Stabilizing compensators for linear time-varying differential systems

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Abstract

In this paper we describe a constructive test to decide whether a given linear time-varying (LTV) differential system admits a stabilizing compensator for the control tasks of tracking, disturbance rejection or model matching and construct and parametrize all of them if at least one exists. In analogy to the linear time-invariant (LTI) case the ring of stable rational functions, noncommutative in the LTV situation, and the Kučera-Youla parametrization play prominent parts in the theory. We transfer Blumthaler's thesis from the LTI to the LTV case and sharpen, complete and simplify the corresponding results in the book 'Linear Time-Varying Systems' by Bourlès and Marinescu.

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1 Introduction

(i) *Results*: In this paper we describe a constructive test to decide whether a given linear time-varying (LTV) differential system admits a *stabilizing compensator for the control tasks of tracking, disturbance rejection or model matching* and construct and parametrize all stabilizing compensators if at least one exists.
(ii) *Algebra and analysis*: It turns out that the famous *Kučera-Youla parametrization and its application to the solution of various control tasks* can be generalized from standard linear time-invariant (LTI) systems to LTV systems. The parametrization is an *algebraic* result and requires the proper choice of the used algebraic data, in particular of the ring $\mathbf{K}$ of time-varying coefficient functions and of the associated ring $\mathbf{A}$ of differential operators with its associated module theory. On the other hand, the components of the system trajectories are either smooth functions of the time-variable t or distributions whose stability refers to the behavior of these trajectories for $t \to \infty$ and is defined by *analytic* conditions. In the LTI case all essential stability results can be reduced to the purely algebraic fact that certain complex polynomials have only zeros with negative real part. In the LTV case the reduction of stability and stabilization

problems to algebraic and algorithmic ones and the choice of the considered systems, algebraic and analytic data are more difficult than in the LTI case.
(iii) *Background*: In this paper we describe systems as suitably defined *behaviors* and use the *weak exponential stability* (w.e.s.), shortly just *stability*, of autonomous behaviors. These notions were introduced and discussed in the recent paper [7] whose main results are recalled in the next lines and, with more details, in Section 2. The trajectories of a stable autonomous behavior converge to 0 for $t \to \infty$ with decay factors $\exp(-\alpha t^{\mu})$ where $\alpha, \mu > 0$. We use the *differential field* $\mathbf{K}$ of locally convergent *Puiseux series* and its associated *skew-polynomial algebra* $\mathbf{A} := \mathbf{K}[\partial]$ of *differential operators with varying coefficients* in $\mathbf{K}$. This ring is *noncommutative* in contrast to the standard commutative polynomial algebra $\mathbb{C}[\partial]$ of differential operators with constant coefficients in $\mathbb{C}$. Nevertheless the algebra $\mathbf{A}$ keeps most of the essential properties of $\mathbb{C}[\partial]$: It is a *left and right euclidean domain*, in particular a left and right principal ideal domain, and is even *simple* [11, Ch. 1]. It admits a variant of the Smith form of matrices (Jacobson-Nakayama-Teichmüller form) and its standard consequences for finitely generated (f.g.) modules [11, Thm. 1.4.7]. As in the LTI case there is a *categorical duality* between f.g. left modules M with a given representation $M = \mathbf{A}^{1\times q}/U$ and their associated behavior $\mathcal{B} := \mathcal{B}(U)$ [7, Thm. 2.2]. The modules U resp. M are called the *equation* resp. *system* module of $\mathcal{B}$. The behavior is autonomous if and only if M is a torsion module and then cyclic of the form $\mathbf{A}/\mathbf{A}F$ with a nonzero differential operator F [11, Lemma 1.4.11]. In contrast to *uniform exponential stability* of LTV state space systems [21, Def. 6.5] w.e.s. is preserved by behavior isomorphisms [7, Lemma 4.11]. The torsion module M is called w.e.s. or just *stable* if for one or, equivalently, for all representations $M = \mathbf{A}^{1\times q}/U$ the behavior $\mathcal{B}(U)$ is stable. The f.g. stable modules form a *Serre category*, i.e., are closed under isomorphisms, submodules, factor modules and extensions [7, Thm. 2.6]. The main Thm. 2.7 of [7] describes an algorithm that permits to test the stability of most f.g. torsion $\mathbf{A}$-modules. Stable modules and stable behaviors with their asymptotically stable trajectories are considered *negligible* in the following considerations.
(iv) *Localization technique*: A nonzero differential operator $s \in \mathbf{A}$ is called *stable* if $\mathbf{A}/\mathbf{A}s$ is stable. The subset $S \subset \mathbf{A}$ of stable differential operators is *multiplicatively closed and saturated* [7, Cor. 4.6], is an *Ore set* and gives rise to the quotient subring $\mathbf{A}_S$ of the quotient field $\mathbf{Q}$ of $\mathbf{A}$ (cf. Section 3.1). The ring $\mathbf{A}_S$ assumes the role of the ring of *stable rational transfer functions* in the language of Vidyasagar [22, Chs. 1,5]. It also induces the *exact quotient module functor* $M \mapsto M_S = \mathbf{A}_S \otimes_{\mathbf{A}} M$ from the abelian category ${}_{\mathbf{A}}\mathbf{Mod}$ of $\mathbf{A}$-left modules to the category ${}_{\mathbf{A}_S}\mathbf{Mod}$ of $\mathbf{A}_S$-left modules. A f.g. module M is stable if and only if $M_S = 0$. Application of this exact functor thus annihilates the negligible or stable modules and simplifies all algebraic stability considerations. In Blumthaler's thesis [4] this *localization technique* was applied to the construction and parametrization of all stabilizing compensators for various control tasks in the LTI case. In this paper we show that the method of this thesis can be transferred to the LTV situation and furnishes the *LTV analogue of the Kučera-Youla parametrization (cf. Thm. 3.14) and the main Thms. 4.4, 4.7 resp. 4.10 of this paper concerning tracking, disturbance rejection resp. model matching.*
(v) *Literature*: The construction of compensators in the LTV case is also treated in [5, Chs. 10,11, pp. 523-562], but under different assumptions and with different methods. The present paper sharpens, completes and simplifies the corresponding results of [5]. In Remark 3.20 we discuss differences and similarities in connection with the Kučera-Youla parametrization [5, Thm. 1143, p. 554]. The first *module theoretic* derivation of the Kučera-Youla parametrization is due to Quadrat [16], [17, Ch. 8, pp.273-308],

[18], but only for commutative rings of functional operators and without the specification of the associated behaviors. We refer to the comprehensive bibliographies of [17] and [5] and also to that of [4] for references to the vast and important literature on the construction of compensators in the LTI case. The paper [1] sums up previous work [19], [14] on the *stabilization of LTV state space systems by static state feedback* and extends and improves it. In [2, §5] the authors prove exponential stabilizability by static state feedback under the assumption that an associated *zero dynamics behavior* is exponentially stable and under additional conditions. In Ex. 3.23 we compare the results of this paper with results from [1] and [2].
(vi) *Interconnection diagram*: We consider the standard interconnection $\mathcal{B}$ of two behaviors $\mathcal{B}_1 := \mathcal{B}(U_1)$ and $\mathcal{B}_2 := \mathcal{B}(U_2)$ according to figure 1 with the corresponding equation resp. system modules $U_i \subseteq \mathbf{A}^{1\times(p+m)}$ resp. $M_i = \mathbf{A}^{1\times(p+m)}/U_i$ for $i = 1, 2$. The behaviors $\mathcal{B}_1$ resp. $\mathcal{B}_2$ are interpreted as the *plant* resp. the *compensator*

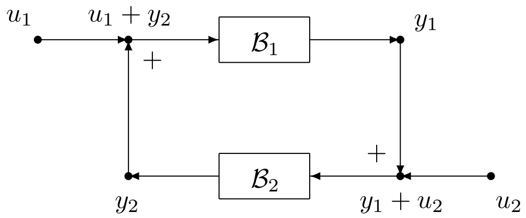

Figure 1: The interconnected behavior $\mathcal{B}$.

or *controller* whose mutual relations are obviously symmetric. We do not assume that the $\mathcal{B}_i$ are input/output (IO) behaviors with input u_i and output y_i. But the dimension p of y_1 has to coincide with that of u_2 and likewise the dimension m of y_2 with that of u_1 in order that the depicted interconnected system $\mathcal{B}$ can be realized. This latter is assumed *well-posed*. This signifies that it is an IO behavior with input $\binom{u_2}{u_1}$ and output $\binom{y_1}{y_2}$. So u_i can be chosen freely as input of $\mathcal{B}$, but not as input of $\mathcal{B}_i$. The plant $\mathcal{B}_1$ is not assumed controllable or, equivalently, the module M_1 is not assumed free. But as in the LTI case the freeness of $M_{1,S}$ as $\mathbf{A}_S$-module is necessary and sufficient for the *stabilizability* of $\mathcal{B}_1$, i.e., the *existence of a stabilizing compensator* $\mathcal{B}_2$ for which the autonomous part of the interconnected IO behavior $\mathcal{B}$ is stable, cf. Cor. 3.11.
(vii) *Base field*: In this paper we use the base field $\mathbb{C}$ of complex numbers, but the results also hold for the real field $\mathbb{R}$. The proofs for this are either special cases of the complex case or analogous.
(viii) *Generalizations*: We are convinced that the construction and parametrization of compensators *can be extended to more general interconnection diagrams* than that in figure 1, cf. [22, Ch. 5], [5, Ch. 11], [3] and the references there. This is, however, not done in the present paper since our principal goal is to show that the LTI localization technique from [4] can be extended to the LTV case with analogous complete results.
(ix) *Constructivity*: The results of [7] and the present paper are constructive. Due to lack of computer skills we cannot implement these algorithms and hope that the younger generation assumes this essential task since calculations by hand in this area are very time-consuming and do not lead very far. Since many different algorithms

from the literature, for instance from [15], [10] and [20], have to be combined this implementation will not be easy.
(x) *Plan*: The main theorems of this paper are proven in Section 4. Thm. 3.14 establishes the Kučera-Youla parametrization for the LTV case and is the main result of Section 3. Section 2 recalls the basic notions and results from the paper [7]. We illustrate the construction and parametrization of all tracking stabilizing compensators in Examples 3.19 and 4.5.
Notations and abbreviations: f.d.=finite-dimensional, f.g.=finitely generated,
IO=input/output, LTI=linear time-invariant, LTV=linear time-varying, p.g.f.= polynomial growth function, resp.=respectively, w.e.s.= weak exponential stability, weakly exponentially stable, w.l.o.g.=without loss of generality, $X^{p\times q}$=set of $p\times q$-matrices with entries in X, $X^{1\times q}$=rows, $X^q:=X^{q\times 1}$=columns , $X^{\bullet\times\bullet}:=\bigcup_{p,q\geq 0}X^{p\times q}$.

2 Differential operators and behaviors

2.1 Differential operators

The differential operators and behaviors of this paper and their properties were introduced and discussed in [7]. We recapitulate them here and refer to [7, §§3,4] for details and references to the literature.
The signals of this paper are defined on open real intervals $(\tau,\infty),\ \tau\geq 0$. Since we study the trajectories for $t\to\infty$ the restriction to $\tau\geq 0$ is no loss of generality. The used signal spaces are the spaces of complex-valued smooth functions or distributions on $(\tau,\infty),\ \tau\geq 0$, i.e.,

$$W(\tau)=\mathrm{C}^\infty(\tau,\infty)\text{ or }W(\tau)=\mathcal{D}'(\tau,\infty). \tag{1}$$

The *valued differential* field of *formal Laurent series* with its *valuation* $v:\mathbb{C}((z))\to\mathbb{Q}\uplus\{\infty\}$ and *derivation* $d/dz:\ \mathbb{C}((z))\to\mathbb{C}((z))$ is given by

$$\begin{aligned}&\mathbb{C}((z)):=\left\{a=\sum_{i=k}^\infty a_iz^i;\ k\in\mathbb{Z},\ a_i\in\mathbb{C}\right\}\text{ with}\\ &v(a):=\begin{cases}k & \text{if } a_k\neq 0\\ \infty & \text{if } a=0\end{cases},\ da/dz:=a'(z):=\sum_{i=k}^\infty a_iiz^{i-1}.\end{aligned} \tag{2}$$

The field $\mathbb{C}((z))$ has the valued differential subfield of *locally convergent* Laurent series given by

$$\mathbb{C}<<z>>:=\left\{a=\sum_{i=k}^\infty a_iz^i\in\mathbb{C}((z));\ \sigma(a):=\limsup_{i\geq 0}\sqrt[i]{|a_i|}<\infty\right\} \tag{3}$$

The inverse $\rho(a):=\sigma(a)^{-1}$ is the convergence radius of a, i.e., $a(z)=\sum_{i=k}^\infty a_iz^i$ converges for $0<|z|<\rho(a)$ and is a holomorphic function in this pointed open disc. Therefore the function $a(t^{-1})$ is contained in $\mathrm{C}^\infty(\sigma(a),\infty)$.
The valued differential field $\mathbf{K}$ is defined as the field of *Puiseux series*

$$\mathbf{K}:=\bigcup_{m\geq 1}\mathbb{C}<<z^{1/m}>>=$$

$$\left\{a(z^{1/m})=\sum_{i=k}^\infty a_iz^{i/m};\ m\in\mathbb{N},\ m\geq 1,\ k\in\mathbb{Z},\ a=\sum_{i=k}^\infty a_iz^i\in\mathbb{C}<<z>>\right\}. \tag{4}$$

This field $\mathbf{K}$ is the *algebraic closure* of $\mathbb{C} << z >>$ and is constructed *algebraically* such that $\left(z^{1/n}\right)^{n/m} = z^{1/m}$ if m divides n. The element $z^{1/m}$ is *not* defined as the function $\exp(m^{-1}\ln(z))$ since $\ln(z)$ is not a Laurent series at 0. The valuation v is extended to $\mathbf{K}$ by $v\left(a(z^{1/m})\right) = v(a)/m \in \mathbf{Q} \uplus \{\infty\}$ and the derivation by $da(z^{1/m})/dz = m^{-1}z^{(1/m)-1}a'(z^{1/m})$. We also define $\sigma(a(z^{1/m})) := \sigma(a)^m$. The field $\mathbf{K}$ has the differential subalgebras

$$\begin{gathered}\mathbf{K}(\tau) := \left\{f = a(z^{1/m}) \in \mathbf{K};\ \tau \geq \sigma(f) = \sigma(a)^m\right\} \text{ with} \\ \mathbf{K}(\tau_0) \subseteq \mathbf{K}(\tau) \text{ for } \tau_0 \leq \tau \text{ and } \mathbf{K} = \bigcup_{\tau \geq 0} \mathbf{K}(\tau).\end{gathered} \tag{5}$$

If

$$\begin{gathered}f = a(z^{1/m}) \in \mathbf{K}(\tau) \text{ and } t > \tau \text{ then } t > \tau \geq \sigma(f) = \sigma(a)^m \Longrightarrow \\ t^{-1/m} < \rho(a) = \sigma(a)^{-1} \Longrightarrow f(t) := a(t^{-1/m}) \in \mathrm{C}^\infty(\tau, \infty),\end{gathered} \tag{6}$$

especially $f(t) \in \mathrm{C}^\infty(\sigma(f), \infty)$. The map

$$\mathbf{K}(\tau) \to \mathrm{C}^\infty(\tau, \infty),\ f = a(z^{1/m}) \mapsto f(t) := a(t^{-1/m}), \tag{7}$$

is an injective algebra homomorphism. The differential field $\mathbf{K}$ gives rise to the *skew-polynomial ring of differential operators*

$$\mathbf{A} = \mathbf{K}[\partial; d/dz] = \oplus_{j=0}^\infty \mathbf{K}\partial^j. \tag{8}$$

The differential operators are polynomials in ∂ whose noncommutative multiplication is determined by

$$\partial a(z^{1/m}) = a(z^{1/m})\partial + m^{-1}z^{(1/m)-1}a'(z^{1/m}). \tag{9}$$

The ring $\mathbf{A}$ is a left and right principal ideal domain, indeed admits euclidean division, and is simple. For any nonzero $h \in \mathbf{K}$ the indeterminate ∂ can be replaced by $h\partial$ and the derivation by hd/dz and one obtains

$$\mathbf{A} = \mathbf{K}[h\partial; hd/dz],\ \text{especially } \mathbf{A} = \mathbf{K}[z\partial; zd/dz] = \mathbf{K}[-z^2\partial; -z^2 d/dz]. \tag{10}$$

The domain $\mathbf{A}$ has the subdomains

$$\begin{gathered}\mathbf{A}(\tau) := \mathbf{K}(\tau)[\partial; d/dz] = \mathbf{K}(\tau)[z\partial; zd/dz] = \mathbf{K}(\tau)[-z^2\partial; -z^2 d/dz] \text{ with} \\ \mathbf{A}(\tau_0) \subseteq \mathbf{A}(\tau) \text{ for } 0 \leq \tau_0 \leq \tau \text{ and } \mathbf{A} = \bigcup_{\tau \geq 0} \mathbf{A}(\tau).\end{gathered} \tag{11}$$

We extend the function σ to differential operators and matrices via

$$\begin{gathered}\sigma(f) := \max\{\sigma(f_j);\ j \in \mathbb{N}) \text{ for } f = \sum_{j \in \mathbb{N}} f_j \partial^j \in \mathbf{A} \text{ where} \\ f_j = a_j(z^{1/m}),\ \sigma(f_j) = \sigma(a_j)^m \\ \sigma(R) := \max\{\sigma(R_{\mu\nu});\ \mu \leq p,\ \nu \leq q\} \text{ for } R = (R_{\mu\nu})_{\mu,\nu} \in \mathbf{A}^{p \times q}. \text{ Then} \\ f \in \mathbf{A}(\tau) \iff \tau \geq \sigma(f) \text{ and } R \in \mathbf{A}(\tau)^{p \times q} \iff \tau \geq \sigma(R).\end{gathered} \tag{12}$$

The signal spaces $W(\tau_0),\ \tau_0 \geq 0$, from (1) are not $\mathbf{A}$-modules, but $\mathbf{A}(\tau_0)$-left modules via the action $f \circ w$, defined by

$$\left(\sum_{j\in\mathbb{N}}^{\infty} f_j(-z^2\partial)^j\right) \circ w := \sum_{j\in\mathbb{N}} f_j(t)w^{(j)},$$
$$\text{where} f_j = a_j(z^{1/m}) \in \mathbf{K}(\tau_0),\ f_j(t) = a_j\left(t^{-1/m}\right) \tag{13}$$
$$a(z^{1/m}) \circ w = a(t^{-1/m})w,\ (-z^2\partial) \circ w = w' := dw/dt,\ \partial \circ w = -t^2 w'.$$

Notice that there is no module action (=scalar multiplication) $\circ'$ with $a(z^{1/m}) \circ' w = a(t^{-1/m})w$ and $\partial \circ' w = w'$. As usual the action (13) is extended to the action $R \circ w$ of a matrix $R \in \mathbf{A}(\tau_0)^{p\times q}$ on a column vector $w \in W(\tau_0)^q$ and defined by

$$R \circ w = \sum_{j\in\mathbb{N}} R_j(t)w^{(j)} \text{ for } R = \sum_{j\in\mathbb{N}} R_j(-z^2\partial)^j \in \mathbf{A}(\tau_0)^{p\times q}$$
$$\text{where } R_j = A_j(z^{1/m}) \in \mathbf{K}(\tau_0)^{p\times q},\ i.e., \tau_0 \geq \sigma(R_j),\ A_j \in \mathbb{C} << z >>^{p\times q},$$
$$R_j(t) = A_j(t^{-1/m}) \in \mathrm{C}^\infty(\tau,\infty)^{p\times q}. \tag{14}$$

Any matrix $R = \sum_{j\in\mathbb{N}} R_j(-z^2\partial)^j \in \mathbf{A}^{p\times q}$ gives rise to the *solution spaces* or *behaviors* $\mathcal{B}(R,\tau),\ \tau \geq \sigma(R)$, defined by

$$\mathcal{B}(R,\tau) := \{w \in W(\tau)^q;\ R \circ w = 0\} = \left\{ w \in W(\tau)^q;\ \sum_{j\in\mathbb{N}} R_j(t)w^{(j)} = 0 \right\}. \tag{15}$$

The dependence of the admissible τ on the matrix R requires a new definition of behaviors that is explained in Section 2.2.

2.2 Modules and behaviors

We assume the data of Section 2.1 and consider matrices $R \in \mathbf{A}^{p\times q}$ and associated behavior families from (15): $(\mathcal{B}(R,\tau))_{\tau\geq\tau_0},\ \tau_0 \geq \sigma(R)$. Two such families $(\mathcal{B}(R_i,\tau))_{\tau\geq\tau_i},\ i = 1, 2$, are called *equivalent* if

$$\exists \tau_3 \geq \max(\tau_1,\tau_2)\forall\tau \geq \tau_3 :\ \mathcal{B}(R_1,\tau) = \mathcal{B}(R_2,\tau). \tag{16}$$

The *equivalence class* is denoted by $\mathrm{cl}\left((\mathcal{B}(R,\tau))_{\tau\geq\tau_0}\right)$, hence especially

$$\mathrm{cl}\left((\mathcal{B}(R,\tau))_{\tau\geq\tau_0}\right) = \mathrm{cl}\left((\mathcal{B}(R,\tau))_{\tau\geq\sigma(R)}\right). \tag{17}$$

For $R = 0 \in \mathbf{A}^{1\times q}$ one especially obtains the classes

$$\mathcal{W} := \mathrm{cl}\left((W(\tau))_{\tau\geq 0}\right) \text{ and } \mathcal{W}^q = \mathrm{cl}\left((W(\tau)^q)_{\tau\geq 0}\right). \tag{18}$$

These replace the standard signal modules in the following behavior theory.
Let ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ denote the category of f.g. $\mathbf{A}$-left modules M *with a given system of generators* or, equivalently, a given representation $M = \mathbf{A}^{1\times q}/U,\ U \subseteq \mathbf{A}^{1\times q}$. The morphisms of the category are the $\mathbf{A}$-linear maps without any additional structure. The category is abelian.
If $M = \mathbf{A}^{1\times q}/U \in {}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ the submodule U is f.g. and therefore the row module

$U = \mathbf{A}^{1\times p}R$ of some nonunique $R \in \mathbf{A}^{\bullet\times q}$ that gives rise to $\mathrm{cl}\left((\mathcal{B}(R,\tau))_{\tau\geq\sigma(R)}\right)$. This class does not depend on the choice of R and is called the *behavior* $\mathcal{B}(U) \subseteq \mathcal{W}^q$ *of* U *or* M [7, Lemma 3.7]. There is a naturally defined abelian category $\mathcal{B}eh$ of behaviors [7, Cor. and Def. 3.9] whose objects are the $\mathcal{B}(U),\ U \subseteq \mathbf{A}^{1\times q},\ q \geq 0$, and that enables a module-behavior duality. This signifies that there is a *contravariant equivalence* or *duality* [7, Thm. 2.2, Cor. 3.14]

$$
{}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}} \cong \mathcal{B}eh
$$

$$
\begin{cases} M = \mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U) \\ \mathrm{Hom}_{\mathbf{A}}(\mathbf{A}^{1\times q_1}/U_1, \mathbf{A}^{1\times q_2}/U_2) \cong \mathrm{Hom}(\mathcal{B}(U_2), \mathcal{B}(U_1)),\ \varphi \mapsto \mathcal{B}(\varphi). \end{cases} \tag{19}
$$

Assume $U_i = \mathbf{A}^{1\times p_i}R_i,\ R_i \in \mathbf{A}^{p_i\times q_i}$. Every φ can be described as

$$
\begin{gathered} \varphi = (\circ P)_{\mathrm{ind}}:\ \mathbf{A}^{1\times q_1}/U_1 \to \mathbf{A}^{1\times q_2}/U_2,\ \overline{\xi} := \xi + U_1 \mapsto \overline{\xi P} := \xi P + U_2, \\ \text{where } P \in \mathbf{A}^{q_1\times q_2},\ U_1P \subseteq U_2,\ \xi \in \mathbf{A}^{1\times q_1},\ \xi P \in \mathbf{A}^{1\times q_2}. \end{gathered} \tag{20}
$$

In particular, there is an $X \in \mathbf{A}^{q_1\times p_2}$ with $R_1P = XR_2$. The morphism $\mathcal{B}(\varphi)$ is defined as the equivalence class of $\mathbb{C}$-linear maps

$$
\begin{gathered} \mathcal{B}(\varphi) = \mathrm{cl}\left((P\circ : \mathcal{B}(R_2,\tau) \to \mathcal{B}(R_1,\tau),\ w_2 \mapsto P \circ w_2)_{\tau\geq\tau_0}\right), \\ \text{where } \tau_0 \geq \max\left(\sigma(R_1), \sigma(R_2), \sigma(P), \sigma(X)\right). \end{gathered} \tag{21}
$$

The equivalence class is defined analogously to that in (17) [7, §3.3]. So objects and morphisms in $\mathcal{B}eh$ are given by the behaviors

$$
\mathcal{B}(R_i,\tau),\ i = 1,2,\ \text{ and maps } P\circ : \mathcal{B}(R_2,\tau) \to \mathcal{B}(R_1,\tau),\ \tau \geq \tau_0, \tag{22}
$$

for sufficiently large τ_0. The module $M = \mathbf{A}^{1\times q}/U$ with $U = \mathbf{A}^{1\times p}R,\ R \in \mathbf{A}^{p\times q}$, is a *torsion module* or, equivalently, $\mathrm{rank}(R) = q$ if and only if $\mathcal{B}(U)$ is *autonomous*. Autonomy signifies that there are $\tau_1 \geq \sigma(R)$ and $d \in \mathbb{N}$ such that for all $t_0 > \tau \geq \tau_1$ the *initial map*

$$
\mathcal{B}(R,\tau) \to \mathbb{C}^{dq},\ w \mapsto (w(t_0), \cdots, w^{(d-1)}(t_0))^\top,\ \text{is injective.} \tag{23}
$$

A function $\varphi : [\tau_0,\infty) \to \mathbb{C},\ \tau \geq 0$, is called *p.g.f.* if it grows at most polynomially for $t \to \infty$. All functions $f(t)$ for $f \in \mathbf{K}$ are p.g.f. on each closed interval $[\tau_0,\infty),\ \tau_0 > \sigma(f)$. The autonomous behavior $\mathcal{B}(U)$, is called *weakly exponentially stable* (w.e.s.) [7, Def. 2.3] if there are $\tau_1 \geq \sigma(R)$, $d \in \mathbb{N}$, $\alpha,\mu > 0$ and, for each $m \in \mathbb{N}$, a p.g.f. $\varphi_m(t) > 0$ on $[\tau_1,\infty)$ such that all trajectories $w \in \mathcal{B}(R,\tau),\ \tau \geq \tau_1$, satisfy the inequalities

$$
\begin{gathered} \|w^{(m)}(t)\| \leq \varphi_m(t_0)\exp\left(-\alpha(t^\mu - t_0^\mu)\right)\|x(t_0)\| \text{ for } t \geq t_0 > \tau \text{ where} \\ x(t_0) := \max(\|w(t_0)\|, \|w'(t_0)\|, \cdots, \|w^{d-1}(t_0)\|). \end{gathered} \tag{24}
$$

3 Kučera-Youla parametrization for LTV systems

3.1 Weakly exponentially stable localization

Since $\mathbf{A}$ is a (left and right) noetherian domain the set $\mathbf{A}\setminus\{0\}$ is an Ore set and gives rise to the quotient field of $\mathbf{A}$ [13, Thm. 2.1.15]:

$$
\mathbf{Q} := \mathrm{quot}(\mathbf{A}) := (\mathbf{A}\setminus\{0\})^{-1}\mathbf{A} = \left\{s^{-1}a = bt^{-1};\ a,b,s,t \in \mathbf{A},\ s \neq 0,\ t \neq 0\right\}. \tag{25}
$$

Lemma 3.1. *Each nonzero ideal is* essential *or* large *in* $\mathbf{A}$ *or, in other words, if the elements* $a_i,\ i = 1, 2$, *are nonzero then also* $\mathbf{A}a_1 \bigcap \mathbf{A}a_2 \neq 0$.

Proof. There are b_1, b_2 such that $a_1 a_2^{-1} = b_1^{-1} b_2$, hence $0 \neq b_1 a_1 = b_2 a_2 \in \mathbf{A}a_1 \bigcap \mathbf{A}a_2$. □

Since the f.g. w.e.s. modules form a Serre category the same holds true for the full subcategory $\mathfrak{C}$ of the category ${}_{\mathbf{A}}\mathbf{Mod}$ of all left $\mathbf{A}$-modules whose f.g. submodules are w.e.s.. Moreover, if $C_i,\ i \in I$, are, possibly infinitely many, submodules in $\mathfrak{C}$ of a left $\mathbf{A}$-module M then also the sum $C := \sum_{i \in I} C_i$ belongs to $\mathfrak{C}$. In particular, each ${}_{\mathbf{A}}M$ has a largest submodule $\mathrm{Ra}_{\mathfrak{C}}(M)$ in $\mathfrak{C}$ that is called the $\mathfrak{C}$*-radical of* M. We also consider the set

$$S := \{s \in \mathbf{A};\ \mathbf{A}/\mathbf{A}s \text{ w.e.s. or } \mathbf{A}/\mathbf{A}s \in \mathfrak{C}\}\,. \tag{26}$$

of all *w.e.s. differential operators*.
In the sequel we call the modules in $\mathfrak{C}$ *and the elements in* S *just* stable *instead of w.e.s..*
No other stability notion will be used.

Corollary 3.2. *([7, Cor. 4.16]) If* $0 \neq s = s_1 s_2 \in \mathbf{A}$ *then* $s \in S$ *if and only if* $s_1, s_2 \in S$. *In other words,* S *is multiplicatively closed and saturated.*

Lemma 3.3. *The set* S *is an Ore set, i.e., for* $b \in \mathbf{A}$ *and* $t \in S$ *there are* $a \in \mathbf{A}$ *and* $s \in T$ *with* $sb = at$ *or* $bt^{-1} = s^{-1}a \in \mathbf{Q}$.

Proof. Consider the linear map $\alpha : \mathbf{A} \to \mathbf{A},\ x \mapsto xb$ and $\mathfrak{a} := \alpha^{-1}(At) = \mathbf{A}s$. The map induces the injection $\alpha_{\mathrm{ind}} : \mathbf{A}/\mathfrak{a} \to \mathbf{A}/\mathbf{A}t \in \mathfrak{C}$, hence $\mathbf{A}/\mathfrak{a} \in \mathfrak{C}$ and $s \in S$. By definition $\alpha(s) = sb \in \mathbf{A}t$ and therefore there is an $a \in \mathbf{A}$ with $sb = at$. □

The Ore set S gives rise to the quotient ring $S^{-1}\mathbf{A} = \mathbf{A}_S \subset \mathbf{Q}$ and to the $\mathbf{A}_S$-quotient module $S^{-1}M = M_S$ of an $\mathbf{A}$-left module M [13, §2.1]. They have the form

$$\begin{gathered}\mathbf{A} \subset S^{-1}\mathbf{A} := \mathbf{A}_S := \left\{s^{-1}a = bt^{-1}; a, b \in \mathbf{A},\ s, t \in S,\ at = sb\right\} \subset \mathbf{Q} := \mathrm{quot}(\mathbf{A}) \\ S^{-1}M := M_S := \left\{s^{-1}x = \frac{x}{s};\ x \in M,\ s \in S\right\}.\end{gathered} \tag{27}$$

There is also the canonical $\mathbf{A}$-linear map [13, Prop. 2.1.17]

$$\begin{gathered}\mathrm{can}_M : M \to M_S,\ x \mapsto \frac{x}{1}, \text{ with kernel} \\ \mathrm{tor}_S(M) := \ker(\mathrm{can}_M) = \{x \in M;\ \exists s \in S \text{ with } sx = 0\} \subseteq \mathrm{tor}(M).\end{gathered} \tag{28}$$

Here $\mathrm{tor}(M)$ resp. $\mathrm{tor}_S(M)$ are the *torsion* resp. S*-torsion submodules* of M. As usual the assignment $M \mapsto M_S$ is extended to the *exact quotient functor*

$$\begin{gathered}{}_{\mathbf{A}}\mathbf{Mod} \to {}_{\mathbf{A}_S}\mathbf{Mod}, \begin{cases} M \mapsto M_S \\ (\varphi : M_1 \to M_2) \mapsto \left(S^{-1}\varphi = \varphi_S :\ M_{1,S} \to M_{2,S}\right) \end{cases} \\ \text{where } \varphi_S\left(\frac{x_1}{s}\right) := \frac{\varphi(x_1)}{s}.\end{gathered} \tag{29}$$

For an $\mathbf{A}$-(left) module $M,\ x \in M$ and annihilator left ideal $\mathbf{A}s = \mathrm{ann}_{\mathbf{A}}(x) := \{a \in \mathbf{A};\ ax = 0\}$ the map $\mathbf{A}/\mathbf{A}s \cong \mathbf{A}x,\ a + \mathbf{A}s \mapsto ax$, is an isomorphism. It implies that

$$s \in S \iff \mathbf{A}/\mathbf{A}s \in \mathfrak{C} \iff \mathbf{A}x \in \mathfrak{C} \iff x \in \mathrm{Ra}_{\mathfrak{C}}(M). \tag{30}$$

Corollary 3.4. *The torsion submodule* $\mathrm{tor}_S(M)$ *of* M *is the largest submodule of* M *in* $\mathfrak{C}$*, i.e.,* $\mathrm{tor}_S(M) = \mathrm{Ra}_{\mathfrak{C}}(M)$*, especially*

$$M \in \mathfrak{C} \iff \mathrm{tor}_S(M) = M \iff M_S = 0. \tag{31}$$

Notice that $\mathbf{A}$ is simple and hence the two-sided annihilator ideal $\mathrm{ann}_{\mathbf{A}}(M) = \{a \in \mathbf{A};\ aM = 0\}$ is zero if M is nonzero.

Lemma 3.5. *Like* $\mathbf{A}$ *itself the quotient ring* $\mathbf{A}_S \subset \mathbf{Q} = \mathrm{quot}(\mathbf{A})$ *is a left and right principal ideal domain and simple.*

Proof. 1. Any left ideal $\mathfrak{b} \subseteq \mathbf{A}_S$ satisfies $\mathfrak{b} = (\mathbf{A} \bigcap \mathfrak{b})_S = \mathbf{A}_S(\mathbf{A} \bigcap \mathfrak{b})$. But $\mathbf{A} \bigcap \mathfrak{b}$ is principal and hence $\mathbf{A} \bigcap \mathfrak{b} = \mathbf{A}a$ and $\mathfrak{b} = \mathbf{A}_S a$.
2. If $\mathfrak{b} \subseteq \mathbf{A}_S$ is a nonzero two-sided ideal of $\mathbf{A}_S$ then so is $\mathfrak{a} := \mathbf{A} \bigcap \mathfrak{b}$ of $\mathbf{A}$. Since $\mathbf{A}$ is simple $\mathfrak{a} = \mathbf{A}$ and hence also $\mathfrak{b} = \mathbf{A}_S$. □

3.2 Stable input/output behaviors

Consider a behavior $\mathcal{B} := \mathcal{B}(U)$ for some $U \subseteq \mathbf{A}^{1\times q}$ and $M = \mathbf{A}^{1\times q}/U$. Recall that U is a free submodule of $\mathbf{A}^{1\times q}$. Then

$$\mathbf{Q}U = \mathbf{Q} \otimes_{\mathbf{A}} U \subseteq \mathbf{Q}^{1\times q} = \mathbf{Q} \otimes_{\mathbf{A}} \mathbf{A}^{1\times q} \text{ and } \mathbf{Q} \otimes_{\mathbf{A}} M \underset{\text{identification}}{=} \mathbf{Q}^{1\times q}/\mathbf{Q}U. \tag{32}$$

As usual we define

$$\begin{aligned} &p := \mathrm{rank}(U) := \dim_{\mathbf{A}}(U) = \dim_{\mathbf{Q}}(\mathbf{Q}U), \\ &m := \mathrm{rank}(M) := \dim_{\mathbf{Q}}(\mathbf{Q} \otimes_{\mathbf{A}} M), \text{ hence p+m=q} \\ &\implies \exists R \in \mathbf{A}^{p\times q} \text{ with } \mathrm{rank}(R) = p,\ U = \mathbf{A}^{1\times p}R. \end{aligned} \tag{33}$$

In general there are various subsets $I \subset \{1, \cdots, q\}$ of p elements such that the projection $\mathrm{proj}:\ \mathbf{Q}^{1\times q} \to \mathbf{Q}^{1\times I}$ induces an isomorphism $\mathrm{proj} : \mathbf{Q}U \cong \mathbf{Q}^{1\times I}$. Such a subset I is called an *input/output (IO) structure* of U, M or $\mathcal{B}$ and $\mathcal{B}$ with this structure is called an *IO behavior*. We assume such a structure and also as usual, after a possible column permutation, that $I = \{1, \cdots, p\}$ and hence the isomorphism

$$\mathbf{Q}^{1\times(p+m)} \supset \mathbf{Q}U \cong \mathbf{Q}^{1\times p},\ (\xi, \eta) \mapsto \xi. \tag{34}$$

This isomorphism, in turn, is equivalent to the isomorphism

$$(\circ(0, \mathrm{id}_m))_{\mathrm{ind}}:\ \mathbf{Q}^{1\times m} \cong \mathbf{Q}^{1\times(p+m)}/\mathbf{Q}U = \mathbf{Q} \otimes_{\mathbf{A}} M,\ \eta \mapsto (0, \eta) + \mathbf{Q}U. \tag{35}$$

With $U^0 = U \left(\begin{smallmatrix} \mathrm{id}_p \\ 0 \end{smallmatrix}\right) \subseteq \mathbf{A}^{1\times p}$ and $M^0 := \mathbf{A}^{1\times p}/U^0$ the preceding isomorphisms are also equivalent to $\mathrm{rank}(M^0) = 0$, i..e., the torsion property of M^0, and the exactness of

$$0 \to \mathbf{A}^{1\times m} \xrightarrow{(\circ(0,\mathrm{id}_m))_{\mathrm{ind}}} M = \mathbf{A}^{1\times(p+m)}/U \xrightarrow{\left(\circ\left(\begin{smallmatrix} \mathrm{id}_p \\ 0 \end{smallmatrix}\right)\right)_{\mathrm{ind}}} M^0 = \mathbf{A}^{1\times p}/U^0 \to 0. \tag{36}$$

For the matrix R the isomorphisms (34) and (35) imply

$$\begin{gathered} R = (P, -Q) \in \mathbf{A}^{p\times(p+m)},\ p = \mathrm{rank}(R) = \mathrm{rank}(P),\ H = P^{-1}Q, \\ U = \mathbf{A}^{1\times p}(P, -Q) \subseteq \mathbf{A}^{1\times(p+m)},\ U^0 = \mathbf{A}^{1\times p}P \subseteq \mathbf{A}^{1\times p}. \end{gathered} \tag{37}$$

Here we used the equivalences

$$\mathrm{rank}(P) = p \iff \mathbf{Q}^{1\times p}P = \mathbf{Q}^{1\times p} \iff P\mathbf{Q}^p = \mathbf{Q}^p \iff P \in \mathrm{Gl}_p(\mathbf{Q}). \quad (38)$$

The matrix H is the *transfer matrix* of the IO behavior. It is also characterized by the property that

$$\mathbf{Q}U = \mathbf{Q}^{1\times p}(P, -Q) = \mathbf{Q}^{1\times p}(\mathrm{id}_p, -H). \quad (39)$$

This shows that H depends on U and the IO structure, but not on the special choice of R. Recall the behavior $\mathcal{W} := \mathrm{cl}\left((W(\tau))_{\tau\geq 0}\right)$. By duality the exactness of (36) is equivalent to the exactness of the behavior sequences

$$\begin{array}{cccccccc} 0 \to & \mathcal{B}(U^0) & \xrightarrow{\binom{\mathrm{id}_p}{0}\circ} & \mathcal{B}(U) & \xrightarrow{(0,\mathrm{id}_m)\circ} & \mathcal{W}^{1\times m} & \to 0 \\ 0 \to & \mathcal{B}(P,\tau) & \xrightarrow{\binom{\mathrm{id}_p}{0}\circ} & \mathcal{B}((P,-Q),\tau) & \xrightarrow{(0,\mathrm{id}_m)\circ} & W(\tau)^m & \to 0 \\ & y & \mapsto & \binom{y}{0}, \binom{y}{u} & \mapsto & u & \end{array} \quad (40)$$
$$\mathcal{B}(P,\tau) = \{y \in W(\tau)^p;\ P \circ y = 0\},$$
$$\mathcal{B}((P,-Q),\tau) = \left\{\binom{y}{u} \in W(\tau)^{p+m};\ P \circ y = Q \circ u\right\},$$

where $\tau \geq \tau_0$ and $\tau_0 \geq 0$ is sufficiently large.

Lemma and Definition 3.6. *For the IO behavior $\mathcal{B} = \mathcal{B}(U)$ from above and its autonomous part $\mathcal{B}^0 = \mathcal{B}(U^0)$ the following properties are equivalent:*
1. The autonomous behavior $\mathcal{B}^0$ or M^0 are stable, i.e., $M^0_S = 0$ or $\mathbf{A}_S^{1\times p}P = \mathbf{A}_S^{1\times p}$ or $P \in \mathrm{Gl}_p(\mathbf{A}_S)$.
2. The quotient module M_S is free and $H \in \mathbf{A}_S^{p\times m}$.
If 1. and 2. are satisfied the IO behavior is called weakly exponentially stable *(w.e.s.) or just* stable.

Proof. 1. $\Longrightarrow$ 2.: The stability of M^0 implies $M^0_S = 0$. We apply the exact quotient functor $M \mapsto M_S$ to (36) and infer the isomorphism $\mathbf{A}_S^{1\times m} \cong M_S$, hence M_S is free. Moreover $P \in \mathrm{Gl}_p(\mathbf{A}_S)$ and hence $H = P^{-1}Q \in \mathbf{A}_S^{p\times m}$.
2. $\Longrightarrow$ 1.: Application of the exact quotient functor $(-)_S$ to the exact sequence

$$0 \to \mathbf{A}^{1\times p} \xrightarrow{\circ(P,-Q)} \mathbf{A}^{1\times(p+m)} \xrightarrow{\mathrm{can}} M = \mathbf{A}^{1\times(p+m)}/\mathbf{A}^{1\times p}(P,-Q) \to 0 \quad (41)$$

with the canonical map can furnishes the exact sequence of $\mathbf{A}_S$-modules

$$0 \to \mathbf{A}_S^{1\times p} \xrightarrow{\circ(P,-Q)} \mathbf{A}_S^{1\times(p+m)} \xrightarrow{\mathrm{can}} M_S \to 0. \quad (42)$$

Since M_S is free this sequence splits and in particular $(P, -Q)$ has a right inverse $\binom{X}{Y}$ with $X, Y \in \mathbf{A}_S^{\bullet\times\bullet}$ and $\mathrm{id}_p = \begin{pmatrix} P & -Q\end{pmatrix}\binom{X}{Y} = PX - QY = P(X - HY)$. Since $H \in \mathbf{A}_S^{\bullet\times\bullet}$ the last equation implies that P has the right inverse and thus inverse $X - HY$, i.e., $P \in \mathrm{Gl}_p(\mathbf{A}_S)$. □

3.3 Stabilizing compensators

We consider the interconnected behavior $\mathcal{B}$ of the plant $\mathcal{B}_1$ and the controller $\mathcal{B}_2$ according to figure 1. The components of $\mathcal{B}_1$ resp. $\mathcal{B}_2$ are

$$\binom{y_1}{u_1},\ \binom{u_2}{y_2} \in W(\tau)^{p+m}. \quad (43)$$

In the interconnected behavior $\mathcal{B}$ the component y_i is added to the component u_{3-i} for $i = 1, 2$. The behaviors are given as

$$\begin{aligned}
&\mathcal{B}_1 = \mathcal{B}(U_1),\ U_1 = \mathbf{A}^{1\times p}R_1,\ R_1 \in \mathbf{A}^{p\times(p+m)},\ \dim_{\mathbf{A}}(U_1) = \operatorname{rank}(R_1) = p\\
&\mathcal{B}_2 = \mathcal{B}(U_2),\ U_2 = \mathbf{A}^{1\times m}R_2,\ R_2 \in \mathbf{A}^{m\times(p+m)},\ \dim_{\mathbf{A}}(U_2) = \operatorname{rank}(R_2) = m\\
&M_1 := \mathbf{A}^{1\times(p+m)}/U_1,\ M_2 := \mathbf{A}^{1\times(p+m)}/U_2.
\end{aligned} \tag{44}$$

As indicated in the Introduction we neither assume that the behaviors $\mathcal{B}_i,\ i = 1, 2$, are IO behaviors with inputs u_i nor that they are controllable (cf. [5, §11.2, p.545]). Nevertheless we decompose the matrices R_i as

$$\begin{aligned}
&R_1 = (P_1, -Q_1) \in \mathbf{A}^{p\times(p+m)},\ R_2 = (-Q_2, P_2) \in \mathbf{A}^{m\times(p+m)},\ \text{hence}\\
&\mathcal{B}(R_1,\tau) = \left\{\left(\begin{smallmatrix} y_1\\ u_1\end{smallmatrix}\right) \in W(\tau)^{p+m};\ P_1 \circ y_1 = Q_1 \circ u_1\right\},\\
&\mathcal{B}(R_2,\tau) = \left\{\left(\begin{smallmatrix} u_2\\ y_2\end{smallmatrix}\right) \in W(\tau)^{p+m};\ P_2 \circ y_2 = Q_2 \circ u_2\right\},\\
&\mathcal{B}(U_1) = \operatorname{cl}\left((\mathcal{B}(R_1,\tau))_{\tau\ge\tau_0}\right),\ \mathcal{B}(U_2) = \operatorname{cl}\left((\mathcal{B}(R_2,\tau))_{\tau\ge\tau_0}\right),\ \tau \ge \tau_0,
\end{aligned} \tag{45}$$

where, as usual, τ_0 is sufficiently large. The interconnected behavior $\mathcal{B}$ is then defined by

$$\begin{aligned}
&\mathcal{B} = \operatorname{cl}\left((\mathcal{B}(\tau))_{\tau\ge\tau_0}\right),\ \mathcal{B}(\tau) = \left\{\left(\begin{smallmatrix} y\\ u\end{smallmatrix}\right) \in W(\tau)^{(p+m)+(p+m)};\ P \circ y = Q \circ u\right\}\ \text{where}\\
&y = \left(\begin{smallmatrix} y_1\\ y_2\end{smallmatrix}\right) \in W(\tau)^{p+m},\ u = \left(\begin{smallmatrix} u_2\\ u_1\end{smallmatrix}\right) \in W(\tau)^{p+m},\ \begin{cases} P_1 \circ y_1 = Q_1 \circ (u_1 + y_2)\\ P_2 \circ y_2 = Q_2 \circ (u_2 + y_1)\end{cases}\\
&P := \left(\begin{smallmatrix} R_1\\ R_2\end{smallmatrix}\right) = \begin{pmatrix} P_1 & -Q_1\\ -Q_2 & P_2\end{pmatrix} \in \mathbf{A}^{(p+m)\times(p+m)},\ Q := \begin{pmatrix} 0 & Q_1\\ Q_2 & 0\end{pmatrix} \in \mathbf{A}^{(p+m)\times(p+m)}.
\end{aligned} \tag{46}$$

The corresponding modules are

$$\begin{aligned}
&U := \mathbf{A}^{1\times(p+m)}(P, -Q) \subseteq \mathbf{A}^{1\times 2(p+m)},\ M := \mathbf{A}^{1\times 2(p+m)}/U,\ \mathcal{B} := \mathcal{B}(U),\\
&U^0 := \mathbf{A}^{1\times(p+m)}P = \mathbf{A}^{1\times p}R_1 + \mathbf{A}^{1\times m}R_2 = U_1 + U_2 \subseteq \mathbf{A}^{1\times(p+m)},\\
&M^0 := \mathbf{A}^{1\times(p+m)}/U^0,\ \mathcal{B}^0 := \mathcal{B}(U^0),\\
&\operatorname{rank}(P) = \dim_{\mathbf{A}}(U^0) = \dim_{\mathbf{A}}(U_1 + U_2) \le \dim_{\mathbf{A}}(U_1) + \dim_{\mathbf{A}}(U_2) = p + m.
\end{aligned} \tag{47}$$

Lemma and Definition 3.7. *(Cf. [5, Thm. and Def. 1136, p.546]) Assume the data from* (44)-(47). *The following properties are equivalent:*

1. $\operatorname{rank}(P) = p + m$ *or, equivalently,* $P \in \mathrm{Gl}_{p+m}(\mathbf{Q})$, *i.e.,* $\mathcal{B}$ *is an IO behavior with input* $u = \left(\begin{smallmatrix} u_2\\ u_1\end{smallmatrix}\right)$ *and output* $y = \left(\begin{smallmatrix} y_1\\ y_2\end{smallmatrix}\right)$.
2. $U^0 = U_1 \oplus U_2$ *or, equivalently,* $U_1 \bigcap U_2 = 0$.

If these conditions are satisfied the behavior $\mathcal{B}$ *is called* well-posed, *its autonomous part is* $\mathcal{B}^0 = \mathcal{B}(U^0)$ *and its transfer matrix is* $H = P^{-1}Q$.

Proof. $\dim_{\mathbf{A}}(U_1 + U_2) = \dim_{\mathbf{A}}(U_1) + \dim_{\mathbf{A}}(U_2) \iff U_1 \bigcap U_2 = 0.$ □

Remark 3.8. In the following theory the behavior $\mathcal{B}$ is always assumed well-posed or, equivalently, an IO behavior with input $\left(\begin{smallmatrix} u_2\\ u_1\end{smallmatrix}\right)$. Since all considerations refer to the interconnected behavior $\mathcal{B}$ the components u_1 and u_2 are free (can be freely chosen) as inputs of $\mathcal{B}$ although they are not free in general as inputs of $\mathcal{B}_1$ resp. $\mathcal{B}_2$. Transfer matrices H_1, H_2 and their usual factorizations $H_1 = P_1^{-1}Q_1$ and $H_2 = P_2^{-1}Q_2$ do not exist in general and are not needed or used in the following.

Corollary and Definition 3.9. *For the behavior $\mathcal{B}$ from* (46) *and* (47) *the following properties are equivalent:*

1. $P \in \mathrm{Gl}_{p+m}(\mathbf{A}_S)$.
2. *The interconnected behavior $\mathcal{B}$ is well-posed, M_S is $\mathbf{A}_S$-free, necessarily of dimension $p+m$, and $H = P^{-1}Q \in \mathbf{A}_S^{(p+m)\times(p+m)}$.*
3. $U_{1,S} \oplus U_{2,S} = \mathbf{A}_S^{1\times(p+m)}$.

These equivalent properties imply the canonical isomorphisms

$$M_{1,S} \cong \mathbf{A}_S^{1\times(p+m)}/U_{1,S} \cong U_{2,S} \tag{48}$$

and thus that $M_{1,S}$ is a free $\mathbf{A}_S$-module of dimension $m = \mathrm{rank}(M_1)$. The behavior $\mathcal{B}_2$ is then called a weakly exponentially stabilizing *or just a* stabilizing *compensator of $\mathcal{B}_1$. Obviously this property holds reciprocally, i.e., $\mathcal{B}_1$ is also a stabilizing compensator for $\mathcal{B}_2$.*

Proof. 1. $\Longleftrightarrow$ 2.: Lemma and Def. 3.6.
In 2. and 3. the well-posedness of the feedback behavior implies

$$U = U_1 \oplus U_2 = \mathbf{A}^{1\times(p+m)}P \text{ and } U_S = U_{1,S} \oplus U_{2,S} = \mathbf{A}_S^{1\times(p+m)}P.$$

1. $\Longleftrightarrow$ 3.:

$$P \in \mathrm{Gl}_{p+m}(\mathbf{A}_S) \iff \mathbf{A}_S^{1\times(p+m)}P = \mathbf{A}_S^{1\times(p+m)} \iff U_{1,S} \oplus U_{2,S} = \mathbf{A}_S^{1\times(p+m)}.$$

□

Definition 3.10. The behavior $\mathcal{B}(U_1)$ or module M_1 are called *weakly exponentially stabilizable* or just *stabilizable* if $\mathcal{B}(U_1)$ admits a stabilizing compensator $\mathcal{B}(U_2)$ according to Cor. and Def. 3.9.

Corollary 3.11. *The behavior $\mathcal{B}_1$ from* (43) *is stabilizable if and only if $M_{1,S}$ is a free $\mathbf{A}_S$-module, necessarily of dimension m.*

Proof. 1. If $\mathcal{B}_1$ is stabilizable then $M_{1,S}$ is free according to Cor. and Def. 3.9 and (48). If, conversely, $M_{1,S}$ is free the canonical map $\mathrm{can} : \mathbf{A}_S^{1\times(p+m)} \to M_{1,S} = \mathbf{A}_S^{1\times(p+m)}/U_{1,S}$ has a section or right inverse σ and hence

$$\mathbf{A}_S^{1\times(p+m)} = \ker(\mathrm{can}) \oplus \mathrm{im}(\sigma),\ U_{1,S} \oplus V,\ V := \mathrm{im}(\sigma).$$

Define

$$U_2 := \mathbf{A}^{1\times(p+m)} \bigcap V, \text{ hence } V = U_{2,S} \text{ and } \mathbf{A}_S^{1\times(p+m)} = U_{1,S} \oplus U_{2,S}. \text{ Then}$$
$$\dim_{\mathbf{A}}(U_2) = \dim_{\mathbf{A}_S}(U_{2,S}) = (p+m) - \dim_{\mathbf{A}_S}(U_{1,S}) = (p+m) - p = m.$$

Equation (44) and Cor. and Def. 3.9 imply that $\mathcal{B}_2 := \mathcal{B}(U_2)$ is a stabilizing compensator of $\mathcal{B}_1$.

□

3.4 The parametrization of stabilizing compensators

This section contains the LTV version of the Kučera-Youla parametrization for the modules and behaviors of this paper. Assume that the behavior $\mathcal{B}_1$ from (44) is stabilizable, i.e., that $M_{1,S} \cong \mathbf{A}_S^{1\times m}$. Omission of $M_{1,S}$ in the exact sequence

$$0 \to \mathbf{A}_S^{1\times p} \overset{\circ(P_1,-Q_1)}{\longrightarrow} \mathbf{A}_S^{1\times(p+m)} \overset{\mathrm{can}}{\longrightarrow} M_{1,S} \cong \mathbf{A}_S^{1\times m} \to 0$$

furnishes a split exact sequence

$$0 \to \mathbf{A}_S^{1\times p} \overset{\circ(P_1,-Q_1)}{\longrightarrow} \mathbf{A}_S^{1\times(p+m)} \overset{\circ\left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right)}{\longrightarrow} \mathbf{A}_S^{1\times m} \to 0 \text{ where } \left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right) \in \mathbf{A}_S^{(p+m)\times m}. \tag{49}$$

Since the sequence splits the matrix $(P_1, -Q_1)$ has a right inverse $\begin{pmatrix} D_2^0 \\ N_2^0 \end{pmatrix} \in \mathbf{A}_S^{(p+m)\times p}$ and $\left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right)$ has a left inverse $(-Q_2^0, P_2^0) \in \mathbf{A}_S^{m\times(p+m)}$ such that also the sequence

$$0 \leftarrow \mathbf{A}_S^{1\times p} \overset{\circ\begin{pmatrix} D_2^0 \\ N_2^0 \end{pmatrix}}{\longleftarrow} \mathbf{A}_S^{1\times(p+m)} \overset{(-Q_2^0,P_2^0)}{\longleftarrow} \mathbf{A}_S^{1\times m} \leftarrow 0 \tag{50}$$

is exact. This is a standard algebraic result, cf. [4, Lemma 2.3]. Moreover

$$\begin{aligned}
&\mathbf{A}_S^{1\times(p+m)} = \mathbf{A}_S^{1\times p}(P_1, -Q_1) \oplus \mathbf{A}_S^{1\times m}(-Q_2^0, P_2^0)\\
&E_1^0 := \begin{pmatrix} D_2^0 \\ N_2^0 \end{pmatrix}(P_1, -Q_1) = (E_1^0)^2,\ E_2^0 = \left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right)(-Q_2^0, P_2^0) = (E_2^0)^2,\\
&\mathrm{id}_{p+m} = \begin{pmatrix} \mathrm{id}_p & 0 \\ 0 & \mathrm{id}_m \end{pmatrix} = E_1 + E_2 = \begin{pmatrix} D_2^0P_1 - N_1Q_2^0, & -D_2^0Q_1 + N_1P_2^0 \\ N_2^0P_1 - D_1Q_2^0, & -N_2^0Q_1 + D_1P_2^0 \end{pmatrix}\\
&\quad = \begin{pmatrix} D_2^0 & N_1 \\ N_2^0 & D_1 \end{pmatrix}\begin{pmatrix} P_1 & -Q_1 \\ -Q_2^0 & P_2^0 \end{pmatrix} \Longrightarrow \begin{pmatrix} P_1 & -Q_1 \\ -Q_2^0 & P_2^0 \end{pmatrix}^{-1} = \begin{pmatrix} D_2^0 & N_1 \\ N_2^0 & D_1 \end{pmatrix}.
\end{aligned} \tag{51}$$

The parametrization of all sequences (50) or, equivalently, all $V \subseteq \mathbf{A}^{1\times(p+m)}$ with $U_{1,S} \oplus V = \mathbf{A}_S^{1\times(p+m)}$ is the analogue of the Kučera-Youla parametrization that was established in the LTI case only. The following lemma is the LTV form of this. Its proof is a standard result on split exact sequences and analogous to that of [4, Lemma 3.10].

Lemma 3.12. *(Cf. [4, Lemma 3.10]) Assume that the behavior $\mathcal{B}(U_1)$ from* (44) *is stabilizable and the ensuing exact sequences* (49) *and* (50)*.*
1. The following bijections hold:

$$\begin{array}{ccc}
\left\{V \subseteq \mathbf{A}_S^{1\times \ell};\ U_{1,S} \oplus V = \mathbf{A}_S^{1\times(p+m)}\right\} & \ni & V \\
\cong & & \updownarrow \\
\left\{\left(\begin{smallmatrix} D_2 \\ N_2 \end{smallmatrix}\right);\ (P_1, -Q_1)\left(\begin{smallmatrix} D_2 \\ N_2 \end{smallmatrix}\right) = \mathrm{id}_p\right\} & \ni & \left(\begin{smallmatrix} D_2 \\ N_2 \end{smallmatrix}\right) \\
\cong & & \updownarrow \\
\left\{(-Q_2, P_2) \in \mathbf{A}_S^{m\times(p+m)};\ (-Q_2, P_2)\left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right) = \mathrm{id}_m\right\} & \ni & (-Q_2, P_2) \\
\cong & & \updownarrow \\
\mathbf{A}^{m\times p} & \ni & X
\end{array} \tag{52}$$

The bijections are given by the equations

$$\begin{gathered}
V = \ker\left(\circ\left(\begin{smallmatrix} D_2 \\ N_2 \end{smallmatrix}\right)\right) = \mathbf{A}_S^{1\times m}(-Q_2, P_2),\ \left(\begin{smallmatrix} D_2 \\ N_2 \end{smallmatrix}\right) = \begin{pmatrix} D_2^0 \\ N_2^0 \end{pmatrix} - \left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right)X \\
(-Q_2, P_2) = (-Q_2^0, P_2^0) + X(P_1, -Q_1),\ P_2 = P_2^0 - XQ_1, Q_2 = Q_2^0 - XP_1.
\end{gathered} \tag{53}$$

2. *In analogy to* (50) *and* (51) *these data give rise to the exact sequence*

$$0 \leftarrow \mathbf{A}_S^{1\times p} \overset{\circ\left(\begin{smallmatrix} D^2 \\ N_2 \end{smallmatrix}\right)}{\longleftarrow} \mathbf{A}_S^{1\times(p+m)} \overset{(-Q_2,P_2)}{\longleftarrow} \mathbf{A}_S^{1\times m} \leftarrow 0 \text{ and to}$$

$$\begin{pmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{pmatrix}^{-1} = \begin{pmatrix} D_2 & N_1 \\ N_2 & D_1 \end{pmatrix}, \tag{54}$$

$$H = \begin{pmatrix} H_{y_1,u_2} & H_{y_1,u_1} \\ H_{y_2,u_2} & H_{y_2,u_1} \end{pmatrix} = \begin{pmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix} = \begin{pmatrix} N_1Q_2 & D_2Q_1 \\ D_1Q_2 & N_2Q_1 \end{pmatrix}.$$

Corollary 3.13. *According to Cor. and Def. 3.9 every stabilizing compensator $\mathcal{B}(U_2)$ of $\mathcal{B}(U_1)$ gives rise to a decomposition $U_{1,S} \oplus U_{2,S} = \mathbf{A}_S^{1\times(p+m)}$. According to Lemma 3.12 $U_{2,S}$ has a unique representation $U_{2,S} = \mathbf{A}_S^{1\times m}(-Q_2, P_2)$ where $(-Q_2, P_2) \in \mathbf{A}_S^{m\times(p+m)}$ is precisely one matrix as described in the lemma.*

Theorem 3.14. (Construction of all stabilizing compensators) *Assume that the behavior $\mathcal{B}(U_1)$ from* (44) *is stabilizable and the ensuing exact sequences* (49), (50) *and data from Lemma 3.12. All stabilizing compensators $\mathcal{B}(U_2)$ of $\mathcal{B}(U_1)$ are obtained in the following fashion:*

(i) Choose a direct complement $V = \mathbf{A}_S^{1\times m}(-Q_2, P_2)$ of $U_{1,S}$ according to Lemma 3.12 and define the matrix $R_2^V = (-Q_2^V, P_2^V) \in \mathbf{A}^{m\times(p+m)}$ by

$$\mathbf{A}^{1\times m} R_2^V = U_2^V := \mathbf{A}^{1\times(p+m)} \bigcap V, \text{ hence } \mathbf{A}_S^{1\times m} R_2^V = U_{2,S}^V = V. \tag{55}$$

In other words, the rows of R_2^V are an $\mathbf{A}_S$-basis of V in $\mathbf{A}^{1\times m}$.

(ii) Choose a matrix $Y \in \mathbf{A}^{m\times m} \bigcap \mathrm{Gl}_m(\mathbf{A}_S)$ (cf. Remark 3.15) and define

$$R_2^{\mathbf{A}} := Y R_2^V = (-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}}) \text{ and } U_2 := \mathbf{A}^{1\times m} R_2^{\mathbf{A}}. \tag{56}$$

For given V the unique controllable compensator is $\mathcal{B}(U_2^V)$.

Proof. 1. *The constructed $\mathcal{B}(U_2)$ is a stabilizing compensator*: Since $Y \in \mathrm{Gl}_m(\mathbf{A}_S)$ we get

$$U_{2,S} = \mathbf{A}_S^{1\times m} Y R_2^V = \mathbf{A}_S^{1\times m} R_2^V = V \text{ and } \dim_{\mathbf{A}}(U_2) = \dim_{\mathbf{A}_S}(V) = m.$$

2. *Every stabilizing compensator has this form*: Let U_2 be the equation module of a stabilizing compensator $\mathcal{B}(U_2)$. We define $V := U_{2,S}$ and obtain, by Cor. and Def. 3.9, $U_{1,S} \oplus V = \mathbf{A}_S^{1\times(p+m)}$. Thus V is a submodule as in item (i). Moreover

$$U_2 \subseteq \mathbf{A}^{1\times(p+m)} \bigcap V = \mathbf{A}^{1\times m} R_2^V, \text{ hence } \exists Y \in \mathbf{A}^{m\times m} \text{ such that}$$
$$U_2 = \mathbf{A}^{1\times m} Y R_2^V \text{ and } \mathbf{A}_S^{1\times m} R_2^V = V = U_{2,S} = \mathbf{A}_S^{1\times m} Y R_2^V.$$

Since $\mathrm{rank}(R_2^V) = m$ the rows of R_2^V are $\mathbf{Q}$-linearly independent and R_2^V can be cancelled as right factor. We conclude $\mathbf{A}_S^{1\times m} Y = \mathbf{A}_S^{1\times m}$ and $Y \in \mathbf{A}^{m\times m} \bigcap \mathrm{Gl}_m(\mathbf{A}_S)$.
3. (i) The inclusion

$$M_2^V := \mathbf{A}^{1\times(p+m)} / \left(\mathbf{A}^{1\times(p+m)} \bigcap V\right) \underset{\text{ident.}}{\subset} \mathbf{A}_S^{1\times(p+m)}/V$$

implies that M_2^V is torsionfree and thus free and hence $\mathcal{B}(U_2^V)$ is controllable.
(ii) If, conversely, $\mathcal{B}(U_2)$ is controllable and $M_2 = \mathbf{A}^{1\times(p+m)}/U_2$ is free and hence torsionfree then the canonical map $M_2 \to M_{2,S} = \mathbf{A}_S^{1\times(p+m)}/U_{2,S} = \mathbf{A}_S^{1\times(p+m)}/V$ is injective. This signifies that $U_2 = \mathbf{A}^{1\times(p+m)} \bigcap V = U_2^V$. □

Remark 3.15. (Cf. [11, Cor. 1.4.8], [5, Thm. and Def. 662]) Every matrix $Y \in \mathbf{A}^{m\times m}$ is equivalent to a diagonal matrix $D = \operatorname{diag}(\overset{1}{1}, \cdots, 1, \overset{r}{a}, 0, \cdots, \overset{m}{0}) \in \mathbf{A}^{m\times m}$ where $0 \le r \le m$ and $a \ne 0$. Then $r = \operatorname{rank}(Y)$ and, of course, $r = m$ if and only if $Y \in \operatorname{Gl}_m(\mathbf{Q})$. The matrix Y can be obtained from D by a finite sequence of elementary row and column operations. This implies

$$Y \in \mathbf{A}^{m\times m} \bigcap \operatorname{Gl}_m(\mathbf{A}_S) \iff D \in \mathbf{A}^{m\times m} \bigcap \operatorname{Gl}_m(\mathbf{A}_S) \iff r = m \text{ and } a \in S. \tag{57}$$

Hence all matrices $Y \in \mathbf{A}^{m\times m} \bigcap \operatorname{Gl}_m(\mathbf{A}_S)$ in Thm. 3.14 can be obtained from diagonal matrices $D = \operatorname{diag}(1, \cdots, 1, \overset{m}{s}),\ s \in S$, by a sequence of elementary row and column operations.

We specialize the preceding considerations to a controllable IO behavior $\mathcal{B}_1$. For this purpose assume an arbitrary nonzero transfer matrix $H_1 \in \mathbf{Q}^{p\times m}$. The matrices $\left(\begin{smallmatrix} H_1 \\ \mathrm{id}_m \end{smallmatrix}\right)$ resp. $(-\operatorname{id}_p, H_1)$ have rank m resp. p and thus give rise to exact sequences

$$\begin{array}{ccccccc} 0 \to & \mathbf{A}^{1\times p} & \overset{\circ(P_1,-Q_1)}{\longrightarrow} & \mathbf{A}^{1\times(p+m)} & \overset{\circ\left(\begin{smallmatrix} H_1 \\ \mathrm{id}_m \end{smallmatrix}\right)}{\longrightarrow} & \mathbf{Q}^{1\times m} & , \\ 0 \to & \mathbf{A}^{m} & \overset{\left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right)\circ}{\longrightarrow} & \mathbf{A}^{p+m} & \overset{(-\operatorname{id}_p,H_1)\circ}{\longrightarrow} & \mathbf{Q}^{p} & \end{array}$$

$$\Longrightarrow \begin{cases} P_1H_1 = Q_1,\ \operatorname{rank}(P_1) = \operatorname{rank}(P_1, -Q_1) = p,\ P_1 \in \operatorname{Gl}_p(\mathbf{Q}),\ H_1 = P_1^{-1}Q_1 \\ H_1D_1 = N_1,\ \operatorname{rank}(D_1) = \operatorname{rank}\left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right) = m,\ D_1 \in \operatorname{Gl}_m(\mathbf{Q}),\ H_1 = N_1D_1^{-1} \end{cases}. \tag{58}$$

Since $\mathbf{Q}^{1\times m}$ and $\mathbf{Q}^p$ are $\mathbf{A}$-torsionfree the modules

$$M_1 := \mathbf{A}^{1\times(p+m)}/U_1,\ U_1 := \mathbf{A}^{1\times p}(P_1, -Q_1), \text{ and } \mathbf{A}^{p+m}/\left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right)\mathbf{A}^m \tag{59}$$

are left resp. right $\mathbf{A}$-free. In particular, $\mathcal{B}_1 := \mathcal{B}(U_1)$ is controllable and an IO behavior due to $P_1 \in \operatorname{Gl}_p(\mathbf{Q})$ and the matrices $(P_1, -Q_1)$ resp. $\left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right)$ have a right inverse resp. a left inverse. The exactness of (58) implies that of (49) with $\mathbf{A}_S$ replaced by $\mathbf{A}$, i.e., of

$$0 \to \mathbf{A}^{1\times p} \overset{\circ(P_1,-Q_1)}{\longrightarrow} \mathbf{A}^{1\times(p+m)} \overset{\circ\left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right)}{\longrightarrow} \mathbf{A}^{1\times m} \to 0 \tag{60}$$

The matrices P_1 resp. D_1 are unique up to multiplication with a matrix of $\operatorname{Gl}_p(\mathbf{A})$ resp. $\operatorname{Gl}_m(\mathbf{A})$ from the left resp. the right. By Lemma and Def. 3.6 the controllable IO behavior $\mathcal{B}_1$ is stable or $P_1 \in \operatorname{Gl}_p(\mathbf{A}_S)$ if and only if $H_1 \in \mathbf{A}_S^{p\times m}$. With an analogous argument one proves

Corollary 3.16. *(Cf. [5, Prop. 1012]) The following properties are equivalent for the controllable behavior $\mathcal{B}_1 = \mathcal{B}(\mathbf{A}^{1\times p}(P_1, -Q_1))$ from* (58)-(60)*:*

$$\mathcal{B}_1 \text{ is stable} \iff H_1 \in \mathbf{A}_S^{p\times m} \iff P_1 \in \operatorname{Gl}_p(A_S) \iff D_1 \in \operatorname{Gl}_m(\mathbf{A}_S). \tag{61}$$

Proof. Only the last equivalence has still to be shown. Obviously

$$D_1 \in \operatorname{Gl}_m(\mathbf{A}_S) \Longrightarrow H_1 = N_1D_1^{-1} \in \mathbf{A}_S^{p\times m}.$$

If, conversely, H_1 belongs to $\mathbf{A}_S^{p\times m}$ localization of the second exact sequence of (58) furnishes the exact sequence

$$0 \to \mathbf{A}_S^m \overset{\left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right)\circ}{\longrightarrow} \mathbf{A}_S^{p+m} \overset{(-\operatorname{id}_p,H_1)\circ}{\longrightarrow} \mathbf{A}_S^p \to 0. \text{ But } \ker\left((-\operatorname{id}_p, H_1)\circ\right) = \left(\begin{smallmatrix} H_1 \\ \mathrm{id}_m \end{smallmatrix}\right)\mathbf{A}_S^m$$
$$\Longrightarrow \exists X \in \operatorname{Gl}_m(\mathbf{A}_S) \text{ with } \left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} H_1 \\ \mathrm{id}_m \end{smallmatrix}\right)X \Longrightarrow D_1 = X \in \operatorname{Gl}_m(\mathbf{A}_S).$$

□

The last equivalence of Cor. 3.16 also follows from the isomorphism

$$(\circ N_1)_{\mathrm{ind}} : \mathbf{A}^{1\times p}/\mathbf{A}^{1\times p}P_1 \cong \mathbf{A}^{1\times m}/\mathbf{A}^{1\times m}D_1, \tag{62}$$

cf. [5, Prop. 1012]. According to Lemma 3.12 there are a right inverse $\begin{pmatrix} D_2^0 \\ N_2^0 \end{pmatrix} \in \mathbf{A}^{p\times(p+m)}$ of $(P_1, -Q_1)$ and a left inverse $(-Q_2^0, P_2^0)$ of $\begin{pmatrix} N_1 \\ D_1 \end{pmatrix}$ such that also

$$0 \leftarrow \mathbf{A}^{1\times p} \xleftarrow{\circ\begin{pmatrix} D_2^0 \\ N_2^0 \end{pmatrix}} \mathbf{A}^{1\times(p+m)} \xleftarrow{\circ(-Q_2^0, P_2^0)} \mathbf{A}^{1\times m} \leftarrow 0 \tag{63}$$

is exact. This is the exact sequence (50) with $\mathbf{A}_S$ replaced by $\mathbf{A}$. The behavior $\mathcal{B}(\mathbf{A}^{1\times m}(-Q_2^0, P_2^0))$ is one stabilizing compensator of $\mathcal{B}_1$. Since $\begin{pmatrix} P_1 & -Q_1 \\ -Q_2^0 & P_2^0 \end{pmatrix} \in \mathrm{Gl}_{p+m}(\mathbf{A})$ the autonomous part of the interconnected behavior is not only stable, but zero.

Corollary 3.17. *Lemma 3.12 and Thm. 3.14 are applicable to the exact sequences* (60) *and* (63) *for the controllable IO behavior* $\mathcal{B}_1$ *from* (58)-(60). *Hence all stabilizing compensators for* $\mathcal{B}_1$ *are obtained by the two steps of Thm. 3.14.*

It is presently not clear which $P_2 = P_2^0 - XQ_1$ belong to $\mathrm{Gl}_m(\mathbf{Q})$ and then make the compensator an IO behavior.

Remark 3.18. *Standard terminology*: The behavior $\mathcal{B}_1$ is called the (unique) *controllable realization* of the transfer matrix H_1. The representations $H_1 = P_1^{-1}Q_1$ resp. $H_1 = N_1D_1^{-1}$ are the essentially unique *left resp. right* coprime factorizations of H_1.

Example 3.19. This example is constructed without computer algebra from the desired outcome instead from a plant of engineering relevance. Consider the stable operators

$$\begin{aligned} &s_1 := -2z^2\partial + z^{1/2},\ s_2 := -z^2\partial + 1 \text{ with the stable solutions} \\ &y_1(t) = \exp\left(-(t^{1/2} - t_0^{1/2})\right) y_1(t_0),\ y_2(t) = \exp\left(-(t - t_0)\right) y_2(t_0),\ t \geq t_0 > 0 \\ &\Longrightarrow s_1, s_2 \in S \Longrightarrow \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \in \mathrm{Gl}_2(\mathbf{A}_S). \end{aligned} \tag{64}$$

We choose $a \in \mathbf{A}$ and define the matrices

$$\begin{aligned} \begin{pmatrix} P_1 & -Q_1 \\ -Q_2^0 & P_2^0 \end{pmatrix} &:= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} = \begin{pmatrix} (1+a)s_1 & s_2 \\ as_1 & s_2 \end{pmatrix} \in \mathrm{Gl}_2(\mathbf{A}_S) \\ \begin{pmatrix} D_2^0 & N_1 \\ N_2^0 & D_1 \end{pmatrix} &:= \begin{pmatrix} P_1 & -Q_1 \\ -Q_2^0 & P_2^0 \end{pmatrix}^{-1} = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} s_1^{-1} & 0 \\ 0 & s_2^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s_1^{-1} & -s_1^{-1} \\ -s_2^{-1}a & s_2^{-1}(a+1) \end{pmatrix} \in \mathrm{Gl}_2(\mathbf{A}_S). \end{aligned} \tag{65}$$

Choose $x \in \mathbf{A}_S$ arbitrarily and define

$$\begin{aligned} (-Q_2, P_2) &:= (-Q_2^0, P_2^0) + x(P_1, -Q_1) = (as_1, s_2) + x((1+a)s_1, s_2) \\ &= ((1+x)as_1 + xs_1, (1+x)s_2) \in \mathbf{A}_S^{1\times 2},\ x \in \mathbf{A}_S. \end{aligned} \tag{66}$$

We conclude that

$$\begin{aligned} \mathcal{B}_1 := \mathcal{B}\left(\mathbf{A}((1+a)s_1, s_2)\right) &\text{ with } \mathcal{B}(((1+a)s_1, s_2), \tau) \\ &= \left\{\begin{pmatrix} y_1 \\ u_1 \end{pmatrix} \in W(\tau)^2;\ (1+a)s_1 \circ y_1 = -s_2 \circ u_1\right\},\ \tau \geq \tau_0, \end{aligned} \tag{67}$$

is stabilizable where, as always, τ_0 is sufficiently large. The behavior $\mathcal{B}_1$ is an IO behavior if and only if $1+a \neq 0$ and additionally stable if and only if $1+a \in S$. The operator $1+a := 1+z^2\partial$ with its solution $y(t) = \exp(t-t_0)y(t_0)$ is not stable.
The stabilizing compensators are obtained as follows: They are IO behaviors if and only if $1+x \neq 0$. The parameter $x \neq -1$ has the form $t^{-1}b$ with arbitrary $t \in S$ and $b \neq -t$ in $\mathbf{A}$. Since $t \in S \subset \mathrm{Gl}_1(\mathbf{A}_S)$ we infer

$$\begin{aligned}\mathbf{A}_S(-Q_2, P_2) = \mathbf{A}_S t(-Q_2, P_2) &= \mathbf{A}_S\left((t+b)as_1 + bs_1, (t+b)s_2\right)\\ &= \left(\mathbf{A}\left((t+b)as_1 + bs_1, (t+b)s_2\right)\right)_S\end{aligned} \tag{68}$$

According to Thm. 3.14 all stabilizing compensators of $\mathcal{B}_1$ have the form

$$\begin{gathered}\mathcal{B}_2 = \mathcal{B}\left(\mathbf{A}r(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}})\right),\ r \in S,\ \text{where}\\ \mathbf{A}(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}}) = \mathbf{A}^{1\times 2} \bigcap \mathbf{A}_S\left((t+b)as_1 + bs_1, (t+b)s_2\right),\ t \in S,\ -t \neq b \in \mathbf{A}.\end{gathered} \tag{69}$$

The computation of $(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}})$ by hand is difficult. Among these $\mathcal{B}_2$ there are the compensators

$$\begin{aligned}&\mathcal{B}_2 := \mathcal{B}\left(\mathbf{A}\left((t+b)as_1 + bs_1, (t+b)s_2\right)\right) \text{ with } t \in S,\ -t \neq b \in \mathbf{A} \text{ and}\\ &\mathcal{B}_2\left(\left((t+b)as_1 + bs_1, (t+b)s_2\right), \tau\right)\\ &= \left\{\left(\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right) \in W(\tau)^2;\ (t+b)s_2 \circ y_2 = -\left((t+b)as_1 + bs_1\right) \circ u_2\right\}.\end{aligned} \tag{70}$$

Remark 3.20. *Comparison with [5, Thm. 1143, p. 554]*: The Kučera-Youla parametrization for differential LTV systems is also derived in the quoted theorem. The correspondence is given by

$$\begin{gathered}(A_L, -B_L) = (P_1, -Q_1),\ \left(\begin{smallmatrix} B_R \\ A_R \end{smallmatrix}\right) = \left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right),\ (-R_L, S_L) = (-Q_2^0, P_2^0),\\ \left(\begin{smallmatrix} S_R \\ R_R \end{smallmatrix}\right) = \begin{pmatrix} D_2^0 \\ N_2^0 \end{pmatrix},\ (-Y', X') = (-Q_2, P_2),\ \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) = \left(\begin{smallmatrix} D_2 \\ N_2 \end{smallmatrix}\right).\end{gathered}$$

The signs differ slightly because the block diagram [5, Fig. 11.1] uses a minus sign at the left upper node. The parameter matrix X from Lemma 3.12 corresponds to $\mathcal{K} = \overline{\mathcal{K}}$. Nevertheless the quoted result differs from the derivations here. To point out the differences the language of this paper is used: (i) The employed behaviors of [5] differ from those in [7] and the present paper. The differential field $\mathbf{K}$ is replaced by an Ore field [5, Lemma and Def. 835]. (ii) The behaviors $\mathcal{B}_1$ and $\mathcal{B}_2$ of [5, Fig. 11.1] are assumed as controllable IO behaviors. This signifies that the sequences (49) and (50) are exact with $\mathbf{A}_S$ replaced by $\mathbf{A}$ and that $P_1 \in \mathrm{Gl}_p(\mathbf{Q})$ and $P_2^0 \in \mathrm{Gl}_m(\mathbf{Q})$. (iii) The quoted theorem parametrizes the transfer matrices H from (54) and not the interconnected behaviors. (iv) The main difference consists in the notion of exponential stability [5, Thm. 998, Def. 1013], called bm-stability here. A torsion module $\mathbf{A}/\mathbf{A}F$ is called bm-stable if F is the product of linear factors $z^2\partial + \gamma(t)$ with $\lim_{t\to\infty} \Re(\gamma(t)) < 0$ [5, Thm. 998]. These linear operators are obviously stable. Such a product decomposition does not exist in general for $F \in \mathbf{A}$ and for Ore fields only very rarely as far as we see. The same problem arises with the definition of a stable transfer matrix in [5, Def. 1013]. Exactly these problems motivated the paper [7] with its general definition of w.e.s. and its main result that permits to decide stability in most cases. (v) That the stable torsion modules form a Serre category implies that the set S of stable differential operators is an Ore set (cf. Lemma 3.3) and that the set $\mathbf{A}_S$ of stable rational functions is a (noncommutative) subring of $\mathbf{Q}$. According to [22, p.3] this is the motivating property for the set of stable rational functions. This property is not shown for bm-stability in [5], but the ensueing equivalence (61) is employed in [5, Prop. 1012, Def. 1013] and the third last line of the proof of [5, Thm. 1143]. It seems to us that the proof of the Kučera-Youla parametrization in [5] is incomplete.

3.5 State space representations

The following state space representation of an arbitrary IO behavior is due to Fliess with respect to its algebraic content. Its LTI analogue is Kalman's famous *realization theorem*. We consider an IO behavior $\mathcal{B}$ with the data

$$\begin{aligned}
&(P,-Q)\in\mathbf{A}^{p\times(p+m)},\ \operatorname{rank}(P)=p,\ U:=\mathbf{A}^{1\times p}(P,-Q),\ U^0:=\mathbf{A}^{1\times p}P,\\
&M:=\mathbf{A}^{1\times(p+m)}/U,\ M^0:=\mathbf{A}^{1\times p}/U^0,\ \mathcal{B}:=\mathcal{B}(U),\ \mathcal{B}^0=\mathcal{B}(U^0),\\
&\operatorname{can}:=\left(\circ\left(\begin{smallmatrix}\mathrm{id}_p\\0\end{smallmatrix}\right)\right)_{\mathrm{ind}}:M\to M^0,\ \overline{(\xi,\eta)}=(\xi,\eta)+U\mapsto\overline{\xi}=\xi+U^0,\\
&\operatorname{inj}:=(\circ(0,\mathrm{id}_m))_{\mathrm{ind}}:\mathbf{A}^{1\times m}\to M,\ \eta\mapsto\overline{(0,\eta)},\ n:=\dim_{\mathbf{K}}(M^0).
\end{aligned}\tag{71}$$

The transfer matrix of $\mathcal{B}$ is $H=P^{-1}Q\in\mathbf{Q}^{p\times m}$. Recall the exact sequence

$$0\to\mathbf{A}^{1\times m}\xrightarrow{\operatorname{inj}}M\xrightarrow{\operatorname{can}}M^0\to 0 \text{ or } \mathbf{A}^{1\times m}\cong\operatorname{im}(\operatorname{inj})=\ker(\operatorname{can}).\tag{72}$$

The behavior $\mathcal{B}$ is the equivalence class of the behavior family

$$\mathcal{B}((P,-Q),\tau):=\left\{\left(\begin{smallmatrix}y\\u\end{smallmatrix}\right)\in W(\tau)^{p+m};\ P\circ y=Q\circ u\right\},\ \tau\geq\tau_0,\tag{73}$$

where, as always, τ_0 is sufficiently large. Let $\delta_i,\ i=1,\cdots,p+m$, denote the standard basis of $\mathbf{A}^{1\times(p+m)}$ and define

$$\begin{gathered}
\mathbf{y}_i=\delta_i+U\in M \text{ for } i=1,\cdots,p \text{ and } \mathbf{u}_j:=\delta_{p+j}+U\in M \text{ for } j=1,\cdots,m,\\
\mathbf{y}:=(\mathbf{y}_1,\cdots,\mathbf{y}_p)^\top\in M^p,\ \mathbf{u}:=(\mathbf{u}_1,\cdots,\mathbf{u}_m)^\top\in M^m,\ \text{ hence}\\
M=\mathbf{A}^{1\times p}\mathbf{y}+\mathbf{A}^{1\times m}\mathbf{u}\ni\overline{(\xi,\eta)}=(\xi,\eta)+U=\xi\mathbf{y}+\eta\mathbf{u},\\
\operatorname{inj}(\eta)=\eta\mathbf{u},\ \mathbf{A}^{1\times m}\mathbf{u}=\ker(\operatorname{can}),\ M^0=\sum_{j=1}^{p}\mathbf{A}\operatorname{can}(\mathbf{y}_j)=\mathbf{A}^{1\times p}\operatorname{can}(\mathbf{y}),
\end{gathered}\tag{74}$$

In the following theorem we consider $\mathbf{A}=\mathbf{K}[\partial;d/dz]=\mathbf{K}[\delta;-z^2d/dz]$ with $\delta:=-z^2\partial$.

Theorem 3.21. *(Cf. [12], [5, Thm. 861]) For the data from* (71) *and* (74) *there are* $\mathbf{x}=(\mathbf{x}_1,\cdots,\mathbf{x}_n)^\top\in M^n$, $A\in\mathbf{K}^{n\times n}$ *and* $B\in\mathbf{K}^{n\times m}$ *such that* $\operatorname{can}(\mathbf{x})$ *is a* $\mathbf{K}$*-basis of* M^0 *and* $\delta\mathbf{x}=A\mathbf{x}+B\mathbf{u}$*. These data imply (*$\delta=-z^2\partial$*):*
(i) There are unique matrices $C\in\mathbf{K}^{p\times n}$ *and* $D\in\mathbf{A}^{p\times m}$ *such that* $\mathbf{y}=C\mathbf{x}+D\circ\mathbf{u}$.
(ii) With $U^s:=\mathbf{A}^{1\times n}(\delta\,\mathrm{id}_n-A,-B)$, $U^{s,0}:=\mathbf{A}^{1\times n}(\delta\,\mathrm{id}_n-A)$, $M^s:=\mathbf{A}^{1\times(n+m)}/U^s$ *and* $M^{s,0}:=\mathbf{A}^{1\times n}/U^{s,0}$ *there is the following commutative diagram with well-defined vertical isomorphisms*

$$\begin{array}{ccccccc}
0\to & \mathbf{A}^{1\times m} & \overset{\operatorname{inj}=(\circ(0,\mathrm{id}_m))_{\mathrm{ind}}}{\longrightarrow} & M & \overset{\operatorname{can}=(\circ\left(\begin{smallmatrix}\mathrm{id}_n\\0\end{smallmatrix}\right))_{\mathrm{ind}}}{\longrightarrow} & M^0 & \to 0\\
 & \| & & \left(\circ\left(\begin{smallmatrix}C & D\\0 & \mathrm{id}_m\end{smallmatrix}\right)\right)_{\mathrm{ind}}\downarrow & & (\circ C)_{\mathrm{ind}}\downarrow & \\
0\to & \mathbf{A}^{1\times m} & \overset{(\circ(0,\mathrm{id}_m))_{\mathrm{ind}}}{\longrightarrow} & M^s & \overset{(\circ\left(\begin{smallmatrix}\mathrm{id}_n\\0\end{smallmatrix}\right))_{\mathrm{ind}}}{\longrightarrow} & M^{s,0} & \to 0
\end{array}\tag{75}$$

(iii) The behavior $\mathcal{B}^s := \mathcal{B}(U^s)$ is a standard state space [21, Ch. 2, (1)] IO behavior with autonomous part $\mathcal{B}^{s,0} = \mathcal{B}(U^{s,0})$. The middle and right vertical isomorphisms from (75) *imply behavior isomorphisms, for $\tau \geq \tau_0$ and sufficiently large τ_0,*

$$\begin{aligned}&\left(\begin{smallmatrix} C & D \\ 0 & \mathrm{id}_m \end{smallmatrix}\right)\circ : \ \mathcal{B}((\delta\,\mathrm{id}_n - A, -B), \tau) = \left\{\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in W(\tau)^{n+m};\ x' = Ax + Bu\right\}\\ &\quad \cong \mathcal{B}((P, -Q), \tau) = \left\{\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in W(\tau)^{p+m};\ P \circ y = Q \circ u\right\},\ \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} Cx + D\circ u \\ u \end{smallmatrix}\right)\\ &C\circ : \ \mathcal{B}(\delta\,\mathrm{id}_n - A, \tau) = \{x \in W(\tau)^n;\ x' = Ax\}\\ &\qquad\qquad \cong \mathcal{B}(P, \tau) = \{y \in W(\tau)^p;\ P \circ y = 0\},\ x \mapsto Cx.\end{aligned} \tag{76}$$

The isomorphism (76) *implies in particular that the state space equations*

$$x' = Ax + Bu,\ y = Cx + D \circ u \tag{77}$$

are observable.

Proof. The existence of $\mathbf{x}$, A and B is derived in the quoted theorems.
(i) Since $\mathrm{can}(\mathbf{x})$ is a K-basis of M^0 there is a unique representation

$$\mathrm{can}(\mathbf{y}) = C\,\mathrm{can}(\mathbf{x}) \Longrightarrow \mathbf{y} - C\mathbf{x} \in \ker(\mathrm{can})^p = \mathbf{A}^{p\times m}\mathbf{u} \Longrightarrow \mathbf{y} = C\mathbf{x} + D \circ \mathbf{u}$$

for some matrix $\mathbf{D} \in \mathbf{A}^{p\times m}$. Since $\mathrm{can}(\mathbf{x})$ resp. $\mathbf{u}$ are $\mathbf{K}$- resp. $\mathbf{A}$-linearly independent the matrices C and D are unique.
(ii) Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & \mathbf{A}^{1\times m} & \xrightarrow{(\circ(0,\mathrm{id}_m))} & A^{1\times(n+m)} & \xrightarrow{(\circ\binom{\mathrm{id}_n}{0})} & A^{1\times n} & \to 0\\ & \| & & \varphi\downarrow & & \psi\downarrow & \\ 0 \to & \mathbf{A}^{1\times m} & \xrightarrow{\mathrm{inj}} & M & \xrightarrow{\mathrm{can}} & M^0 & \to 0 \end{array} \tag{78}$$
$$\text{where } \varphi(\xi,\eta) := (\xi,\eta)\left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{u} \end{smallmatrix}\right) = \xi\mathbf{x} + \eta\mathbf{u},\ \psi(\xi) = \xi\,\mathrm{can}(\mathbf{x}).$$

Since $M = \mathbf{A}^{1\times(n+m)}\left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{u} \end{smallmatrix}\right)$ the vertical maps are epimorphisms. But

$$\begin{aligned}\delta\mathbf{x} = A\mathbf{x} + B\mathbf{u} \Longrightarrow (\delta\,\mathrm{id}_n - A, -B)\left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{u} \end{smallmatrix}\right) = 0 \Longrightarrow\\ \varphi(U^s) = 0 \Longrightarrow \exists\varphi_{\mathrm{ind}} :\ M^s \to M,\ (\xi,\eta) + U^s \mapsto (\xi,\eta)\left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{u} \end{smallmatrix}\right).\end{aligned}$$

Likewise ψ induces the map $\psi_{\mathrm{ind}} :\ M^{s,0} \to M^0$. The induced diagram

$$\begin{array}{ccccccc} 0 \to & \mathbf{A}^{1\times m} & \xrightarrow{(\circ(0,\mathrm{id}_m))} & M^s & \xrightarrow{(\circ\binom{\mathrm{id}_n}{0})} & M^{s,0} & \to 0\\ & \| & & \varphi_{\mathrm{ind}}\downarrow & & \psi_{\mathrm{ind}}\downarrow & \\ 0 \to & \mathbf{A}^{1\times m} & \xrightarrow{(\circ(0,\mathrm{id}_m))_{\mathrm{ind}}} & M & \xrightarrow{(\circ\binom{\mathrm{id}_n}{0})_{\mathrm{ind}}} & M^0 & \to 0 \end{array} \tag{79}$$

is obviously commutative with vertical epimorphisms. But $\dim_{\mathbf{K}}(M^{s,0}) \leq n = \dim_{\mathbf{K}}(M^0)$ and hence ψ_{ind} is an isomorphism. Standard algebra, for instance the *Snake Lemma*, implies that also φ_{ind} is an isomorphism. Let

$$\begin{aligned}&\overline{(\gamma,\eta)} = (\gamma,\eta) + U \in M \text{ and } (\xi,\eta) := (\gamma,\eta)\left(\begin{smallmatrix} C & D \\ 0 & \mathrm{id}_m \end{smallmatrix}\right)\\ &\Longrightarrow \varphi_{\mathrm{ind}}\left((\xi,\eta) + U^s\right) = (\gamma,\eta)\left(\begin{smallmatrix} C & D \\ 0 & \mathrm{id}_m \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{u} \end{smallmatrix}\right) = (\gamma,\eta)\left(\begin{smallmatrix} C\mathbf{x} + D\mathbf{u} \\ \mathbf{u} \end{smallmatrix}\right)\\ &\qquad\qquad = (\gamma,\eta)\left(\begin{smallmatrix} \mathbf{y} \\ \mathbf{u} \end{smallmatrix}\right) = \overline{(\gamma,\eta)}.\end{aligned}$$

But this implies

$$\varphi_{\mathrm{ind}}^{-1}(\overline{(\gamma,\eta)}) = (\gamma,\eta)\left(\begin{smallmatrix} C & D \\ 0 & \mathrm{id}_m \end{smallmatrix}\right) + U^s \text{ and } \varphi_{\mathrm{ind}}^{-1} = \left(\circ\left(\begin{smallmatrix} C & D \\ 0 & \mathrm{id}_m \end{smallmatrix}\right)\right)_{\mathrm{ind}}.$$

This isomorphism together with the diagram (79) implies the commutative diagram (75) with vertical isomorphisms and the behavior isomorphisms (76).

□

Since $M = \mathbf{A}^{1\times(p+m)}\left(\begin{smallmatrix}\mathbf{y}\\ \mathbf{u}\end{smallmatrix}\right)$ there is a matrix

$$\begin{aligned}&(F,G)\in\mathbf{A}^{n\times(p+m)} \text{ such that } \mathbf{x}=(F,G)\left(\begin{smallmatrix}\mathbf{y}\\ \mathbf{u}\end{smallmatrix}\right),\ \left(\begin{smallmatrix}\mathbf{x}\\ \mathbf{u}\end{smallmatrix}\right)=\left(\begin{smallmatrix}F & G\\ 0 & \mathrm{id}_m\end{smallmatrix}\right)\left(\begin{smallmatrix}\mathbf{y}\\ \mathbf{u}\end{smallmatrix}\right)\\ &\Longrightarrow \varphi_{\mathrm{ind}}(\overline{(\xi,\eta)})=(\xi,\eta)\left(\begin{smallmatrix}\mathbf{x}\\ \mathbf{u}\end{smallmatrix}\right)=(\xi,\eta)\left(\begin{smallmatrix}F & G\\ 0 & \mathrm{id}_m\end{smallmatrix}\right)\left(\begin{smallmatrix}\mathbf{y}\\ \mathbf{u}\end{smallmatrix}\right)\\ &\Longrightarrow \varphi_{\mathrm{ind}}=\left(\circ\left(\begin{smallmatrix}F & G\\ 0 & \mathrm{id}_m\end{smallmatrix}\right)\right)_{\mathrm{ind}},\ \psi_{\mathrm{ind}}=(\circ F)_{\mathrm{ind}}\,.\end{aligned}\tag{80}$$

Corollary 3.22. *The inverse isomorphisms of* (76) *for* $\tau\geq\tau_1\geq\tau_0$ *are*

$$\begin{aligned}&\left(\begin{smallmatrix}F & G\\ 0 & \mathrm{id}_m\end{smallmatrix}\right)\circ:\ \left\{\left(\begin{smallmatrix}y\\ u\end{smallmatrix}\right)\in W(\tau)^{p+m};\ P\circ y=Q\circ u\right\}\\ &\qquad\cong\left\{\left(\begin{smallmatrix}x\\ u\end{smallmatrix}\right)\in W(\tau)^{n+m};\ x'=Ax+Bu\right\},\ \left(\begin{smallmatrix}y\\ u\end{smallmatrix}\right)\mapsto\left(\begin{smallmatrix}F\circ y+G\circ u\\ u\end{smallmatrix}\right),\\ &F\circ:\ \{y\in W(\tau)^p;\ P\circ y=0\}\cong\{x\in W(\tau)^n;\ x'=Ax\},\ y\mapsto F\circ x\end{aligned}\tag{81}$$

where, as always, τ_1 *is sufficiently large.*

Example 3.23. (Compare [1, Thm. 4.1, (4.1), (4.2)]) Assume that $\mathcal{B}_1=\mathcal{B}(U_1)$ with $U_1=\mathbf{A}^{p+m}(P_1,-Q_1)$ from Thm. 3.14 is an IO system with an observable state space representation $\mathcal{B}_1^s$ in the sense of Thm. 3.21 and the inverse isomorphism from Cor. 3.22 where $\delta=-z^2\partial$:

$$\begin{aligned}&\left(\begin{smallmatrix}C_1 & D_1\\ 0 & \mathrm{id}_m\end{smallmatrix}\right)\circ:\ \mathcal{B}((\delta\,\mathrm{id}_{n_1}-A_1,-B_1),\tau)=\left\{\left(\begin{smallmatrix}x_1\\ u_1\end{smallmatrix}\right)\in W(\tau)^{n_1+m};\ x_1'=A_1x_1+B_1u_1\right\}\\ &\qquad\cong\mathcal{B}((P_1,-Q_1),\tau)=\left\{\left(\begin{smallmatrix}y_1\\ u_1\end{smallmatrix}\right)\in W(\tau)^{p+m};\ P_1\circ y_1=Q_1\circ u_1\right\},\\ &\qquad\left(\begin{smallmatrix}x_1\\ u_1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}y_1\\ u_1\end{smallmatrix}\right),\ y_1=C_1x_1+D_1\circ u_1,\ x_1=F_1\circ y_1+G_1\circ u_1.\end{aligned}\tag{82}$$

By duality,

$$\begin{aligned}&\left(\circ\left(\begin{smallmatrix}C_1 & D_1\\ 0 & \mathrm{id}_m\end{smallmatrix}\right)\right)_{\mathrm{ind}}:\ M=\mathbf{A}^{1\times(p+m)}/\mathbf{A}^{1\times p}(P_1,-Q_1)\\ &\qquad\cong M_1^s=\mathbf{A}^{1\times(n_1+m)}/\mathbf{A}^{1\times n_1}(\delta\,\mathrm{id}_{n_1}-A_1,-B_1)\end{aligned}\tag{83}$$

is an isomorphism with inverse

$$\left(\circ\left(\begin{smallmatrix}F_1 & G_1\\ 0 & \mathrm{id}_m\end{smallmatrix}\right)\right)_{\mathrm{ind}}=\left(\circ\left(\begin{smallmatrix}C_1 & D_1\\ 0 & \mathrm{id}_m\end{smallmatrix}\right)\right)_{\mathrm{ind}}^{-1}\tag{84}$$

The isomorphism (83) induces the isomorphism $M_{1,S}\cong M_{1,S}^s$, in particular $M_{1,S}$ is $\mathbf{A}_S$-free if and only if $M_{1,S}^s$ is or, equivalently, $\mathcal{B}_1$ is stabilizable if and only if $\mathcal{B}_1^s$ is. Notice that this can be checked constructively
Now assume this and let $\mathcal{B}_2=\mathcal{B}(U_2)$ with $U_2=\mathbf{A}^{1\times m}(-Q_2^{\mathbf{A}},P_2^{\mathbf{A}})$ be *one* stabilizing compensator according to Thm. 3.14 where indeed *all* stabilizing compensators are constructed. Let

$$\begin{aligned}&y:=\left(\begin{smallmatrix}y_1\\ y_2\end{smallmatrix}\right)\in W(\tau)^{p+m},\ P:=\begin{pmatrix}P_1 & -Q_1\\ -Q_2^{\mathbf{A}} & P_1^{\mathbf{A}}\end{pmatrix}\Longrightarrow\\ &\mathcal{B}^0(\tau):=\left\{y\in W(\tau)^{p+m};\ P\circ y=0\right\}\end{aligned}\tag{85}$$

is the autonomous part of the interconnection of $\mathcal{B}_1$ and $\mathcal{B}_2$ and thus stable. The isomorphims (82) and (84) induce the behavior isomorphisms, for sufficiently large τ,

$$\begin{aligned}&\mathcal{B}^{s,0}(\tau) := \left\{ \left(\begin{smallmatrix} x_1 \\ y_2 \end{smallmatrix}\right) \in W(\tau)^{n_1+p+m};\ \begin{cases} x_1' = A_1x_1 + B_1y_2, \\ (P_2^{\mathbf{A}} - Q_2^{\mathbf{A}}D_1) \circ y_2 = Q_2^{\mathbf{A}}C_1 \circ x_1 \end{cases} \right\} \\ &\qquad \cong \mathcal{B}^0(\tau) = \left\{ y \in W(\tau)^{p+m};\ P \circ y = 0 \right\},\ \left(\begin{smallmatrix} x_1 \\ y_2 \end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right), \\ &\qquad \text{where } y_1 = C_1x_1 + D_1 \circ y_2,\ x_1 = F_1 \circ y_1 + G_1 \circ y_2. \end{aligned} \tag{86}$$

Since $\mathcal{B}^0$ is stable so is the isomorphic autonomous behavior $\mathcal{B}^{s,0}$ and in particular its trajectories $\left(\begin{smallmatrix} x_1 \\ y \end{smallmatrix}\right)$ decrease exponentially with suitable decay factors $\exp(-\alpha t^\mu)$ where $\alpha, \mu > 0$. The equations of $\mathcal{B}^{s,0}$ thus furnish the stabilization of state space equations $x_1' = A_1x_1 + B_1u_1,\ y_1 = C_1x_1 + D_1 \circ u_1$ by *dynamic output feedback*.
In [1, Sect. (4.1), (4.2)] the authors consider *dynamic state feedback* with different assumptions for the matrices A_1, B_1. For the preceding data this signifies

$$x_1 = y_1,\ C_1 = F_1 = \mathrm{id}_p,\ D_1 = G_1 = 0. \tag{87}$$

The stabilizing equations of $\mathcal{B}^{s,0}(\tau)$ simplify to

$$x_1' = A_1x_1 + B_1y_2,\ P_2^{\mathbf{A}} \circ y_2 = Q_2^{\mathbf{A}} \circ x_1. \tag{88}$$

The authors of [1] assume that $\mathcal{B}_2$ can be described by state space equations

$$x_2' = A_2x_2 + B_2u_2,\ y_2 = C_2x_2 + D_2u_2,\ A_2, B_2, C_2, D_2 \in \mathbf{K}^{\bullet\times\bullet}. \tag{89}$$

Then the stabilizing equations (88) obtain the form [1, (4.2)]

$$x_1' = (A_1 + B_1D_2)x_1 + B_1C_2x_2,\ x_2' = A_2x_2 + B_2x_1. \tag{90}$$

4 Stabilizing compensators for three control goals

In this section we describe necessary and sufficient conditions for the existence of stabilizing compensators for three standard control tasks, viz. *tracking, disturbance rejection and model matching*. If such compensators exist we construct and parametrize all of them. We do this by generalizing the LTI methods of [4] to the LTV situation. It is remarkable that BIBO stability of stable LTV behaviors (cf. [21, Ch. 12]) does neither hold in the general situation of this paper nor is it needed anywhere.

4.1 Tracking and disturbance rejection

We consider a stabilizable behavior $\mathcal{B}(U_1)$, its parametrized stabilizing compensators as in Thm. 3.14 and the corresponding interconnected behavior $\mathcal{B}$ from Cor. and Def. 3.9, especially

$$\begin{aligned} U_1 = \mathbf{A}^{1\times p}(P_1, -Q_1),\ U_2 = \mathbf{A}^{1\times m}(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}}), \\ U_{2,S} = \mathbf{A}_S^{1\times m}(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}}) = \mathbf{A}_S^{1\times m}(-Q_2, P_2). \end{aligned} \tag{91}$$

In addition we consider an autonomous system

$$\mathcal{B}_3 = \mathcal{B}(U_3) \subseteq \mathcal{W}^p,\ U_3 = \mathbf{A}^{1\times p}R_3,\ R_3 \in \mathbf{A}^{p\times p},\ \mathrm{rank}(R_3) = p. \tag{92}$$

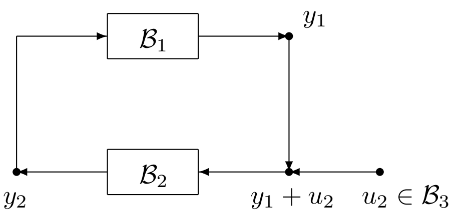

Figure 2: Tracking resp. disturbance rejection interconnection.

The system $\mathcal{B}_3$ is called a *signal generator*. Its trajectories are called *reference signals* and are fed into $\mathcal{B}$ as u_2 whereas we choose $u_1 = 0$, hence $u = \binom{u_2}{0}$ with $R_3 \circ u_2 = 0$ is the input of the feedback system. Figure 2 shows the block diagram of this interconnection. The error signal is $e_2 := y_1 + u_2$. The $+$-sign in e_2 comes from the convention that signals are always added at nodes of two incoming arrows.

Definition 4.1. The stabilizing compensator $\mathcal{B}_2$ is said to be a *tracking compensator* for $\mathcal{B}_3$ if the signals $e_2 := y_1 + u_2$ are w.e.s. for all $u_2 \in \mathcal{B}_3$. Then one also says that the signal y_1 tracks the reference signal $-u_2$.

The equations of the block diagram are

$$P_1 \circ y_1 = Q_1 \circ y_2,\ P_2^{\mathbf{A}} \circ y_2 = Q_2^{\mathbf{A}} \circ (y_1 + u_2),\ R_3 \circ u_2 = 0 \tag{93}$$

The corresponding behavior and modules are

$$\begin{aligned}
&\mathcal{B}_4 := \mathcal{B}(U_4),\ U_4 = \mathbf{A}^{1\times(p+m+p)} R_4^{\mathbf{A}} \subseteq \mathbf{A}^{1\times(p+m+p)},\ R_4^{\mathbf{A}} := \begin{pmatrix} P_1 & -Q_1 & 0 \\ -Q_2^{\mathbf{A}} & P_2^{\mathbf{A}} & -Q_2^{\mathbf{A}} \\ 0 & 0 & R_3 \end{pmatrix} \\
&U_4 = \mathbf{A}^{1\times p}(P_1, -Q_1, 0) + \mathbf{A}^{1\times m}(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}}, -Q_2^{\mathbf{A}}) + \mathbf{A}^{1\times p}(0, 0, R_3) \\
&U_{4,S} = \mathbf{A}_S^{1\times p}(P_1, -Q_1, 0) + \mathbf{A}_S^{1\times m}(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}}, -Q_2^{\mathbf{A}}) + \mathbf{A}_S^{1\times p}(0, 0, R_3) \\
&\quad = \mathbf{A}_S^{1\times p}(P_1, -Q_1, 0) + \mathbf{A}_S^{1\times m}(-Q_2, P_2, -Q_2) + \mathbf{A}_S^{1\times p}(0, 0, R_3) \\
&\quad = \mathbf{A}_S^{1\times(p+m+p)} R_4,\ R_4 := \begin{pmatrix} P_1 & -Q_1 & 0 \\ -Q_2 & P_2 & -Q_2 \\ 0 & 0 & R_3 \end{pmatrix}.
\end{aligned} \tag{94}$$

For the last equation in (94) we used (91) that implies

$$\begin{aligned}
&(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}}) = Y(-Q_2, P_2),\ Y \in \mathrm{Gl}_m(\mathbf{A}_S) \\
&\Longrightarrow (-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}}, -Q_2^{\mathbf{A}}) = Y(-Q_2, P_2, -Q_2) \\
&\Longrightarrow \mathbf{A}_S^{1\times m}(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}}, -Q_2^{\mathbf{A}}) = \mathbf{A}_S^{1\times m}(-Q_2, P_2, -Q_2).
\end{aligned} \tag{95}$$

Moreover we get

$$\begin{aligned}
&\mathrm{rank}(R_4^{\mathbf{A}}) = \mathrm{rank}(R_4) = \mathrm{rank}\begin{pmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{pmatrix} + \mathrm{rank}(R_3) = p + m + p \Longrightarrow \\
&M_4 := \mathbf{A}^{1\times(p+m+p)}/U_4 \text{ and } M_{4,S} := \mathbf{A}^{1\times(p+m+p)}/U_{4,S}
\end{aligned} \tag{96}$$

are torsion modules and $\mathcal{B}_4$ is thus autonomous.
The *error behavior* and its equation modules and system modules are

$$\begin{aligned}
&\mathcal{B}_{\mathrm{err}} := (\mathrm{id}_p, 0, \mathrm{id}_p) \circ \mathcal{B}_4 = \mathcal{B}(U_{\mathrm{err}}) \subseteq \mathcal{W}^p, \\
&U_{\mathrm{err}} := (\circ(\mathrm{id}_p, 0, \mathrm{id}_p))^{-1}(U_4) \subseteq \mathbf{A}^{1\times p},\ M_{\mathrm{err}} := \mathbf{A}^{1\times p}/U_{\mathrm{err}} \underset{\text{ident.}}{\subseteq} M_4, \\
&U_{\mathrm{err},S} := (\circ(\mathrm{id}_p, 0, \mathrm{id}_p))^{-1}(U_{4,S}) \subseteq \mathbf{A}_S^{1\times p},\ M_{\mathrm{err},S} := \mathbf{A}_S^{1\times p}/U_{\mathrm{err},S}.
\end{aligned} \tag{97}$$

The second line in (97) follows from the exactness of the behavior functor $\mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U)$. As a submodule of M_4 the module M_{err} is torsion and therefore $\mathcal{B}_{\mathrm{err}}$ is autonomous. By definition the behavior $\mathcal{B}_2$ is a tracking behavior for $\mathcal{B}_3$ if and only $\mathcal{B}_{\mathrm{err}}$ is stable or, equivalently, $M_{\mathrm{err},S} = 0$ or $U_{\mathrm{err},S} = \mathbf{A}_S^{1\times p}$.

Remark 4.2. For the computation of $U_{\mathrm{err},S} = (\circ(\mathrm{id}_p, 0, \mathrm{id}_p))^{-1}(U_{4,S})$ we apply the following standard algorithm: Assume a noetherian ring $\mathbf{B}$ and matrices and modules

$$\begin{gathered}
R_1 \in \mathbf{B}^{k_1\times \ell_1},\ P \in \mathbf{B}^{\ell_2\times \ell_1}, \\
U_1 := \mathbf{B}^{1\times k_1}R_1 \subseteq \mathbf{B}^{1\times \ell_1},\ U_2 := (\circ P)^{-1}(U_1) \subseteq \mathbf{B}^{1\times \ell_2}.
\end{gathered} \tag{98}$$

Let $(X, R_2) \in \mathbf{B}^{k_2\times(k_1+\ell_2)}$ be a universal left annihilator of $\left(\begin{smallmatrix} R_1 \\ -P \end{smallmatrix}\right)$. This signifies that

$$\mathbf{B}^{1\times k_2} \xrightarrow{\circ(X,R_2)} \mathbf{B}^{1\times(k_1+\ell_2)} \xrightarrow{\circ\left(\begin{smallmatrix} R_1 \\ -P \end{smallmatrix}\right)} \mathbf{B}^{1\times \ell_2} \tag{99}$$

is exact or that X and R_2 are universal matrices with $XR_1 = R_2P$. Then

$$U_2 = \mathbf{B}^{1\times k_2}R_2. \tag{100}$$

For the computation of $U_{\mathrm{err},S} = (\circ(\mathrm{id}_p, 0, \mathrm{id}_p))^{-1}(U_{4,S})$ let $(X_1, X_2, X_3, R_{\mathrm{err}}) \in \mathbf{A}_S^{k_{\mathrm{err}}\times(p+m+p+p)}$ be a universal left annihilator of

$$\begin{gathered}
\begin{pmatrix} R_4 \\ -(\mathrm{id}_p,0,\mathrm{id}_p) \end{pmatrix} = \begin{pmatrix} P_1 & -Q_1 & 0 \\ -Q_2 & P_2 & -Q_2 \\ 0 & 0 & R_3 \\ -\mathrm{id}_p & 0 & -\mathrm{id}_p \end{pmatrix}, \text{hence} \\
(X_1, X_2)\begin{pmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{pmatrix} = (R_{\mathrm{err}}, 0),\ -X_2Q_2 + X_3R_3 - R_{\mathrm{err}} = 0.
\end{gathered} \tag{101}$$

By means of (54) the preceding equations are equivalent to

$$\begin{gathered}
(X_1, X_2) = (R_{\mathrm{err}}, 0)\begin{pmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{pmatrix}^{-1} = (R_{\mathrm{err}}, 0)\left(\begin{smallmatrix} D_2 & N_1 \\ N_2 & D_1 \end{smallmatrix}\right) = (R_{\mathrm{err}}D_2, R_{\mathrm{err}}N_1) \\
\text{and } R_{\mathrm{err}}(\mathrm{id}_p + N_1Q_2) - X_3R_3 = 0.
\end{gathered} \tag{102}$$

Moreover

$$\begin{aligned}
&\left(\begin{smallmatrix} D_2 & N_1 \\ N_2 & D_1 \end{smallmatrix}\right)\begin{pmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_p & o \\ 0 & \mathrm{id}_m \end{pmatrix} \\
&\Longrightarrow \mathrm{id}_p = D_2P_1 - N_1Q_2 \Longrightarrow D_2P_1 = \mathrm{id}_p + N_1Q_2 \underset{(54)}{=} \mathrm{id}_p + H_{y_1,u_2} \\
&\Longrightarrow (R_{\mathrm{err}}, X_3)\left(\begin{smallmatrix} D_2P_1 \\ -R_3 \end{smallmatrix}\right) = R_{\mathrm{err}}D_2P_1 - X_3R_3 = 0.
\end{aligned} \tag{103}$$

Since $(X_1, \cdots, R_{\mathrm{err}})$ is a *universal* left annihilator of $\begin{pmatrix} R_4 \\ (-\mathrm{id}_p,0,-\mathrm{id}_p) \end{pmatrix}$ the matrix $(R_{\mathrm{err}}, X_3) \in \mathbf{A}_S^{k_{\mathrm{err}}\times(p+p)}$ is a *universal* left annihilator of $\left(\begin{smallmatrix} D_2P_1 \\ -R_3 \end{smallmatrix}\right) \in \mathbf{A}_S^{(p+p)\times p}$;

this follows directly from (102). Since $\operatorname{rank}(R_3) = p$ and thus also $\operatorname{rank}\left(\begin{smallmatrix} D_2P_1 \\ -R_3 \end{smallmatrix}\right) = p$ the kernel

$$\mathbf{A}_S^{1\times k_{\mathrm{err}}}(R_{\mathrm{err}}, X_3) = \ker\left(\circ \left(\begin{smallmatrix} D_2P_1 \\ -R_3 \end{smallmatrix}\right) :\ \mathbf{A}_S^{1\times(p+p)} \to \mathbf{A}_S^{1\times p}\right) \tag{104}$$

is free of dimension $p = (p+p) - p$ and therefore we assume $k_{\mathrm{err}} = p$ w.l.o.g. By construction the matrix R_{err} generates $U_{\mathrm{err},S}$, hence

$$U_{\mathrm{err},S} = \mathbf{A}_S^{1\times p} R_{\mathrm{err}},\ M_{\mathrm{err},S} = \mathbf{A}_S^{1\times p}/\mathbf{A}_S^{1\times p} R_{\mathrm{err}}. \tag{105}$$

Corollary 4.3. *For the stabilizable behavior $\mathcal{B}_1$, its stabilizing compensator $\mathcal{B}_2$, the signal generator $\mathcal{B}_3$, the interconnected behavior $\mathcal{B}$ from figure 2 and the derived data from above the component y_1 of $\mathcal{B}_1$ tracks the reference signals $-u_2 \in \mathcal{B}_3$ if and only if $R_{\mathrm{err}} \in \mathrm{Gl}_p(\mathbf{A}_S)$.*

Theorem 4.4. *Assume a stabilizable behavior $\mathcal{B}_1 \subset \mathcal{W}^{p+m}$ and a signal generator $\mathcal{B}_3 \subset \mathcal{W}^p$. Also consider the parametrization from Lemma 3.12 and the construction of all stabilizing compensators $\mathcal{B}_2$ of $\mathcal{B}_1$ in Thm. 3.14 and the associated data.*
(i) There is a tracking stabilizing compensator for reference signals in $\mathcal{B}_3$ if and only if the following inhomogeneous linear equation has a solution (X, Y) with entries in $\mathbf{A}_S$:

$$D_2^0 P_1 = N_1 X P_1 + Y R_3,\ X \in \mathbf{A}_S^{m\times p},\ Y \in \mathbf{A}_S^{p\times p}. \tag{106}$$

(ii) If (106) *is solvable then* all *tracking stabilizing compensators $\mathcal{B}(U_2)$ are obtained by the following steps:*

1. *Solve equation* (106)*. Only the component X is used for the further construction.*
2. *According to* (53) *define*

$$\begin{gathered}(-Q_2, P_2) = (-Q_2^0, P_2^0) + X(P_1, -Q_1),\ \left(\begin{smallmatrix} D_2 \\ N_2 \end{smallmatrix}\right) = \begin{pmatrix} D_2^0 \\ N_2^0 \end{pmatrix} - \left(\begin{smallmatrix} N_1 \\ D_1 \end{smallmatrix}\right) X,\ \textit{hence} \\ U_{1,S} \oplus \mathbf{A}_S^{1\times m}(-Q_2, P_2) = \mathbf{A}_S^{1\times(p+m)},\ \begin{pmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{pmatrix} \in \mathrm{Gl}_{p+m}(\mathbf{A}_S).\end{gathered} \tag{107}$$

3. *Construct a stabilizing compensator $\mathcal{B}(U_2)$ from $(-Q_2, P_2)$ according to Thm. 3.14.*

The equations imply

$$\mathrm{id}_p + H_{y_1,u_2} = D_2 P_1 = Y R_3. \tag{108}$$

Proof. 1. *Necessity of* (106): Assume that $\mathcal{B}_2$ is such a compensator so that the derived data from (93)- (105) are defined and $R_{\mathrm{err}} \in \mathrm{Gl}_p(\mathbf{A}_S)$. From (52) and (103) we get

$$\begin{gathered}D_2 = D_2^0 - N_1 X \text{ and } R_{\mathrm{err}} D_2 P_1 = X_3 R_3 \text{ with } X, X_3 \in \mathbf{A}_S^{\bullet\times\bullet} \Longrightarrow \\ D_2 P_1 = R_{\mathrm{err}}^{-1} X_3 R_3 \Longrightarrow D_2^0 P_1 = N_1 X P_1 + Y R_3,\ Y := R_{\mathrm{err}}^{-1} X_3.\end{gathered} \tag{109}$$

2. *Sufficiency of* (104): With X from (106) we define the matrices from (107). According to Lemma 3.12 and Thm. 3.14 the matrix $(-Q_2, P_2)$ gives rise to several stabilizing compensators $\mathcal{B}(U_2)$ with $U_{2,S} = \mathbf{A}_S^{1\times m}(-Q_2, P_2)$. The compensator $\mathcal{B}_2$ gives rise to the data (97)-(105), in particular (R_{err}, X_3) is a *universal* left annihilator of $\left(\begin{smallmatrix} D_2P_1 \\ -R_3 \end{smallmatrix}\right)$. Equation (106) furnishes $D_2 P_1 = Y R_3$ and therefore $(\mathrm{id}_p, Y) \in \mathbf{A}_S^{p\times(p+p)}$ is *one* left annihilator of $\left(\begin{smallmatrix} D_2P_1 \\ -R_3 \end{smallmatrix}\right)$. This implies that there is a matrix $Z \in \mathbf{A}_S^{p\times p}$ such that $Z(R_{\mathrm{err}}, X_3) = (\mathrm{id}_p, Y)$ and especially $Z R_{\mathrm{err}} = \mathrm{id}_p$ and $R_{\mathrm{err}} \in \mathrm{Gl}_p(\mathbf{A}_S)$. According to Cor. 4.3 this signifies that $\mathcal{B}(U_2)$ is a tracking compensator for $\mathcal{B}_3$. It is clear that this construction furnishes all tracking stabilizing compensators. $\square$

Example 4.5. We consider the data from Example 3.19, especially

$$P_1 = (1+a)s_1,\ D_2^0 = s_1^{-1},\ N_1 = -s_1^{-1}. \tag{110}$$

Let $\mathcal{B}_3 = \mathcal{B}\left(\mathbf{A}R_3\right),\ 0 \neq R_3 \in \mathbf{A}$, be an *arbitrary* autonomous signal generator. With $(x,\eta) \in \mathbf{A}_S^{1\times 2}$ instead of (X,Y) in equation (106) the latter gets the form

$$\begin{aligned} D_2^0 P_1 = N_1 X P_1 + Y R_3 \iff & s_1^{-1}(1+a)s_1 = -s_1^{-1}x(1+a)s_1 + \eta R_3 \\ \iff & s_1^{-1}(1+x)(1+a)s_1 = \eta R_3. \end{aligned} \tag{111}$$

The intersection

$$\mathbf{A}_S(1+a)s_1 \bigcap \mathbf{A}_S R_3 = \mathbf{A}_S \sigma \tag{112}$$

is a nonzero principal ideal. Hence there are nonzero $\xi, \eta \in \mathbf{A}_S$ such that

$$\begin{aligned} &\sigma = \xi(1+a)s_1 = \eta R_3 \iff s_1^{-1}(1+x)(1+a)s_1 = \eta R_3 \text{ with } x := s_1\xi - 1 \\ &\iff D_2^0 P_1 = N_1 x P_1 + \eta R_3. \end{aligned} \tag{113}$$

The stabilizing compensators from Example 3.19 constructed with this x or, more generally, with $x = s_1\zeta\xi - 1,\ \zeta \in \mathbf{A}_S$, track the reference signals from $\mathcal{B}_3$.

Disturbance rejection deals with the same stabilizable behavior $\mathcal{B}_1$, signal generator $\mathcal{B}_3$ and interconnected behavior $\mathcal{B}_4$ as in the tracking case, but the control goal and error behavior are different.

Definition and Corollary 4.6. The error behavior for disturbance rejection is $\mathcal{B}_{\mathrm{err}} := (\mathrm{id}_p, 0, 0) \circ \mathcal{B}_4$. The compensator $\mathcal{B}_2$ is said to *reject disturbances from* $\mathcal{B}_3$ if $\mathcal{B}_{\mathrm{err}}$ is stable. This signifies that inputs $u_1 = 0$ and $u_2 \in \mathcal{B}_3$ generate stable output components y_1.

We state the corresponding theorem without its proof that is fully analogous to the tracking case.

Theorem 4.7. (Disturbance rejection) *(i) The stabilizing compensator $\mathcal{B}(U_2)$ of $\mathcal{B}(U_1)$ rejects disturbances from $\mathcal{B}_3$ if and only if the following inhomogeneous linear equation*

$$N_1 Q_2^0 = N_1 X P_1 + Y R_3,\ X \in \mathbf{A}_S^{m\times p},\ Y \in \mathbf{A}_S^{p\times p}, \tag{114}$$

has a solution (X,Y) with entries in $\mathbf{A}_S$.
(ii) If (114) *is solvable* all *stabilizing compensators $\mathcal{B}_2$ of $\mathcal{B}_1$ for the rejection of signals $u_2 \in \mathcal{B}_3$ are obtained by the following construction steps:*

1. *Solve the equation* (114) *where only X is needed for the further construction.*
2. *Define $(-Q_2, P_2) := (-Q_2^0, P_2^0) + X(P_1, -Q_1)$ according to* (53). *Use Thm. 3.14 to construct the matrix*

$$\begin{aligned} &(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}}) \in \mathbf{A}^{m\times(p+m)} \text{ and } U_2 := \mathbf{A}^{1\times m}(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}}) \subseteq \mathbf{A}^{1\times(p+m)} \\ &\text{such that } U_{2,S} = \mathbf{A}_S^{1\times m}(-Q_2, P_2) \text{ and } U_{1,S} \oplus U_{2,S} = \mathbf{A}_S^{1\times(p+m)}. \end{aligned} \tag{115}$$

The behaviors $\mathcal{B}(U_2)$ are the desired compensators that are essentially parametrized by the components $X \in \mathbf{A}^{m\times p}$ of the solutions of (114)*. The equations imply*

$$H_{y_1,u_2} \underset{(54)}{=} N_1 Q_2 = Y R_3. \tag{116}$$

4.2 Model matching

We consider a stabilizable behavior $\mathcal{B}_1 = \mathcal{B}(U_1) \subseteq \mathcal{W}^{p+m}$ and its parametrized family of stabilizing compensators $\mathcal{B}(U_2)$ with $U_{1,S} \oplus U_{2,S} = \mathbf{A}_S^{1\times(p+m)}$ as described in Sections 3.3 and 3.4. In addition we consider a model input/output behavior

$$\mathcal{B}(U_m),\ U_m = \mathbf{A}^{1\times p}(P_m, -Q_m),\ (P_m, -Q_m) \in \mathbf{A}^{p\times(p+p)},\ H_m = P_m^{-1}Q_m, \tag{117}$$

with transfer matrix H_m. The IO property signifies that $P_m \in \mathrm{Gl}_p(\mathbf{Q})$. The interconnection of these behaviors is realized by taking $u_2 \in W(\tau)^p$ as input both for $\mathcal{B}_2$ and for $\mathcal{B}_m$ where we choose $u_1 = 0$ as in the tracking case. This interconnection is thus described by the trajectories $(y_1, y_2, u_2, y_m)^\top$ and the equations

$$P_1 \circ y_1 = Q_1 \circ y_2,\ P_2^{\mathbf{A}} \circ y_2 = Q_2^{\mathbf{A}} \circ (u_2 + y_1),\ P_m \circ y_m = Q_m \circ u_2 \tag{118}$$

that give rise to the behavior

$$\begin{gathered}\mathcal{B}_4 = \mathcal{B}(U_4),\ U_4 = \mathbf{A}^{1\times(p+m+p+p)}R_4,\ R_4^{\mathbf{A}} := \begin{pmatrix} P_1 & -Q_1 & 0 & 0 \\ -Q_2^{\mathbf{A}} & P_2^{\mathbf{A}} & -Q_2^{\mathbf{A}} & 0 \\ 0 & 0 & -Q_m & P_m \end{pmatrix},\ \text{hence} \\ U_{4,S} = \mathbf{A}^{1\times(p+m+p+p)}R_4,\ R_4 := \begin{pmatrix} P_1 & -Q_1 & 0 & 0 \\ -Q_2 & P_2 & -Q_2 & 0 \\ 0 & 0 & -Q_m & P_m \end{pmatrix}\ \text{with} \\ \mathrm{rank}(R_4) = p+m+p\ \text{since}\ \begin{pmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{pmatrix} \in \mathrm{Gl}_{p+m}(\mathbf{Q})\ \text{and}\ P_m \in \mathrm{Gl}_p(\mathbf{Q}).\end{gathered} \tag{119}$$

The *error signal* is defined as $e = y_1 - y_m$ and the *error behavior* as $\mathcal{B}_{\mathrm{err}} = (\mathrm{id}_p, 0, 0, -\mathrm{id}_p) \circ \mathcal{B}_4$.

Definition 4.8. The stabilizing compensator is called *model matching* for $\mathcal{B}_m$ if $\mathcal{B}_{\mathrm{err}}$ is autonomous and stable. One also says that y_1 *matches the model* y_m *for all inputs* u_2.

The equation module of $\mathcal{B}_{\mathrm{err}}$ is $U_{\mathrm{err}} = ((\mathrm{id}_p, 0, 0, -\mathrm{id}_p)^{-1}(U_4) \subseteq \mathbf{A}^{1\times p}$ and $\mathcal{B}_2$ is model matching if and only $U_{\mathrm{err},S} = \mathbf{A}_S^{1\times p}$ where

$$U_{\mathrm{err},S} = ((\mathrm{id}_p, 0, 0, -\mathrm{id}_p))^{-1}(U_{4,S}) = \mathbf{A}_S^{1\times k_{\mathrm{err}}}R_{\mathrm{err}}. \tag{120}$$

Analogous computations to those for Cor. 4.3 furnish

$$\begin{gathered}U_{\mathrm{err},S} = \mathbf{A}^{1\times k_{\mathrm{err}}}R_{\mathrm{err}},\ R_{\mathrm{err}} = -X_3P_m\ \text{where} \\ \mathbf{A}^{1\times k_{\mathrm{err}}}X_3 = \ker\left(\circ(P_mN_1Q_2 - Q_m):\ \mathbf{A}_S^{1\times p} \to \mathbf{A}_S^{1\times p}\right).\end{gathered} \tag{121}$$

Corollary 4.9. *The stabilizing compensator $\mathcal{B}_2$ is model-matching for the IO behavior $\mathcal{B}_m$, i.e., $U_{\mathrm{err},S} = \mathbf{A}_S^{1\times p}$, if and only if $\mathcal{B}_m$ is stable and $H_m = N_1Q_2 \underset{(54)}{=} H_{y_1,u_2}$.*

Proof. $\Longrightarrow$: The equations $\mathbf{A}^{1\times k_{\mathrm{err}}}R_{\mathrm{err}} = \mathbf{A}_S^{1\times p}$ and $R_{\mathrm{err}} = -X_3P_m$ imply $\mathbf{A}^{1\times p}P_m = \mathbf{A}_S^{1\times p}$ and hence $P_m \in \mathrm{Gl}_p(\mathbf{A}_S)$ and $\mathrm{rank}(X_3) = p$. According to Lemma and Def. 3.6 $\mathcal{B}_m$ is stable and X_3 can be cancelled as left factor. But

$$X_3(P_mN_1Q_2 - Q_m) = 0 \Longrightarrow P_m(N_1Q_2 - H_m) = 0 \Longrightarrow H_m = N_1Q_2.$$

$\Longleftarrow$: If

$$\begin{gathered}H_m = N_1Q_2 \Longrightarrow P_m(N_1Q_2 - H_m) = P_mN_1Q_2 - Q_m = 0 \Longrightarrow \\ k_{\mathrm{err}} = p,\ X_3 = \mathrm{id}_p \Longrightarrow R_{\mathrm{err}} = -P_m \underset{P_m \in \mathrm{Gl}_p(\mathbf{A}_S)}{\Longrightarrow} R_{\mathrm{err}} \in \mathrm{Gl}_p(\mathbf{A}_S).\end{gathered}$$

□

Theorem 4.10. *Assume a stabilizable behavior* $\mathcal{B}_1 = \mathcal{B}(U_1)$ *where* $U_1 = \mathbf{A}^{1\times p}(P_1, -Q_1)$ *and* $(P_1, -Q_1) \in \mathbf{A}^{p\times(p+m)}$ *such that the exact sequences* (49) *and* (50) *exist. Also assume a stable model IO behavior* $\mathcal{B}_m = \mathcal{B}(U_m)$ *where* $U_m = \mathbf{A}^{1\times p}(P_m, -Q_m)$, $(P_m, -Q_m) \in \mathbf{A}^{p\times(p+p)}$ *and* $P_m \in \mathrm{Gl}_p(\mathbf{A}_S)$.
(i) Then $\mathcal{B}_1$ *admits a model-matching stabilizing compensator for* $\mathcal{B}_m$ *if and only the inhomogeneous linear equation*

$$N_1 Q_2^0 - H_m = N_1 X P_1,\ X \in \mathbf{A}_S^{m\times p}, \tag{122}$$

has a solution X *with entries in* $\mathbf{A}_S$.
(ii) If (122) *is solvable then* all *stabilizing compensators for matching the model* $\mathcal{B}_m$ *are obtained with the following steps:*

1. *Solve* (122) *and use* X *to define* $(-Q_2, P_2) = (-Q_2^0, P_2^0) + X(P_1, -Q_1)$ *according to* (53).
2. *Use Thm. 3.14 to construct a module* $U_2 = \mathbf{A}^{1\times m}(-Q_2^{\mathbf{A}}, P_2^{\mathbf{A}})$ *with* $U_{2,S} = \mathbf{A}_S^{1\times m}(-Q_2, P_2)$.

Then $\mathcal{B}(U_2)$ *is the desired compensator.*

Proof. (i)1. *Necessity of* (122): From Cor. 4.9 we get $H_m = N_1 Q_2$. From $(-Q_2, P_2) = (-Q_2^0, P_2^0) + X(P_1, -Q_1)$ we infer $Q_2 = Q_2^0 - XP_1$ and hence (122).
2. *Sufficiency of* (122): Perform the steps above. The equations (122) and $Q_2 = Q_2^0 - XP_1$ imply $N_1 Q_2 = H_m$. By Cor. 4.9 the latter equation signifies that $\mathcal{B}(U_2)$ is model matching.
(ii) is a simple consequence of (i).

□

5 Algorithms

Almost all steps of the preceding constructions can be carried out algorithmically. We describe the algorithms, but leave to experts in *Computer Algebra* [10], [20] to implement them. We do not discuss addition and multiplication in $\mathbf{K} \subset \mathbf{A} \subset \mathbf{Q}$. Since $\mathbf{K}$ contains arbitrary Laurent series it is clear that in practical examples one has to consider rational coefficient functions with coefficients in $\mathbb{Q}(i)$ or standard functions like $z^k \exp(z^{-1}),\ k \in \mathbb{Z}$. Euclidean division in $\mathbf{A}$ is possible. Also the theory of f.g. $\mathbf{A}$-modules is constructive. In particular, the freeness (=projectivity) and the torsion property of a f.g. $\mathbf{A}$-left module can be decided. The main theorem 2.7 of [6] describes a test for weak exponential stability of *most* differential operators and therefore a test for the inclusion $s \in S$. If $q = ba^{-1},\ 0 \neq a, b \in \mathbf{A}$, is a nonzero element of $\mathbf{Q}$ then

$$\mathbf{A}s := \{f \in \mathbf{A};\ fq \in \mathbf{A}\} = (\circ b)^{-1}(\mathbf{A}a) \text{ and } (q \in \mathbf{A}_S \iff s \in S) \tag{123}$$

can be computed, hence $q \in \mathbf{A}_S$ can be decided. A f.g. torsion module M is constructively isomorphic to $\mathbf{A}/\mathbf{A}F$ where $0 \neq F \in \mathbf{A}$. Since f.g. stable modules are closed under isomorphisms the module M is stable if and only $\mathbf{A}/\mathbf{A}F$ is stable and this is equivalent to $F \in S$ that can be decided (in most cases). The construction of compensators in Thms. 4.4, 4.7 and 4.10 depends on the solution of inhomogeneous linear equations

$$AXB = C,\ A, B, C, X \in \mathbf{A}_S^{\bullet\times\bullet}, \tag{124}$$

where A, B, C are given and a solution X is sought. These equations can always be solved. By multiplying A from the left and B from the right by common denominators in S we may and do assume wlog that A and B, but not necessarily C have entries in $\mathbf{A}$. Every matrix A with entries in $\mathbf{A}$ is equivalent to a diagonal matrix. One can construct invertible matrices U_1 and U_2 and a square diagonal matrix $D = \mathrm{diag}(a_1, \cdots, a_r)$ with $r = \mathrm{rank}(A)$ and nonzero a_i such that

$$U_1 A U_2 = \widehat{A} := \left(\begin{smallmatrix} D(a) & 0 \\ 0 & 0 \end{smallmatrix}\right),\ U_1, U_2 \in \mathrm{Gl}_\bullet(\mathbf{A}),\ D(a) = \mathrm{diag}(a_1, \cdots, a_r),\ 0 \neq a_i \in \mathbf{A}. \tag{125}$$

One can even choose $a_1 = \cdots = a_{r-1} = 1$. Consider the analogous decomposition for B, i.e.,

$$V_1 B V_2 = \widehat{B} := \left(\begin{smallmatrix} D(b) & 0 \\ 0 & 0 \end{smallmatrix}\right),\ U_1, U_2 \in \mathrm{Gl}_\bullet(\mathbf{A}),\ D = \mathrm{diag}(b_1, \cdots, b_s),\ 0 \neq b_j \in \mathbf{A}. \tag{126}$$

Then

$$\begin{gathered} AXB = C \iff \widehat{A}\widehat{X}\widehat{B} = \widehat{C} \iff \left(\begin{smallmatrix} D(a) & 0 \\ 0 & 0 \end{smallmatrix}\right) \widehat{X} \left(\begin{smallmatrix} D(b) & 0 \\ 0 & 0 \end{smallmatrix}\right) = \widehat{C} \\ \text{where } \widehat{X} := U_2^{-1} X V_1^{-1},\ \widehat{C} := U_1 C V_2. \end{gathered} \tag{127}$$

The last equation implies

$$\widehat{C}_{ij} = \begin{cases} a_i \widehat{X}_{ij} b_j & \text{if } i \leq r, j \leq s \\ 0 & \text{otherwise} \end{cases} \text{ and } \forall i \leq r \forall j \leq s:\ \widehat{X}_{ij} = a_i^{-1} \widehat{C}_{ij} b_j^{-1}. \tag{128}$$

Lemma 5.1. *(i) The linear system $AXB = C$ is solvable in $\mathbf{A}_S^{\bullet\times\bullet}$ if and only if*

$$\begin{cases} \forall i > r:\ \widehat{C}_{i-} = 0 \textit{ and } \forall j > s:\ \widehat{C}_{-j} = 0 \\ \forall i \leq r \forall j \leq s:\ a_i^{-1} \widehat{C}_{ij} b_j^{-1} \in \mathbf{A}_S \end{cases}. \tag{129}$$

(ii) If the system is solvable then

$$\begin{gathered} \widehat{L} = \left\{ \widehat{X};\ \widehat{A}\widehat{X}\widehat{B} = \widehat{C} \right\} = \left\{ \widehat{X} \in \mathbf{A}_S^{\bullet\times\bullet};\ \forall i \leq r \forall j \leq s;\ \widehat{X}_{ij} = a_i^{-1} \widehat{C}_{ij} b_j^{-1} \right\} \\ L := \{X;\ AXB = C\} = U_2 \widehat{L} V_1 = \left\{ U_2 \widehat{X} V_1;\ \widehat{X} \in \widehat{L} \right\}. \end{gathered} \tag{130}$$

(iii) For the homogeneous systems this implies

$$\begin{gathered} \widehat{L}^0 := \left\{ \widehat{X};\ \widehat{A}\widehat{X}\widehat{B} = 0 \right\} = \left\{ \widehat{X} \in \mathbf{A}_S^{\bullet\times\bullet};\ \forall i \leq r \forall j \leq s:\ \widehat{X}_{ij} = 0 \right\}, \\ L^0 := \{X;\ AXB = 0\} = U_2 \widehat{L}^0 V_1, \textit{ hence also } L = X^1 + L^0,\ X^1 \in L. \end{gathered} \tag{131}$$

The conditions in (129) can be checked and therefore the linear systems $AXB = C$ with matrices with entries in $\mathbf{A}_S$ can be solved.

6 Conclusion

In this paper we present a new *constructive* theory of stabilization and control for LTV differential systems whose computer implementation is left to younger colleagues. Examples 3.19 and 4.5 computed by hand illustrate the theory. The paper extends the LTI theories from, for instance, [8], [9, §9.6, pp. 488-506], [22, §5.7, pp. 151-155], to the

LTV situation. Most papers (google search) on this subject in the literature deal with Kalman's state space systems (cf. [21, pp. 280-284]) of the form

$$x'(t) = A(t)x(t) + B(t)u(t),\ y(t) = C(t)x(t) + D(t)u(t), \tag{132}$$

whereas we consider general *implicit* linear differential systems *of arbitrary order*. Arbitrary analytic or even smooth coefficient functions are unsuitable for such systems. It has turned out that smooth functions derived from the field of locally convergent Puiseux-series have all necessary properties. These coefficient functions approximate large classes of continuous functions and this enables the application of the present theory to even more general systems. The important problem of robustness in this context has, however, not yet been solved. We also employ a new kind of behaviors and their weak exponential stability that were introduced in the recent paper [7]. The results of the latter paper are essential for the proofs of the present one. In Remark 3.20 we explain differences and similarities to [5, Ch. 11, pp. 545-554] that, to our knowledge, is the only work where tracking and disturbance rejection of LTV systems have been discussed in the same generality.
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