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Habilitation thesis

Structure and Topology in non-separable Banach Spaces

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Contents

Abstract

The present Habilitation thesis comprises works of the author, collected in 11 published papers and 2 additional accepted papers. This material constitutes the core of the author's research from his first Post-Doc position started in October 2018 till today and all the published papers presented here appeared in print in the years between 2020 and 2024. All papers revolve around the study of the structure of Banach spaces, particularly non-separable ones. For convenience, we separated the thesis in four mutually connected parts, corresponding to the papers indicated in the subsequent section. An introduction to the contents of each part, together with an explanation of the connection between different parts, will be given in the Introduction below.

Keywords: Non-separable Banach spaces; Smoothness and Renorming; Dense subspaces; Structure theory of non-separable Banach spaces; norming Markushevich bases; Descriptive Topology; Metric geometry; Lineability; Lipschitz-free spaces; Banach spaces of Lipschitz functions; Schauder bases; Approximation properties; Higher-order smoothness; Fundamental biorthogonal systems; Operator ranges; Asplund Banach spaces; Weakly compactly generated Banach spaces; Banach spaces of continuous functions; Eberlein, Corson, Valdivia compact spaces; weakly Corson compact spaces; Coarse-wedge topology on trees; Overcomplete sequences; Complemented subspaces; Banach spaces of homogeneous polynomials; Hyperbolic spaces; Tilings.

Author's publications

Smoothness in dense subspaces

[DHR'20] S. Dantas, P. Hájek, and T. Russo
Smooth norms in dense subspaces of Banach spaces
J. Math. Anal. Appl. **487** (2020), 123963
DOI: 10.1016/j.jmaa.2020.123963

[DHR'23] S. Dantas, P. Hájek, and T. Russo
Smooth and polyhedral norms via fundamental biorthogonal systems
Int. Math. Res. Not. IMRN **2023** (2023), 13909–13939
DOI: 10.1093/imrn/rnac211

[HR'20] P. Hájek and T. Russo
On densely isomorphic normed spaces
J. Funct. Anal. **279** (2020), 108667
DOI: 10.1016/j.jfa.2020.108667

[DHR'24] S. Dantas, P. Hájek, and T. Russo
Smooth norms in dense subspaces of $\ell_p(\Gamma)$ *and operator ranges*
Rev. Mat. Complut. **37** (2024), 723–734
DOI: 10.1007/s13163-023-00479-w

Norming Markushevich bases

[HR'25+] P. Hájek and T. Russo
Norming Markushevich bases: recent results and open problems
Pure Appl. Funct. Anal. (in press)
DOI: arXiv:2402.18442

[HRST'21] P. Hájek, T. Russo, J. Somaglia, and S. Todorčević
An Asplund space with norming Markuševič basis that is not weakly compactly generated
Adv. Math. **392** (2021), 108041
DOI: 10.1016/j.aim.2021.108041

[RS'23] T. Russo and J. Somaglia
Banach spaces of continuous functions without norming Markushevich bases
Mathematika **69** (2023), 992–1010
DOI: 10.1112/mtk.12217

Topological aspects

[CRS'21] C. Correa, T. Russo, and J. Somaglia
Small semi-Eberlein compacta and inverse limits
Topology Appl. **302** (2021), 107835
DOI: 10.1016/j.topol.2021.107835

[RS'22] T. Russo and J. Somaglia
Weakly Corson compact trees
Positivity **26** (2022), 33
DOI: 10.1007/s11117-022-00874-5

Additional results

[RS'21] T. Russo and J. Somaglia
Overcomplete sets in non-separable Banach spaces
Proc. Amer. Math. Soc. **149** (2021), 701–714
DOI: 10.1090/proc/15213

[LRS'23] P. Leonetti, T. Russo, and J. Somaglia
Dense lineability and spaceability in certain subsets of ℓ_∞
Bull. Lond. Math. Soc. **55** (2023), 2283–2303
DOI: 10.1112/blms.12858

[HR'22] P. Hájek and T. Russo
Projecting Lipschitz functions onto spaces of polynomials
Mediterr. J. Math. **19** (2022), 190
DOI: 10.1007/s00009-022-02075-6

[BLR'25+] C. Bargetz, F. Luggin, and T. Russo
Tilings of the hyperbolic space and Lipschitz functions
Canad. J. Math. (online first)
DOI: 10.4153/S0008414X24000804

Author’s contribution

The articles contained in the present Habilitation thesis were written with co-authors. The contributions and percentage shares to the different papers of the Habilitation applicant are listed below (in alphabetic order).

[BLR’25+] Applicant’s contributions: Scientific conception and literature search, Theoretical results, Writing of the manuscript, Proofreading.
Applicant’s share: 33%.

[CRS’21] Applicant’s contributions: Scientific conception and literature search, Theoretical results, Writing of the manuscript, Proofreading.
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[HR’25+] Applicant’s contributions: Scientific conception and literature search, Theoretical results, Writing of the manuscript, Proofreading.
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[HRST’21] Applicant’s contributions: Scientific conception and literature search, Theoretical results, Writing of the manuscript, Proofreading.
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[LRS’23] Applicant’s contributions: Scientific conception and literature search, Theoretical results, Writing of the manuscript, Proofreading.
Applicant’s share: 33%.

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Introduction

The study of structural properties of Banach spaces becomes radically more complicated when passing from finite-dimensional Banach spaces, to (infinite-dimensional) separable ones, and then to non-separable spaces. For instance, in finite-dimensional Banach spaces the compactness of the unit ball forces the existence of a unique linear topology on the space; in particular, one deduces the equivalence of all norms and the continuity of all linear transformations. One more aspect of this is the existence of a Euclidean structure (*i.e.*, an orthonormal basis) on every space. Moreover, the existence of Lebesgue measure permits to perform convolution arguments which then result in a complete smoothness theory: the existence of C^∞-smooth partitions of unity allows for smooth approximation of continuous functions and the existence of smooth norms.

In the infinite-dimensional framework all these properties break down. The unit ball $B_{\mathcal{X}}$ of a Banach space $\mathcal{X}$ is no longer compact, which results in the existence of discontinuous linear functionals and hence of non-equivalent norms. Moreover, one is then led to consider different natural linear topologies, such as the w topology on $\mathcal{X}$ and the w^* topology on $\mathcal{X}^*$; it then becomes a non-trivial matter to understand which is the correct linear topology to consider for a certain problem and which are the connections between different such topologies. Additionally, orthonormal bases only exist in Hilbert spaces and, while (unconditional) Schauder bases are an effective replacement of orthonormal ones, they also fail to exist in some separable Banach spaces (even in classical spaces, in case of unconditional bases). The fact that every separable Banach space admits a Markushevich basis (M-basis, for short) illustrates another instance where the choice of the correct system of coordinates is part of the problem at hand. Concerning the renorming theory, it remains valid that several renorming options are still available, for example let us just mention here the existence of a locally uniformly rotund norm, or of a Gâteaux differentiable one, on every separable Banach space. On the other hand, the existence of a Fréchet smooth norm on a separable Banach space $\mathcal{X}$ forces the separability of $\mathcal{X}^*$; to wit, there is no (Fréchet) smoothness theory for a general separable Banach space. Further, in infinite-dimensions Gâteaux and Fréchet smoothness are distinct notions also for Lipschitz functions.

When considering non-separable Banach spaces the situation becomes even more intricate. For separable $\mathcal{X}$, the dual unit ball $B_{\mathcal{X}^*}$ is a compact metrisable space in the w^* topology. For non-separable $\mathcal{X}$, $(B_{\mathcal{X}^*}, w^*)$ remains compact but now fails to be metrisable, which leads to several additional complications; for example the dual ball might fail to be sequentially compact in the w^* topology. Concerning coordinate systems, there are natural examples of Banach spaces that do not admit any M-basis, such as ℓ_∞. It is even equi-consistent with the usual ZFC axioms that there exist non-separable Banach spaces that admit no uncountable biorthogonal systems; hence, they admit no coordinate system whatsoever that could capture the non-separable structure. The renorming theory is also weaker, as witnessed for instance by some rigidity results asserting that every renorming of a certain Banach space $\mathcal{X}$ contains a fixed Banach space $\mathcal{Z}$ isometrically. For example we quote Partington's results that every renorming of $\ell_\infty^c(\Gamma)$, Γ uncountable, contains $\ell_\infty^c(\omega_1)$ isometrically and every renorming of ℓ_∞/c_0 contains ℓ_∞ isometrically. As a simpler instance, ℓ_∞ doesn't admit any LUR norm.

Because of these reasons, the research in non-separable Banach space theory usually focuses on distilling 'large' classes of 'well-behaved' non-separable Banach spaces, where stronger results could be obtained. These classes usually consist of generalisations of the class of separable Banach spaces with the hope that they would retain certain properties of separable spaces. As it happens, these classes are frequently defined by some topological property of the w^* compact dual unit ball, which are then proved to imply some structure property on the space, such as good coordinate systems. In turn, such coordinate systems enhance renorming possibilities. As a sample, let us mention the class of weakly Lindelöf determined Banach spaces, that are defined by stipulating the dual ball to be a Corson compact and can be characterised by the existence of an M-basis that countably supports the dual space. In turn, the existence of such an M-basis yields a LUR renorming.

The topics of this thesis also witness this interaction between topological properties, structural results (such as existence of coordinate systems), and renorming results. The exposition is divided in four parts, the first three being dedicated to these topics and the fourth one collecting some additional results that are less directly connected with the above parts. As it should be expected the first three parts are tightly connected one another; for instance some sections of papers in one part would more naturally belong to a different part of the thesis.

Smoothness in dense subspaces

It is by now a well-known fact that the existence of a smooth bump (namely, a non-zero function with bounded support) on a Banach space $\mathcal{X}$ carries several deep structural consequences for $\mathcal{X}$. For example, the presence of a C^1-smooth bump implies that $\mathcal{X}$ is Asplund, [Fa'87b], while the presence of an LFC bump yields that $\mathcal{X}$ is a c_0-saturated Asplund space, [FZ'97, PWZ'81]. If $\mathcal{X}$ admits a C^2-smooth bump, then either it contains a copy of c_0, or it is super-reflexive with type 2, [FWZ'83]. Finally, if $\mathcal{X}$ admits a C^∞-smooth bump and it contains no copy of c_0, then it has exact cotype $2k$, for some $k \in \mathbb{N}$, and it contains ℓ_{2k}, [De'89]. As it turns out, the proofs of the above results involve at some point the completeness of $\mathcal{X}$, frequently via the appeal to some form of variational principles, such as the Ekeland variational principle [Ek'74], or the compact variational principle [DF'89]. Moreover, results concerning the best order of smoothness for concrete Banach spaces also involve completeness in their proofs, [DGZ'93].

On the other hand, there seems to be no obvious reason why completeness should play an essential role in the above results. Accordingly, several authors asked different version of the general question whether the existence of a smooth norm on a large subset of $\mathcal{X}$ should imply some structural consequences for the space. Quoting [BL'00, p. 96], it is unknown if the above result on C^1-smooth bumps '*can be strengthened to say that if $\mathcal{X}$ is separable and the set where the norm fails to be Fréchet differentiable is "small" in some sense, then $\mathcal{X}^*$ is separable*'. For instance, it is unknown if there is a norm on ℓ_1 that is Fréchet differentiable outside a countable union of hyperplanes, [GMZ'16, Problem 148]. Another open problem is whether $\mathcal{X}$ must be Asplund provided that the restriction of its norm to every closed subspace admits points of Fréchet differentiability.

A good testing ground for the above problems could be to try constructing smooth norms in normed spaces of countable dimension, since those are the farthest possible to an infinite-dimensional Banach space. Such an attempt was indeed considered in the Nineties, leading to Vanderwerff's construction of a C^1-smooth norm in such spaces [Va'95] and to Hájek's construction of a C^∞-smooth one [Há'95]. In particular, for a separable normed space $\mathcal{X}$ the existence of a C^1-smooth norm does not imply that $\mathcal{X}^*$ is separable, a result that is quite surprising at first sight. Moreover, under the same conditions, $\mathcal{X}$ also admits a polyhedral norm, [DFH'98]. A slight reformulation of these results is that every separable Banach space admits a dense subspace with a C^∞-smooth norm as well as a polyhedral one. Having such rather strong results for separable spaces, it was natural to investigate the non-separable counterpart to these results. This was for instance suggested in [GMZ'16, Problem 149], where the problem is asked for $\ell_1(\Gamma)$, and undertook by Dantas, Hájek, and the author in the papers [DHR'20, DHR'23, DHR'24].

The paper [DHR'20], published in the *Journal of Mathematical Analysis and Applications* in 2020, is the one where the problem was stated for the first time in its full generality as follows: given a Banach space $\mathcal{X}$, is there a dense subspace $\mathcal{Z}$ of $\mathcal{X}$ that admits a C^k-smooth norm, for some $k \in \mathbb{N} \cup \{\infty, \omega\}$? (By definition, C^ω means analytic.) The first obtained result is a sharpening of the separable results from the Nineties, by obtaining that normed spaces of countable dimension admit analytic norms (and that analytic norms are dense in the set of equivalent norms). The same technique also proves that the space of finitely supported vectors in $\ell_1(\mathfrak{c})$ admits an analytic norm; however, it remained open whether this result extends to $\ell_1(\Gamma)$ for all sets Γ. The second main result of [DHR'20] is that the linear span $\mathcal{Z}$ of a long unconditional basis in a Banach space $\mathcal{X}$ also admits a C^∞-smooth norms. In particular, this answers [GMZ'16, Problem 149], by getting a C^∞-smooth norm in the space of finitely supported vectors in $\ell_1(\Gamma)$.

Even though these results were already fairly promising, they present at least three drawbacks. First and foremost, the assumption that $\mathcal{X}$ admits a long unconditional basis is quite restrictive; secondly, in this last result, the argument wasn't flexible enough to additionally give the density of smooth norms; last, the existence of smooth norms could only be assured on one specific dense subspace. The first two problems were successfully handled in the subsequent paper [DHR'23], while the third problem motivated the results from [HR'20] and [DHR'24].

Let us follow the logical order instead of the chronological one and pass to the presentation of [DHR'23], published in *International Mathematics Research Notices* in 2023. The main motivation for the paper was to extend the above results beyond the case of long unconditional bases and to obtain a theory of smoothness in dense subspaces for certain well-known classes of non-separable Banach spaces. The main output of the paper is the following result. If $\mathcal{X}$ is a Banach space that admits a fundamental biorthogonal system, then the linear span $\mathcal{Z}$ of such a system admits a C^∞-smooth norm that locally depends on finitely many coordinates (LFC, for short) as well as a polyhedral norm that is also LFC. Moreover, such norms are dense in the space of all equivalent norms (here and throughout, when talking about norm approximation we understand uniform convergence on bounded sets). Combining this result with known techniques from smoothness, such as Haydon's method for constructing partitions of unity [Ha'96] and Hájek–Procházka's construction of C^1-smooth LUR norms [HPr'14], one also obtains the following. The space $\mathcal{Z}$ also admits locally finite, σ-uniformly discrete C^∞-smooth and LFC partitions of unity; moreover, $\mathcal{Z}$ admits a C^1-smooth LUR norm and every equivalent norm can be approximated by such norms.

Consequently, the existence of a fundamental biorthogonal system leads to a complete smoothness theory on a dense subspace, thereby substantially strengthening the conclusion of results from [DHR'20]. Moreover, the assumption on $\mathcal{X}$ is now considerably more general, as the existence of a fundamental biorthogonal system is assured for many classes of non-separable Banach spaces. Here we thus have a connection to the topic of the second part, as structural properties of certain classes of Banach spaces enter the theory of smoothness. For a detailed description of the classes of spaces that the result applies to, we refer to the introduction of [DHR'23], where this is explained at large. Here we just mention that every Plichko space admits a fundamental biorthogonal system, hence the result covers reflexive spaces, weakly compactly generated ones, $\ell_1(\Gamma)$ for every set Γ, every $L_1(\mu)$ space, and every $\mathcal{C}(\mathcal{K})$ space, where K is a Valdivia compactum or an Abelian compact group.

Throughout the papers [DHR'20] and [DHR'23] the dense subspace where a smooth norm is defined always has the form of the linear span of a (fundamental) biorthogonal system; in particular, it has countable dimension in the case when $\mathcal{X}$ is separable. The natural question then arose whether the same results could be pushed to hold for some different dense subspaces, for instance if every dense subspace could contain a further dense subspace where the same results would be true. Another natural quest would be to find a counterexample to the general problem, namely a Banach space $\mathcal{X}$ such that none of its dense subspaces admits a smooth norm. Heuristically, if one could prove some result asserting that all dense subspaces of a Banach space behave similarly in some respect, then the above questions would simplify considerably. For instance, it would be conceivable that there exists a dense subspace $\mathcal{Z}$ such that every dense subspace of $\mathcal{X}$ would contain an isomorphic copy of $\mathcal{Z}$; such a $\mathcal{Z}$ could be seen as a common kernel of all dense subspaces. Then, the answer to the above questions would only depend on the answer for the specific subspace $\mathcal{Z}$.

This question was the explicit motivation for the paper [HR'20], published in the *Journal of Functional Analysis* in 2020, where the notion of densely isomorphic normed spaces is introduced. Two normed spaces $\mathcal{X}$ and $\mathcal{Y}$ are densely isomorphic if there are dense subspaces $\mathcal{X}_0$ and $\mathcal{Y}_0$ of $\mathcal{X}$ and $\mathcal{Y}$ respectively that are isomorphic. The existence of a common kernel $\mathcal{Z}$ as before would then imply that all dense subspaces of $\mathcal{X}$ are mutually densely isomorphic. However, the main result in [HR'20] is that in general there are dense subspaces of a Banach space that fail to be densely isomorphic. More precisely, every non-separable WLD Banach space $\mathcal{X}$ contains two dense subspaces that fail to be densely isomorphic. Moreover, if the density character of $\mathcal{X}$ is at most $\mathfrak{c}$, there are two dense subspaces with no non-separable mutually isomorphic subspaces. In other words, two dense subspaces of a Banach space, not only might fail to admit a dense kernel, but even might not admit a non-separable kernel, thus witnessing the extreme diversity of dense subspaces of a Banach space.

A different approach to the same result is also given in the paper, via the following result (which can be considered to be the main output of the paper). Under the assumption of the Continuum Hypothesis, every WLD Banach space $\mathcal{X}$ with density equal to $\mathfrak{c}$ contains a dense subspace without any uncountable biorthogonal system. It is then easy to show that such a subspace is not densely isomorphic to the linear span of an M-basis of $\mathcal{X}$. Besides, the result yields a counterpart for normed spaces to several consistent constructions of non-separable Banach spaces without uncountable biorthogonal systems.

Notwithstanding the pessimistic conclusions obtained in [HR'20], in some particular cases it has been possible to construct smooth norms in dense subspaces that are not the linear span of a biorthogonal system, but are in some sense larger. This has been achieved in [DHR'24], published in *Revista Matemática Complutense* in 2024. More precisely, it is proved there that for $p \in [1, \infty)$, the dense subspace $\mathcal{Z}_p$ of $\ell_p(\Gamma)$ comprising all vectors that are q-summable for some $q \in (0, p)$ also admits a C^∞-smooth and LFC norm. Interestingly, when $p > 1$, the subspace $\mathcal{Z}_p$ contains a dense operator range, given by a continuous injection of $\ell_1(\Gamma)$; thus C^∞-smooth norms can also exist in dense subspaces that are operator ranges. The relevance of this remark is that operator ranges bear a certain form of completeness, that is not shared by all normed spaces. Therefore, building smooth norms on (dense) operator ranges is in principle harder than building those on normed spaces that are the linear span of a biorthogonal system; accordingly, the results presented here require stronger assumptions on the space.

Norming Markushevich bases

Among the various classes of non-separable Banach spaces that have been considered in the literature the most investigated and well-known one is the class of weakly compactly generated (or WCG) Banach spaces, introduced by Amir and Lindenstrauss in their *Annals* paper [AL'68]. WCG spaces constitute a common generalisation of separable and reflexive Banach spaces; besides these examples, also $c_0(\Gamma)$ for every set Γ and $L_1(\mu)$ for every finite measure μ are canonical examples of WCG Banach spaces. Among the most important properties of WCG spaces is the existence of a projectional resolution of the identity (PRI, for short), a well-ordered collection of norm-one projections onto smaller subspaces, that satisfy certain continuity and compatibility conditions. In essence, a PRI yields a canonical way to decompose a Banach space into smaller well-behaved pieces, from which information on the space can be reconstructed, arguing by transfinite induction on the density character of the space. For instance, the existence of a PRI in a WCG space $\mathcal{X}$ was used in [AL'68] to build an M-basis in $\mathcal{X}$; hence, $\mathcal{X}$ admits an injection into $c_0(\Gamma)$ and therefore a strictly convex renorming. Shortly after, PRI's were used by Troyanski [Tr'70] and subsequently Zizler [Zi'84] in the celebrated renorming theorem giving an LUR norm in every WCG Banach space.

The papers presented in this part [HR'25+, HRST'21, RS'23] are all dedicated to the study of the connection between WCG Banach spaces and norming M-bases. We decided not to present them in chronological order, because the paper [HR'25+] surveys recent results on norming M-bases and in particular contains a detailed overview of the problem and the results contained in [HRST'21] and [RS'23], therefore it fits best as opening to this part. Let us now briefly describe some results from the literature indicating a connection between WCG Banach spaces and norming M-bases; for a more detailed outline we refer to [HR'25+, Section 4].

We already mentioned the results of Amir–Lindenstrauss [AL'68] and Troyanski [Tr'70] that every WCG Banach space admits a PRI and an LUR norm. Subsequently, John and Zizler [JZ'74] modified the construction of a PRI from [AL'68] and obtained the existence of a PRI in every Banach space with 1-norming M-basis; hence, one infers that Banach spaces with norming M-basis admit an LUR renorming. It was then natural to conjecture the existence of an implication between the notions WCG spaces and norming M-bases. However, the example of $\ell_1(\Gamma)$, that admits a norming M-basis and fails to be WCG, immediately dashed the most optimistic hope that a Banach space is WCG if and only if it admits a norming M-basis. In the same paper the authors asked whether every WCG Banach space must admit a norming M-basis; the problem eventually became one of the classical problems in Banach space theory and was reiterated multiple times in several articles and books, see, *e.g.*, [GMZ'16, Problem 111], until its recent negative solution due to Hájek [Há'19].

As we said, the converse implication is easily seen to be false in general. However, several results since the Seventies strongly hinted towards the fact that the converse might be true under the additional assumption that the space is Asplund. In fact, Fabian [Fa'87a] proved that every WCD (weakly countably determined) Asplund space admits a shrinking M-basis and hence it is WCG. Fabian's argument consists in working in $\mathcal{X}$ and in $\mathcal{X}^*$ simultaneously to build a PRI in $\mathcal{X}$ such that the adjoints of the projections form a PRI in $\mathcal{X}^*$ (such a PRI is usually called shrinking). Then this combines with a result of Vašák [Va'81] who built a shrinking M-basis from a shrinking PRI. In a similar vein, Valdivia built a shrinking PRI in WLD Asplund spaces, [Va'88, Lemma 2]. Moreover, at the same time of [Fa'87a], Fabian and Godefroy [FG'88] built a PRI in the dual of every Asplund space. Therefore, the Asplund assumption is responsible for the existence of a PRI in the dual space, while the WCD (or WLD) one yields a PRI in the space. Fabian's method from [Fa'87a] actually consisted in merging the two techniques to obtain compatible projections in the space and in the dual simultaneously (see the explanation in [Fa'97, Chapter 8] or [DGZ'93, Sections VI.2–VI.4]). However, as we said, John and Zizler [JZ'74] built a PRI in presence of a 1-norming M-basis in $\mathcal{X}$. Therefore, it was natural to conjecture that one could also glue the techniques from John–Zizler [JZ'74] with these from Fabian–Godefroy [FG'88] to build a shrinking PRI in Asplund spaces with a norming M-basis. That it were the case was indeed conjectured in the early Nineties by Godefroy, see, *e.g.*, [GMZ'16, Problem 112]; equivalently, the conjecture was that Asplund spaces with norming M-bases are WCG. The negative answer to this conjecture has recently been given by Hájek, Somaglia, Todorčević, and the author in [HRST'21].

Before we explain the contents of [HRST'21] in more detail, let us discuss the survey paper [HR'25+], that has recently been accepted for publication in *Pure and Applied Functional Analysis*. The main part of the paper can be considered to be Section 4 there that explains the context to the problems of John–Zizler and Godefroy that we mentioned before and explains some ingredients in their solutions from [Há'19] and [HRST'21]. Before this, the first three sections give a rather self-contained introduction to the theory of norming M-bases and to some classes of non-separable Banach spaces, such as WCG, WLD, and Plichko spaces. In particular, Section 2 gives several elementary remarks and examples concerning norming M-bases and their connection to (long) Schauder bases; this part requires essentially no prerequisites and can be seen as an introduction to the area for the readers who wouldn't be familiar with it. The third section gathers the definition and some characterisations of the above-mentioned classes, with particular focus on topological properties of the dual ball, existence of M-bases with additional properties, and topological characterisations of those $\mathcal{C}(\mathcal{K})$ spaces that belong to the class. This section is thus closely connected to the topological aspects in non-separable Banach spaces, that we will describe further in the third part of this thesis. As a sample, let us mention here that a Banach space $\mathcal{X}$ is WCG if and only if it admits a weakly compact M-basis, *i.e.*, an M-basis $\{e_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ such that $\{e_\alpha\}_{\alpha\in\Gamma}\cup\{0\}$ is weakly compact in $\mathcal{X}$. This is easily seen to be equivalent to the fact that the evaluation map

$$\mathsf{ev}\colon \mathcal{X}^*\to\mathbb{R}^\Gamma\colon \qquad \varphi\mapsto(\langle\varphi,e_\alpha\rangle)_{\alpha\in\Gamma}$$

has values in $c_0(\Gamma)$. Note that one can assume without loss of generality that $\|e_\alpha\|=1$, whence $\mathsf{ev}[B_{\mathcal{X}^*}]\subseteq c_0(\Gamma)\cap[-1,1]^\Gamma$. As a consequence of this, one infers that $\mathcal{X}$ is a subspace of a WCG space if and only if $(B_{\mathcal{X}^*},w^*)$ is an Eberlein compactum. When $\mathcal{X}$ is a $\mathcal{C}(\mathcal{K})$ space, this characterisation simplifies to the fact that $\mathcal{C}(\mathcal{K})$ is WCG if and only if $\mathcal{K}$ is Eberlein, if and only if $(B_{\mathcal{C}(\mathcal{K})^*},w^*)$ is Eberlein.

The second part of the paper, comprising Sections 5 to 7, highlights some directions for future research and points out explicitly some particular open problems. The first direction analyses the open problem if an Asplund $\mathcal{C}(\mathcal{K})$ space with norming M-basis is WCG, namely Godefroy's problem for $\mathcal{C}(\mathcal{K})$ spaces. This problem has been considered first in [RS'23], where a positive answer has been given in some particular cases; we shall describe these results below. The section also contains an outline of a possible strategy for solving the problem together with some open problems that originate from this strategy. Section 6 is dedicated to the heredity problem for norming M-bases, namely the problem whether the existence of a norming M-basis passes from a Banach space to its closed subspaces. In the section we revise some results giving a positive answer under additional assumptions on the subspace and mention some specific spaces for which the problem is open. Finally, Section 7 is dedicated to open problems concerning semi-Eberlein compacta, a class of compact spaces connected to WCG spaces. Such a topic really belongs to the topological theory, hence we will say more about it in the third part of the thesis.

As we mentioned, the negative answer to Godefroy's problem for a general Banach space has been given by Hájek, Somaglia, Todorčević, and the author in [HRST'21], published in *Advances in Mathematics* in 2021. As it turns out, the result is actually stronger, because the Banach space $\mathcal{X}_\varrho$ that we constructed, besides being an Asplund space with norming M-basis and not being WCG, also admits the following stronger property: $\mathcal{X}_\varrho$ admits a 1-norming M-basis that is additionally an Auerbach basis and a long monotone Schauder basis. The construction of the space is crucially based upon Todorčević's theory of walks on ordinals, developed in [To'87] and [To'07]. In essence, the core of the argument consists in the construction of a compact space $\mathcal{K}_\varrho$, depending upon a certain ordinal metric ϱ, and then the desired counterexample is a rather explicit subspace of $\mathcal{C}(\mathcal{K}_\varrho)$. The topological counterpart to the construction, namely the construction of $\mathcal{K}_\varrho$, also led to a solution to some open question, as $\mathcal{K}_\varrho$ is a (scattered) semi-Eberlein compact space that admits a weak P-point, whose existence solved a problem due to Kubiś and Leiderman, [KL'04].

At first sight the above counterexample seems quite close to being a $\mathcal{C}(\mathcal{K})$ space, since $\mathcal{C}(\mathcal{K}_\varrho)$ is an Asplund space that fails to be WCG. However, it is not known whether $\mathcal{C}(\mathcal{K}_\varrho)$ admits a norming M-basis, therefore an actual $\mathcal{C}(\mathcal{K})$ counterexample is still not available. On the other hand Hájek's counterexample from [Há'19] is a $\mathcal{C}(\mathcal{K})$ space, which motivated the question whether Godefroy's problem might possibly have a positive answer for $\mathcal{C}(\mathcal{K})$ spaces. This problem has been faced in [RS'23], published in *Mathematika* in 2023. The most natural compact space $\mathcal{K}$ to consider for this problem is $\mathcal{K} = [0, \omega_1]$, for which space an answer has been given by Alexandrov and Plichko [AP'06]: the Banach space $\mathcal{C}([0, \omega_1])$ does not admit any norming M-basis. The simplest reason why such a space would be a good candidate to consider is that $\mathcal{C}([0, \omega_1])$ is an Asplund space that is not WCG; however there is much more than this, as explained in [HR'25+, Section 5]. The main motivation for [RS'23] was to generalise this result in order to get a positive solution to Godefroy's problem for $\mathcal{C}(\mathcal{K})$ spaces. In short, the outcome of this research is the following: Alexandrov–Plichko's method cannot solve the problem in full generality, but it does give an answer in some interesting particular cases. In particular, the main result of the paper is that $\mathcal{C}([0, \omega_1])$ does not embed in a Banach space with a norming M-basis. As a consequence, one obtains the result that $\mathcal{C}(\mathcal{K})$ does not admit a norming M-basis whenever $[0, \omega_1]$ is a continuous image of $\mathcal{K}$. By itself, this result is not sufficient to conclude an answer to the main problem, because there exist compact spaces $\mathcal{K}$ such that $\mathcal{C}(\mathcal{K})$ is Asplund and not WCG (namely

$\mathcal{K}$ is scattered and not Eberlein), but admitting no continuous image onto $[0, \omega_1]$. One such an example is actually the space $\mathcal{K}_\varrho$. On the other hand, when $\mathcal{K}$ is a compact tree (endowed with the coarse-wedge topology), the result gives a positive answer to Godefroy's problem. From this discussion one clearly evinces the importance of topological considerations in the structural theory of non-separable Banach spaces, that we will describe in the next part.

However, before we move to those we would like to point out the connection of these results with the subspace problem for norming M-bases from [HR'25+, Section 6]. The generalisation obtained in [RS'23] is obviously formally stronger than Alexandrov–Plichko's result; however, it is apparently not known whether admitting a norming M-basis and embedding in a Banach space with norming M-bases are distinct conditions in general. Therefore, the open problem considered in [HR'25+, Section 6] is motivated by the question whether one could actually deduce the result of [RS'23] directly from [AP'06], of whether the former is a strictly stronger result.

Topological aspects

The connection between compact topological spaces and Banach spaces has been mentioned already several times in these pages. To each Banach space $\mathcal{X}$ one associates the compact space $(B_{\mathcal{X}^*}, w^*)$ and conversely for every compact space $\mathcal{K}$ one can consider the Banach space $\mathcal{C}(\mathcal{K})$. These two operations form a sort of duality due to the facts that $\mathcal{K}$ canonically embeds (homeomorphically) in $(B_{\mathcal{C}(\mathcal{K})^*}, w^*)$ and $\mathcal{X}$ embeds isometrically in $\mathcal{C}(B_{\mathcal{X}^*}, w^*)$. Accordingly, several analytic properties on the Banach space level can be deduced from topological properties of the compact spaces. Perhaps the simplest instance is that $\mathcal{X}$ is separable if and only if $(B_{\mathcal{X}^*}, w^*)$ is metrisable and that $\mathcal{K}$ is metrisable if and only if $\mathcal{C}(\mathcal{K})$ is separable. Similarly, we mentioned that a Banach space $\mathcal{X}$ is a subspace of a WCG space if and only if its dual ball $(B_{\mathcal{X}^*}, w^*)$ is Eberlein and a compact space $\mathcal{K}$ is Eberlein if and only if $\mathcal{C}(\mathcal{K})$ is WCG, if and only if $(B_{\mathcal{C}(\mathcal{K})^*}, w^*)$ is Eberlein. Notice that the first result is not as symmetric as the second one, because the class of WCG spaces is not closed under taking subspaces [Ro'74], while Eberlein compacta are closed under continuous images [BRW'77].

More generally, one could mention the classes of Talagrand, Gul'ko, Corson, and Valdivia compact spaces, that correspond respectively to weakly K-analytic, WCD (or Vašák), WLD, and Plichko Banach spaces. More precisely, a Banach space is weakly K-analytic (resp. WCD) if and only if $(B_{\mathcal{X}^*}, w^*)$ is a Talagrand (resp. Gul'ko) compactum and a compact space $\mathcal{K}$ is Talagrand (resp. Gul'ko) if and only if $\mathcal{C}(\mathcal{K})$ is weakly K-analytic (resp. WCD), [Fa'97]. Moreover, a Banach space is WLD if and only if $(B_{\mathcal{X}^*}, w^*)$ is Corson, [AM'93, Section 1], [VWZ'94, Theorem 1.1] and references therein. Further, a $\mathcal{C}(\mathcal{K})$ space is WLD if and only if $\mathcal{K}$ is Corson and it has property (M), namely every measure on $\mathcal{K}$ has separable support, [AMN'88]; the assumption of property (M) can be dispensed with under the assumption of Martin Axiom, but not in general, [AMN'88]. The situation concerning Plichko spaces and Valdivia compacta is less symmetric. If a

Banach space $\mathcal{X}$ is 1-Plichko, then $(B_{\mathcal{X}^*}, w^*)$ is Valdivia. However, it is not true that the dual ball of $\mathcal{X}$ is Valdivia when $\mathcal{X}$ is merely Plichko, [Ka'99a]. Further, it is also not true that $\mathcal{X}$ must be 1-Plichko if $(B_{\mathcal{X}^*}, w^*)$ is Valdivia, [Ka'02]. Finally, when $\mathcal{X}$ is a $\mathcal{C}(\mathcal{K})$ space, $\mathcal{C}(\mathcal{K})$ is 1-Plichko for $\mathcal{K}$ Valdivia, [Or'92] (see [Ka'00b, Proposition 5.1]), while the converse fails, [BK'06], [Ka'09, Section 4]. Moreover, if $\mathcal{K}$ has a dense set of G_δ points, $\mathcal{K}$ is Valdivia if and only if $\mathcal{C}(\mathcal{K})$ is 1-Plichko, if and only if $(B_{\mathcal{C}(\mathcal{K})^*}, w^*)$ is Valdivia, [Ka'00b, Theorem 5.3].

The class of Asplund spaces that played an important role in the previous part also admits topological descriptions. The cleanest characterisation is for $\mathcal{C}(\mathcal{K})$ spaces and asserts that $\mathcal{C}(\mathcal{K})$ is Asplund if and only if $\mathcal{K}$ is scattered. An entirely topological characterisation in terms of the dual ball is not possible, because, for instance, the dual unit balls $(B_{\ell_1^*}, w^*)$ and $(B_{\ell_2^*}, w^*)$ are homeomorphic by Keller theorem, but the former is not Asplund. However, one still has the characterisation that $\mathcal{X}$ is Asplund if and only if $(B_{\mathcal{X}^*}, w^*)$ is fragmented by the norm, [Ph'89]. Moreover, when $\mathcal{X}$ is Asplund, its dual ball $(B_{\mathcal{X}^*}, w^*)$ is a Radon-Nikodým compactum, which class can be characterised by being fragmented by a lower semi-continuous metric, [Na'87]. Let us point out that the class of Banach spaces with Radon-Nikodým dual ball is much larger than the Asplund class. In fact, by the Davis–Figiel–Johnson–Pełczyński factorisation theorem [DFJP'74], every Eberlein compact is homeomorphic to a weakly compact subset of a reflexive Banach space. Therefore, every Eberlein compact space is Radon-Nikodým, thus every subspace of a WCG space has Radon-Nikodým dual ball.

This duality has effectively being used to obtain results both on the Banach space and on the topological level. For instance, from the fact that every Eberlein compact is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ [AL'68] one deduces that Eberlein compacta are sequentially compact, thus obtaining an alternative proof of the Eberlein–Šmulian theorem. Moreover, Benyamini, Rudin, and Wage [BRW'77] proved the topological claim that Eberlein compacta are closed under continuous images and deduced that $(B_{\mathcal{X}^*}, w^*)$ is Eberlein when $\mathcal{X}$ is a subspace of a WCG space (the result for WCG $\mathcal{X}$ was already known from [AL'68]).

For this reason it is especially important to study the stability under continuous images and under subspaces of the above classes of compacta (if $\mathcal{Y}$ is a subspace of $\mathcal{X}$, $(B_{\mathcal{Y}^*}, w^*)$ is a continuous image of $(B_{\mathcal{X}^*}, w^*)$ and, likewise, if $\mathcal{Y}$ is a quotient of $\mathcal{X}$ $(B_{\mathcal{Y}^*}, w^*)$ is a subspace of $(B_{\mathcal{X}^*}, w^*)$). By definition, Eberlein, Talagrand, Gul'ko, Corson, scattered and Radon-Nikodým compacta are closed under taking subspaces, while Valdivia compacta are not closed under subspaces, because every cube $[0,1]^\Gamma$ is Valdivia.

The corresponding results for continuous images are typically harder to prove directly. Therefore a simple way to obtain some such results is to derive them from Banach space properties: if $\mathcal{L}$ is a continuous image of a scattered compact $\mathcal{K}$, then $\mathcal{C}(\mathcal{L})$ is a subspace of the Asplund space $\mathcal{C}(\mathcal{K})$. Thus $\mathcal{C}(\mathcal{L})$ is Asplund and $\mathcal{L}$ is scattered (the original topological proof is due to Rudin [Ru'57]). The same reasoning applies to Talagrand, Gul'ko, and Corson compacta. That Eberlein compacta are stable under continuous image is the deep result from [BRW'77] that we quoted already. The stability of Radon-Nikodým compacta has been a long-standing open problem until his negative solution [AK'13]. Concerning Valdivia compacta, it is not hard to see that the quotient of $[0, \omega_1]$ obtained by identifying ω and ω_1 is not Valdivia; Kalenda [Ka'99b] generalised this result by proving that every Valdivia compactum that is not Corson admits a non Valdivia continuous image. An important particular case is the question of stability under open mappings. Then an open image of a Valdivia compactum is Valdivia provided that it has a dense set of G_δ points [Ka'00a], but not in general [KU'05].

Before we introduce the papers [CRS'21] and [RS'22], let us say a few words about [HRST'21, Section 4.3] that is dedicated to semi-Eberlein compacta (that are also introduced in [HR'25+, Section 7]). Semi-Eberlein compacta can be seen as the topological counterpart to Banach spaces admitting a norming M-basis, in the sense that if $\mathcal{X}$ admits a 1-norming M-basis, then the dual ball is semi-Eberlein. However, the converse implication fails to hold, because the Banach space constructed in [Há'19] has Eberlein dual ball, but cannot be embedded in a space with norming M-basis. An important and non-trivial example of a compact space that is not semi-Eberlein is the ordinal interval $[0, \omega_1]$; in fact, Kubiś and Leiderman proved that semi-Eberlein compacta do not admit P-points [KL'04, Theorem 4.2], while ω_1 is an easy example of a P-point in $[0, \omega_1]$. Motivated by this result, they asked the natural question whether semi-Eberlein compacta can admit weak P-points, which, as we said above, has been answered positively in [HRST'21]. This result triggered a new interest for this class of spaces, resulting in the papers [CCS'22, CRS'21, MPZ'24].

The paper [CRS'21], published in *Topology and its Applications* in 2021, is dedicated to the investigation of properties of semi-Eberlein compacta that are related to inverse limits, inspired by similar results for Valdivia compacta from [Ku'06, KM'06]. Perhaps the first result in this direction, from [AMN'88], is that every Valdivia compact space can be obtained as the inverse limit of a continuous inverse system of Valdivia compacta and with retractions as bonding maps. The result was generalised in [KM'06, Theorem 4.2], asserting that if the bonding retractions are so called simple retractions, then it is enough to assume that the first compact space in the inverse system is Valdivia. In particular, a compact space of weight at most ω_1 is Valdivia if and only if it is the limit of an inverse system of compact metric spaces, whose bonding maps are retractions. Such a characterisation was for instance used to deduce that every retract of a generalised cube (namely, a product of compact metric spaces) is Valdivia. This was strengthened in [KL'04] to say that every such a retract is even semi-Eberlein. The first part of [CRS'21] contains a similar decomposition as [AMN'88] for semi-Eberlein compacta: every semi-Eberlein compact space is the inverse limit of an inverse system of semi-Eberlein compacta with bonding retractions. The result depends upon some analogue results in terms of projectional skeletons [CCS'22].

The second part of the paper is inspired by a stability result for semi-Eberlein compact spaces from [KL'04]. More precisely, it was shown in [KL'04, Theorem 4.2] that every inverse limit of a continuous inverse system of compact metric spaces whose bonding maps are semi-open retractions is semi-Eberlein. In the paper we introduce and investigate the properties of the class $\mathcal{RS}$ comprising all such inverse limits. Having in mind the aforementioned characterisation of Valdivia compacta of weight at most ω_1, it was quite natural to conjecture that every semi-Eberlein compactum of weight at most ω_1 belongs to $\mathcal{RS}$. However, it turns out that this is not the case and actually $\mathcal{RS}$ contains no scattered Eberlein compact space of weight ω_1. Further, we show that $\mathcal{RS}$ is closed under clopen subsets and study stability under products and continuous images.

The paper [RS'22], published by *Positivity* in 2022, is instead dedicated to one example relevant to the theory of continuous images of Valdivia compacta. Even though the class of Valdivia compacta is not closed under continuous images, as we explained above, continuous images of Valdivia compacta share several properties with Valdivia ones, [Ka'99c, Ka'00b, Ka'03]. Kalenda asked in [Ka'03] if a compact space $\mathcal{K}$ is weakly Corson provided that each of its closed subsets is weakly Valdivia. Let us just mention here that weakly Valdivia means, by definition, continuous image of a Valdivia compactum and that weakly Corson compacta roughly are the analogue in weakly Valdivia compacta of Σ-subsets of Valdivia compacta. The interval $[0,\omega_1]$ is a simple example of a space that is not Corson even though all its closed subsets are Valdivia; however, ordinal intervals can't provide a counterexample to Kalenda's question, [Ka'03, Theorem 3.5], which was the motivation for the problem.

The question was the main motivation at the origin of [RS'22], where a negative answer is given. As it turns out, as a counterexample one may take the full binary tree of height $\omega_1 + 1$, endowed with the coarse wedge topology. While the construction of the example is extremely simple, a certain amount of work is needed in order to show that it admits the desired properties. Given the need to study Corson countably compact spaces, an important part of the paper consists in the introduction of a topology, called the countably coarse wedge topology, that makes every chain complete, rooted tree into a countably compact topological space. This topology is a variation of the coarse wedge topology and we also study in some detail the connection between these topologies and their role in the study of weakly Corson and weakly Valdivia compact trees.

Additional results

Loosely speaking, one could view a common theme in the papers [RS'21, LRS'23, HR'22, BLR'25+] presented in this part in the attempt to generalise to the non-separable setting some known results or theories from the separable theory. It has been mentioned several times in this introduction that these attempts most frequently require structural assumptions on the non-separable Banach spaces. This phenomenon is present in [RS'21], while the other papers are dedicated to one specific Banach space and therefore the structural theory is not present. As it turns out, most considerations in [LRS'23, HR'22, BLR'25+] are essentially finite-dimensional in nature. Because of these reasons, the papers in this part will be presented independently.

The paper [RS'21], published by the *Proceedings of the American Mathematical Society* in 2021, is devoted to the introduction of a non-separable counterpart to the notion of overcomplete sequence. A sequence in a Banach space $\mathcal{X}$ is overcomplete if every of its subsequences has linearly dense span in $\mathcal{X}$. Such sequences were first introduced by Klee in [Kl'58] (let us also refer to [BP'75, p. 283] for some historical comments) in his proof of the existence of uncountably many mutually non homeomorphic separable normed spaces. Subsequently they were thoroughly investigated by Terenzi, *e.g.*, [Te'78, Te'81a, Te'81b, Te'83] in particular in connection with the existence of basic sequences. Indeed, while an overcomplete sequence cannot admit any basic subsequence, Terenzi [Te'83] found a surprising dichotomy implying that, in a sense, being overcomplete and being basic are complementary notions. More recently they have been studied in [CFP'06, FZ'14]. The latter paper also contains the dual notion of overtotal sequence and generalisations of both notions, via the concepts of almost overcomplete and almost overtotal sequences. Perhaps the main result in [FZ'14] is the fact that every bounded almost overcomplete sequence is relatively compact. In the same article, this property has been used to provide a short and unified approach to some spaceability results.

Even though the concept has attracted the attention of many mathematicians and it admits a natural counterpart for non-separable Banach spaces, the notion in a general Banach space was only introduced in [RS'21]. A set S in a Banach space $\mathcal{X}$ is called overcomplete when it has cardinality equal to the density of $\mathcal{X}$ and every subset of S having the same cardinality as S is linearly dense in $\mathcal{X}$. The main results in the paper focus on existence and non-existence results: for instance, under the assumption of the Continuum Hypothesis, every Banach space $\mathcal{X}$ such that both $\mathcal{X}$ and $\mathcal{X}^*$ have density ω_1 contains overcomplete sets. On the other hand, as a consequence of Hajnal's theorem, every Banach space of density character at least ω_2 and with a fundamental biorthogonal system fails to admit overcomplete sets. Further, the Banach space $\ell_1(\Gamma)$ admits no overcomplete set, for every uncountable set Γ. One more contribution of the paper consists in detecting a possible analogue to the relative compactness result from [FZ'14] quoted before. For instance, it is proved that under CH a Banach space $\mathcal{X}$ of density ω_1 is WCG if and only if it contains a relatively weakly compact overcomplete set and it is reflexive if and only if every overcomplete set is relatively weakly compact. To conclude, let us mention that the notion of overcomplete sets already attracted the interest of the community, as witnessed by the recent papers [GK'22, Ko'21].

The second paper [LRS'23], published in the *Bulletin of the London Mathematical Society* in 2023, is dedicated to lineability problems in ℓ_∞. Lineability problems have been considered since Gurariy's result [Gu'66] that the set of nowhere differentiable functions in $\mathcal{C}([0,1])$ contains, except for 0, an infinite-dimensional vector space. This research area is now very active and too wide to be summarised here, therefore we limit ourselves to indicating [ABPS'16, BPS'14] as possible references. A particular case that has been considered in the literature is the lineability of sets of the the form $\mathcal{X} \setminus \mathcal{Y}$, where $\mathcal{Y}$ is a closed subspace of $\mathcal{X}$. In this setting there are simple and complete results, see [BO'14, KT'11, Wi'75]. In particular, $\mathcal{X} \setminus \mathcal{Y}$ is spaceable if and only if $\mathcal{X} \setminus \mathcal{Y}$ is lineable, if and only if $\mathcal{Y}$ has infinite codimension [Wi'75]. Moreover, for separable $\mathcal{X}$, these conditions are equivalent to $\mathcal{X} \setminus \mathcal{Y}$ being densely lineable in $\mathcal{X}$ [BO'14]. For non-separable spaces, Papathanasiou [Pa'22] recently proved that $\ell_\infty \setminus c_0$ is densely lineable in ℓ_∞. Our first observation in the paper [LRS'23] is that essentially the same proof of [BO'14] actually gives the complete characterisation that $\mathcal{X} \setminus \mathcal{Y}$ is densely lineable in $\mathcal{X}$ if and only if the linear dimension of the quotient $\mathcal{X}/\mathcal{Y}$ is at least as large as the density of $\mathcal{X}$. In this sense, the paper can be considered as a contribution to the non-separable theory.

However, the main results in the paper are of a different nature and focus on ℓ_∞. As it turns out, the proof in [Pa'22] yields that the set of sequences in ℓ_∞ with exactly continuum many accumulation points is densely lineable in ℓ_∞. This result was the starting point of the research in [LRS'23], as we were pondering lineability results for subsets of ℓ_∞ with a prescribed number of accumulation points (some results in a similar direction could also be found in [BG'13, MP'24]). For a cardinal number κ, denote by $\mathrm{L}(\kappa)$ the set of all $x \in \ell_\infty$ that have exactly κ accumulation points. Similarly as the result from [Pa'22], we show that $\mathrm{L}(\omega)$ and $\bigcup_{2\leqslant n<\omega} \mathrm{L}(n)$ are densely lineable in ℓ_∞. The situation for spaceability is different: in fact, while we show that $\mathrm{L}(\mathfrak{c})$ and $\mathrm{L}(\omega)$ are spaceable, it turns out that $\bigcup_{2\leqslant n<\omega} \mathrm{L}(n)$ is not spaceable. This result could be seen as a motivation for the research, as it nowadays seems harder to disprove lineability properties rather than prove them. As we mentioned, finite-dimensional arguments are involved in the above proofs. This is particularly apparent in one intermediate result needed for the proof of the above mentioned theorem, namely that $\mathrm{L}(n) \cup \cdots \cup \mathrm{L}(n+d)$ is $(d+1)$-lineable, but not $(d+2)$-lineable. One more direction that is investigated in the paper is the corresponding problem in the space $\mathbb{R}^\omega$ of all sequences, with the pointwise topology. While the results concerning dense lineability carry over immediately, the situation for spaceability is different. In fact, while it remains true that $\mathrm{L}(\mathfrak{c})$ is spaceable, $\bigcup_{\kappa\leqslant\omega} \mathrm{L}(\kappa)$ is not spaceable in $\mathbb{R}^\omega$. In particular, also $\mathrm{L}(\omega)$ and $\bigcup_{2\leqslant n<\omega} \mathrm{L}(n)$ fail to be spaceable. More results in this direction have been recently obtained in [MP'24], but several natural and simply stated problems remain open.

The paper [HR'22] was published by the *Mediterranean Journal of Mathematics* in 2022 and is dedicated to one problem relevant to the theory of Lipschitz-free Banach spaces and the study of approximation properties. Lipschitz-free Banach spaces constitute perhaps the most active topic of research in Banach space theory within the last 20 years, in particular since the seminal paper [GK'03] by Godefroy and Kalton. As a small instance of some recent result, let us mention for example [AACD'21, AACD'22a, AGPP'22, AP'20, AP'23, DK'22, Ve'23]. Among the most investigated questions concerning Lipschitz-free spaces are problems concerning the existence of Schauder bases and approximation properties. In this direction, Godefroy and Kalton proved in [GK'03] that the Lipschitz-free Banach space $\mathcal{F}(\mathcal{X})$ over a Banach space $\mathcal{X}$ admits the bounded approximation property if and only if $\mathcal{X}$ itself has the property. On the other hand, much less is known about the dual Banach space $\mathrm{Lip}_0(\mathcal{X})$ of $\mathcal{F}(\mathcal{X})$. For instance, it is known that $\mathrm{Lip}_0(\mathbb{R}) = \ell_\infty$ and that $\mathrm{Lip}_0(\mathcal{X})$ is not isomorphic to ℓ_∞, whenever $\mathcal{X}$ has dimension at least 2. However, there are no known examples of two separable infinite-dimensional Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathrm{Lip}_0(\mathcal{X})$ and $\mathrm{Lip}_0(\mathcal{Y})$ aren't isomorphic. It is not even known if $\mathrm{Lip}_0(\mathcal{X})$ can be isomorphic to $\mathrm{Lip}_0(\mathbb{R}^2)$ for some infinite-dimensional $\mathcal{X}$, or if $\mathrm{Lip}_0(\mathbb{R}^n)$ is isomorphic to $\mathrm{Lip}_0(\mathbb{R}^k)$ for some distinct $k, n \geqslant 2$. A natural approach to these problems would be to understand approximation properties in Banach spaces of Lipschitz functions. For instance, finding an infinite-dimensional separable Banach space $\mathcal{X}$ such that $\mathrm{Lip}_0(\mathcal{X})$ admits the approximation property would solve the first problem (by taking a separable $\mathcal{Y}$ without the approximation property). For instance, an open problem due to Gilles Godefroy asks whether $\mathrm{Lip}_0(\ell_2)$ admits the approximation property. Once more, notice that this is a non-separable problem, because the Banach spaces $\mathrm{Lip}_0(\mathcal{X})$ are non-separable for infinite-dimensional $\mathcal{X}$.

The paper [HR'22] originated from an attempt to solve this problem in the negative, by finding a complemented subspace of $\mathrm{Lip}_0(\ell_2)$ that fails the approximation property. Indeed, the space $\mathcal{P}(^2\mathcal{X})$ of bounded 2-homogeneous polynomials on $\mathcal{X}$ admits a natural embedding in $\mathrm{Lip}_0(B_\mathcal{X})$ and it is known that $\mathcal{P}(^2\ell_2)$ fails the approximation property, [Fl'97, p. 173] (see also [DM'15]). If $\mathcal{P}(^2\ell_2)$ were a complemented subspace of $\mathrm{Lip}_0(B_{\ell_2})$, it would follow that $\mathrm{Lip}_0(B_{\ell_2})$ and $\mathrm{Lip}_0(\ell_2)$ fail the approximation property (note that these two spaces are isomorphic [Ka'14]). Moreover, a classical result of Lindenstrauss [Li'64] asserts that $\mathcal{X}^*$ is always 1-complemented in $\mathrm{Lip}_0(\mathcal{X})$. Therefore, the question if $\mathcal{P}(^2\mathcal{X})$ could be a complemented subspace of $\mathrm{Lip}_0(B_\mathcal{X})$ would also be a polynomial counterpart to Lindenstrauss' result. The main result in the paper is that actually $\mathcal{P}(^2\mathcal{X})$ is not complemented in $\mathrm{Lip}_0(B_\mathcal{X})$, for all Banach spaces $\mathcal{X}$ with non-trivial type, in particular for ℓ_2. Additionally, the result is also valid for the space $\mathcal{P}(^n\mathcal{X})$ of n-homogeneous polynomials and for the space $\mathcal{P}_0^n(\mathcal{X})$ of polynomials of degree at most n (that vanish at 0). On the other hand, when $\mathcal{X}$ is a separable $\mathscr{L}_1$-space, $\mathcal{P}(^k\mathcal{X})$ is isomorphic to ℓ_∞, for every $k \geqslant 1$, hence complemented in $\mathrm{Lip}_0(B_\mathcal{X})$ [AF'96, AS'76].

The last paper in this part, [BLR'25+], has recently been accepted for publication in the *Canadian Journal of Mathematics* and it is also dedicated to some aspects of the theory of Lipschitz-free spaces. Among the most well-studied Lipschitz-free spaces is the Lipschitz-free space $\mathcal{F}(\mathbb{R}^d)$ over $\mathbb{R}^d$. For $d = 1$ a standard computation shows that $\mathcal{F}(\mathbb{R})$ is isometric to $L_1(\mathbb{R})$, while it is a famous result due to Naor and Schechtman [NS'07] that $\mathcal{F}(\mathbb{R}^d)$ does not embed in L_1 for $d \geqslant 2$. Consequently, $\mathcal{F}(\mathbb{R})$ is not isomorphic to $\mathcal{F}(\mathbb{R}^2)$; on the other hand, it is still not known whether for distinct $d, k \geqslant 2$ $\mathcal{F}(\mathbb{R}^d)$ is isomorphic to $\mathcal{F}(\mathbb{R}^k)$. Concerning the isomorphic structure of $\mathcal{F}(\mathbb{R}^d)$, it is known that such a space admits a Schauder basis [HPe'14] and the metric approximation property [PS'15]. Furthermore, if M is a subset of $\mathbb{R}^d$, then $\mathcal{F}(M)$ admits a finite-dimensional decomposition [B-L'12], hence the bounded approximation property [LP'13], and it is weakly sequentially complete [CDW'16]. Some of these results have also been extended to the case of Lipschitz-free p-spaces for $p \in (0, 1)$, [AACD'22b]. In addition, there also is an isometric representation of the space $\mathcal{F}(\mathbb{R}^d)$, as the quotient of $L_1(\mathbb{R}^d, \mathbb{R}^d)$ by the subspace of vector fields with zero distributional divergence [CKK'17, GL'18].

Recently, several examples of metric spaces $\mathcal{M}$ such that $\mathcal{F}(\mathcal{M})$ is isomorphic to $\mathcal{F}(\mathbb{R}^d)$ have been found. The first result in this direction is due to Kaufmann [Ka'14] who proved that $\mathcal{F}(\mathcal{M})$ is isomorphic to $\mathcal{F}(\mathbb{R}^d)$, whenever $\mathcal{M}$ is a subset of $\mathbb{R}^d$ with non-empty interior. Furthermore, $\mathcal{F}(\mathbb{S}^d)$ is also isomorphic to $\mathcal{F}(\mathbb{R}^d)$ [AACD'21] and, more generally, $\mathcal{F}(\mathcal{M})$ is isomorphic to $\mathcal{F}(\mathbb{R}^d)$ for every compact d-dimensional Riemannian manifold $\mathcal{M}$, [FG'23]. In absence of compactness the situation is less defined and in particular the space $\mathcal{F}(\mathbb{H}^d)$ has been subject of little investigation prior to [BLR'25+]. In fact, the only paper where $\mathcal{F}(\mathbb{H}^d)$ has been considered is [DK'22], where it is proved that $\mathcal{F}(\mathcal{M})$ admits a Schauder basis for every net $\mathcal{M}$ in $\mathbb{H}^d$ and it is asked if $\mathcal{F}(\mathbb{H}^d)$ has a Schauder basis and whether it is isomorphic to $\mathcal{F}(\mathbb{R}^d)$.

The paper [BLR'25+] attempted to tackle these questions and obtained as a main result that $\mathcal{F}(\mathbb{H}^d)$ is isomorphic to $\mathcal{F}(\mathbb{R}^d) \oplus \mathcal{F}(\mathcal{M})$, where $\mathcal{M}$ is a net in $\mathbb{H}^d$ and $d = 2, 3, 4$. As a consequence of this isomorphism and results mentioned above, one deduces that $\mathcal{F}(\mathbb{H}^d)$ admits a Schauder basis, for said values of d. While our paper was under review, Gartland [Ga'25+] proved with entirely different methods that $\mathcal{F}(\mathbb{H}^d)$ is isomorphic to $\mathcal{F}(\mathbb{R}^d)$ for every value of d. The approach in [BLR'25+] is based on a regular, orthogonal tiling of $\mathbb{H}^d$ to construct an isomorphism between $\mathcal{F}(\mathbb{H}^d)$ and $\mathcal{F}(\mathbb{R}^d) \oplus \mathcal{F}(\mathcal{M})$ that is almost completely explicit. More precisely, we build an explicit isomorphism between $\mathrm{Lip}_0(\mathbb{H}^d)$ and $\mathcal{Z} \oplus \mathrm{Lip}_0(\mathcal{M})$, where $\mathcal{Z}$ is a direct sum of certain subspaces of $\mathrm{Lip}_0(\mathcal{P})$ and $\mathcal{P}$ is a polytope in $\mathbb{R}^d$. This gives us an explicit (w^*-to-w^* continuous) procedure to reduce the study of $\mathrm{Lip}_0(\mathbb{H}^d)$ to the discrete case of $\mathrm{Lip}_0(\mathcal{M})$ and to a space of Lipschitz functions on $\mathbb{R}^d$. It should be pointed out that such regular, orthogonal tilings of $\mathbb{H}^d$ only exist in dimensions $d = 2, 3, 4$, by classical results from the theory of reflection groups, [Da'08, PV'05, VS'88, Vi'84]. The same methods also permit us to prove that the space $\mathrm{Lip}(\mathbb{H}^d)$ of bounded Lipschitz functions on $\mathbb{H}^d$ is isomorphic to $\mathrm{Lip}(\mathbb{R}^d)$, this time by giving an entirely explicit isomorphism. Combining this with standard arguments, it can be concluded that the spaces of bounded Lipschitz functions on the model spaces of metric geometry, namely $\mathrm{Lip}(\mathbb{H}^d)$, $\mathrm{Lip}(\mathbb{R}^d)$, and $\mathrm{Lip}(\mathbb{S}^d)$, are all isomorphic.

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