A of a separable Banach space X is <u>Haar null</u> if there is a regular Borel probability measure μ on X such that all the translates of A are μ-null.