Exponentially stable linear time-varying discrete behaviors

SIAM J. Control Optim. 53(2015), 2725-2761, DOI: 10.1137/140988498

H. Bourlès [*], B. Marinescu [†], U. Oberst[‡]

September 13, 2015

Abstract

We study implicit systems of linear time-varying (LTV) difference equations with rational coefficients of arbitrary order and their solution spaces, called discrete LTV-behaviors. The signals are sequences, i.e. functions from the discrete time set of natural numbers into the complex numbers. The difference field of rational functions with complex coefficients gives rise to a noncommutative skew-polynomial algebra of difference operators that act on sequences via left shift. For this paper it is decisive that the ring of operators is a principal ideal domain and that nonzero rational functions have only finitely many poles and zeros and grow at most polynomially. Due to the poles a new definition of behaviors is required. For the latter we derive the important categorical duality between finitely generated left modules over the ring of operators and behaviors. The duality theorem implies the usual consequences for Willems' elimination, the fundamental principle, input/output decompositions and controllability. The generalization to autonomous discrete LTV-behaviors of the standard definition of uniformly exponentially stable (u.e.s.) state space systems is unsuitable since u.e.s. is not preserved by behavior isomorphisms. We define exponentially stable (e.s.) discrete LTV-behaviors by a new analytic condition on its trajectories. These e.s. behaviors are autonomous and asymptotically stable. Our principal result states that e.s. behaviors form a Serre category, i.e., are closed under isomorphisms, subbehaviors, factor behaviors and extensions or, equivalently, that the series connection of two e.s. input/output behaviors is e.s. if and only the two blocks are. As corollaries we conclude various stability and instability results for autonomous behaviors. There is presently no algebraic characterization and test for e.s. of behaviors, but otherwise the results are constructive.

AMS-classification: 93D20, 93C55, 93C05
Key-words: exponential stability, discrete behavior, time-varying, duality

[*]SATIE, ENS Cachan/CNAM, 61 Avenue President Wilson, F-94230, Cachan, France. email: henri.bourles@satie.ens-cachan.fr

[†]École Centrale de Nantes, B.P. 92101, 1, rue de la Noë, FR-44321 Nantes Cedex 3, France. email: bogdan.marinescu@irccyn.ec-nantes.fr

[‡]Institut für Mathematik, Universität Innsbruck, Technikerstrasse 13, A-6020 Innsbruck, Austria. email: ulrich.oberst@uibk.ac.at

1 Introduction

Stability theory for linear time-varying (LTV) discrete systems has been mainly developed for the discrete time set $\mathbb{N} = \{\text{natural numbers}\}$ and Kalman's state space equations

$$\begin{gathered} x(t+1) = A(t)x(t) + B(t)u(t),\ y(t) = C(t)x(t) + D(t)u(t),\ t \in \mathbb{N}, \\ x(t) \in \mathbb{C}^n,\ u(t) \in \mathbb{C}^m,\ y(t) \in \mathbb{C}^p \text{ with matrices } A(t), B(t), C(t), D(t) \in \mathbb{C}^{\bullet\times\bullet} \end{gathered} \tag{1}$$

of suitable sizes. The complex field $\mathbb{C}$ is often replaced by the real field $\mathbb{R}$. We write $\mathbb{C}^n := \mathbb{C}^{n\times 1}$ resp. $\mathbb{C}^{1\times n}$ for the space of column- resp. row-vectors. The vectors $x(t)$, $u(t)$ resp. $y(t)$ are the *state, input* resp. *output* at the time instant $t \in \mathbb{N}$. If an initial time t_0, an initial state $x(t_0)$ and an input $(u(t))_{t\geq t_0}$ are chosen all $x(t)$ and $y(t)$ for $t \geq t_0$ can be computed [25, (21) on p. 392] via

$$\begin{gathered} x(t) = \Phi(t,t_0)x(t_0) + \sum_{i=t_0}^{t-1} \Phi(t,i+1)B(i)u(i),\ y(t) = C(t)x(t) + D(t)u(t), \\ \text{with } \Phi(t,t_0) = A(t-1)\cdots A(t_0). \end{gathered} \tag{2}$$

For excellent surveys of the stability theory of equations (1) and its history we refer to the books [25, Chs. 22-24, pp. 423-461] and [15, Ch. 3, pp. 193-368], cf. also [17]. In the present paper we treat higher order and implicit linear systems of difference equations with rational coefficients

$$\sum_{j=0}^{d} R_j(t)w(t+j) = u(t),\ t \geq n_0,\ R_j \in \mathbb{C}(t)^{p\times\ell},\ w(t) \in \mathbb{C}^\ell,\ u(t) \in \mathbb{C}^p, \tag{3}$$

that may be homogeneous ($u = 0$) or inhomogeneous ($u \neq 0$). Here the entries of the R_j belong to the field $\mathbb{C}(t)$ of rational functions in the indeterminate t and it is assumed that no $t \geq n_0$ is a pole of any R_j so that $R_j(t) \in \mathbb{C}^{p\times\ell}$ for all $t \geq n_0$. For all $n \geq n_0$ we identify $R_j = (R_j(t))_{t\geq n}$ and therefore use the same letter for the indeterminate and the time instants. For $n \geq n_0$ the interval $n + \mathbb{N} = [n,\infty) := \{t \in \mathbb{N};\ t \geq n\}$ is the time-set with initial time n and the space of sequences

$$\mathbb{C}^{n+\mathbb{N}} = \{a = (a(n), a(n+1), \cdots);\ \forall t \geq n:\ a(t) \in \mathbb{C}\} \tag{4}$$

is interpreted as the space of signals starting at time n. We identify

$$\begin{gathered} (\mathbb{C}^{p\times\ell})^{n+\mathbb{N}} = (\mathbb{C}^{n+\mathbb{N}})^{p\times\ell} \ni X = (X_{ij})_{i\leq p, j\leq\ell} = (X(n), X(n+1), \cdots), \\ X_{ij} \in \mathbb{C}^{n+\mathbb{N}},\ X(t) \in \mathbb{C}^{p\times\ell},\ X_{ij}(t) = X(t)_{ij}. \end{gathered} \tag{5}$$

The homogeneous equations (3) give rise to the *solution spaces* or *behaviors*

$$\forall n \geq n_0:\ \mathcal{B}(R,n) := \left\{ w \in (\mathbb{C}^{n+\mathbb{N}})^\ell;\ \forall t \geq n:\ \sum_{j=0}^{d} R_j(t)w(t+j) = 0 \right\}. \tag{6}$$

Stability theory of these solution spaces concerns the behavior of the trajectories $w(t)$ for $t \to \infty$.

Remark 1.1. *We give some arguments for the suitability of $F := \mathbb{C}(t)$ as coefficient field. The case of periodic coefficients is not discussed here since it can be reduced to the LTI-theory.*
(a) The following properties of F are decisive: (i) F is a field or, at least, a noetherian domain. (ii) If $a(t)$ belongs to F then so does $a(t+1)$. (iii) For nonzero $a \in F$ there is $n \geq 0$ such that no $t \geq n$ is a pole or zero of a. (iv) A rational function grows at most polynomially.
(b) Rational functions have obvious advantages for numerical computations since they are given by finitely many numbers. They appear as Padé approximants of more general functions. They are often used in engineering models, cf. [2, (8.14), (8.15)], [27, Appendix].
Assume that $f(t) = t^k g(t) \in \mathrm{C}^0[n_0, \infty),\ n_0 > 0,\ k \in \mathbb{Z}$, is any continuous coefficient function such that $g(\infty) := \lim_{t\to\infty} g(t)$ exists and define $g_1(t) := g(t^{-1}) \in \mathrm{C}^0[0, n_0^{-1}]$. By the Stone-Weierstrass theorem there is a polynomial $a_1(t)$ that approximates g_1 arbitrarily. Then the rational function $a(t) := t^k a_1(t^{-1})$ is a good approximation for f on $[n_0, \infty)$. Hence rational functions approximate a large class of more general coefficient functions, but not all, for instance $f(t) = 2 + \sin(t)$. Such approximations raise the problem of robustness, of course. Note moreover that for the questions of stability the time instant n_0 can be chosen as large as desired. *So arbitrary LTV-systems with continuous coefficient functions $f(t) = t^k g(t),\ k \in \mathbb{Z}$, and existing $g(\infty)$ can be approximated for stability problems by the systems of this paper.*
Linearization of a nonlinear system in the neighborhood of a nominal trajectory leads to LTV-systems. Since this is only an approximation process the further approximation of the coefficients by rational functions seems suitable.
In item 7. of Section 4 and more detailed in [4] we describe another larger coefficient field with the properties (i)-(iv) [22, Ex. 1.2].
(c) For scalar *state space systems* $x(t+1) = a(t)x(t)$ or, more generally, those of (1) one may choose arbitrary $a = (a(t))_{t\geq n_0} \in \mathbb{C}^{n_0+\mathbb{N}}$ or A [25, p. 383]. It is surprising that Ehrenpreis' fundamental principle holds for arbitrary discrete, even multidimensional behaviors with arbitrary varying coefficients [3, Thm. 2.1]. For the behavioral stability theory such general coefficients are not suitable. We explain this for the continuous case where the effects are clearer. So consider differential equations for smooth signals, the coefficient field of meromorphic coefficients [26], [16] and the differential equation $\cos^2(t)x'(t) - x(t) = 0$ with its solution $x(t) = c\exp(\tan(t))$. The infinitely many zeros $(n + 1/2)\pi,\ n \in \mathbb{Z}$, of $\cos^2(t)$ or poles of $\tan(t)$ are those time instants where the system explodes. There is no reasonable asymptotic behavior of this system. This suggests that the condition (iii) is essential for a reasonable stability theory. Due to these singularities the quoted authors, see also [2, §5.4.2.2], omit the generally infinite, discrete set of singularities from the time domain of the signals. This procedure, however, does not solve the problem because a time domain with infinitely many gaps is beyond engineering reality. Hence holomorphic or even continuous coefficients are suitable for the stability theory of state space systems [25], [15], [14] , but not for that of general behaviors.
(d) Coefficient rings of *smooth functions* are neither domains nor noetherian in general and this is inherited by the associated rings of difference or differential operators. Algebraic properties of these rings are not known, a behavioral duality theory cannot be developed and there are no algebraic algorithms that are so important in the standard LTI (linear time-invariant) systems theories.

In contrast to the LTI-case and in analogy to, for instance, [25, Defs. 22.1, 22.5] the whole family $(\mathcal{B}(R,n))_{n\geq n_0}$ of behaviors and not just $\mathcal{B}(R,n_0)$ has to be investigated where n_0 depends on the equations. For the comparison of different systems of equations (3) we introduce the equivalence relation of the behavior families from (6) by

$$\begin{aligned}&(\mathcal{B}(R,n))_{n\geq n_0}\equiv(\mathcal{B}(R',n))_{n\geq n_0'}:\Longleftrightarrow\\ &\exists n_1\geq\max(n_0,n_0')\forall n\geq n_1:\ \mathcal{B}(R,n)=\mathcal{B}(R',n).\end{aligned}\tag{7}$$

The equivalence class is denoted by $\mathrm{cl}\left((\mathcal{B}(R,n))_{n\geq n_0}\right)$ (cl for *class*, not for closure) and is called the *behavior defined by* (3), cf. Example 1.5.

Remark 1.2. To investigate $\mathrm{cl}\left((\mathcal{B}(R,n))_{n\geq n_0}\right)$ for given equations (3) means to study $\mathcal{B}(R,n)$ for $n\geq n_1\geq n_0$ where n_1 is a possibly large initial time. The transient behavior of trajectories up to the time n_1 is disregarded. This set-up is very suitable for stability questions where the limits $\lim_{t\to\infty}w(t)$ play a dominant part.

Principal Results 1.3. We prove a *module-behavior duality* for the new behaviors. It implies the standard consequences for Willems' *elimination*, the *fundamental principle*, *input/output decompositions* and *controllability*. We characterize *autonomous behaviors* and show that they are isomorphic to state space behaviors. Therefore the examples in [25] are typical also for the autonomous LTV-behaviors of this paper. We introduce a new notion of *exponential stability* (e.s.) of autonomous behaviors since *uniform exponential stability* (u.e.s.) [25, Def. 22.5] is not preserved by behavior isomorphisms (cf. [25, Thm. 6.15] and Example 3.2) and therefore unsuitable for the behavioral theory. We show that e.s. autonomous behaviors form a *Serre subcategory* of the category of all behaviors. As corollaries we prove various stability and instability results for autonomous behaviors.

Definition 1.4. A class of objects or a full subcategory $\mathfrak{S}$ of an abelian category $\mathfrak{C}$ is called a *Serre subcategory* if it is closed under isomorphisms, subobjects, factor objects and extensions.

We first introduce the operator algebra. The field $\mathbb{C}(t)$ is a *difference field* with its natural automorphism α defined by $\alpha(h)(t):=h(t+1)$ for $h\in\mathbb{C}(t)$. It gives rise to the noncommutative *skew-polynomial* $\mathbb{C}$*-algebra* $\mathbf{A}$ in an indeterminate q [19, §1.2]:

$$\begin{aligned}&\mathbf{A}:=\mathbb{C}(t)[q;\alpha]=\oplus_{j\in\mathbb{N}}\mathbb{C}(t)q^j\ni f=\sum_{j\in\mathbb{N}}f_jq^j,\ f_j\in\mathbb{C}(t),\\ &\text{with the multiplication } (h_1q^{j_1})(h_2q^{j_2})=h_1\alpha^{j_1}(h_2)q^{j_1+j_2}\text{ for}\\ &h_1,h_2\in\mathbb{C}(t),\ \alpha^{j_1}(h_2)(t)=h_2(t+j_1),qh(t)=h(t+1)q.\end{aligned}\tag{8}$$

The $q^j, j\in\mathbb{N}$, are a $\mathbb{C}(t)$-basis of $\mathbf{A}$. By definition almost all (up to finitely many) coefficients f_j of f are zero. The algebra $\mathbf{A}$ is a left and right principal ideal domain and its finitely generated (f.g.) modules are precisely known [19, Thm. 1.2.9, §5.7, Cor. 5.7.19]. The category of left $\mathbf{A}$-modules is denoted by ${}_{\mathbf{A}}\mathbf{Mod}$. The category of f.g. left $\mathbf{A}$-modules M *with a given list of generators* or, equivalently, a given representation $M=\mathbf{A}^{1\times\ell}/U$ as factor module of a free module $\mathbf{A}^{1\times\ell}$ by a submodule U and with the $\mathbf{A}$-linear maps as morphisms is denoted by ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$. Fliess [9], [10] calls a *module* M with the additional structure $M=\mathbf{A}^{1\times\ell}/U$ a *linear dynamic* or *(discrete LTV-)system.*

If the rows of $R=\sum_{j\in\mathbb{N}}R_jq^j\in\mathbf{A}^{p\times\ell},\ R_j\in\mathbb{C}(t)^{p\times\ell}$, generate U, i.e., $U=$

$\mathbf{A}^{1\times p}R$, and if no $t \geq n_0$ is a pole of any R_j we obtain the behaviors

$$\forall n \geq n_0: \ \mathcal{B}(R,n) = \left\{ w \in (\mathbb{C}^{n+\mathbb{N}})^\ell; \ \forall t \geq n: \ \sum_{j\in\mathbb{N}} R_j(t)w(t+j) = 0 \right\} \text{ and}$$
$$\mathcal{B}(U) := \mathrm{cl}\left((\mathcal{B}(R,n)_{n\geq n_0}\right). \tag{9}$$

Lemma 2.5 shows that $\mathcal{B}(U)$ depends on U only and not on the special choice of R. We call $\mathcal{B}(U)$ *the behavior defined by* U *or associated to* $\mathbf{A}^{1\times q}/U$, see Remark 1.2.

Example 1.5. Consider

$$\begin{gathered} U := \mathbf{A}q = \mathbf{A}(tq) \subset \mathbf{A}, \text{ hence } \mathcal{B}(\mathbf{A}q) = \mathcal{B}(\mathbf{A}(tq)), \text{ indeed} \\ \forall n \geq 1: \ \mathcal{B}(q,n) = \mathcal{B}(tq,n) = \left\{w \in \mathbb{C}^{n+\mathbb{N}}; \ \forall k \geq n+1: \ w(k) = 0\right\}, \text{ but} \\ \mathcal{B}(q,0) = \left\{w \in \mathbb{C}^{\mathbb{N}}; \ \forall k \geq 1: \ w(k) = 0\right\} \subsetneq \\ \mathcal{B}(tq,0) = \left\{w \in \mathbb{C}^{\mathbb{N}}; \ \forall k \geq 2: \ w(k) = 0\right\}. \end{gathered} \tag{10}$$

Also $\mathcal{B}(\mathbf{A}(t-2)^{-1}q) = \mathcal{B}(\mathbf{A}q)$, but $\mathcal{B}((t-2)^{-1}q,n)$ is not defined for $n \leq 2$. This motivates the introduction of the equivalence relation (7).

In Cor. and Def. 2.7 we extend the construction of $\mathcal{B}(U)$ to a contravariant functor

$$\begin{gathered} \mathbf{A}^{1\times\ell}/U \mapsto \mathcal{B}(U), \\ \left(\varphi: \mathbf{A}^{1\times\ell_1}/U_1 \to \mathbf{A}^{1\times\ell_2}/U_2\right) \mapsto \left(\mathcal{B}(\varphi): \mathcal{B}(U_2) \to \mathcal{B}(U_1)\right), \\ \mathrm{Hom}(\mathcal{B}(U_2),\mathcal{B}(U_1)) := \left\{\mathcal{B}(\varphi); \ \varphi: \mathbf{A}^{1\times\ell_1}/U_1 \to \mathbf{A}^{1\times\ell_2}/U_2\right\}. \end{gathered} \tag{11}$$

Notice that no $\mathbb{C}^{n+\mathbb{N}}$ is canonically an $\mathbf{A}$-module and that the behavior $\mathcal{B}(U)$ is *not* of the form $\mathrm{Hom}_{\mathbf{A}}(\mathbf{A}^{1\times\ell}/U, W)$ for a natural signal module ${}_{\mathbf{A}}W$.

Theorem 1.6. *The functor* (11) *is a duality (contravariant equivalence). More precisely the following properties hold:*

1. *It transforms exact sequences of modules into exact sequences of behaviors.*
2. *For all* $\mathbf{A}^{1\times\ell_1}/U_1, \mathbf{A}^{1\times\ell_2}/U_2 \in_{\mathbf{A}} \mathbf{Mod}^{\mathrm{fg}}$ *there is the* $\mathbb{C}$*-linear isomorphism*

$$\mathrm{Hom}_{\mathbf{A}}(\mathbf{A}^{1\times\ell_1}/U_1, \mathbf{A}^{1\times\ell_2}/U_2) \cong \mathrm{Hom}(\mathcal{B}(U_2),\mathcal{B}(U_1)), \ \varphi \mapsto \mathcal{B}(\varphi). \tag{12}$$

3. *For all* $U_1, U_2 \subseteq \mathbf{A}^{1\times\ell}$*:*

$$U_1 \subseteq U_2 \iff \mathcal{B}(U_2) \subseteq \mathcal{B}(U_1), \text{ especially } U_1 = U_2 \iff \mathcal{B}(U_2) = \mathcal{B}(U_1). \tag{13}$$

The injectivity of the map (12) replaces the *cogenerator property* of the signal module ${}_{\mathbb{C}[q]}\mathbb{C}^{\mathbb{N}}$ in the standard discrete LTI-systems theory. Section 2 is devoted to the proof of Thm. 1.6 in several steps. The last step is contained in Cor. 2.12 where the injectivity of (12) is proven. The surjectivity holds by definition in (11).
The following definition of e.s. of $\mathcal{B}(U)$ from (9) is justified by Lemma 3.7 and Example 3.2 that show that e.s. is preserved by behavior isomorphisms, but u.e.s. is not. A sequence $(\varphi(n))_{n\geq n_0} \in \mathbb{C}^{n_0+\mathbb{N}}$ is called a *sequence of at most polynomial growth* (p.g.s.) if

$$\exists c \geq 1 \exists m \in \mathbb{N} \forall n \geq n_0: \ |\varphi(n)| \leq c n^m. \tag{14}$$

This p.g.s. is called *positive*, $\varphi > 0$, if $\varphi(n) > 0$ for all $n \geq n_0$. On all finite-dimensional vector spaces $\mathbb{C}^\ell$, $\mathbb{C}^{1\times\ell}$ we use the maximum norm

$$\|v\| := \max\{|v_i|;\ i = 1, \cdots, \ell\},\ v = (v_1, \cdots, v_\ell) \in \mathbb{C}^{1\times\ell}. \tag{15}$$

Definition 1.7. The behavior $\mathcal{B}(U)$ from (9) is called *exponentially stable* if

$$\begin{aligned} &\exists n_1 \geq n_0 \exists d \in \mathbb{N} \exists \rho \text{ with } 0 < \rho < 1 \exists \text{ p.g.s. } \varphi \in \mathbb{C}^{n_1+\mathbb{N}} \text{ with } \varphi > 0 \\ &\qquad \forall t \geq n \geq n_1 \forall w \in \mathcal{B}(R, n):\ \|w(t)\| \leq \varphi(n)\rho^{t-n}\|x(n)\| \\ &\text{where } x(n) := (w(n), \cdots, w(n+d-1)). \end{aligned} \tag{16}$$

It is called *uniformly exponentially stable* if $(\varphi(n))_{n\geq n_1}$ can be chosen constant.

An e.s. behavior $\mathcal{B}(U)$ is *asymptotically stable* in the sense that

$$\forall n \geq n_1 \forall w \in \mathcal{B}(R, n):\ \lim_{t\to\infty} w(t) = 0. \tag{17}$$

Nonuniform e.s. state space systems with a nondecreasing factor $\varphi(n)$ are also defined in [21, §3]. An e.s. behavior is always autonomous, but the trajectories w are not uniquely determined by $w(n)$ alone, but only by the initial vector $x(n)$. Therefore $d \in \mathbb{N}$ is required. E.s. like u.e.s. are analytic properties of the trajectories w of the components $\mathcal{B}(R, n)$ of $\mathcal{B}(U)$ and are not defined by algebraic properties of the module $\mathbf{A}^{1\times\ell}/U$. At present there is no algebraic characterization of the modules $\mathbf{A}^{1\times\ell}/U$ with e.s. $\mathcal{B}(U)$ nor is there such a characterization of Rugh's u.e.s. state space equations [25, Def. 22.5]. In the simplest case of state space equations $x(t+1) = Ax(t)$ with a constant matrix $A \in \mathbb{C}^{n\times n}$, however, e.s. of the corresponding behavior means that A is asymptotically stable, i.e., that the spectrum $\mathrm{spec}(A)$, i.e., the set of eigenvalues of A, belongs to the open unit disc $\mathbf{D} := \{\lambda \in \mathbb{C};\ |\lambda| < 1\}$. More generally, an autonomous LTI-behavior is asymptotically stable if and only if it is e.s. in the sense of this paper.

Theorem 1.8. *The exponentially stable behaviors form a Serre subcategory of the category of all LTV-behaviors. This means that for an exact sequence of modules and its dual exact behavior sequence*

$$\begin{array}{ccccccc} 0 \to & \mathbf{A}^{1\times\ell_1}/U_1 & \xrightarrow{\varphi} & \mathbf{A}^{1\times\ell_2}/U_2 & \xrightarrow{\psi} & \mathbf{A}^{1\times\ell_3}/U_3 & \to 0 \\ 0 \leftarrow & \mathcal{B}(U_1) & \xleftarrow{\mathcal{B}(\varphi)} & \mathcal{B}(U_2) & \xleftarrow{\mathcal{B}(\psi)} & \mathcal{B}(U_3) & \leftarrow 0 \end{array} \tag{18}$$

the behavior $\mathcal{B}(U_2)$ is e.s. if and only $\mathcal{B}(U_1)$ and $\mathcal{B}(U_3)$ are e.s..

Corollary 1.9. *The series interconnection of two input/output behaviors is e.s. if and only if both building blocks are (cf. Section 4, item 5).*

Thm. 1.8 and Cor. 1.9 are equivalent in the sense that the theorem also follows easily from the corollary. Thm. 1.8 also holds for discrete LTI-behaviors where e.s. behaviors are defined as autonomous behaviors whose characteristic variety (=set of characteristic values) is contained in the open unit disc. The proof of the LTI-result is algebraic and much simpler than that of Thm. 1.8.

Due to [7], for instance, most results of this paper are constructive. However, there is presently no algorithm to check exponential stability in general. Continuous LTV-systems have been treated more often and in more detail, see, for instance, the books

[25] and [2] and the papers [12], [26], [14].
The Sections 2 resp. 3 are devoted to the proof of the main Theorems 1.6 resp. 1.8. In Section 2.5 we moreover characterize *autonomous* LTV-behaviors. The Sections 3.5 and 3.6 are devoted to various stability and instability results for autonomous behaviors. In particular we also discuss the existence and properties of *quasi-poles* of an autonomous behavior (cf. [2, §6.7.1]). In Section 4 we use the duality Thm. 1.6 to embed standard LTI-results into our LTV-frame-work and to derive LTV-analogues of *Willems' elimination*, the *fundamental principle*, *input/output decompositions* and *controllability*. We refer to [2, Part 1, pp. 3-268] for algebraic background material.
Abbreviations: e.s.= exponentially stable, f.d.=finite-dimensional, f.g.=finitely generated, p.g.s.=sequence of at most polynomial growth, u.e.s.= uniformly e.s., w.l.o.g.= without loss of generality, $\mathbf{A}^{\bullet\times\bullet}$= the set of matrices with entries in $\mathbf{A}$ of all (suitable) sizes,

2 LTV-systems

2.1 Complements of the basic data

We complete the general data of the Introduction.

Remark 2.1. The derivations of Section 2 hold for any base field instead of the complex field $\mathbb{C}$. The definition of e.s. needs analysis and therefore Section 3 can be carried out over the fields $\mathbb{R}$ or $\mathbb{C}$ only. The signals $w(t)$ are always functions of the *real* time variable t, but in this paper the values of the signals may be complex. Since $\mathbb{R} \subset \mathbb{C}$ the complex theory contains the real one. Equations like $e^{it} = \cos(t) + i\sin(t)$ and the complex eigenvalues of real matrices suggest to use complex coefficients and to use $\mathbf{A} = \mathbb{C}(t)[q;\alpha]$ instead of $\mathbb{R}(t)[q;\alpha]$, and this is done in this paper.

To write (3) as operator equation we also consider $\mathbb{C}^{n+\mathbb{N}}$ as difference algebra. Its multiplication, one-element and algebra endomorphism $\alpha : \mathbb{C}^{n+\mathbb{N}} \to \mathbb{C}^{n+\mathbb{N}}$ are given as

$$(ab)(t) := a(t)b(t),\ 1_{\mathbb{C}^{n+\mathbb{N}}} := (\overset{n}{1}, 1, \cdots),\ \alpha(a)(t) := a(t+1),\ a, b \in \mathbb{C}^{\mathbb{N}},\ t \geq n. \tag{19}$$

The endomorphism α is the standard forward shift. As in (8) the difference ring $(\mathbb{C}^{n+\mathbb{N}}, \alpha)$ gives rise to the noncommutative skew-polynomial algebra [19, §1.2.3]

$$\begin{gathered}\mathbf{B}(n) := \mathbb{C}^{n+\mathbb{N}}[q;\alpha] = \oplus_{j\in\mathbb{N}}\mathbb{C}^{n+\mathbb{N}}q^j \ni f = \sum_{j\in\mathbb{N}} f_j q^j,\\ (f_i q^i)(g_j q^j) = f_i \alpha^i(g_j) q^{i+j},\ f_i, g_j \in \mathbb{C}^{n+\mathbb{N}},\ (\alpha^i(g))(t) = g(t+i).\end{gathered} \tag{20}$$

The $\mathbb{C}$-algebra $\mathbf{B}(n)$ is neither a domain nor noetherian and, in contrast to $\mathbf{A}$, little is known about its algebraic properties and modules. There is the canonical action $f \circ w$ of $f = \sum_j f_j q^j \in \mathbf{B}(n)$ on $w \in \mathbb{C}^{n+\mathbb{N}}$, defined by

$$(f \circ w)(t) := \sum_j f_j(t) w(t+j),\ (q \circ w)(t) = w(t+1),\ f \in \mathbf{B}(n),\ w \in \mathbb{C}^{n+\mathbb{N}}. \tag{21}$$

It makes $\mathbb{C}^{n+\mathbb{N}}$ a $\mathbf{B}(n)$-left module that is denoted by ${}_{\mathbf{B}(n)}\mathbb{C}^{n+\mathbb{N}}$. It is the most general natural signal module for discrete LTV-systems theory. The action $\circ$ is extended to an action of a matrix $R \in \mathbf{B}(n)^{p\times\ell}$ on a vector $w = (w_1, \cdots, w_\ell)^\top \in (\mathbb{C}^{n+\mathbb{N}})^\ell$ by

$$R = (R_{\mu,\nu})_{1\le\mu\le p,1\le\nu\le\ell} = \sum_{j=0}^{d} R_j q^j \in \mathbf{B}(n)^{p\times\ell},\ R_{\mu,\nu} \in \mathbf{B}(n),\ R_j \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{p\times\ell},$$
$$R \circ w := \left(\sum_{\nu=1}^{\ell} R_{\mu,\nu} \circ w_\nu\right)_{\mu=1,\cdots,p} \in (\mathbb{C}^{n+\mathbb{N}})^p,$$
$$\forall t \ge n:\ (R \circ w)(t) = \sum_j R_j(t) w(t+j). \tag{22}$$

Note that there is no action of $\mathbf{A}$ on $\mathbb{C}^{n+\mathbb{N}}$ since, for instance $(t-n)^{-1} \circ w,\ w \in \mathbb{C}^{n+\mathbb{N}}$, is not defined.
Recall the poles and zeros of a nonzero rational function $h \in \mathbb{C}(t)$: Write $h = fg^{-1},\ f, g \in \mathbb{C}[t],\ f, g \ne 0$, with coprime f and g. A *pole* resp. a *zero* $z \in \mathbb{C}$ of h is characterized by

$$\begin{gathered} f(z) \ne 0, g(z) = 0,\ h(z) := \infty \text{ resp. } f(z) = 0, g(z) \ne 0,\ h(z) = 0. \\ \text{Then } \operatorname{dom}(h) = \mathbb{C} \setminus \{z \in \mathbb{C};\ h(z) = \infty\} \end{gathered} \tag{23}$$

is the open domain of definition of h as function. For almost all n (up to finitely many) the lattice $n + \mathbb{N}$ is contained in $\operatorname{dom}(h)$ and we identify

$$\begin{gathered} h \underset{\text{ident.}}{=} (h(t))_{t\ge n} \in \mathbb{C}^{n+\mathbb{N}},\ n + \mathbb{N} \subseteq \operatorname{dom}(h), \text{ since} \\ \forall n \in \mathbb{N} \forall h_1, h_2 \in \mathbb{C}(t) \text{ with } n + \mathbb{N} \subseteq \operatorname{dom}(h_i),\ i = 1, 2: \\ \left(h_1 = h_2 \iff (h_1(t))_{t\ge n} = (h_2(t))_{t\ge n}\right). \end{gathered} \tag{24}$$

For

$$\begin{gathered} R = \sum_j R_j q^j \in \mathbf{A}^{p\times\ell},\ R_j = (R_{j,\mu,\nu})_{1\le\mu\le p,1\le\nu\le\ell} \in \mathbb{C}(t)^{p\times\ell}, \text{ define} \\ \operatorname{dom}(R_j) := \bigcap_{\mu,\nu} \operatorname{dom}(R_{j,\mu,\nu}),\ \operatorname{dom}(R) := \bigcap_j \operatorname{dom}(R_j). \end{gathered} \tag{25}$$

If $n_0 + \mathbb{N} \subseteq \operatorname{dom}(R)$ then

$$\begin{gathered} \forall n \ge n_0:\ R_j \underset{\text{ident.}}{=} (R_j(t))_{t\ge n} \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{p\times\ell},\ R = \sum_j R_j q^j \in \mathbf{B}(n)^{p\times\ell} \text{ and} \\ \mathcal{B}(R, n) \underset{(6)}{=} \left\{w \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^\ell;\ R \circ w = 0\right\}. \end{gathered} \tag{26}$$

The last equation is the usual operator description of the behavior. The elements in $\mathbb{C} \setminus \operatorname{dom}(R_j)$ resp. in $\mathbb{C} \setminus \operatorname{dom}(R)$ are called the *poles* of R_j resp. of R. The behaviors $\mathcal{B}(R, n)$ are defined for all $n \ge n_0$ if and only if no $t \ge n_0$ is a pole of R. Since the ring $\mathbf{B}(n)$ is noncommutative the behavior $\mathcal{B}(R, n)$ is a $\mathbb{C}$-space only and not a $\mathbb{C}^{n+\mathbb{N}}$ or $\mathbf{B}(n)$-module.

2.2 A directed system category

We formalize the equivalence relation from (7) *in a more general situation with good algebraic properties and introduce a new category* $\mathfrak{B}$. *The basic example for our approach is Example 2.2 below.*

Consider $\mathbb{N}$ as directed ordered set. A directed system over $\mathbb{N}$ of $\mathbb{C}$-vector spaces is a countable family

$$V = (V_i, g_i)_{i\in\mathbb{N}} = \left(V_0 \xrightarrow{g_0} V_1 \xrightarrow{g_1} V_2 \xrightarrow{g_2} ....\right) \tag{27}$$

of $\mathbb{C}$-spaces V_i and $\mathbb{C}$-linear maps g_i. We identify a directed system $(V_i, g_i)_{i\geq n}$ with the longer system

$$(V_i, g_i)_{i\geq n} \underset{\text{ident.}}{=} (V_i, g_i)_{i\geq 0} := \left(0 \to \cdots \to \overset{n-1}{0} \to V_n \xrightarrow{g_n} V_{n+1} \to \cdots\right). \tag{28}$$

A morphism from one such system to another is a family of $\mathbb{C}$-linear maps

$$\begin{gathered}\Phi = (\Phi_i)_{i\in\mathbb{N}} : V = (V_i, g_i)_{i\in\mathbb{N}} \longrightarrow V' = (V_i', g_i')_{i\in\mathbb{N}} \text{ with} \\ \forall i \in \mathbb{N} :\ \Phi_i : V_i \to V_i',\ \Phi_{i+1} g_i = g_i' \Phi_i.\end{gathered} \tag{29}$$

The set $\mathrm{Hom}(V, V')$ of all these morphisms is naturally a $\mathbb{C}$-space. The composition of morphisms is, of course, the componentwise one and with this the directed systems form a category. It is abelian where kernels, cokernels etc. are formed componentwise. We form the *new category* $\mathfrak{B}$ as the quotient category of the direct system category modulo the following equivalence relation $\equiv$:

$$\begin{gathered}V = (V_i, g_i)_{i\in\mathbb{N}} \equiv V' = (V_i', g_i')_{i\in\mathbb{N}} :\Longleftrightarrow \\ \exists n \forall i \geq n :\ V_i = V_i',\ g_i = g_i'.\end{gathered} \tag{30}$$

The equivalence class is denoted by $\mathrm{cl}(V)$. These $\mathrm{cl}(V)$ are the objects of $\mathfrak{B}$. With the identification from (28) we obtain

$$\mathrm{cl}\left((V_i, g_i)_{i\geq 0}\right) = \mathrm{cl}\left((V_i, g_i)_{i\geq n}\right) \tag{31}$$

The study of $\mathrm{cl}\left((V_i, g_i)_{i\geq 0}\right)$ means that of $(V_i, g_i)_{i\geq n}$ for possibly large n. For two objects $\mathrm{cl}\left((V_i, g_i)_{i\geq n_0}\right)$ and $\mathrm{cl}\left((V_i', g_i')_{i\geq n_0'}\right)$ we consider direct system morphisms

$$\begin{aligned}\Phi &:= (\Phi_i)_{i\geq n_1} : (V_i, g_i)_{i\geq n_1} \to (V_i', g_i')_{i\geq n_1}, \\ \Psi &:= (\Psi_i)_{i\geq n_2} : (V_i, g_i)_{i\geq n_2} \to (V_i', g_i')_{i\geq n_2}\end{aligned} \tag{32}$$

where $n_1, n_2 \geq \max(n_0, n_0')$ and define the equivalence relation

$$\Phi \equiv \Psi :\Longleftrightarrow \exists n \geq \max(n_1, n_2) \forall i \geq n :\ \Phi_i = \Psi_i. \tag{33}$$

The equivalence class is denoted by $\mathrm{cl}(\Phi)$. Then the set of morphisms from $\mathrm{cl}(V)$ to $\mathrm{cl}(V')$ is defined as

$$\begin{gathered}\mathfrak{B}(\mathrm{cl}(V), \mathrm{cl}(V')) := \mathrm{Hom}\left(\mathrm{cl}(V), \mathrm{cl}(V')\right) := \\ \left\{\mathrm{cl}(\Phi);\ \Phi = (\Phi_i)_{i\geq n} : (V_i, g_i)_{i\geq n} \to (V_i', g_i')_{i\geq n}\right\}.\end{gathered} \tag{34}$$

With the componentwise $\mathbb{C}$-linear structure and composition we obtain the category $\mathfrak{B}$ of equivalence classes of directed systems. This is abelian too, kernels, cokernels and images being also formed componentwise.

Example 2.2. The signal spaces $\mathbb{C}^{n+\mathbb{N}},\ n \in \mathbb{N}$, give rise to the directed system

$$\begin{aligned}
&\left(\mathbb{C}^{0+\mathbb{N}} \to \cdots \to \mathbb{C}^{n+\mathbb{N}} \xrightarrow{\mathrm{proj}_n} \mathbb{C}^{(n+1)+\mathbb{N}} \xrightarrow{\mathrm{proj}_{n+1}} \cdots\right) \text{ where}\\
&\mathrm{proj}_n : \mathbb{C}^{n+\mathbb{N}} \to \mathbb{C}^{n+1+\mathbb{N}},\ w = (w(t))_{t\geq n} \mapsto w|_{n+1+\mathbb{N}} := (w(t))_{t\geq n+1}, \quad (35)\\
&\text{and } \mathcal{W} := \mathrm{cl}\left(\mathbb{C}^{0+\mathbb{N}} \to \cdots \to \mathbb{C}^{n+\mathbb{N}} \xrightarrow{\mathrm{proj}_n} \mathbb{C}^{(n+1)+\mathbb{N}} \xrightarrow{\mathrm{proj}_{n+1}} \cdots\right).
\end{aligned}$$

This directed system consists of $\mathbb{C}$-algebras and $\mathbb{C}$-algebra homomorphisms. Under the assumptions of (9) we obtain the subsystems

$$\begin{aligned}
\mathcal{B}(R) := (\mathcal{B}(R,n), \mathrm{proj}_n)_{n\geq n_0} \subseteq ((\mathbb{C}^{n+\mathbb{N}})^\ell, \mathrm{proj}_n)_{n\geq n_0} \text{ and}\\
\mathrm{cl}\,(\mathcal{B}(R)) \subseteq \mathcal{W}^\ell = \mathrm{cl}\left(((\mathbb{C}^{n+\mathbb{N}})^\ell, \mathrm{proj}_n)_{n\geq n_0}\right).
\end{aligned} \quad (36)$$

Definition 2.3. The equivalence class $\mathrm{cl}(\mathcal{B}(R))$ from (36) is called the LTV-behavior associated with the matrix $R \in \mathbf{A}^{p\times\ell}$.

2.3 The functor $\mathrm{Mod}_{\mathbf{A}}^{\mathrm{fg}} \to \mathfrak{B},\ \mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U)$

We are going to show that in (9) *the behavior* $\mathcal{B}(U) \subseteq \mathcal{W}^\ell$ *is well-defined and that the assignment* $\mathbf{A}^{1\times\ell}/U \mapsto \mathcal{B}(U)$ *from the objects of* ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ *to those of* $\mathfrak{B}$ *can be canonically extended to a functor* ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}} \to \mathfrak{B}$.
Assume the data from (9), ie.,

$$\begin{aligned}
&R = \sum_j R_j q^j \in \mathbf{A}^{p\times\ell},\ U = \mathbf{A}^{1\times p}R,\ M = \mathbf{A}^{1\times\ell}/U,\ n_0 + \mathbb{N} \subseteq \mathrm{dom}(R),\\
&\forall n \geq n_0 :\ \mathcal{B}(R,n) := \left\{ w \in (\mathbb{C}^{n+\mathbb{N}})^\ell;\ \forall t \geq n :\ \sum_j R_j(t)w(t+j) = 0 \right\}.
\end{aligned} \quad (37)$$

The behaviors $\mathcal{B}(R,n)$ require the knowledge of U, the knowledge of M alone does not determine the representation $M = \mathbf{A}^{1\times q}/U$. The standard basis $\delta = (\delta_1, \cdots, \delta_\ell)^\top \in (\mathbf{A}^{1\times\ell})^\ell$ gives rise to the column

$$\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_\ell)^\top \in M^\ell,\ \mathbf{w}_i := \delta_i + U, \quad (38)$$

of generators of M. Conversely, the epimorphism

$$\varphi_{\mathbf{w}} : \mathbf{A}^{1\times\ell} \to M,\ \xi = \xi\delta \mapsto \xi\mathbf{w} = \sum_{i=1}^{\ell} \xi_i\mathbf{w}_i, \text{ with } \varphi_{\mathbf{w}}(\delta_i) = \mathbf{w}_i,\ \ker(\varphi_{\mathbf{w}}) = U, \quad (39)$$

shows that the system of generators $\mathbf{w}$ of M determines both the dimension of $\mathbf{A}^{1\times\ell}$ and its submodule U. Therefore the category ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ of f.g. $\mathbf{A}$-modules is defined as indicated in the Introduction: The objects of the category are pairs $(M, \mathbf{w})$ of f.g. modules M with a given list $\mathbf{w}$ of generators or a given representation $M = \mathbf{A}^{1\times\ell}/U$. Notice that in $M = \mathbf{A}^{1\times\ell}/U$ a special system of generators of U, i.e., a representation $U = \mathbf{A}^{1\times p}R$ or finite presentation (=exact sequence)

$$\mathbf{A}^{1\times p} \xrightarrow{\circ R} \mathbf{A}^{1\times\ell} \xrightarrow{\mathrm{can}} M \to 0,\ R \in \mathbf{A}^{p\times\ell}, \quad (40)$$

is not assumed or part of the structure. A morphism $\varphi : M = \mathbf{A}^{1\times\ell}/U \to M' = \mathbf{A}^{1\times\ell'}/U'$ is just an $\mathbf{A}$-linear map without additional structure, i.e.,

$$\mathrm{Hom}(\mathbf{A}^{1\times\ell}/U, \mathbf{A}^{1\times\ell'}/U') := \mathrm{Hom}_{\mathbf{A}}(M, M'), \tag{41}$$

and the composition of morphisms is also just that in ${}_{\mathbf{A}}\mathbf{Mod}$. If $\mathbf{w}$ and $\mathbf{w}'$ are two lists of generators of a f.g. M of possibly different lengths then $(M, \mathbf{w})$ and $(M, \mathbf{w}')$ are different objects in ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ and $\mathrm{id}_M : (M, \mathbf{w}) \to (M, \mathbf{w}')$ is an isomorphism, but not the identity. Exactness in ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ is defined as that in ${}_{\mathbf{A}}\mathbf{Mod}$. A kernel of a map $\varphi : (M, \mathbf{w}) \to (M', \mathbf{w}')$ is $(\ker(\varphi : M \to M'), \mathbf{k})$ where $\mathbf{k}$ is any generating system of $\ker(\varphi)$. The category ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ is abelian and the kernel of a morphism is unique up to isomorphism as in any abstract abelian category.

Example 2.4. This example explains the structural necessity of $\mathbf{w}$ or U already in the LTI-theory.

$$\begin{aligned} &M := \mathbf{A} = \mathbf{A}1 = \mathbf{A}\mathbf{w}_+ + \mathbf{A}\mathbf{w}_-,\ \mathbf{w}_+ = 1+q,\ \mathbf{w}_- = 1-q. \\ &U_1 := \ker(\mathbf{A} \to \mathbf{A},\ 1 \mapsto 1) = 0, \\ &U_2 := \ker(\mathbf{A}^{1\times 2} \to \mathbf{A},\ (\xi_+, \xi_-) \mapsto \xi_+\mathbf{w}_+ + \xi_-\mathbf{w}_-) = \mathbf{A}(1-q, -1-q). \end{aligned} \tag{42}$$

Then $\mathcal{B}(U_1) \cong \mathcal{B}(U_2)$, but $\mathcal{B}(U_1) \neq \mathcal{B}(U_2)$ where

$$\begin{aligned} W(n) = \mathcal{B}(1, n) &\cong \mathcal{B}((1-q, -1-q), n) \\ &= \left\{ \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right) \in W(n)^2;\ w_1(t) - w_1(t+1) - w_2(t) - w_2(t+1) = 0 \right\}, \\ w \leftrightarrow \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right),&\ w(t) = w_1(t) + w_1(t+1) + w_2(t) - w_2(t+1). \end{aligned} \tag{43}$$

Lemma 2.5. *Assume a submodule $U \subseteq \mathbf{A}^{1\times\ell}$, a matrix $R \in \mathbf{A}^{p\times\ell}$ with $U = \mathbf{A}^{1\times p}R$ and the data from* (37). *Then the object*

$$\mathcal{B}(U) :\underset{\text{Def. (30)}}{=} \mathrm{cl}\left((\mathcal{B}(R, n), \mathrm{proj}_n)_{n\geq n_0}\right) \in \mathfrak{B} \tag{44}$$

depends on U only and not on the special choice of R, and hence (9) *is justified. Moreover $U_1 \subseteq U_2$ implies $\mathcal{B}(U_2) \subseteq \mathcal{B}(U_1)$.*

Proof. Assume that $U_1 = \mathbf{A}^{1\times p_1}R_1 \subseteq U_2 = \mathbf{A}^{1\times p_2}R_2 \subseteq \mathbf{A}^{1\times\ell}$. Then there is a matrix $X \in \mathbf{A}^{p_1\times p_2}$ such that $R_1 = XR_2$. Choose $n_1 \geq n_0$ such that $n_1 + \mathbb{N} \subseteq \mathrm{dom}(R_1) \cap \mathrm{dom}(R_2) \cap \mathrm{dom}(X)$. Then

$$\begin{aligned} &\forall n \geq n_1 :\ R_1 = XR_2 \in \mathbf{B}(n)^{p_1\times\ell} \\ &\qquad \text{and } \mathcal{B}(R_2, n) := \left\{ w \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^\ell;\ R_2 \circ w = 0 \right\} \subseteq \mathcal{B}(R_1, n) \\ &\Longrightarrow \mathrm{cl}\left((\mathcal{B}(R_2, n), \mathrm{proj}_n)_{n\geq n_0}\right) = \mathrm{cl}\left((\mathcal{B}(R_2, n), \mathrm{proj}_n)_{n\geq n_1}\right) \\ &\qquad\qquad \subseteq \mathrm{cl}\left((\mathcal{B}(R_1, n), \mathrm{proj}_n)_{n\geq n_1}\right). \end{aligned} \tag{45}$$

If $U_1 = U_2$ the reverse inclusion follows likewise and implies the independence of $\mathcal{B}(U)$ in (44) of the choice of R. Eq. (45) implies $\mathcal{B}(U_2) \subseteq \mathcal{B}(U_1)$ if $U_1 \subseteq U_2$. ☐

Next we extend the assignment $\mathbf{A}^{1\times\ell}/U \mapsto \mathcal{B}(U)$ to a contravariant functor. Let $M_i = \mathbf{A}^{1\times\ell_i}/U_i,\ i = 1, 2$, be two f.g. modules and $\varphi : M_1 \to M_2$ an $\mathbf{A}$-linear map.

Since $\mathbf{A}^{1\times\ell_1}$ is free φ can be embedded into various commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_1 & \xrightarrow{\subseteq} & \mathbf{A}^{1\times\ell_1} & \xrightarrow{\mathrm{can}_1} & M_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (\circ P)|_{U_1}} & & \downarrow{\scriptstyle \circ P} & & \downarrow{\scriptstyle \varphi=(\circ P)_{\mathrm{ind}}} & & \\ 0 & \longrightarrow & U_2 & \xrightarrow{\subseteq} & \mathbf{A}^{1\times\ell_2} & \xrightarrow{\mathrm{can}_2} & M_2 & \longrightarrow & 0 \end{array} \tag{46}$$
$$\text{where } P \in \mathbf{A}^{\ell_1\times\ell_2},\ U_1P \subseteq U_2,\ \varphi(\xi + U_1) = \xi P + U_2.$$

The following corollary is a standard result from module theory and follows easily from the diagram in (46). It was used by Cluzeau and Quadrat in systems theory [8].

Corollary 2.6. *(i) The map $P \mapsto (\circ P)_{\mathrm{ind}}$ induces the isomorphism*

$$\left\{P \in \mathbf{A}^{\ell_1\times\ell_2};\ U_1P \subseteq U_2\right\} / \left\{P \in \mathbf{A}^{\ell_1\times\ell_2};\ \mathbf{A}^{1\times\ell_1}P \subseteq U_2\right\} \cong \mathrm{Hom}_{\mathbf{A}}(M_1, M_2). \tag{47}$$

(ii) The map $(\circ P)_{\mathrm{ind}} : \mathbf{A}^{1\times\ell_1}/U_1 \to \mathbf{A}^{1\times\ell_2}/U_2$ is an isomorphism if and only if it is bijective or $(\circ P)_{\mathrm{ind}}^{-1} = (\circ Q)_{\mathrm{ind}} : \mathbf{A}^{1\times\ell_2}/U_2 \to \mathbf{A}^{1\times\ell_1}/U_1$ exists. The necessary and sufficient conditions for $Q \in \mathbf{A}^{\ell_2\times\ell_1}$ to satisfy $(\circ Q)_{\mathrm{ind}} = (\circ P)_{\mathrm{ind}}^{-1}$ are

$$U_2Q \subseteq U_1,\ \mathbf{A}^{1\times\ell_1}(PQ - \mathrm{id}_{\ell_1}) \subseteq U_1,\ \mathbf{A}^{1\times\ell_2}(QP - \mathrm{id}_{\ell_2}) \subseteq U_2. \tag{48}$$

So the additional structure $M = \mathbf{A}^{1\times\ell}/U$ implies canonical matrix representations of the morphisms in ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$, a fact that is well-known from f.d. vector spaces with given bases. For the data from (46) and (47) we additionally assume that $U_i = \mathbf{A}^{1\times p_i}R_i$. The condition $\mathbf{A}^{1\times p_1}R_1P = U_1P \subseteq U_2 = \mathbf{A}^{1\times p_2}R_2$ implies the existence of $X \in \mathbf{A}^{p_1\times p_2}$ with $R_1P = XR_2$. Again we choose n_1 sufficiently large such that $R_1, R_2, P, X \in \mathbf{B}(n)^{\bullet\times\bullet}$ for $n \geq n_1$. For $w \in \mathcal{B}(R_2, n)$ this implies

$$\begin{gathered} R_1 \circ (P \circ w) = XR_2 \circ w = X \circ (R_2 \circ w) = X \circ 0 = 0 \text{ and hence} \\ P\circ : \mathcal{B}(R_2, n) = \left\{w \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_1};\ R_2 \circ w = 0\right\} \to \mathcal{B}(R_1, n),\ w \mapsto P \circ w. \end{gathered} \tag{49}$$

Corollary 2.7. *For an $\mathbf{A}$-linear map*

$$\begin{aligned} &\varphi = (\circ P)_{\mathrm{ind}} : \mathbf{A}^{1\times\ell_1}/U_1 \to \mathbf{A}^{1\times\ell_2}/U_2 \textit{ define} \\ &\quad P\circ := \mathcal{B}(\varphi) := \mathcal{B}\left((\circ P)_{\mathrm{ind}}\right) := \mathrm{cl}\left((P\circ : \mathcal{B}(R_2, n)) \to \mathcal{B}(R_1, n))_{n\geq n_1}\right) : \\ &\qquad\qquad \mathcal{B}(U_2) = \mathrm{cl}\left((\mathcal{B}(R_2, n), \mathrm{proj}_n)_{n\geq n_1}\right) \to \mathcal{B}(U_1). \end{aligned} \tag{50}$$

Then $\mathcal{B}(\varphi)$ is well-defined, i.e., independent of the choice of P, and the assignment

$${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}} \to \mathfrak{B},\ \mathbf{A}^{1\times\ell}/U \mapsto \mathcal{B}(U),\ \varphi = (\circ P)_{\mathrm{ind}} \mapsto \mathcal{B}(\varphi) = P\circ, \tag{51}$$

is a contravariant additive functor.

Proof. Equation (49) implies $P\circ : \mathcal{B}(U_2) \to \mathcal{B}(U_1)$. If also $\varphi = (\circ P_1)_{\mathrm{ind}}$ equation (47) implies $\mathbf{A}^{1\times\ell_1}(P_1 - P) \subseteq U_2 = \mathbf{A}^{1\times\ell_2}R_2$ or, equivalently, the existence of a matrix X such that $P_1 - P = XR_2$. For sufficiently large $n_2 \geq n_1$ this implies $P_1, P, X \in \mathbf{B}(n)^{\ell_1\times\ell_2}$ for $n \geq n_2$ and hence

$$\forall w \in \mathcal{B}(R_2, n) :\ P_1 \circ w = P \circ w + X \circ (R_2 \circ w) = P \circ w \Longrightarrow P\circ = P_1\circ. \tag{52}$$

Thus $\mathcal{B}(\varphi)$ is well-defined. The functorial property and the additivity of this assignment follow directly from the explicit construction of $\mathcal{B}(U)$ and $\mathcal{B}(\varphi)$, cf. the definition of the category $\mathfrak{B}$ in Section 2.2. □

Remark 2.8. If ${}_{\mathbf{A}}W$ is any signal module and $M=\mathbf{A}^{1\times\ell}/U,\ U=\mathbf{A}^{1\times p}R$, then

$$\mathcal{B}_W(U):=U^{\perp}:=\left\{w\in W^{\ell};\ R\circ w=0\right\}\underset{\text{Malgrange}}{\cong}\mathrm{Hom}_{\mathbf{A}}(M,W). \tag{53}$$

This shows that $\mathcal{B}(U)$ is the analogue of $\mathrm{Hom}_{\mathbf{A}}(M,W)$ in standard behavioral systems theory, but $\mathcal{B}(U)$ is not of this form for a natural W. Recall that ${}_{\mathbf{A}}W$ is *injective* if and only if $\mathrm{Hom}_{\mathbf{A}}(-,W)$ is exact and a *cogenerator* if and only if

$$\mathrm{Hom}_{\mathbf{A}}(M_1,M_2)\to\mathrm{Hom}_{\mathbb{C}}\left(\mathrm{Hom}_{\mathbf{A}}(M_2,W),\mathrm{Hom}_{\mathbf{A}}(M_1,W)\right),\ \varphi\mapsto\mathrm{Hom}(\varphi,W), \tag{54}$$

is a monomorphism for all $M_1,M_2\in_{\mathbf{A}}\mathbf{Mod}$.

2.4 The exactness of $\mathbf{A}^{1\times\ell}/U\mapsto\mathcal{B}(U)$

In this section we prove the exactness of the functor $\mathbf{A}^{1\times\ell}/U\mapsto\mathcal{B}(U)$. This is the analogue of the injectivity of the signal modules in the standard LTI-theory.

Consider f.g. modules $M_i:=\mathbf{A}^{1\times\ell_i}/U_i\in_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}},\ i=1,2,3$, and a sequence of $\mathbf{A}$-linear maps

$$M_1\xrightarrow{\varphi=(\circ P)_{\mathrm{ind}}}M_2\xrightarrow{\psi=(\circ Q)_{\mathrm{ind}}}M_3,\ U_1P\subseteq U_2,\ U_2Q\subseteq U_3. \tag{55}$$

Application of the functor $\mathbf{A}^{1\times\ell}/U\mapsto\mathcal{B}(U)$ furnishes the sequence of behaviors

$$\mathcal{B}(U_1)\xleftarrow{\mathcal{B}(\varphi)=P\circ}\mathcal{B}(U_2)\xleftarrow{\mathcal{B}(\psi)=Q\circ}\mathcal{B}(U_3). \tag{56}$$

First we prove that $P\circ:\mathcal{B}(U_2)\to\mathcal{B}(U_1)$ is an epimorphism if $\varphi=(\circ P)_{\mathrm{ind}}$ is injective. The simplest case is that $\ell_1=\ell_2=1, U_1=U_2=0, 0\neq P=P_dq^d+\cdots+P_0\in\mathbf{A}$ and $P_d\neq 0$. If n_0 is chosen such that no $t\geq n_0$ is a pole of any P_i or a zero of P_d then

$$\begin{aligned}P\circ:\mathbb{C}^{n+\mathbb{N}}=\mathcal{B}(0,n)\to\mathbb{C}^{n+\mathbb{N}}=\mathcal{B}(0,n),\ w\mapsto P\circ w=u,\ n\geq n_0,\\ \text{with } (P\circ w)(t)=P_d(t)w(t+d)+\cdots+P_0(t)w(t)=u(t),\end{aligned} \tag{57}$$

is surjective since the last equation can be solved inductively. Therefore

$$P\circ:\mathcal{W}=\mathcal{B}(0)=\mathrm{cl}((\mathbb{C}^{n+\mathbb{N}})_{n\geq n_0})\to\mathcal{W} \tag{58}$$

is an epimorphism. In the general case let $\mathbf{w}=(\mathbf{w}_1,\cdots,\mathbf{w}_{\ell_1})^{\top}\in M_1^{\ell_1}$ be the generating system of M_1. Then there is $\mathbf{v}=(\mathbf{v}_1,\cdots,\mathbf{v}_\ell)^{\top}\in M_2^{\ell}$ such that

$$\begin{aligned}M_2=\sum_{i=1}^{\ell_1}\mathbf{A}\varphi(\mathbf{w}_i)+\sum_{j=1}^{\ell}\mathbf{A}\mathbf{v}_j\Longrightarrow M_2=\mathbf{A}^{1\times\ell_2'}/U_2'\\ \text{where } \ell_2'=\ell_1+\ell,\ U_2'=\ker\left(\mathbf{A}^{1\times(\ell_1+\ell)}\to M_2,\ (\xi,\eta)\mapsto\varphi(\xi\mathbf{w})+\eta\mathbf{v}\right).\end{aligned} \tag{59}$$

We obtain a commutative diagram

$$\begin{array}{ccccc}\mathbf{A}^{1\times\ell_1} & \xrightarrow{\circ(\mathrm{id}_{\ell_1},0)} & \mathbf{A}^{1\times(\ell_1+\ell)} & \xrightarrow{\circ Q} & \mathbf{A}^{1\times\ell_2}\\ \downarrow{\scriptstyle\mathrm{can}_1} & & \downarrow{\scriptstyle\mathrm{can}_2'} & & \downarrow{\scriptstyle\mathrm{can}_2}\\ M_1=\mathbf{A}^{1\times\ell_1}/U_1 & \xrightarrow{\varphi} & M_2=\mathbf{A}^{1\times(\ell_1+\ell)}/U_2' & \xrightarrow{\mathrm{id}_{M_2}} & M_2=\mathbf{A}^{1\times\ell_2}/U_2\end{array} \tag{60}$$

where Q is a suitable matrix that induces the identity isomorphism, i.e., $(\circ Q)_{\rm ind} = {\rm id}_{M_2}$, and therefore also the isomorphism $Q\circ : \mathcal{B}(U_2) \cong \mathcal{B}(U_2')$. It therefore suffices to prove that $({\rm id}_{\ell_1}, 0)\circ : \mathcal{B}(U_2') \to \mathcal{B}(U_1)$ is an epimorphism, i.e., surjective for large n. By induction on ℓ we assume $\ell = 1$ wlog.

Lemma 2.9. *If*

$$(\circ P)_{\rm ind} : \mathbf{A}^{1\times\ell_1}/U_1 \to \mathbf{A}^{1\times(\ell_1+1)}/U_2 \tag{61}$$

is injective then $P\circ : \mathcal{B}(U_2) \to \mathcal{B}(U_1)$ *is an epimorphism.*

Proof. By the preceding reduction steps we may assume that a special injective linear map

$$(\circ({\rm id}_{\ell_1}, 0))_{\rm ind} : \mathbf{A}^{1\times\ell_1}/U_1 \to \mathbf{A}^{1\times(\ell_1+1)}/U_2 \tag{62}$$

is given. We have to show that

$$({\rm id}_{\ell_1}, 0)\circ = {\rm proj} : \mathcal{B}(U_2) \to \mathcal{B}(U_1),\ w = (w_1, \cdots, w_{\ell_1}, w_{\ell_1+1})^\top \mapsto (w_1, \cdots, w_{\ell_1})^\top, \tag{63}$$

is an epimorphism. Let

$$U_2 = \mathbf{A}^{p_2\times(\ell_1+1)}R_2,\ R_2 = (R_2', R_2'') \in \mathbf{A}^{p_2\times(\ell_1+1)}. \tag{64}$$

Wlog we assume $R_2'' \neq 0$. Then $\mathfrak{a} := \mathbf{A}^{1\times p_2}R_2''$ is a nonzero left ideal of $\mathbf{A}$ and cyclic of the form

$$\mathfrak{a} = \mathbf{A}f,\ 0 \neq f = f_d q^d + \cdots + f_0 \in \mathbf{A},\ f_d \neq 0,\ \text{hence } R_2'' = Y_2 f,\ \mathbf{A}^{1\times p_2}Y_2 = \mathbf{A}.$$

The relation module

$$K := \left\{\xi \in \mathbf{A}^{1\times p_2};\ \xi R_2'' = 0\right\} = \{\xi \in \mathbf{A};\ \xi Y_2 = 0\} \subseteq \mathbf{A}^{1\times p_2}$$

is free of dimension $p_2 - 1$. Let the rows of $X_1 \in \mathbf{A}^{(p_2-1)\times p_2}$ be a basis of K. We obtain the exact sequence of free modules

$$0 \to \mathbf{A}^{1\times(p_2-1)} \xrightarrow{\circ X_1} \mathbf{A}^{1\times p_2} \xrightarrow{\circ Y_2} \mathbf{A} \to 0. \tag{65}$$

In particular, X_1 is a universal left annihilator of R_2'' or Y_2. Standard arguments furnish a retraction $Y_1 \in \mathbf{A}^{p_2\times(p_2-1)}$ of X_1 with $X_1Y_1 = {\rm id}_{p_2-1}$ and a section $X_2 \in \mathbf{A}^{1\times p_2}$ of Y_2 with $X_2Y_2 = 1$ such that

$$\begin{gathered} 0 \leftarrow \mathbf{A}^{1\times(p_2-1)} \xleftarrow{\circ Y_1} \mathbf{A}^{1\times p_2} \xleftarrow{\circ X_2} \mathbf{A} \leftarrow 0 \text{ is exact too and} \\ E_1 := Y_1X_1 = E_1^2,\ E_2 := Y_2X_2 = E_2^2,\ E_1 + E_2 = Y_1X_1 + Y_2X_2 = {\rm id}_{p_2}. \end{gathered} \tag{66}$$

A simple computation yields

$$\begin{aligned} U_1 = (\circ({\rm id}_{\ell_1}, 0))^{-1}(U_2) = \mathbf{A}^{1\times(p_2-1)}R_1 \text{ with } R_1 := X_1R_2' \in \mathbf{A}^{(p_2-1)\times\ell_1} \\ \text{and } R_2' = {\rm id}_{p_2} R_2' = Y_1X_1R_2' + Y_2X_2R_2' = Y_1R_1 + Y_2X_2R_2'. \end{aligned} \tag{67}$$

Choose n_0 such that none of the constructed matrices has a pole $t \geq n_0$ and that $f_d(t) \neq 0$ for $t \geq n_0$. Then

$$\begin{gathered} \mathcal{B}(U_2) = {\rm cl}\left((\mathcal{B}(R_2, n))_{n\geq n_0}\right),\ \mathcal{B}(U_1) = {\rm cl}\left((\mathcal{B}(R_1, n))_{n\geq n_0}\right), \\ ({\rm id}_{\ell_1}, 0) : \mathcal{B}(U_2) \to \mathcal{B}(U_1),\ \forall n \geq n_0 :\ ({\rm id}_{\ell_1}, 0) : \mathcal{B}(R_2, n) \to \mathcal{B}(R_1, n). \end{gathered} \tag{68}$$

It now suffices to show the surjectivity of the maps in the last row: Let

$$\begin{aligned} &v=(v_1,\cdots,v_{\ell_1})^\top\in\mathcal{B}(R_1,n)\Longrightarrow R_1\circ v=X_1R_2'\circ v=0\underset{(67)}{\Longrightarrow}\\ &R_2'\circ v=Y_1R_1\circ v+Y_2X_2R_2'\circ v=Y_2X_2R_2'\circ v=Y_2\circ(X_2R_2'\circ v). \end{aligned} \tag{69}$$

According to (57) there is an $u\in\mathbb{C}^{n+\mathbb{N}}$ with $f\circ u=X_2R_2'\circ v$. We infer

$$\begin{aligned} &R_2'\circ v=R_2'\circ(v_1,\cdots,v_{\ell_1})^\top=Y_2\circ f\circ u=Y_2f\circ u=R_2''\circ u\\ &\Longrightarrow R_2\circ(v_1,\cdots,v_{\ell_1},-u)^\top=(R_2',R_2'')\circ(v_1,\cdots,v_{\ell_1},-u)^\top=R_2'\circ v-R_2''\circ u=0\\ &\Longrightarrow(v_1,\cdots,v_{\ell_1},-u)^\top\in\mathcal{B}(R_2,n). \end{aligned} \tag{70}$$

As required we have thus shown the surjectivity of

$$(\mathrm{id}_{\ell_1},0)\circ:\mathcal{B}(R_2,n)\to\mathcal{B}(R_1,n),\ (v_1,\cdots,v_{\ell_1},-u)^\top\mapsto v. \tag{71}$$

□

Theorem 2.10. *The functor*

$${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}\to\mathfrak{B},\ M=\mathbf{A}^{1\times\ell}/U\mapsto\mathcal{B}(U),$$

is exact, i.e., (56) *is exact if* (55) *is exact.*

Proof. The exactness of (55) implies

$$\begin{aligned} &U_3':=(\circ Q)^{-1}(U_3)=\mathbf{A}^{1\times\ell_1}P+U_2\\ &\Longrightarrow\mathbf{A}^{1\times\ell_1}/U_1\xrightarrow{(\circ P)_{\mathrm{ind}}}\mathbf{A}^{1\times\ell_2}/U_2\xrightarrow{(\circ\,\mathrm{id}_{\ell_2})_{\mathrm{ind}}}\mathbf{A}^{1\times\ell_2}/U_3'\to0 \end{aligned}$$

is exact and ψ factorizes as

$$\begin{aligned} &\psi=(\circ Q)_{\mathrm{ind}}:\mathbf{A}^{1\times\ell_2}/U_2\xrightarrow{(\circ\,\mathrm{id}_{\ell_2})_{\mathrm{ind}}}\mathbf{A}^{1\times\ell_2}/U_3'\xrightarrow{(\circ Q)_{\mathrm{ind},2}}\mathbf{A}^{1\times\ell_3}/U_3\\ &\Longrightarrow Q\circ:\ \mathcal{B}(U_3)\xrightarrow{Q\circ}\mathcal{B}(U_3')\subseteq\mathcal{B}(U_2). \end{aligned}$$

But

$$\begin{aligned} &\mathbf{A}^{1\times\ell_2}/U_3'\xrightarrow{(\circ Q)_{\mathrm{ind},2}}\mathbf{A}^{1\times\ell_3}/U_3\text{ is injective}\\ &\underset{\text{Lemma 2.9}}{\Longrightarrow}Q\circ:\mathcal{B}(U_3)\to\mathcal{B}(U_3')\text{ is an epimorphism}\\ &\Longrightarrow\mathrm{im}\,(Q\circ:\mathcal{B}(U_3)\to\mathcal{B}(U_2))=\mathcal{B}(U_3'). \end{aligned}$$

Hence it remains to show that $\mathcal{B}(U_3')=\ker(P\circ:\mathcal{B}(U_2)\to\mathcal{B}(U_1))$. Let

$$U_2=\mathbf{A}^{1\times p_2}R_2\Longrightarrow U_3'=\mathbf{A}^{1\times\ell_1}P+\mathbf{A}^{1\times p_2}R_2=\mathbf{A}^{1\times(\ell_1+p_2)}\left(\begin{smallmatrix}P\\R_2\end{smallmatrix}\right).$$

As usual choose n_0 such that none of the constructed matrices has a pole $t\geq n_0$. Then

$$\begin{aligned} &\mathcal{B}(U_2)=\mathrm{cl}\left((\mathcal{B}(R_2,n))_{n\geq n_0}\right),\ \mathcal{B}(U_3')=\mathrm{cl}\left((\mathcal{B}(\left(\begin{smallmatrix}P\\R_2\end{smallmatrix}\right),n))_{n\geq n_0}\right)\\ &\forall n\geq n_0:\ \mathcal{B}(R_2,n)=\left\{w\in(\mathbb{C}^{n+\mathbb{N}})^{\ell_2};\ R_2\circ w=0\right\},\\ &\forall n\geq n_0:\ \mathcal{B}(\left(\begin{smallmatrix}P\\R_2\end{smallmatrix}\right),n)=\left\{w\in(\mathbb{C}^{n+\mathbb{N}})^{\ell_2};\ R_2\circ w=0,\ P\circ w=0\right\}=\\ &\qquad=\ker\left(P\circ:\mathcal{B}(R_2,n)\to(\mathbb{C}^{n+\mathbb{N}})^{\ell_1}\right)\\ &\Longrightarrow\mathcal{B}(U_3')=\mathrm{cl}\left(\mathcal{B}(\left(\begin{smallmatrix}P\\R_2\end{smallmatrix}\right),n)\right)=\ker\left(P\circ:\mathcal{B}(U_2)\to\mathcal{B}(U_1)\right). \end{aligned}$$

□

2.5 Autonomous behaviors

We prove the analogue of the cogenerator property of the standard signal modules for the behaviors of this paper and simultaneously characterize autonomous behaviors. A finitely generated $\mathbf{A}$-module $M_1 = \mathbf{A}^{1\times\ell_1}/U_1 = \mathbf{A}^{1\times\ell_1}/\mathbf{A}^{1\times p_1}R_1$ is isomorphic to a direct sum of cyclic modules [19, Cor. 5.7.19]. Hence there is an isomorphism

$$\begin{gathered} M_1 = \mathbf{A}^{1\times\ell_1}/U_1 \cong \mathbf{A}/\mathbf{A}f_1 \times \cdots \times \mathbf{A}/\mathbf{A}f_r \times \mathbf{A}^{1\times(\ell_2-r)} = \mathbf{A}^{1\times\ell_2}/U_2 =: M_2 \\ \text{where } r\geq 0,\ f_i\in\mathbf{A},\ \deg_q(f_i)>0,\ U_2=\mathbf{A}^{1\times\ell_2}R_2, \\ R_2 = \operatorname{diag}(f_1,\cdots,f_r,0,\cdots,0)\in\mathbf{A}^{\ell_2\times\ell_2}. \end{gathered} \tag{72}$$

A special matrix R_2 is called the *Jacobson/Smith/Teichmüller/Nakayama*-form of R_1 and is computed with the help of euclidean division that is applicable in $\mathbf{A}$ and makes it a *euclidean ring*. If $R_1 \in \mathbb{Q}(t)[q;\alpha]^{p_1\times\ell_1} \subset \mathbf{A}^{p_1\times\ell_1} = \mathbb{C}(t)[q;\alpha]^{p_1\times\ell_1}$ then R_2 and thus the f_i can be computed with the the *Jacobson package* of [7]. The functor $\mathbf{A}^{1\times\ell}/U \mapsto \mathcal{B}(U)$ is applied to the first line of (72) and implies the isomorphism

$$\begin{gathered} \mathcal{B}(U_1)\cong\mathcal{B}(U_2)=\mathcal{B}(\mathbf{A}f_1)\times\cdots\times\mathcal{B}(\mathbf{A}f_r)\times\mathcal{W}^{\ell_2-r} \\ \text{where } \mathcal{W}=\mathcal{B}(0)=\operatorname{cl}\left(\cdots\mathbb{C}^{n+\mathbb{N}}\xrightarrow{\operatorname{proj}_n}\mathbb{C}^{n+1+\mathbb{N}}\to\cdots\right). \end{gathered} \tag{73}$$

The systems $\mathcal{B}(\mathbf{A}f_j)$ are particularly simple: Consider, more generally, any

$$\begin{aligned} &g = g_dq^d+\cdots+g_0\in\mathbf{A},\ \deg_q(g)=d,\ \text{ i.e., } g_d\neq 0 \\ &\Longrightarrow \mathbb{C}(t)^{1\times d}\underset{\mathbb{C}(t)}{\cong}\mathbf{A}/\mathbf{A}g,\ (a_0,\cdots,a_{d-1})\mapsto\sum_{i=0}^{d-1}a_iq^i+\mathbf{A}g,\ a_i\in\mathbb{C}(t), \\ &\Longrightarrow d=\dim_{\mathbb{C}(t)}(\mathbf{A}/\mathbf{A}g)<\infty. \end{aligned} \tag{74}$$

The preceding isomorphism follows via euclidean division. Choose n_0 such that no $t\geq n_0$ is a pole of any g_i or a zero of g_d. For all $n\geq n_0$ we obtain the isomorphisms

$$\begin{gathered} \mathcal{B}(g,n)=\left\{w\in\mathbb{C}^{n+\mathbb{N}};\ \forall t\geq n:\ g_d(t)w(t+d)+\cdots+g_0(t)w(t)=0\right\}\cong\mathbb{C}^d \\ w\mapsto(w(0),\cdots,w(d-1))^\top. \end{gathered} \tag{75}$$

We conclude

$$\begin{gathered} \forall n\geq n_0:\ \dim_{\mathbb{C}}(\mathcal{B}(g,n))=d \text{ and} \\ \left(\mathcal{B}(\mathbf{A}g)=\operatorname{cl}\left((\mathcal{B}(g,n))_{n\geq n_0}\right)=0\iff d=0\right). \end{gathered} \tag{76}$$

Theorem 2.11. *If $M_1=\mathbf{A}^{1\times\ell_1}/U_1$ is nonzero then so is $\mathcal{B}(U_1)$.*

Proof. In (73) $\mathcal{W}$ is nonzero and so are the behaviors $\mathcal{B}(\mathbf{A}f_j)$ of $\mathbb{C}$-dimension $\deg_q(f_j)>0$. Hence $\mathcal{B}(U_1)$ is zero if and only if $\ell_2=0$. Equation (72) implies likewise that $M_1=0$ if and only if $\ell_2=0$. □

Corollary 2.12. *For $M_i=\mathbf{A}^{1\times\ell_i}/U_i,\ i=1,2,$ the $\mathbb{C}$-linear map*

$$\operatorname{Hom}_{\mathbf{A}}(M_1,M_2)\to\mathfrak{B}(\mathcal{B}(U_2),\mathcal{B}(U_1)),\ \varphi=(\circ P)_{\text{ind}}\mapsto\mathcal{B}(\varphi)=P\circ, \tag{77}$$

is injective, and therefore

$$\operatorname{Hom}_{\mathbf{A}}(M_1,M_2)\cong\operatorname{Hom}(\mathcal{B}(U_2),\mathcal{B}(U_1)):=\{\mathcal{B}(\varphi);\ \varphi:M_1\to M_2\}. \tag{78}$$

Therefore the exact functor ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}} \to \mathfrak{B},\ \mathbf{A}^{1\times\ell}/U \mapsto \mathcal{B}(U)$, *induces a duality between* ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ *and the subcategory* $\{$*LTV-behaviors*$\}$ *of* $\mathcal{B}$ *whose objects are the behaviors and whose morphisms are the behavior morphisms* $\mathcal{B}(\varphi)$. *The proof of Thm. 1.6 is thus complete.*

Proof. Let $\varphi : M_1 = \mathbf{A}^{1\times\ell_1}/U_1 \to M_2 = \mathbf{A}^{1\times\ell_2}/U_2$ and $\mathcal{B}(\varphi) = 0$. The linear map φ can be factorized as

$$M_1 \xrightarrow{\varphi_1} M_3 = \mathbf{A}^{1\times\ell_3}/U_3 \xrightarrow{\varphi_2} M_2,\ \varphi = \varphi_2\varphi_1,$$

where φ_1 is an epimorphism and φ_2 a monomorphism. This factorization is obtained by the corresponding one in ${}_{\mathbf{A}}\mathbf{Mod}$ with $M_3 := \varphi(M_1)$ and by the choice of a list $\mathbf{w}_3$ of generators of M_3 that induces the representation $M_3 = \mathbf{A}^{1\times\ell_3}/U_3$. This factorization implies $0 = \mathcal{B}(\varphi) = \mathcal{B}(\varphi_1)\mathcal{B}(\varphi_2)$ with an epimorphism $\mathcal{B}(\varphi_2)$ and a monomorphism $\mathcal{B}(\varphi_1)$ since $\mathbf{A}^{1\times\ell}/U \mapsto \mathcal{B}(U)$ is exact. We infer $\mathcal{B}(U_3) = 0$, hence $M_3 = 0$ by Thm. 2.11 and $\varphi = 0$.

□

The torsion submodule $\mathrm{tor}(M)$ of M is the set of all elements $x \in M$ that are annihilated by some nonzero $g \in \mathbf{A}$ ($gx = 0$). The isomorphism (72) then implies the isomorphisms

$$\begin{aligned}&\mathrm{tor}(M_1) \underset{\mathbf{A}}{\cong} \mathrm{tor}(M_2) = \oplus_{j=1}^{r}\mathbf{A}/\mathbf{A}f_j \underset{\mathbb{C}(t)}{\cong} \mathbb{C}(t)^{1\times d},\ d := \sum_{j=1}^{r}\deg_q(f_j),\ \text{and}\\ &M_1/\,\mathrm{tor}(M_1) \underset{\mathbf{A}}{\cong} M_2/\,\mathrm{tor}(M_2) \underset{\mathbf{A}}{\cong} \mathbf{A}^{\ell_2-r}.\end{aligned} \tag{79}$$

The module M_1 is called *torsion* (adjective) or a *torsion module* if $M_1 = \mathrm{tor}(M_1)$ and *torsionfree* if $\mathrm{tor}(M_1) = 0$. In the latter case $M_1 \cong \mathbf{A}^{\ell_2-r}$ is free.
Since $\mathbf{A}$ is a noetherian domain it has the quotient skew-field [19, Thm. 2.1.15]

$$\mathbf{K} := \mathrm{quot}(\mathbf{A}) := \left\{a^{-1}b;\ a,b \in \mathbf{A},\ a \neq 0\right\} = \left\{ba^{-1};\ a,b \in \mathbf{A},\ a \neq 0\right\} \supset \mathbf{A}. \tag{80}$$

The rank of a matrix $R \in \mathbf{K}^{k\times\ell}$ is defined by

$$\mathrm{rank}(R) = \dim\left({}_{\mathbf{K}}(\mathbf{K}^{1\times k}R)\right) = \dim\left((R\mathbf{K}^{\ell})_{\mathbf{K}}\right) \tag{81}$$

where the row space resp. column space of R are a left resp. a right $\mathbf{K}$-space. For $R \in \mathbf{A}^{k\times\ell}$, $U := \mathbf{A}^{1\times k}R$ and $M := \mathbf{A}^{1\times\ell}/U$ there is also the quotient module

$$\begin{aligned}&\mathbf{K}\otimes_{\mathbf{A}} M \underset{\text{ident.}}{=} \left\{a^{-1}x;\ 0 \neq a \in \mathbf{A},\ x \in M\right\} \underset{\text{ident.}}{=} \mathbf{K}^{1\times\ell}/\mathbf{K}U,\\ &x = \xi + U,\ \xi \in \mathbf{A}^{1\times\ell} \subset \mathbf{K}^{1\times\ell},\ a^{-1}x = a^{-1}\otimes x = a^{-1}\xi + \mathbf{K}U.\end{aligned} \tag{82}$$

The canonical map $\mathrm{can} : M \to \mathbf{K}\otimes_{\mathbf{A}} M,\ x \mapsto 1\otimes x$, has the kernel $\mathrm{tor}(M)$. The rank of M is defined by

$$\begin{aligned}&\mathrm{rank}(M) := \dim_{\mathbf{K}}(\mathbf{K}\otimes_{\mathbf{A}} M),\ \text{hence}\\ &\mathrm{rank}(R) = \dim({}_{\mathbf{A}}U) = \dim({}_{\mathbf{K}}\mathbf{K}U)\ \text{and}\ \mathrm{rank}(R) + \mathrm{rank}(M) = \ell.\end{aligned} \tag{83}$$

Lemma 2.13. *For the data of* (72) *and* (73) *the following properties are equivalent for the module* $M_1 = \mathbf{A}^{1\times\ell_1}/U_1,\ U_1 = \mathbf{A}^{1\times p_1}R_1$, *and behavior* $\mathcal{B}(U_1) = \mathrm{cl}\left((\mathcal{B}(R_1,n))_{n\geq n_0}\right)$*:*

(i) $\operatorname{rank}(M_1) = 0$ *or* $\operatorname{rank}(R_1) = \ell_1$.

(ii) M_1 *is a torsion module.*

(iii) $d := \dim_{\mathbb{C}(t)}(M_1) < \infty$.

(iv) *There are* $n_0, d \in \mathbb{N}$ *such that* $\forall n \geq n_0 : \ \dim_{\mathbb{C}}(\mathcal{B}(R_1, n)) = d$.

Proof. $(i) \iff (ii)$: obvious. $(ii) \iff (iii)$: (79) with $d = \sum_{j=1}^{r} \deg_q(f_j)$. $(iii) \iff (iv)$: For sufficiently large n_0 and $n \geq n_0$ we have

$$\mathcal{B}(R_1, n) \cong \mathcal{B}(R_2, n) = \mathcal{B}(f_1, n) \times \cdots \mathcal{B}(f_r, n) \times \mathcal{B}(0, n)^{q_2 - r},$$
$$\dim_{\mathbb{C}} \left(\mathcal{B}(f_1, n) \times \cdots \mathcal{B}(f_r, n)\right) \underset{(76)}{=} \sum_{j=1}^{r} \deg_q(f_j) = d, \quad \text{but} \dim_{\mathbb{C}}(\mathcal{B}(0, n)) = \infty. \tag{84}$$

□

Definition 2.14. If the conditions of Lemma 2.13 are satisfied the behavior $\mathcal{B}(U_1)$ is called *autonomous*.

Definition 2.15. Consider a f.g. module $M = \mathbf{A}^{1 \times \ell}/U$ with $U = \mathbf{A}^{1 \times p} R$, $R \in \mathbf{A}^{p \times \ell}$, $n_0 + \mathbb{N} \subseteq \operatorname{dom}(R)$ and the associated behaviors

$$\begin{gathered}\forall n \geq n_0 : \ \mathcal{B}(R, n) := \left\{ w \in \left(\mathbb{C}^{n + \mathbb{N}}\right)^{\ell}; \ \forall t \geq n : \ \sum_{j=0}^{k} R_j(t) w(t + j) = 0 \right\}, \\ \mathcal{B}(U) = \operatorname{cl}\left((\mathcal{B}(R, n))_{n \geq n_0}\right).\end{gathered} \tag{85}$$

The behavior $\mathcal{B}(U)$ is called *trajectory-autonomous* (t-autonomous) of *memory size* d if there are $n_1 \geq n_0$ and $d \in \mathbb{N}$ such that

$$\forall n \geq n_1 : \ \mathcal{B}(R, n) \to \mathbb{C}^{dq}, \ w \mapsto (w(n), \cdots, w(n + d - 1)), \ \text{is injective,} \tag{86}$$

but not necessarily bijective. This means that for sufficiently large n all trajectories $w \in \mathcal{B}(R, n)$ with initial time n are uniquely determined by the *initial data* $x(n) := (w(n), \cdots, w(n + d - 1))$. The number d is obviously not unique.

Corollary 2.16. *The behaviors* $\mathcal{B}(\mathbf{A}g)$ *from* (74) *resp.* $\mathcal{B}(\mathbf{A}f_1) \times \cdots \times \mathcal{B}(\mathbf{A}f_r)$ *from* (73) *are obviously t-autonomous of memory sizes*

$$\deg_q(g) \ \textit{resp.} \ \max\left\{\deg_q(f_j); \ j = 1, \cdots, r\right\}. \tag{87}$$

Lemma 2.17. *Trajectory-autonomy is preserved by isomorphisms.*

Proof. Consider two isomorphic f.g. modules and their associated isomorphic behaviors (cf. Cor. 2.6):

$$\begin{gathered}M_i := \mathbf{A}^{1 \times \ell_i}/U_i, \ U_i = \mathbf{A}^{1 \times p_i} R_i, \ R_i \in \mathbf{A}^{p_i \times \ell_i}, \\ \varphi = (\circ P)_{\text{ind}} : M_1 \cong M_2, \ \mathcal{B}(\varphi) : \mathcal{B}(U_2) \cong \mathcal{B}(U_1).\end{gathered} \tag{88}$$

Let $P = \sum_{i=0}^{k} P_j q^j$. Assume that $\mathcal{B}(U_1)$ is t-autonomous with memory size d_1 and define $d_2 := d_1 + k$. There is an n_1 such that

$$\begin{aligned} &\forall n \geq n_1 :\ P\circ : \mathcal{B}(R_2, n) \cong \mathcal{B}(R_1, n),\ w_2 \mapsto w_1 := P \circ w_2, \\ &\text{and } \forall t \geq n \geq n_1 :\ w_1(t) = \sum_{j=0}^{k} P_j(t) w_2(t+j),\ \text{especially} \\ &\forall i = 0, \cdots, d_1 - 1 :\ w_1(n+i) = \sum_{j=0}^{k} P_j(n+i) w_2(n+i+j). \end{aligned} \tag{89}$$

If $0 \leq i \leq d_1 - 1$ and $0 \leq j \leq k$ then $n \leq n + i + j \leq n + d_1 - 1 + k = n + d_2 - 1$. If $w_2(n) = \cdots = w_2(n + d_2 - 1) = 0$ equation (89) implies $w_1(n) = \cdots = w_1(n + d_1 - 1)$. Since $\mathcal{B}(U_1)$ has memory size d_1 this implies $w_1 = 0$ and hence $w_2 = 0$ since $P\circ$ in (89)is bijective. We conclude that $\mathcal{B}(U_2)$ has memory size d_2. □

Theorem 2.18. *A behavior $\mathcal{B}(U_1)$ is t-autonomous if and only if it is autonomous.*

Proof. This follows directly from the isomorphism (73) and the preceding lemma since $\mathcal{B}(\mathbf{A}f_1) \times \cdots \times \mathcal{B}(\mathbf{A}f_r)$ is t-autonomous according to Cor. 2.16, but $\mathcal{W} = \mathcal{B}(0)$ is obviously not. □

3 Exponentially stable (e.s.) behaviors

The main goal of Section 3 is the proof of Thm. 1.8.

3.1 Exponential stability for state space behaviors

We first recall the notion of *uniform exponential stability* for state space systems. We endow all $\mathbb{C}^\ell,\ \ell \geq 0$, and matrix spaces $\mathbb{C}^{\ell \times \ell}$ with the maximum norm

$$\begin{aligned} \forall v = (v_1, \cdots, v_\ell)^\top \in \mathbb{C}^\ell :\ \|v\| := \max\{|v_i|;\ 1 \leq i \leq \ell\}, \\ \forall A \in \mathbb{C}^{\ell \times \ell} :\ \|A\| := \max\left\{\|Av\|;\ v \in \mathbb{C}^\ell,\ \|v\| = 1\right\}, \\ \text{hence } \forall v \in \mathbb{C}^\ell :\ \|Av\| \leq \|A\|\|v\|. \end{aligned} \tag{90}$$

Definition 3.1. (cf. [25, Def. 22.5], (2)) A state space system

$$\begin{aligned} w(t+1) = A(t)w(t),\ A \in \left(\mathbb{C}^{n_0+\mathbb{N}}\right)^{\ell \times \ell}, \text{ or} \\ \forall t \geq n \geq n_0 :\ w(t) = \Phi(t,n)w(n),\ \Phi(t,n) := A(t-1)\cdots A(n), \end{aligned} \tag{91}$$

is called *uniformly exponentially stable* (u.e.s.) if

$$\begin{aligned} \exists c \geq 1 \exists \rho \in \mathbb{R} \text{ with } 0 < \rho < 1 \forall t \geq n \geq n_0 \forall w \in \mathcal{B}(\mathrm{id}_\ell\, q - A, n) : \\ \|w(t)\| \leq c\rho^{t-n}\|w(n)\| \text{ or, equivalently, } \|\Phi(t,n)\| \leq c\rho^{t-n}. \end{aligned} \tag{92}$$

Notice that Rugh [25] admits arbitrary $A \in \left(\mathbb{C}^{n_0+\mathbb{N}}\right)^{\ell \times \ell}$ (mostly $n_0 = 0$), hence $\mathrm{id}_\ell\, q - A \in \mathbf{B}(n_0)^{\ell \times \ell}$ (cf. (20)). The behavioral theory of this paper cannot be extended from the field $\mathbb{C}(t)$ to the nonnoetherian ring $\mathbb{C}^{n_0+\mathbb{N}}$ with many zero-divisors. The following example shows that u.e.s. is not preserved by behavior isomorphisms and is therefore unsuitable for the behavioral LTV-theory of this paper.

Example 3.2. Let

$$\begin{aligned}
&\ell := 2,\ A := \left(\begin{smallmatrix}\rho_1 & 0\\ 0 & \rho_2\end{smallmatrix}\right),\ 0<\rho_1<\rho_2<1,\ \ R := q\,\mathrm{id}_2 - A,\ n\geq 0,\\
&\mathcal{B}_1(n) := \mathcal{B}_1(R,n) := \left\{w = \left(\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right)\in \left(\mathbb{C}^{n+\mathbb{N}}\right)^2; w(t+1) = Aw(t)\right\} = \mathbb{C}\rho_1^{t-n}\oplus\mathbb{C}\rho_2^{t-n}\\
&X(t) := \left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)\in \mathrm{Gl}_2(\mathbb{C}[t])\subset \mathrm{Gl}_2(\mathbf{A}),\ B(t) := X(t+1)AX(t)^{-1}\\
&v(t) := X(t)w(t),\ \mathcal{B}_2 := X\circ\mathcal{B}_1,\\
&\mathcal{B}_2(n) := X\circ\mathcal{B}_1(n) = \{v(t) = X(t)w(t);\ w(t+1) = Aw(t)\} =\\
&\qquad = \left\{v\in\left(\mathbb{C}^{n+\mathbb{N}}\right)^2;\ v(t+1) = B(t)v(t)\right\}.
\end{aligned} \tag{93}$$

Notice that both $\mathcal{B}_1$ and $\mathcal{B}_2$ are state space behaviors as in [25], $\mathcal{B}_1$ is an LTI-behavior with an asymptotically stable matrix and $\mathcal{B}_2$ is an LTV-behavior. Obviously

$$X\circ : \mathcal{B}_1(n)\cong\mathcal{B}_2(n),\ w\mapsto X\circ w = Xw,$$

is an isomorphism of state space behaviors. But

$$\begin{aligned}
&X(t)^{-1} = \left(\begin{smallmatrix}1 & -t\\ 0 & 1\end{smallmatrix}\right)\Longrightarrow B(t) = \left(\begin{smallmatrix}1 & t+1\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\rho_1 & 0\\ 0 & \rho_2\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & -t\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}\rho_1 & \rho_2+(\rho_2-\rho_1)t\\ 0 & \rho_2\end{smallmatrix}\right)\\
&\quad\Longrightarrow \forall t\geq n\geq 1\forall v\in\mathcal{B}_2(n):\ v_1(t+1) = \rho_1 v_1(t)+\rho_2 v_2(t)+(\rho_2-\rho_1)tv_2(t)\\
&\quad\Longrightarrow \forall n\geq 1\forall v\in\mathcal{B}_2(n):\ v_1(n+1) = \rho_1 v_1(n)+\rho_2 v_2(n)+(\rho_2-\rho_1)nv_2(n).
\end{aligned} \tag{94}$$

For any $n\geq 1$ let $v^n := (v_1^n, v_2^n)^\top\in\mathcal{B}_2(n)$ be the unique trajectory with

$$\begin{aligned}
&v_1^n(n) := v_2^n(n) := 1 \text{ and hence } \|v^n(n)\| = \max(|v_1^n(n)|,|v_2^n(n)|) = 1\\
&\quad\Longrightarrow v_1^n(n+1) = \rho_1+\rho_2+(\rho_2-\rho_1)n \Longrightarrow (\rho_2-\rho_1)n\leq\rho_1+\rho_2+|v_1^n(n+1)|.
\end{aligned} \tag{95}$$

Assume that $\mathcal{B}_2$ is u.e.s. Then there are $c\geq 1$ and $\rho,\ 0<\rho<1$, with

$$\begin{aligned}
&\forall t\geq n\geq 1:\ |v_1^n(t)|\leq\|v^n(t)\|\leq c\rho^{t-n}\|v^n(n)\| = c\rho^{t-n}\\
&\Longrightarrow |v_1^n(n+1)|\leq c\rho \underset{(96)}{\Longrightarrow} \forall n\geq 1:\ (\rho_2-\rho_1)n\leq\rho_1+\rho_2+c\rho.
\end{aligned} \tag{96}$$

This is a contradiction and thus $\mathcal{B}_2$ is not u.e.s, but, of course, e.s.. A nontrivial computation shows that for all ρ_3 with $\rho_2<\rho_3<1$ there is a $c_3\geq 1$ such that

$$\forall t\geq n\geq 1:\ \|\Phi_B(t,n)\| = \|B(t-1)\cdots B(n)\|\leq\varphi(n)\rho_3^{t-n} \text{ with } \varphi(n) := c_3 n. \tag{97}$$

The initial condition $x(n) = (w(n),\cdots,w(n+d-1)) = 0$ (cf. the definition) and (16) imply $w = 0$. So an e.s. behavior is autonomous. For $p\in\mathbb{C}[t]$ and $h\in\mathbb{C}(t)$ with $n_0+\mathbb{N}\subseteq\mathrm{dom}(h)$ the sequences $(p(n))_{n\geq 0}\in\mathbb{C}^{\mathbb{N}}$ and $(h(n))_{n\geq n_0}$ are obviously p.g.s.. This implies that for any matrix $A\in\mathbb{C}(t)^{p\times\ell}$ and $n_0+\mathbb{N}\subseteq\mathrm{dom}(A)$ also the norm sequence $(\|A(n)\|)_{n\geq n_0}$ is a p.g.s.. The sum and product of p.g.s. are again such.

Corollary 3.3. *Consider a matrix $A(t)\in\mathbb{C}(t)^{\ell\times\ell}$ that has no poles $t\geq n_0$, $R := q\,\mathrm{id}_q - A\in\mathbf{A}^{\ell\times\ell}$ and for all $n\geq n_0$ the associated state space behaviors*

$$\mathcal{B}(R,n) := \left\{w\in\left(\mathbb{C}^{n+N}\right)^\ell;\ \forall t\geq n:\ w(t+1) = A(t)w(t)\right\}\cong\mathbb{C}^\ell,\ w\mapsto w(n).$$

Then $\mathcal{B} = \operatorname{cl}((\mathcal{B}(R,n))_{n\geq n_0})$ *is e.s. if and only if there are* $n_1 \geq n_0$*, a p.g.s.* $\varphi_2 > 0$ *in* $\mathbb{R}^{n_1+\mathbb{N}}$ *and* ρ_2 *with* $0 < \rho_2 < 1$ *such that*

$$\begin{aligned} &\forall t \geq n \geq n_1 \forall w \in \mathcal{B}(R,n): \ \|w(t)\| \leq \rho^{t-n}\varphi_2(n)\|w(n)\| \text{ or} \\ &\|\Phi(t,n)\| \leq \varphi_2(n)\rho^{t-n}, \ \Phi(t,n) := A(t-1)\cdots A(n). \end{aligned} \tag{98}$$

The e.s. here differs from u.e.s. in Def. 3.1 by the additional p.g. factor $\varphi_2(n)$ instead of a constant.

Proof. It has only to be shown that (98) is necessary. With $x(n) := (w(n), \cdots, w(n+d-1))$ and $\|x(n)\| = \max\{\|w(n+i)\|;\ 0 \leq i \leq d-1\}$ equation (16) furnishes

$$\forall t \geq n \geq n_1 \geq n_0 \forall w \in \mathcal{B}(R,n): \ \|w(t)\| \leq \rho^{t-n}\varphi_1(n)\|x(n)\|.$$

The norm sequence $(\|A(n)\|)_{n\geq n_1}$ is a p.g.s. and so are

$$\begin{aligned} &\|\Phi(n+i,n)\| \leq \|A(n+i-1)\|\cdots\|A(n)\| \text{ and} \\ &\varphi(n) := \max\{\|\Phi(n+i,n)\|;\ 0 \leq i \leq d-1\}. \end{aligned}$$

But

$$\begin{aligned} &w(n+i) = \Phi(n+i,n)w(n) \\ &\Longrightarrow \|w(n+i)\| \leq \|\Phi(n+i,n)\|\|w(n)\| \leq \varphi(n)\|w(n)\| \Longrightarrow \|x(n)\| \leq \varphi(n)\|w(n)\| \\ &\Longrightarrow \forall t \geq n \geq n_1: \ \|w(t)\| \leq \rho^{t-n}\varphi_1(n)\|x(n)\| \leq \rho^{t-n}\varphi_1(n)\varphi(n)\|w(n)\| \\ &= \rho^{t-n}\varphi_2(n)\|w(n)\| \text{ with the p.g.s. } \varphi_2(n) := \varphi_1(n)\varphi(n). \end{aligned}$$

□

Corollary 3.4. *If in the preceding corollary the matrix* A *is constant, i.e.,* $A \in \mathbb{C}^{\ell\times\ell} \subset \mathbb{C}(t)^{\ell\times\ell}$*, then* $w(t+1) = Aw(t)$ *is e.s. if and only if the spectrum* $\operatorname{spec}(A)$ *is contained in the open unit disc* $\mathbf{D} := \{z \in \mathbb{C};\ |z| < 1\}$.

Proof. It is well-known that $\operatorname{spec}(A) \subset \mathbf{D}$ implies (98) with a constant φ_2. Let, conversely, (98) be satisfied and assume that λ is an eigenvalue of A with nonzero eigenvector $w(n)$. Then

$$\begin{aligned} w(t) = A^{t-n}w(n) = \lambda^{t-n}w(n) \text{ and } |\lambda|^{t-n}\|w(n)\| = \|w(t)\| \leq \varphi(n)\rho^{t-n}\|w(n)\| \\ \text{with } 0 < \rho < 1 \Longrightarrow \lim_{t\to\infty} \lambda^t = 0 \Longrightarrow |\lambda| < 1. \end{aligned}$$

□

We apply Cor. 3.3 to any $f := f_d q^d + \cdots + f_0 \in \mathbf{A},\ \deg_q(f) = d$, and $n_0 \in \mathbb{N}$ such that no $t \geq n_0$ is a pole of any f_i or a zero of f_d. We first construct the usual isomorphic state space system. Define

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -f_d^{-1}f_0 & -f_d^{-1}f_1 & \cdots & \cdots & \cdots & \cdots & -f_d^{-1}f_{d-1} \end{pmatrix} \in \mathbb{C}(t)^{d\times d},\ R := q\,\mathrm{id}_d - A \in \mathbf{A}^{d\times d}. \tag{99}$$

With $\mathbf{q}_d := (1, q, \cdots, q^{d-1})^\top$ and $\delta_{0,d} := (\overset{0}{1}, 0, \cdots, \overset{d-1}{0})$ these data imply the standard $\mathbf{A}$-linear isomorphism

$$\begin{aligned}
&(\circ\delta_{0,d})_{\mathrm{ind}} : \mathbf{A}/\mathbf{A}f \overset{\cong}{\longleftrightarrow} \mathbf{A}^{1\times d}/\mathbf{A}^{1\times d}R : (\circ\mathbf{q}_d)_{\mathrm{ind}},\\
&\eta + \mathbf{A}f \to \eta\delta_{0,d} + \mathbf{A}^{1\times d}R = (\overset{0}{\eta}, 0, \cdots, \overset{d-1}{0}) + \mathbf{A}^{1\times d}R\\
&\xi\mathbf{q}_d + \mathbf{A}f = \sum_{i=0}^{d-1}\xi_i q^i + \mathbf{A}f \leftarrow \xi + \mathbf{A}^{1\times d}(q\,\mathrm{id}_d - A).
\end{aligned} \tag{100}$$

For $n \geq n_0$ this module isomorphism gives rise to the behavior isomorphism

$$\begin{aligned}
&\mathcal{B}(f,n) = \left\{w \in \mathbb{C}^{n+\mathbb{N}};\ \forall t \geq n:\ f_d(t)w(t+d) + \cdots + f_0(t)w(t) = 0\right\} \cong\\
&\mathcal{B}(q\,\mathrm{id}_d - A, n) = \left\{x \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^d;\ \forall t \geq n:\ x(t+1) = A(t)x(t)\right\} \cong \mathbb{C}^d,\\
&w(t) = x_0(t) \longleftrightarrow x(t) = (x_0(t), \cdots, x_{d-1}(t))^\top = (w(t), \cdots, w(t+d-1))^\top\\
&\longleftrightarrow x(n) = (w(n), \cdots, w(n+d-1))^\top.
\end{aligned} \tag{101}$$

The preceding isomorphisms and Cor. 3.3 imply

Corollary 3.5. *For $f := f_d q^d + \cdots + f_0,\ f_d \neq 0$, the behavior $\mathcal{B}(\mathbf{A}f)$ is e.s. if and only if there are n_1, a p.g.s. φ with $\varphi(n) > 0$ and ρ ($0 < \rho < 1$) such that*

$$\forall t \geq n \geq n_1 \forall w \in \mathcal{B}(f,n):\ |w(t)| \leq \varphi(n)\rho^{t-n}\max\left\{|w(n+i)|; 0 \leq i \leq d-1\right\}. \tag{102}$$

The isomorphisms (100) and (101) can be generalized in the following fashion. Consider an arbitrary torsion module

$$M = \mathbf{A}^{1\times\ell}/\mathbf{A}^{1\times p}R \cong \bigoplus_{j=1}^{r}\mathbf{A}/\mathbf{A}f_j,\ 0 \neq f_j \in \mathbf{A} = \mathbb{C}(t)[q;\alpha],\ d_j := \deg_q(f_j) > 0, \tag{103}$$

and let $A_j \in \mathbb{C}(t)^{d_j\times d_j}$ be derived from f_j like A from f in (99). We define $R_1 := \mathrm{diag}(f_1, \cdots, f_r) \in \mathbf{A}^{r\times r}$, $d := \sum_{j=1}^{r} d_j$, $A := \mathrm{diag}(A_1, \cdots, A_r) \in \mathbb{C}(t)^{d\times d}$ and $R_2 = q\,\mathrm{id}_d - A$ and obtain the isomorphisms

$$M = \mathbf{A}^{1\times\ell}/\mathbf{A}^{p\times q}R \cong \mathbf{A}^{1\times r}/\mathbf{A}^{1\times r}R_1 \cong \mathbf{A}^{1\times d}/\mathbf{A}^{1\times d}R_2 \tag{104}$$

where the second isomorphism in (104) is explicitly given by

$$\begin{array}{rcl}
(\eta_1, \cdots, \eta_r) + \mathbf{A}^{1\times r}R_1 & \to & (\eta_1\delta_{0,d_1}, \cdots, \eta_r\delta_{0,d_r}) + \mathbf{A}^{1\times d}R_2,\\
(\xi_1\mathbf{q}_{d_1}, \cdots, \xi_r\mathbf{q}_{d_r}) + \mathbf{A}^{1\times r}R_1 & \leftarrow & (\xi_1, \cdots, \xi_r) + \mathbf{A}^{1\times d}R_2.
\end{array} \tag{105}$$

Theorem 3.6. *For the torsion module M from* (103) *and the derived data from* (104) *there are matrices $P \in \mathbf{A}^{\ell\times d}, Q \in \mathbf{A}^{d\times\ell}$ that induce an isomorphism $(\circ P)_{\mathrm{ind}}$ and its inverse $(\circ Q)_{\mathrm{ind}}$ as follows:*

$$\begin{array}{rccll}
(\circ P)_{\mathrm{ind}}: & M = \mathbf{A}^{1\times\ell}/\mathbf{A}^{1\times p}R & \overset{\cong}{\longleftrightarrow} & \mathbf{A}^{1\times d}/\mathbf{A}^{1\times d}(q\,\mathrm{id}_d - A) & : (\circ Q)_{\mathrm{ind}}\\
& \omega + \mathbf{A}^{1\times p}R & \longmapsto & \omega P + \mathbf{A}^{1\times d}(q\,\mathrm{id}_d - A) &\\
& \xi Q + \mathbf{A}^{1\times p}R & \longleftarrow & \xi + \mathbf{A}^{1\times d}(q\,\mathrm{id}_d - A). &
\end{array} \tag{106}$$

For sufficiently large n_0 and $n \geq n_0$ these isomorphisms induce behavior isomorphisms

$$\begin{aligned} &Q\circ : \mathcal{B}(R,n) := \left\{ w \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell};\ R \circ w = 0 \right\} \overset{\cong}{\longleftrightarrow} \\ &\qquad \mathcal{B}(q\,\mathrm{id}_d - A, n) = \left\{ x \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{d};\ x(t+1) = A(t)x(t) \right\} : P\circ, \\ &\qquad\qquad w = P \circ x \longleftrightarrow x = Q \circ w, \text{ and hence} \\ &Q\circ : \mathcal{B}(U) := \mathrm{cl}\left((\mathcal{B}(R,n))_{n\geq n_0}\right) \overset{\cong}{\longleftrightarrow} \\ &\qquad\qquad \mathcal{B}\left(\mathbf{A}^{1\times d}(q\,\mathrm{id}_d - A)\right) := \mathrm{cl}\left((\mathcal{B}(q\,\mathrm{id}_d - A, n))_{n\geq n_0}\right) : P\circ . \end{aligned} \tag{107}$$

Moreover there are the $\mathbb{C}$-linear isomorphisms

$$\begin{aligned} &\mathcal{B}(q\,\mathrm{id}_d - A, n) \cong \mathbb{C}^d,\ x \mapsto x(n),\ x(t) := \Phi(t,n)x(n), \text{ where} \\ &\forall t \geq n:\ \Phi(t,n) := A(t-1)\cdots A(n),\ \Phi(n,n) = \mathrm{id}_d . \end{aligned} \tag{108}$$

The isomorphism (108) *means that for a given $v \in \mathbb{C}^d$ there is a unique trajectory x with initial vector $x(n) = v$ and $x(t) = \Phi(t,n)x(n)$. The behaviors $\mathcal{B}(q\,\mathrm{id}_d - A, n)$ are d-dimensional over $\mathbb{C}$.*

The preceding theorem shows that autonomous behaviors are isomorphic to state space behaviors that are the main subject of [25, Ch. 20-22].

3.2 Preservation of e.s. under behavior isomorphisms

Lemma 3.7. *Exponential stability is preserved by isomorphisms, i.e., if $\mathbf{A}^{1\times\ell_1}/U_1 \cong \mathbf{A}^{1\times\ell_2}/U_2$ and if $\mathcal{B}(U_1)$ is e.s. then so is $\mathcal{B}(U_2)$ ($\cong \mathcal{B}(U_1)$).*

Proof. Let $U_i = \mathbf{A}^{1\times p_i}R_i,\ i = 1,2$, and consider an isomorphism and its inverse

$$(\circ P_1)_{\mathrm{ind}} : \mathbf{A}^{1\times\ell_1}/U_1 \overset{\cong}{\longleftrightarrow} \mathbf{A}^{1\times\ell_2}/U_2 : (\circ P_2)_{\mathrm{ind}}. \tag{109}$$

Let $P_i = \sum_{j=0}^{d_i} P_{ij}(t)q^j,\ i = 1,2$. Then there is an n_0 such that for all $t \geq n \geq n_0$

$$\begin{array}{ccc} P_1\circ : \mathcal{B}(R_2,n) & \overset{\cong}{\longleftrightarrow} & \mathcal{B}(R_1,n) : (P_1\circ)^{-1} = P_2\circ \\ w_2(t) = \sum_{j=0}^{d_2} P_{2j}(t)w_1(t+j) & \longleftrightarrow & w_1(t) = \sum_{j=0}^{d_1} P_{1j}(t)w_2(t+j). \end{array} \tag{110}$$

Assume that $\mathcal{B}(U_1)$ is e.s. and that for a p.g.s. $\varphi_1 > 0$ and $\rho_1,\ 0 < \rho_1 < 1$,

$$\begin{aligned} &\forall t \geq n \geq n_0 \forall w_1 \in \mathcal{B}(R_1,n):\ \|w_1(t)\| \leq \rho_1^{t-n}\varphi_1(n)\|x_1(n)\| \\ &\qquad \text{where } x_1(t) = (w_1(t), \cdots, w_1(t+d-1)). \end{aligned} \tag{111}$$

For $w_2 \in \mathcal{B}(R_2,n)$ define $x_2(t) := (w(t), \cdots, w(t+d+d_1-1))$. Now let

$$\begin{aligned} &w_2 \in \mathcal{B}(R_2,n),\ w_1 := P_1 \circ w_2 \Longrightarrow w_2 = P_2 \circ w_1, \\ &\qquad \text{hence } w_1(t) = \sum_{j=0}^{d_1} P_{1j}(t)w_2(t+j),\ w_2(t) = \sum_{i=0}^{d_2} P_{2i}(t)w_1(t+i). \end{aligned} \tag{112}$$

Since the P_{ij} are rational they are p.g. and therefore there are $m_i \in \mathbb{N}$ and $c_i \geq 1$ such that $\|P_{ij}(t)\| \leq c_i t^{m_i}$ for $i = 1, 2$, and $0 \leq j \leq d_i$, hence

$$\|w_2(t)\| \leq \sum_{i=0}^{d_2} \|P_{2i}(t)\| \|w_1(t+i)\| \leq (d_2+1)c_2 t^{m_2} \max\left\{\|w_1(t+i)\|;\ 0 \leq i \leq d_2\right\}.$$

$$\begin{aligned} &\text{Moreover } \|w_1(t+i)\| \leq \rho_1^{t+i-n}\varphi_1(n)\|x_1(n)\| \leq \rho_1^{t-n}\varphi_1(n)\|x_1(n)\| \\ &\quad \Longrightarrow \|w_2(t)\| \leq (d_2+1)c_2 t^{m_2}\rho_1^{t-n}\varphi_1(n)\|x_1(n)\|. \end{aligned} \tag{113}$$

Likewise

$$\begin{aligned} &\|x_1(n)\| = \max\left\{\|w_1(n+i)\|;\ 0 \leq i \leq d-1\right\} \text{ and} \\ &\quad \|w_1(n+i)\| \leq (d_1+1)c_1(n+i)^{m_1} \max\left\{\|w_2(n+i+j)\|; 0 \leq j \leq d_1\right\} \\ &\Longrightarrow \|x_1(n)\| \leq (d_1+1)c_1(n+d-1)^{m_1}\|x_2(n)\| \end{aligned} \tag{114}$$

Inserting (114) into (113) furnishes

$$\begin{aligned} \|w_2(t)\| &\underset{(113)}{\leq} (d_2+1)c_2 t^{m_2}\rho_1^{t-n}\varphi_1(n)\|x_1(n)\| \\ &\leq (d_2+1)c_2 t^{m_2}\rho_1^{t-n}\varphi_1(n)(d_1+1)c_1(n+d-1)^{m_1}\|x_2(n)\| \\ &\underset{\text{if } t>n}{=} c_3(t-n)^{m_2}\rho_1^{t-n}\left(t/(t-n)\right)^{m_2}(n+d-1)^{m_1}\varphi_1(n)\|x_2(n)\| \end{aligned} \tag{115}$$

with $c_3 := (d_1+1)(d_2+1)c_1c_2 \geq 1$. We choose ρ_2 with $\rho_1 < \rho_2 < 1$ and $c_4 \geq 1$ such that $t^{m_2}\rho_1^t \leq c_4\rho_2^t$. Moreover $t/(t-n) = 1 + n/(t-n) \leq 1 + n$ for $t > n$ and hence

$$\begin{aligned} &\forall t > n \geq n_0 :\ \|w_2(t)\| \leq \rho_2^{t-n}\varphi_2(n)\|x_2(n)\| \\ &\quad \text{where } \varphi_2(n) := c_3c_4(1+n)^{m_2}(n+d-1)^{m_1}\varphi_1(n) > 0 \end{aligned} \tag{116}$$

is also p.g.. We choose $\varphi_1(n) \geq 1$. This implies $\varphi_2(n) \geq 1$ and thus (116) also for $t = n$ and the e.s. of $\mathcal{B}(U_2)$. □

Thm. 3.6 and Lemma 3.7 imply that in connection with exponential stability of a behavior one may assume that it is a state space behavior. It is an open question which e.s. state space behaviors are isomorphic to u.e.s. ones.

3.3 The standard form of short exact sequences

We derive standard forms of short exact sequences (117) *under isomorphism that essentially simplify the proof of Thm. 1.8.*
Consider the exact sequence

$$0 \to M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \to 0 \tag{117}$$

of f.g. $\mathbf{A}$-modules. After the choice of presentations $M_i = \mathbf{A}^{1\times\ell_i}/U_i$ the exact sequence (117) induces an exact behavior sequence

$$0 \leftarrow \mathcal{B}_1 \overset{\mathcal{B}(\varphi)}{\longleftarrow} \mathcal{B}_2 \overset{\mathcal{B}(\psi)}{\longleftarrow} \mathcal{B}_3 \leftarrow 0 \tag{118}$$

Let

$$\begin{aligned} &M_i = \mathbf{A}^{1\times\ell_i}/U_i,\ U_i = \mathbf{A}^{1\times p_i}R_i,\ R_i \in \mathbf{A}^{p_i\times\ell_i},\ i = 1, 3, \\ &\mathrm{can}_i : \mathbf{A}^{1\times\ell_i} \to M_i,\ \xi \mapsto \xi + U_i,\ \mathbf{w}_{ij} := \delta_j + U_i, \mathbf{w}_i := (\mathbf{w}_{i1}, \cdots, \mathbf{w}_{i\ell_i})^\top \in M_i^{\ell_i}, \end{aligned} \tag{119}$$

be arbitrarily chosen such presentations where, as usual, the δ_j are the standard basis vectors. Choose an inverse image $\mathbf{v}\in M_2^{\ell_3}$ of $\mathbf{w}_3$ under ψ, $\psi(\mathbf{v})=\mathbf{w}_3$. Then $\mathbf{w}_2 := \left(\begin{smallmatrix}\mathbf{v}\\ \varphi(\mathbf{w}_1)\end{smallmatrix}\right)\in M_2^{\ell_3+\ell_1}$ is a generating system of M_2 with its associated presentation, i.e.,

$$
\begin{aligned}
U_2 &:= \ker\Big(\mathrm{can}_2 : \mathbf{A}^{1\times(\ell_3+\ell_1)} \to M_2,\ (\xi,\eta)\mapsto \xi\mathbf{v}+\eta\varphi(\mathbf{w}_1)\Big),\\
&\qquad M_2 \underset{\text{ident.}}{=} \mathbf{A}^{1\times(\ell_3+\ell_1)}/U_2.
\end{aligned}
\tag{120}
$$

Remark 3.8. The following Lemma 3.9 is a special case of [6, Prop. V.2.2]. It was rediscovered and applied to systems theory by Quadrat and Robertz in [24, Thm. 7, §5] and [23, Thm. 7, §4]. The proof given here is more direct.

Lemma 3.9. *For the data of* (119) *and* (120) *there is a matrix* $R\in\mathbf{A}^{p_3\times\ell_1}$ *such that*

$$
U_2=\mathbf{A}^{1\times(p_3+p_1)}R_2,\ R_2:=\left(\begin{smallmatrix}R_3 & R\\ 0 & R_1\end{smallmatrix}\right)\in\mathbf{A}^{(p_3+p_1)\times(\ell_3+\ell_1)}. \tag{121}
$$

Proof. The choice of $\mathbf{w}_2$ induces a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & U_1 & \longrightarrow & U_2 & \longrightarrow & U_3 & \longrightarrow & 0\\
 & & \downarrow\subseteq & & \downarrow\subseteq & & \downarrow\subseteq & & \\
0 & \longrightarrow & \mathbf{A}^{1\times\ell_1} & \xrightarrow{\circ(0,\mathrm{id}_{\ell_1})} & \mathbf{A}^{1\times(\ell_3+\ell_1)} & \xrightarrow{\circ\left(\begin{smallmatrix}\mathrm{id}_{\ell_3}\\ 0\end{smallmatrix}\right)} & \mathbf{A}^{1\times\ell_3} & \longrightarrow & 0\\
 & & \downarrow\mathrm{can}_1 & & \downarrow\mathrm{can}_2 & & \downarrow\mathrm{can}_3 & & \\
0 & \longrightarrow & M_1 & \xrightarrow{\varphi} & M_2 & \xrightarrow{\psi} & M_3 & \longrightarrow & 0
\end{array}
\tag{122}
$$

The exactness of the first row is a consequence of the snake lemma [6, L. III.3.2-3].

By definition the rows $(R_3)_{i-},\ i=1,\cdots,p_3$, belong to U_3. Since $U_2\to U_3$ is surjective there are rows $R_{i-}\in\mathbf{A}^{1\times\ell_1}$ such that

$$
((R_3)_{i-},R_{i-})\in U_2\subseteq\mathbf{A}^{1\times(\ell_3+\ell_1)},\ \text{i.e., } (R_3)_{i-}\mathbf{v}+R_{i-}\varphi(\mathbf{w}_1)=0. \tag{123}
$$

Let $R\in\mathbf{A}^{p_3\times\ell_1}$ be the matrix with rows R_{i-} and $R_2:=\left(\begin{smallmatrix}R_3 & R\\ 0 & R_1\end{smallmatrix}\right)\in\mathbf{A}^{(p_3+p_1)\times(\ell_3+\ell_1)}$. Then

$$
\begin{aligned}
R_2\mathbf{w}_2 &= \left(\begin{smallmatrix}R_3 & R\\ 0 & R_1\end{smallmatrix}\right)\left(\begin{smallmatrix}\mathbf{v}\\ \varphi(\mathbf{w}_1)\end{smallmatrix}\right)=\left(\begin{smallmatrix}R_3\mathbf{v}+R\varphi(\mathbf{w}_1)\\ \varphi(R_1\mathbf{w}_1))\end{smallmatrix}\right)=\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\right)=0\\
&\Longrightarrow R_3\mathbf{v}+R\varphi(\mathbf{w}_1)=0,\ \mathbf{A}^{1\times(p_3+p_1)}R_2\subseteq U_2.
\end{aligned}
\tag{124}
$$

Let, conversely,

$$
\begin{aligned}
&(\xi,\eta)\in U_2\subseteq\mathbf{A}^{1\times(\ell_3+\ell_1)}\Longrightarrow \xi\mathbf{v}+\eta\varphi(\mathbf{w}_1)=0\\
&\quad\Longrightarrow \xi\mathbf{w}_3 \underset{\psi(\mathbf{v})=\mathbf{w}_3,\ \psi\varphi=0}{=} \psi\left(\xi\mathbf{v}+\eta\varphi(\mathbf{w}_1)\right)=0\Longrightarrow \xi\in U_3=\mathbf{A}^{1\times p_3}R_3\\
&\quad\Longrightarrow \exists\zeta_1 \text{ with } \xi=\zeta_1R_3\Longrightarrow \zeta_1R_3\mathbf{v}+\varphi(\eta\mathbf{w}_1)=0.
\end{aligned}
\tag{125}
$$

With the last equation of (124) this implies

$$
\begin{aligned}
&\varphi\left((\eta-\zeta_1R)\mathbf{w}_1\right)=0 \underset{\varphi\ \text{injective}}{\Longrightarrow} (\eta-\zeta_1R)\mathbf{w}_1=0\\
&\quad\Longrightarrow \eta-\zeta_1R\in U_1=\mathbf{A}^{1\times p_1}R_1\Longrightarrow \exists\zeta_2:\ \eta-\zeta_1R=\zeta_2R_1\\
&\quad\Longrightarrow (\zeta_1,\zeta_2)R_2=(\zeta_1,\zeta_2)\left(\begin{smallmatrix}R_3 & R\\ 0 & R_1\end{smallmatrix}\right)=(\zeta_1R_3,\zeta_1R+\zeta_2R_1)=(\xi,\eta)\\
&\quad\Longrightarrow (\xi,\eta)\in\mathbf{A}^{1\times(p_3+p_1)}R_2\Longrightarrow U_2\subseteq\mathbf{A}^{1\times(p_3+p_1)}R_2.
\end{aligned}
\tag{126}
$$

□

In the sequel we therefore assume w.l.o.g. that the exact sequence (117) has the special form

$$\begin{gathered}0\to \mathbf{A}^{1\times \ell_1}/U_1 \overset{(\circ(0,\mathrm{id}_{\ell_1}))_{\mathrm{ind}}}{\longrightarrow} \mathbf{A}^{1\times \ell_2}/U_2 \overset{(\circ\left(\begin{smallmatrix}\mathrm{id}_{\ell_3}\\ 0\end{smallmatrix}\right))_{\mathrm{ind}}}{\longrightarrow} \mathbf{A}^{1\times \ell_3}/U_3\to 0\\ \text{where } U_i=\mathbf{A}^{1\times p_i}R_i,\ R_i\in \mathbf{A}^{p_i\times \ell_i},\ i=1,2,3,\\ \ell_2:=\ell_3+\ell_1,\ p_2:=p_3+p_1,\ R_2=\left(\begin{smallmatrix}R_3 & R\\ 0 & R_1\end{smallmatrix}\right).\end{gathered}\tag{127}$$

The corresponding exact sequences of behaviors are given by

$$0\leftarrow \mathcal{B}(U_1) \overset{(0,\mathrm{id}_{\ell_1})\circ}{\longleftarrow} \mathcal{B}(U_2) \overset{\left(\begin{smallmatrix}\mathrm{id}_{\ell_3}\\ 0\end{smallmatrix}\right)\circ}{\longleftarrow} \mathcal{B}(U_3)\leftarrow 0 \tag{128}$$

and for sufficiently large n_0 and $n\geq n_0$

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_1} & \overset{(0,\mathrm{id}_{\ell_1})\circ}{\longleftarrow} & \left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_3+\ell_1} & \overset{\left(\begin{smallmatrix}\mathrm{id}_{\ell_3}\\ 0\end{smallmatrix}\right)\circ}{\longleftarrow} & \left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_3} & \longleftarrow & 0\\ & & \uparrow\subseteq & & \uparrow\subseteq & & \uparrow\subseteq & & \\ 0 & \longleftarrow & \mathcal{B}(R_1,n) & \longleftarrow & \mathcal{B}(R_2,n) & \longleftarrow & \mathcal{B}(R_3,n) & \longleftarrow & 0\\ & & w_1 & \longleftarrow\!\shortmid & \left(\begin{smallmatrix}w_3\\ w_1\end{smallmatrix}\right),\ \left(\begin{smallmatrix}w_3\\ 0\end{smallmatrix}\right) & \longleftarrow\!\shortmid & w_3 & & \end{array}$$

$$\begin{gathered}\mathcal{B}(R_i,n)=\left\{w_i\in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_i};\ R_i\circ w_i=0\right\},\ i=1,2,3,\\ \mathcal{B}(R_2,n)=\left\{\left(\begin{smallmatrix}w_3\\ w_1\end{smallmatrix}\right)\in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_3+\ell_1};\ R_3\circ w_3+R\circ w_1=0,\ R_1\circ w_1=0\right\}.\end{gathered}\tag{129}$$

Moreover we always assume w.l.o.g. that the matrices R_1 and R_3 have the state space form from Thm. 3.6:

$$\begin{aligned}&R_i=q\,\mathrm{id}_{\ell_i}-A_i,\ A_i\in\mathbb{C}(t)^{\ell_i\times\ell_i}, i=1,3,\\ &\forall n\geq n_0:\mathcal{B}(R_i,n)=\left\{w_i\in\left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_i};\ w_i(t+1)=A_i(t)w_i(t)\right\}.\end{aligned}\tag{130}$$

For $\mathcal{B}(R_2,n)$ this implies

$$\begin{aligned}&\mathcal{B}(R_2,n)=\left\{\left(\begin{smallmatrix}w_3\\ w_1\end{smallmatrix}\right)\in\left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_3+\ell_1};\ (\star)\right\}\\ &\quad\text{where } (\star)\quad w_3(t+1)=A_3(t)w_3(t)-(R\circ w_1)(t),\ w_1(t+1)=A_1(t)w_1(t)\}.\end{aligned}\tag{131}$$

The behaviors $\mathcal{B}(R_i,n)$ are ℓ_i-dimensional over $\mathbb{C}$ and indeed

$$\begin{aligned}&\mathcal{B}(R_1,n)\cong\mathbb{C}^{\ell_1},\ w_1\mapsto w_1(n),\ \mathcal{B}(R_2,n)\cong\mathbb{C}^{\ell_3},\ w_3\mapsto w_3(n),\\ &\mathcal{B}(R_2,n)\cong\mathbb{C}^{\ell_3+\ell_1},\ \left(\begin{smallmatrix}w_3\\ w_1\end{smallmatrix}\right)\mapsto\begin{pmatrix}w_3(n)\\ w_1(n)\end{pmatrix}\\ &\qquad\text{with } w_1(t)=\Phi_1(t,n)w_1(n),\ \Phi_1(t,n)=A_1(t-1)\cdots A_1(n)\text{ and}\\ &\qquad w_3(t+1)=A_3(t)w_3(t)-(R\circ w_1)(t).\end{aligned}\tag{132}$$

3.4 The proof of Thm. 1.8

More generally than in (132) consider an inhomogeneous equation

$$w(t+1)=A(t)w(t)+u(t),\ A(t)\in\mathbb{C}(t)^{\ell\times\ell},\ t\geq n\geq n_0,\ w,u\in\left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell},\tag{133}$$

where, as always, $A(t)$ has no poles $t \geq n_0$. From (2) we know

$$\begin{aligned} \forall t \geq n \geq n_0 :\ w(t) &= \Phi(t,n)w(n) + \sum_{i=n}^{t-1} \Phi(t,i+1)u(i) \\ \Longrightarrow \|w(t)\| &\leq \|\Phi(t,n)\|\|w(n)\| + \sum_{i=n}^{t-1} \|\Phi(t,i+1)\|\|u(i)\|. \end{aligned} \tag{134}$$

Lemma 3.10. *Assume in* (133) *that* $\mathcal{B} := \mathcal{B}\left(\mathbf{A}^{1\times\ell}(q\,\mathrm{id}_\ell - A)\right)$ *is e.s. and that also the sequence* u *is e.s. in the sense that there are* $n_1 \geq n_0$*, a p.g.s.* $\varphi > 0$ *in* $\mathbb{C}^{n_1+\mathbb{N}}$ *and a positive sequence* $a \in \mathbb{C}^{n_1+\mathbb{N}}$ *and* ρ *with* $0 < \rho < 1$ *such that*

$$\forall t \geq n \geq n_1 :\ \|u(t)\| \leq \rho^{t-n}\varphi(n)a(n).$$

Then every solution w *of* (133) *is e.s. in the sense that there are* $n_2 \geq n_1$*, a p.g.s.* $\varphi_2 > 0$ *in* $\mathbb{C}^{n_2+\mathbb{N}}$ *and* ρ_2 *(*$0 < \rho_2 < 1$*) such that*

$$\forall t \geq n \geq n_2 :\ \|w(t)\| \leq \rho_2^{t-n}\varphi_2(n)\max(\|w(n)\|, a(n)). \tag{135}$$

Proof. By enlarging n_1, $\varphi > 0$ and $\rho,\ 0 < \rho < 1$, we may assume w.l.o.g. that

$$\forall t \geq n \geq n_1 :\ \|\Phi(t,n)\| \leq \rho^{t-n}\varphi(n),\ \|u(t)\| \leq \rho^{t-n}\varphi(n)a(n). \tag{136}$$

Define $n_2 := n_1$ and $b(n) := \max(\|w(n)\|, a(n))$. We insert the inequalities from (136) into (134) and obtain

$$\forall t \geq n \geq n_1 :\ \|w(t)\| \leq \rho^{t-n}\varphi(n)\|w(n)\| + \sum_{i=n}^{t-1} \rho^{t-i-1}\varphi(i+1)\rho^{i-n}\varphi(n)a(n). \tag{137}$$

Since φ is p.g. there are $c_1 \geq 1$ and $m \in \mathbb{N}$ such that $|\varphi(t)| \leq c_1 t^m$ and hence also $|\varphi(i+1)| \leq c_1 t^m$ for $i \leq t-1$. For $t > n \geq n_1$ equation (137) implies

$$\begin{aligned} \|w(t)\| &\leq \varphi(n)\rho^{t-n}\left(1 + (t-n)\rho^{-1}c_1 t^m\right)b(n) \\ &\leq (t-n)^{m+1}\rho^{t-n}\varphi(n)\left(1 + c_1\rho^{-1}\left(\frac{t}{t-n}\right)^m\right)b(n). \end{aligned} \tag{138}$$

Now choose ρ_2 with $\rho < \rho_2 < 1$ and $c_2 \geq 1$ such that $t^{m+1}\rho^t \leq c_2\rho_2^t$. Moreover $t/(t-n) = 1 + n/(t-n) \leq 1 + n$ for $t > n$ and hence for $t > n \geq n_1$

$$\begin{aligned} \|w(t)\| &\leq \rho_2^{t-n}\left(1 + c_1\rho^{-1}(1+n)^m\right)c_2\varphi(n)b(n) = \rho_2^{t-n}\varphi_2(n)b(n) \\ &\text{where } \varphi_2(n) = \left(1 + c_1\rho^{-1}(1+n)^m\right)c_2\varphi(n) \geq 1 \text{ is a p.g.s..} \end{aligned} \tag{139}$$

Since $\varphi_2(n) \geq 1$ and $\|w(n)\| \leq b(n)$ (139) also holds for $t = n$. □

The next theorem coincides with Thm. 1.8 and is the main result of this paper.

Theorem 3.11. *Exponentially stable behaviors form a Serre subcategory of the abelian category of all LTV-behaviors. For the data from* (117), (118) *or, w.l.o.g., from* (127) -(132) *this means that* $\mathcal{B}_2$ *is e.s. if and only if* $\mathcal{B}_1$ *and* $\mathcal{B}_3$ *are.*

Proof. We assume (127)-(133). The time n_0 below is always chosen sufficiently large.
1. $\mathcal{B}_2$ e.s. $\Longrightarrow \mathcal{B}_3$ e.s.: There are a p.g.s. φ and ρ as usual such that

$$\begin{gathered}\forall t \geq n \geq n_0 \forall w_2 = \left(\begin{smallmatrix} w_3 \\ w_1 \end{smallmatrix}\right) \in \mathcal{B}(R_2, n) : \\ \|w_2(t)\| = \max(\|w_3(t)\|, \|w_1(t)\|) \leq \rho^{t-n}\varphi(n)\|w_2(n)\|.\end{gathered} \tag{140}$$

For

$$\begin{aligned}&w_3 \in \mathcal{B}(R_3, n) \text{ define } w_2 := \left(\begin{smallmatrix} w_3 \\ 0 \end{smallmatrix}\right) \in \mathcal{B}(R_2, n) \\ &\quad\Longrightarrow \|w_3(t)\| = \|w_2(t)\|,\ \|w_3(n)\| = \|w_2(n)\| \Longrightarrow \|w_3(t)\| \leq \rho^{t-n}\varphi(n)\|w_3(n)\|.\end{aligned}$$

This means that $\mathcal{B}_3$ is e.s..
2. $\mathcal{B}_2$ e.s. $\Longrightarrow \mathcal{B}_1$ e.s.: Again (140) is assumed. Let $w_1 \in \mathcal{B}(R_1, n)$ and let $w_2 := \left(\begin{smallmatrix} w_3 \\ w_1 \end{smallmatrix}\right)$ be the unique $w_2 \in \mathcal{B}(R_2, n)$ with $w_3(n) = 0$ (cf. (132)) or

$$\begin{aligned}&w_3(t+1) = A_3(t)w_3(t) - (R \circ w_1)(t),\ w_3(n) = 0 \\ &\Longrightarrow \|w_1(n)\| = \max(0, \|w_1(n)\|) = \max(\|w_3(n), \|w_1(n)\|) = \|w_2(n)\| \text{ and} \\ &\qquad \|w_1(t)\| \leq \max(\|w_3(t)\ \|w_1(t)\|) = \|w_2(t)\| \leq \rho^{t-n}\varphi(n)\|w_2(n)\| \\ &\Longrightarrow \|w_1(t)\| \leq \rho^{t-n}\varphi(n)\|w_1(n)\| \Longrightarrow \mathcal{B}(U_1) \text{ e.s..}\end{aligned}$$

3. $\mathcal{B}_1, \mathcal{B}_3$ e.s. $\Longrightarrow \mathcal{B}_2$ e.s.: For $i = 1, 3$ there are, as usual, p.g.s. φ_i, $i = 1, 3$, and ρ_i such that

$$\forall t \geq n \geq n_0 \forall w_i \in \mathcal{B}(R_i, n) :\ \|w_i(t)\| \leq \rho_i^{t-n}\varphi_i(n)\|w_i(n)\|. \tag{141}$$

Let

$$\begin{aligned}&w_2 = \left(\begin{smallmatrix} w_3 \\ w_1 \end{smallmatrix}\right) \in \mathcal{B}(R_2, n),\ R = \left(\begin{smallmatrix} R_3 & R \\ 0 & R_1 \end{smallmatrix}\right) \\ &\qquad\Longrightarrow w_1 \in \mathcal{B}(R_1, n) \text{ and } \|w_1(t)\| \leq \rho_1^{t-n}\varphi_1(n)\|w_1(n)\| \text{ and} \\ &\qquad\qquad \forall t \geq n :\ w_3(t+1) = A_3(t)w_3(t) + u(t),\ u := -R \circ w_1.\end{aligned} \tag{142}$$

Let $R = \sum_{i=0}^{d} B_i(t)q^i,\ B_i \in \mathbb{C}(t)^{p_3 \times q_1}$ and assume that no $B_i(t)$ has a pole $t \geq n_0$. The $B_i(t)$ are rational and therefore of at most polynomial growth. Hence there are $c_1 \geq 1$ and $m \in \mathbb{N}$ such that $\|B_i(t)\| \leq c_1 t^m$ for all $t \geq n_0$. We conclude

$$\begin{aligned}\forall t \geq n :\ -u(t) = (R \circ w_1)(t) &= \sum_{i=0}^{d} B_i(t)w_1(t+i) \\ \Longrightarrow \|u(t)\| = \|(R \circ w_1)(t)\| &\leq \sum_{i=0}^{d} \|B_i(t)\|\|w_1(t+i)\| \\ &\leq (d+1)c_1 t^m \rho_1^{t-n}\varphi_1(n)\|w_1(n)\|.\end{aligned} \tag{143}$$

We choose ρ_1' with $\rho_1 < \rho_1' < 1$ and $c_2 \geq 1$ such that $t^m \rho_1^t \leq c_2 \rho_1'^t$. Moreover

$$\begin{gathered}\text{for } t > n :\ t^m < (t-n)^m(1+n)^m \underset{(143)}{\Longrightarrow} \|u(t)\| \leq \rho_1'^{t-n}\varphi_1'(n)\|w_1(n)\| \\ \text{with } \varphi_1'(n) := (d+1)c_1c_2(n+1)^m\varphi_1(n)\end{gathered} \tag{144}$$

This also holds for $t = n$ due to (143). Obviously φ_1' is a p.g.s.. Thus $u = -R \circ w_1$ is e.s. in the sense of Lemma 3.10 with $a(n) = \|w_1(n)\|$ and the lemma therefore implies that there are a p.g.s. φ_2 and ρ_2 as usual such that for all $t \geq n \geq n_0$

$$\begin{aligned}&\|w_3(t)\| \leq \rho_2^{t-n}\varphi_2(n)\max(\|w_3(n)\|, \|w_1(n)\|) \\ &\qquad\Longrightarrow \|w_2(t)\| = \max(\|w_3(t)\|, \|w_1(t)\|) \leq \varphi_4(n)\rho_4^{t-n}\|w_2(n)\|\end{aligned} \tag{145}$$

where $\varphi_4 = \max(\varphi_2, \varphi_1),\ \rho_4 := \max(\rho_2, \rho_1)$. Hence $\mathcal{B}_2$ is e.s.. $\square$

Corollary 3.12. *Let $f = f_1 f_2$ be a nonzero product in $\mathbf{A}$, hence $0 \neq \mathbf{A}f \subseteq \mathbf{A}f_2$. Then f is exponentially stable if and only if f_1 and f_2 are.*

Proof. The application of Thm. 3.11 to the exact sequences

$$\begin{array}{c} 0 \to \mathbf{A}/\mathbf{A}f_1 \xrightarrow{(\circ f_2)_{\mathrm{ind}}} \mathbf{A}/\mathbf{A}f \xrightarrow{\mathrm{can}} \mathbf{A}/\mathbf{A}f_2 \to 0 \\ a + \mathbf{A}f_1 \longmapsto af_2 + \mathbf{A}f,\ b + \mathbf{A}f \longmapsto b + \mathbf{A}f_2 \\ \\ 0 \leftarrow \mathcal{B}(\mathbf{A}f_1) \xleftarrow{\mathcal{B}(f_2)} \mathcal{B}(\mathbf{A}f) \xleftarrow{\supseteq} \mathcal{B}(\mathbf{A}f_2) \leftarrow 0 \end{array} \tag{146}$$

furnishes the result. □

Consider any torsion module $M = \mathbf{A}^{1\times\ell}/U$ with $U = \mathbf{A}^{1\times p}R$. The module M is of finite length, i.e., artinian and noetherian, and admits a composition series

$$\begin{array}{c} M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{\ell-1} \supsetneq M_\ell = 0 \\ \text{with simple factors } S_i := M_{i-1}/M_i \cong \mathbf{A}/\mathbf{A}p_i, 0 \subsetneq \mathbf{A}p_i \subsetneq \mathbf{A},\ p_i \text{ irreducible,} \\ d(i) := \deg_q(p_i) = \dim_{\mathbb{C}(t)}(M_{i-1}/M_i). \end{array} \tag{147}$$

An element $p \in \mathbf{A}$ is *irreducible* if and only if $\mathbf{A}p$ is a *maximal* left ideal or if and only if $\mathbf{A}/\mathbf{A}p$ is a *simple* module. By the Jordan-Hölder theorem the simple factors are unique up to their numbering and up to isomorphism. Hence the p_i are unique up to their numbering and up to *similarity* where f and g in $\mathbf{A}$ are called similar if $\mathbf{A}/\mathbf{A}f \underset{\mathbf{A}}{\cong} \mathbf{A}/\mathbf{A}g$.

Corollary 3.13. *With the data from* (147) *the behavior $\mathcal{B}(U)$ is e.s. if and only if all behaviors $\mathcal{B}(\mathbf{A}p_i),\ i = 1, \cdots, \ell$, are e.s..*

Proof. Induction and Thm. 3.11 furnish the result by means of the exact sequences

$$0 \to M_i \xrightarrow{\subset} M_{i-1} \to S_i \to 0,\ S_i \cong \mathbf{A}/\mathbf{A}p_i. \tag{148}$$

□

Modulo the Jacobson package of [7] for the algebra $\mathbf{A}_{\mathbb{Q}} := \mathbb{Q}(t)[q;\alpha] \supset \mathbb{Q}[q]$ and (72) the computation of a composition series of a f.g. module over $\mathbf{A}_{\mathbb{Q}}$ reduces to the factorization of a nonzero $f \in \mathbf{A}_{\mathbb{Q}}$ into irreducible factors. For $\mathbb{C}$ instead of $\mathbb{Q}$ such a factorization can only be approximated as is already the case for polynomials in $\mathbb{C}[q] \supset \mathbb{Q}[q]$.
Call a f.g. module M e.s. if for one and then all (cf. Lemma 3.7) representations $M = \mathbf{A}^{1\times\ell}/U$ the behavior $\mathcal{B}(U)$ is e.s.. Due to Thms. 1.8, 3.11 the e.s. modules form a Serre subcategory of the category of f.g. $\mathbf{A}$-modules that are all noetherian.

Corollary 3.14. *Every f.g. $\mathbf{A}$-module M has a largest e.s. submodule $\mathrm{Ra}_{es}(M)$, and moreover $\mathrm{Ra}_{es}(M/\mathrm{Ra}_{es}(M)) = 0$.*

3.5 Special stability results

In Sections 3.5 and 3.6 we describe cases where e.s. or lack of e.s. can be checked algebraically.

The proof of the following result in [13, Satz 11] on a disturbed state space system seems to contain an error and therefore we give a simple different proof similar to that of [25, Thm. 24.7].

Lemma 3.15. *(cf. [13, Satz 11]),[25, Thm. 24.7]) Consider the difference equation*

$$w(t+1) = A(t)w(t) + f(w(t),t),\ t \geq n_0,\ A \in (\mathbb{C}^{n_0+\mathbb{N}})^{\ell\times\ell},\ f : \mathbb{C}^\ell \times \mathbb{N} \to \mathbb{C}^\ell. \tag{149}$$

Assume that $w(t+1) = A(t)w(t)$ *is uniformly exponentially stable (u.e.s.), i.e., that there are* ρ_1 *with* $0 < \rho_1 < 1$ *and* $c_1 \geq 1$ *such that*

$$\forall t \geq n \geq n_0 :\ \|\Phi(t,n)\| \leq c_1\rho_1^{t-n} \text{ where } \Phi(t,n) := A(t-1)\cdots A(n). \tag{150}$$

Also assume that $\|f(v,t)\| \leq \epsilon\|v\|,\ \epsilon > 0,$ *for* $v \in \mathbb{C}^\ell$ *and* $t \geq n_0$.
If ϵ *is sufficiently small then also* (149) *is u.e.s., i.e. there are* $\rho,\ 0 < \rho < 1,$ *and* $c \geq 1$ *such that*

$$\begin{aligned} &\forall t \geq n \geq n_0 \forall w \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^\ell \text{ with } w(t+1) = A(t)w(t) + f(w(t),t) : \\ &\qquad \|w(t)\| \leq c\rho^{t-n}\|w(n)\|. \end{aligned} \tag{151}$$

Proof. The number ϵ is suitably chosen below. With $u(t) := f(w(t),t)$ we obtain

$$\begin{aligned} &w(t+1) = A(t)w(t) + u(t) \\ &\underset{(2)}{\Longrightarrow} \forall t \geq n \geq n_0 :\ w(t) = \Phi(t,n)w(n) + \sum_{i=n}^{t-1}\Phi(t,i+1)u(i) \\ &\underset{(2)}{\Longrightarrow} \forall t \geq n \geq n_0 :\ \ \|w(t)\| \leq \|\Phi(t,n)\|\|w(n)\| + \sum_{i=n}^{t-1}\|\Phi(t,i+1)\|\|u(i)\|. \end{aligned} \tag{152}$$

We insert $\|u(i)\| \leq \epsilon\|w(i)\|$ and (150) into (152) and obtain

$$\forall t \geq n \geq n_0 :\ \|w(t)\| \leq c_1\rho_1^{t-n}\|w(n)\| + \sum_{i=n}^{t-1}c_1\rho_1^{t-(i+1)}\epsilon\|w(i)\|. \tag{153}$$

With $y(t) := \rho_1^{-t}\|w(t)\|$ and $\lambda := c_1\rho_1^{-1}\epsilon$ the preceding inequality implies

$$y(t) \leq c_1y(n) + \sum_{i=n}^{t-1}c_1\rho_1^{-1}\epsilon y(i) = c_1y(n) + \lambda\sum_{i=n}^{t-1}y(i). \tag{154}$$

This suggests to define inductively

$$\forall t \geq n \geq n_0 :\ z(t) := c_1y(n) + \lambda\sum_{i=n}^{t-1}z(i). \tag{155}$$

Since $c_1 \geq 1$ this gives $y(n) \leq z(n) = c_1y(n)$ and inductively, by (154), $y(t) \leq z(t)$ for all $t \geq n$. The sequence $z(t),\ t \geq n,$ satisfies the difference equation

$$\begin{aligned} &z(n) = c_1y(n),\ \forall t \geq n :\ z(t+1) = (1+\lambda)z(t) \Longrightarrow \\ &z(t) = (1+\lambda)^{t-n}z(n) = c_1(1+\lambda)^{t-n}y(n) \Longrightarrow y(t) \leq z(t) = c_1(1+\lambda)^{t-n}y(n) \\ &\underset{\rho_1^t y(t)=\|w(t)\|}{\Longrightarrow} \|w(t)\| \leq c_1\left(\rho_1(1+\lambda)\right)^{t-n}\|w(n)\|. \end{aligned} \tag{156}$$

But $\rho_1 < 1$ and $\rho_1(1+\lambda) = \rho_1 + c_1\epsilon$. Choose $\epsilon > 0$ such that $\rho := \rho_1 + c_1\epsilon < 1$, Equation (156) implies $\|w(t)\| \le c_1\rho^{t-n}\|w(n)\|$ for all $t \ge n \ge n_0$, i.e., (151).

□

Example 3.16. This example shows that Lemma 3.15 does not hold if $w(t+1) = A(t)w(t)$ is only e.s. and not u.e.s. From Ex. 3.2 and Lemma 3.7 we know that for $0 < \rho_1 < \rho_2 < 1$ the system

$$w(t+1) = B(t)w(t),\ t \ge 0,\ B(t) := \begin{pmatrix} \rho_1 & \rho_2+(\rho_2-\rho_1)t \\ 0 & \rho_2 \end{pmatrix} \in \mathbb{C}[t]^{2\times 2} \subset \mathbb{C}(t)^{2\times 2} \tag{157}$$

is e.s., but not u.e.s. Define

$$\begin{aligned} C(t) &:= \begin{pmatrix} 0 & 0 \\ 2(\rho_2+(\rho_2-\rho_1)t)^{-1} & 0 \end{pmatrix}, \\ A_1(t) &:= B(t) + C(t) = \begin{pmatrix} \rho_1 & \rho_2+(\rho_2-\rho_1)t \\ 2(\rho_2+(\rho_2-\rho_1)t)^{-1} & \rho_2 \end{pmatrix} \\ \Phi_1(t,n) &:= A_1(t-1) * \cdots * A_1(n). \end{aligned} \tag{158}$$

We conclude

$$\begin{aligned} &\det(A_1(t)) = \rho_1\rho_2 - 2,\ |\det(\Phi_1(t,n))| = (2-\rho_1\rho_2)^{t-n} \\ &\Longrightarrow \lim_{t\to\infty} C(t) = 0,\ \lim_{t\to\infty} |\det(\Phi_1(t,n))| = \infty. \end{aligned} \tag{159}$$

Hence the system $w(t+1) = B(t)w(t)$ is e.s. and the disturbed system $w(t+1) = (B(t)+C(t))w(t),\ t > 0$, is not although $\lim_{t\to\infty} C(t) = 0$.

In the following corollary we consider a state space equation

$$w(t+1) = A(t)w(t),\ A \in \mathbb{C}(t)^{\ell\times\ell},\ n_0 + \mathbb{N} \subseteq \mathrm{dom}(A),\ t \ge n_0. \tag{160}$$

Moreover we assume that the rational matrix $A(t)$ is proper, i.e., that

$$\begin{aligned} &A(\infty) := \lim_{t\to\infty} A(t) \text{ exists } \Longrightarrow \\ &\lim_{t\to\infty}(A(t) - A(\infty)) = 0 \text{ and } w(t+1) = A(\infty)w(t) + (A(t) - A(\infty))w(t). \end{aligned} \tag{161}$$

The matrices $A(t)$ resp. $A(\infty)$ give rise to an LTV- resp. LTI-state space system.

Corollary 3.17. *If $A(t)$ in (160) is proper then $w(t+1) = A(t)w(t)$ is u.e.s. if and only if $w(t+1) = A(\infty)w(t)$ is (u.)e.s. or, in other terms,* $\mathrm{spec}(A(\infty)) \subset \mathbf{D} = \{z \in \mathbb{C};\ |z| < 1\}$.

Proof. Recall that u.e.s. and e.s. are equivalent for constant matrices. For any

$$\epsilon > 0 \exists n_1 \ge n_0 \forall t \ge n_1 :\ \|A(t) - A(\infty)\| \le \epsilon \Longrightarrow \|(A(t) - A(\infty))w(t)\| \le \epsilon\|w(t)\|.$$

This and Lemma 3.15 now imply the corollary. □

It is open whether the e.s. of $w(t+1) = A(t)w(t)$ instead of its u.e.s. and the existence of $A(\infty)$ also imply $\mathrm{spec}(A(\infty)) \subset \mathbf{D}$.

Corollary 3.18. *Let $f = q^d + a_{d-1}q^{d-1} + \cdots + a_0 \in \mathbf{A} = \mathbb{C}(t)[q;\alpha]$ and assume that all $a_j,\ j = 0, \cdots, d-1$, are proper. Define*

$$a_j(\infty) := \lim_{t\to\infty} a_j(t) \text{ and } f_\infty := q^d + a_{d-1}(\infty)q^{d-1} + \cdots + a_0(\infty) \in \mathbb{C}[q]. \tag{162}$$

Then the behavior $\mathcal{B}(\mathbf{A}f)$ is u.e.s. if and only if all roots of f_∞ belong to $\mathbf{D}$.

If in $f = q - b,\ b \in \mathbb{C}(t)$, the coefficient b is not proper then $\lim_{t\to\infty} |b(t)| = \infty$ and $\mathcal{B}(\mathbf{A}f)$ is not e.s. If $\mathcal{B}(\mathbf{A}f)$ is e.s., $w \in \mathcal{B}(f, n)$ and hence $w(t) = b(t-1)\cdots b(n)w(n)$ the e.s. of w implies $|b(\infty)| < 1$ and hence the u.e.s. of $(q-b)\circ w = 0$. Thus $\mathcal{B}(\mathbf{A}(q-b))$ is e.s. if and only $|b(\infty)| < 1$.

Corollary 3.19. (cf. [2, Thm. 1037]) *Assume the data of* (147) *and in addition that all coefficients $a_{ij}(t) \in \mathbb{C}(t)$ of the $p_i = q^{d(i)} + \sum_{j=0}^{d(i)-1} a_{ij}q^j \in \mathbf{A}$ with $M_{i-1}/M_i \cong \mathbf{A}/\mathbf{A}p_i$ are proper. If all roots of all polynomials $p_{i,\infty} = q^{d(i)} + \sum_{j=0}^{d(i)-1} a_{ij}(\infty)q^j \in \mathbb{C}[q]$ belong to $\mathbf{D}$ then the behavior $\mathcal{B}(U)$ is e.s. .*
If all $d(i) = 1$ or $p_i = q - b_i$, $b_i := -a_{i0}$, then $\mathcal{B}(U)$ is e.s. if and only if $|b_i(\infty)| < 1$ for all i.

Definition 3.20. If the b_i in Cor. 3.19 exist they are called *quasi-poles* of $\mathcal{B}(U)$.

3.6 Special instability results

The following theorem is an unstable counter-part of Cor. 3.17. Its proof is an adaption of that of [5, Thm. 2] where the authors prove an analogue for nonlinear difference equations of an instability result of Chetaev for differential equations; cf. also [25, Thm. 23.6]. In the following let $\|y\|$ resp. $\|y\|_2$ denote the maximum norm resp. the 2-norm on $\mathbb{C}^q$. They obviously satisfy $\|y\| \le \|y\|_2 \le q^{1/2}\|y\|$.

Theorem 3.21. *Consider the system* (160) *with proper $A(t)$ and assume that $A(\infty)$ has at least one eigenvalue α with $|\alpha| > 1$. Then*

$$\begin{aligned}&\exists n_1 \ge n_0 \exists \rho_1 > 1 \forall n \ge n_1 \exists w(n) \in \mathbb{C}^\ell,\ w(n) \ne 0, \forall t \ge n :\\ &\|w(t)\|_2 \ge \rho_1^{t-n}\|w(n)\|_2,\ \textit{hence also } \|w(t)\| \ge q^{-1/2}\rho_1^{t-n}\|w(n)\|.\end{aligned} \tag{163}$$

In particular, the system $w(t+1) = A(t)w(t)$ is not e.s.

Proof. 1. The proof needs several steps and uses ideas from Lyapunov's stability theory. For a matrix $H = (H_{ij})_{i,j} \in \mathbb{C}^{\ell\times\ell}$ let H^* with $(H^*)_{ij} := \overline{H_{ji}}$ be its adjoint. For $y \in \mathbb{C}^\ell$ this implies $y^*y = \|y\|_2^2$. The matrix H is *hermitian* if $H = H^*$. We choose $\rho > 0$ such that

$$\begin{aligned}&|\alpha|^{-1} < \rho < 1 \text{ and } \forall \lambda, \mu \in \operatorname{spec}(A(\infty)) : \rho^2\overline{\lambda}\mu \ne 1\\ &\text{and define } A_1(t) := \rho A(t),\ B := A_1(\infty) = \rho A(\infty).\end{aligned} \tag{164}$$

Then $\operatorname{spec}(B) = \rho \operatorname{spec}(A(\infty))$ contains $\rho\alpha$ with $|\rho\alpha| > 1$ and thus $A_1(t)$ satisfies the same hypotheses as $A(t)$. Moreover $\overline{\lambda}\mu \ne 1$ for all $\lambda, \mu \in \operatorname{spec}(B)$. By [18, Thm. 5.2.3] we infer the existence of a hermitian matrix P with

$$\begin{aligned}&B^*PB - P + I = 0,\ I := \mathrm{id}_\ell,\ P = P^*. \text{ Define}\\ &V(y) = -y^*Py \in \mathbb{R} \text{ for } y \in \mathbb{C}^\ell \Longrightarrow V(By) = V(y) + \|y\|_2^2.\end{aligned} \tag{165}$$

The function V is a quadratic form. Choose an $\epsilon > 0$ such that $I + \epsilon(I - B^*B)$ is positive definite. Then

$$B^*(P+\epsilon I)B - (P+\epsilon I) + I + \epsilon(I - B^*B) = 0. \tag{166}$$

Since B has the eigenvalue $\rho\alpha$ with $|\rho\alpha| > 1$ the Lyapunov criterion implies that $P+\epsilon I$ is not positive definite and hence

$$\exists y \in \mathbb{C}^\ell,\ y \ne 0,\ \text{with } y^*(P+\epsilon I)y = -V(y) + \epsilon\|y\|_2^2 \le 0 \Longrightarrow V(y) > 0. \tag{167}$$

2. Consider any hermitian matrix $H = H^* \in \mathbb{C}^{\ell\times\ell}$. A standard matrix result says that

$$\begin{aligned}\lambda_{\max} &:= \max(\operatorname{spec}(H)) = \max_{0\neq y\in\mathbb{C}^\ell} \|y\|_2^{-2}(y^*Hy) \text{ and}\\ |y^*Hy| &\le \rho(H)\|y\|_2^2 \text{ with } \rho(H) := \max\left\{|\lambda|;\ \lambda\in\operatorname{spec}(H)\right\}.\end{aligned} \tag{168}$$

For $H = -P$ from above equation (167) implies that $\lambda_{\max} > 0$ and hence

$$\forall y\in\mathbb{C}^\ell:\ V(y) = y^*(-P)y \le \lambda_{\max}\|y\|_2^2,\ \exists y_m\in\mathbb{C}^\ell:\ V(y_m) = \lambda_{\max}\|y_m\|_2^2 > 0. \tag{169}$$

We now compute $V(A_1(t)y)$ for $t \ge n_0$. Since $A_1(t)$ is proper and $B = A_1(\infty)$ we write $A_1(t) = B + t^{-1}C(t)$ with proper and thus bounded $C(t)$ and conclude

$$\begin{aligned}V(A_1(t)y) &= y^*(B+t^{-1}C(t))^*(-P)(B+t^{-1}C(t))y =\\ &y^*B^*(-P)By + t^{-1}y^*H(t)y = V(By) + t^{-1}y^*H(t)y\\ &\underset{(165)}{=} V(y) + \|y\|_2^2 + t^{-1}y^*H(t)y\end{aligned} \tag{170}$$

where $H(t)$ is rational, proper, hermitian and bounded. Since $H(t)$ is bounded so is $\operatorname{spec}(H(t))$. Define

$$\begin{aligned}&\sigma := \sup\left\{|\lambda|;\ \lambda\in\bigcup_{t\ge n_0}\operatorname{spec}(H(t))\right\} < \infty \underset{(168)}{\Longrightarrow}\\ &\forall y\forall t\ge n_0:\ |y^*H(t)y| \le \sigma\|y\|_2^2 \underset{(170)}{\Longrightarrow} V(A_1(t)y) \ge V(y) + \left(1 - t^{-1}\sigma\right)\|y\|_2^2.\end{aligned} \tag{171}$$

Choose $n_1 \ge n_0$ such that

$$\begin{aligned}&n_1 \ge \max(n_0, 2\sigma) \Longrightarrow \forall t\ge n_1: 1 - t^{-1}\sigma \ge 1/2\\ &\quad\Longrightarrow \forall t\ge n_1\forall y:\ V(A_1(t)y) \underset{(171)}{\ge} V(y) + 2^{-1}\|y\|_2^2 \underset{(169)}{\ge} (1 + 2^{-1}\lambda_{\max}^{-1})V(y).\end{aligned} \tag{172}$$

3. According to (169) choose a nonzero $y \in \mathbb{C}^\ell$ with $V(y) = \lambda_{\max}\|y\|_2^2 > 0$. Let $n \ge n_1$ and consider the system $y(t+1) = A_1(t)y(t),\ t\ge n,\ y(n) = y$. Equation (172) furnishes

$$\begin{aligned}&\forall t\ge n:\ V(y(t+1)) \ge \left(1+2^{-1}\lambda_{\max}^{-1})\right)V(y(t)),\ V(y(n)) = \lambda_{\max}\|y(n\|_2^2 > 0\\ &\quad\underset{\text{induction}}{\Longrightarrow} V(y(t)) \ge \left(1+2^{-1}\lambda_{\max}^{-1}\right)^{t-n}V(y(n))\\ &\quad\underset{(169)}{\Longrightarrow} \lambda_{\max}\|y(t)\|_2^2 \ge \left(1+2^{-1}\lambda_{\max}^{-1}\right)^{t-n}\lambda_{\max}\|y(n)\|_2^2\\ &\quad\Longrightarrow \forall t\ge n:\ \|y(t)\|_2 \ge \left((1+2^{-1}\lambda_{\max}^{-1})^{1/2}\right)^{t-n}\|y(n)\|_2.\end{aligned} \tag{173}$$

Finally consider the system

$$w(t+1) = A(t)w(t),\ t\ge n,\ \text{with } V(w(n)) = \lambda_{\max}\|w(n)\|_2^2 > 0 \tag{174}$$

and define $y(t) := \rho^t w(t)$ with ρ from (164), especially $y(n) = \rho^n w(n)$. Then

$$\begin{aligned}
&y(t+1) = \rho^{t+1}w(t+1) = \rho A(t)\rho^t w(t) = A_1(t)y(t),\ y(n) = \rho^n w(n) \text{ and}\\
&\quad V(y(n)) = \lambda_{\max}\|y(n)\|_2^2\\
&\quad \underset{(173)}{\Longrightarrow} \forall t \geq n:\ \|y(t)\|_2 \geq \left((1+2^{-1}\lambda_{\max}^{-1})^{1/2}\right)^{t-n}\|y(n)\|_2\\
&\quad \Longrightarrow \forall t \geq n:\ \|w(t)\|_2 \geq \rho_1^{t-n}\|w(n)\|_2 \text{ with } \rho_1 := (1+2^{-1}\lambda_{\max}^{-1})^{1/2}\rho^{-1} > 1.
\end{aligned} \tag{175}$$

□

Definition 3.22. Let $R \in \mathbf{A}^{p\times\ell}$, $U := \mathbf{A}^{1\times p}R$, $\mathcal{B} := \mathcal{B}(U)$ and $n_0 + \mathbb{N} \subseteq \mathrm{dom}(R)$. The behavior $\mathcal{B}$ is called *exponentially unstable* (e.unst.) if

$$\begin{gathered}
\exists n_1 \geq n_0 \forall n \geq n_1 \exists w \in \mathcal{B}(R,n)\exists d \in \mathbb{N}\exists \rho > 1 \exists c > 0 \forall t \geq n:\ c\rho^t \leq \|x(t)\|\\
\text{where } x(t) := (w(t),\cdots,w(t+d)).
\end{gathered} \tag{176}$$

The trajectory w from (176) is also called e.unst..

It is obvious that in Thm. 3.21 the behavior $\mathcal{B}(\mathbf{A}^{1\times\ell}(qI-A))$ is e.unst..

Lemma 3.23. *Exponential instability is preserved by isomorphisms.*

Proof. We use the data from Lemma 3.7 and obtain for sufficiently large $n_1 \geq n_0$ and $n \geq n_1$ surjections $P_1\circ : \mathcal{B}(R_2,n) \to \mathcal{B}(R_1,n)$. Assume that $w_1 \in \mathcal{B}(R_1,n)$ is an e.unst. trajectory, i.e.,

$$\forall t \geq n:\ c_1\rho_1^t \leq \|x_1(t)\|,\ x_1(t) := (w_1(t),\cdots,w_1(t+d)),\ d \in \mathbb{N},\ c_1 > 0,\ \rho_1 > 1.$$

Let $w_2 \in \mathcal{B}(R_2,n)$ be an inverse image with $w_1 = P_1 \circ w_2$. As in the proof of Lemma 3.7 we derive the existence of $c_2 > 0$ and $m \in \mathbb{N}$ such that

$$\begin{gathered}
\forall t \geq n:\ \|w_1(t)\| \leq c_2 t^m \max(\|w_2(t)\|,\cdots,\|w_2(t+d_1)\|) \Longrightarrow\\
\forall t \geq n:\ \|x_1(t)\| \leq c_2 t^{m+d}\|x_2(t)\|,\ x_2(t) := (w_2(t),\cdots,w_2(t+d+d_1)) \Longrightarrow\\
\forall t \geq n:\ \frac{c_1\rho_1^t}{c_2 t^{m+d}} \leq \|x_2(t)\|.
\end{gathered}$$

For any $1 < \rho_2 < \rho_1$ there is a $c_3 > 0$ such that

$$\begin{gathered}
c_2 t^{m+d} \leq c_3\rho_2^t,\ t \geq 1 \Longrightarrow \forall t \geq n:\ \frac{c_1\rho_1^t}{c_3\rho_2^t} \leq \|x_2(t)\| \Longrightarrow\\
c_4\rho_3^t \leq \|x_2(t)\|,\ c_4 := c_1 c_3^{-1},\ \rho_3 := \rho_1\rho_2^{-1} > 1.
\end{gathered}$$

□

Corollary 3.24. *If under the assumptions of* (117) *and* (118) *the behaviors* $\mathcal{B}_1$ *or* $\mathcal{B}_3$ *are exponentially unstable then so is* $\mathcal{B}_2$.

Proof. Due to Lemma 3.23 the proof proceeds like 1. and 2. of that of Thm. 3.11. □

Corollary 3.25. *If under the assumptions of* (147) *all* p_i *are proper and at least one* $p_{i,\infty}$ *(cf. Cor. 3.19) has a root of absolute value* > 1 *then the behavior* $\mathcal{B}(U)$ *is exponentially unstable.*

4 Standard consequences of the duality theorem

We show that the duality theorem Thm. 1.6 and especially the exactness of the duality functor imply various important results well-known from LTI-systems theory. The proofs are slight variants of those of the corresponding LTI-results.

1. *Connection with the LTI-theory*: The relevant LTI-theory is that with the signal module ${}_{\mathbb{C}[q]}\mathbb{C}^{\mathbb{N}}$ where $\mathbb{C}[q](\subset \mathbf{A})$ is the commutative polynomial algebra of difference operators with constant coefficients. The partial fraction decomposition furnishes

$$\begin{aligned}\mathbb{C}(t) = \oplus_{i=0}^{\infty}\mathbb{C}t^i \oplus \oplus_{z\in\mathbb{C}} \oplus_{i=1}^{\infty} \mathbb{C}(t-z)^{-i}, \text{ hence also}\\ \mathbf{A} = \oplus_{j=0}^{\infty}\mathbb{C}(t)q^j = \oplus_{i=0}^{\infty} t^i\mathbb{C}[q] \oplus \oplus_{z\in\mathbb{C}} \oplus_{i=1}^{\infty} (t-z)^{-i}\mathbb{C}[q].\end{aligned} \tag{177}$$

This implies that the right $\mathbb{C}[q]$-module $\mathbf{A}_{\mathbb{C}[q]}$ is free and therefore faithfully flat [1, Prop. I.3.9], i.e. the functor

$${}_{\mathbb{C}[q]}\mathbf{Mod} \to_{\mathbf{A}} \mathbf{Mod},\ M \mapsto \mathbf{A}\otimes_{\mathbb{C}[q]} M, \tag{178}$$

preserves and reflects exact sequences. In particular, [1, Prop. I.3.9]

$$\begin{aligned}\forall V \subseteq_{\mathbb{C}[q]} \mathbb{C}[q]^{1\times\ell}:\ V = \mathbb{C}[q]^{1\times\ell}\cap \mathbf{A}V \text{ where}\\ \mathbf{A}V = \mathbf{A}\otimes_{\mathbb{C}[q]} V \subseteq \mathbf{A}\otimes_{\mathbb{C}[q]}\mathbb{C}[q]^{1\times\ell} = \mathbf{A}^{1\times\ell}.\end{aligned} \tag{179}$$

If $V = \mathbb{C}[q]^{1\times k}R,\ R\in\mathbb{C}[q]^{k\times\ell}$, then $\mathbf{A}V = \mathbf{A}^{1\times k}R$ and the associated LTI- resp. LTV-behaviors are

$$V^{\perp} := \left\{w\in(\mathbb{C}^{\mathbb{N}})^{\ell};\ R\circ w = 0\right\} = \mathcal{B}(R,0) \text{ resp. } \mathcal{B}(\mathbf{A}V) = \mathrm{cl}\left((\mathcal{B}(R,n))_{n\geq 0}\right). \tag{180}$$

If $V_1, V_2 \subseteq \mathbb{C}[q]^{1\times\ell}$ are two submodules the cogenerator property of ${}_{\mathbb{C}[q]}\mathbb{C}^{\mathbb{N}}$, Thm. 1.6, (3), and (179) imply

$$V_1\subseteq V_2 \iff V_2^{\perp}\subseteq V_1^{\perp} \iff \mathbf{A}V_1\subseteq\mathbf{A}V_2 \iff \mathcal{B}(\mathbf{A}V_2)\subseteq\mathcal{B}(\mathbf{A}V_1). \tag{181}$$

These equivalences also follow from the isomorphisms

$$q^n\circ:\ \mathbb{C}^{n+\mathbb{N}}\cong\mathbb{C}^{\mathbb{N}},\ q^n\circ:\ \mathcal{B}(R,n)\cong\mathcal{B}(R,0). \tag{182}$$

Hence the map

$$\{\text{LTI-behaviors}\}\to\{\text{LTV-behaviors}\},\ V^{\perp}\mapsto\mathcal{B}(\mathbf{A}V), \tag{183}$$

is injective and preserves and reflects inclusions. Therefore we identify

$$\forall R\in\mathbb{C}[q]^{k\times\ell},\ V := \mathbb{C}[q]^{1\times k}R:\ \mathcal{B}(R,0) = V^{\perp} = \mathcal{B}(\mathbf{A}V) = \mathrm{cl}\left((\mathcal{B}(R,n))_{n\geq 0}\right). \tag{184}$$

Due to (178) and (184) the LTI-theory of ${}_{\mathbb{C}[q]}\mathbb{C}^{\mathbb{N}}$-behaviors is fully embedded into the LTV-theory of this paper.

2. *Exponential stability in the LTI- resp. LTV-theory*: As in item 1. consider $R\in\mathbb{C}[q]^{k\times\ell}$ and $V = \mathbb{C}[q]^{1\times k}R$. Assume that $V^{\perp} = \mathcal{B}(R,0)\subset\left(\mathbb{C}^{\mathbb{N}}\right)^{\ell}$ is autonomous, i.e., $\mathrm{rank}(R) = \ell$. The *characteristic variety* of the torsion module $M = \mathbb{C}[q]^{1\times\ell}/V$ or of the behavior $V^{\perp}$ is

$$\begin{aligned}\mathrm{char}(M) := \mathrm{char}(V^{\perp}) = \{\lambda\in\mathbb{C};\ \mathrm{rank}(R(\lambda)) < \mathrm{rank}(R) = \ell\} =\\ \left\{\lambda\in\mathbb{C};\ V^{\perp}\bigcap\mathbb{C}[t]^{\ell}\lambda^t \neq 0\right\} \text{ where } \mathbb{C}[t]\lambda^t := \mathbb{C}^{(\mathbb{N})} \text{ for } \lambda = 0.\end{aligned} \tag{185}$$

The characteristic variety gives rise to the *modal decomposition*

$$V^\perp = \bigoplus_{\lambda\in\mathrm{char}(V^\perp)} \left(V^\perp \bigcap \mathbb{C}[t]^\ell \lambda^t\right). \tag{186}$$

It implies that $V^\perp$ is asymptotically or exponentially stable if and only if $\mathrm{char}(V^\perp) \subset \mathbf{D} := \{z \in \mathbb{C};\ |z| < 1\}$. The Smith form implies an isomorphism

$$\begin{aligned} M &\cong \prod_{i=1}^r \mathbb{C}[q]/\mathbb{C}[q]f_i,\ f_i \in \mathbb{C}[q],\ \deg_q(f_i) > 0 \Longrightarrow \\ \mathrm{char}(M) &= \bigcup_{i=1}^r V_{\mathbb{C}}(f_i),\ V_{\mathbb{C}}(f_i) := \{\lambda \in \mathbb{C};\ f_i(\lambda) = 0\}. \end{aligned} \tag{187}$$

Hence $V^\perp$ is asymptotically stable if and only if all roots of all f_i lie in $\mathbf{D}$. The isomorphism in (187) also implies

$$\mathbf{A}^{1\times\ell}/\mathbf{A}V \cong \mathbf{A} \otimes_{\mathbb{C}[q]} M \cong \prod_{i=1}^r \mathbf{A}/\mathbf{A}f_i \Longrightarrow \mathcal{B}(\mathbf{A}V) \cong \prod_{i=1}^r \mathcal{B}(\mathbf{A}f_i). \tag{188}$$

According to Cor. and Def. 3.19 $\mathcal{B}(\mathbf{A}V)$ is e.s. in the sense of this paper if and only if all roots of all f_i lie in $\mathbf{D}$. Hence *asymptotic stability of the LTI-behavior $V^\perp$ and the LTV-e.s. of $\mathcal{B}(\mathbf{A}V)$ coincide*.
3. *Willems' elimination*: Let

$$\begin{aligned} P \in \mathbf{A}^{\ell_2\times\ell_1},\ U_1 \subseteq \mathbf{A}^{1\times\ell_1},\ \mathcal{B}_1 := \mathcal{B}(U_1), \\ U_2 := (\circ P)^{-1}(U_1) := \left\{\eta \in \mathbf{A}^{1\times\ell_2};\ \eta P \in U_1\right\},\ \mathcal{B}_2 := \mathcal{B}(U_2). \end{aligned} \tag{189}$$

The monomorphism $(\circ P)_{\mathrm{ind}}:\ \mathbf{A}^{1\times\ell_2}/U_2 \to \mathbf{A}^{1\times\ell_1}/U_1,\ \eta + U_2 \mapsto \eta P + U_1$, and Thm. 1.6,(1), imply the epimorphism

$$P\circ : \mathcal{B}(U_1) \to \mathcal{B}(U_2) \text{ or } P \circ \mathcal{B}(U_1) = \mathcal{B}(U_2). \tag{190}$$

Hence the image of a behavior under a difference operator $P \in \mathbf{A}^{\ell_2\times\ell_1}$ is again a behavior. In Willems' language the behaviors of this paper admit elimination. Note that Willems considered projections of the form

$$P = (\mathrm{id}_{\ell_2}, 0) \in \mathbf{A}^{\ell_2\times(\ell_2+n)} \text{ and } P\circ : \left(\begin{smallmatrix} w \\ x \end{smallmatrix}\right) \mapsto w \tag{191}$$

only and thus eliminated the n so-called *latent variables* x_i, for instance the *state* x.
4. *Ehrenpreis' fundamental principle*: Consider the behavior $\mathcal{W} = \mathrm{cl}\left((\mathbb{C}^{n+\mathbb{N}})_{n\geq 0}\right) = \mathcal{B}(0)$ and an exact sequence of modules and its dual exact sequence of behaviors

$$\begin{aligned} &\mathbf{A}^{1\times\ell_3} \xrightarrow{\circ Q} \mathbf{A}^{1\times\ell_2} \xrightarrow{\circ P} \mathbf{A}^{1\times\ell_1},\ \text{hence } \mathbf{A}^{1\times\ell_3}Q = \left\{\eta \in \mathbf{A}^{1\times\ell_2};\ \eta P = 0\right\}, \\ &\mathcal{W}^{\ell_1} \xrightarrow{P\circ} \mathcal{W}^{\ell_2} \xrightarrow{Q\circ} \mathcal{W}^{\ell_3},\ \text{hence } P \circ \mathcal{W}^{\ell_1} = \ker(Q\circ), \\ &\quad \Longrightarrow \exists n_1 \forall n \geq n_1 :\ P \circ (\mathbb{C}^{n+\mathbb{N}})^{\ell_1} = \left\{u \in (\mathbb{C}^{n+\mathbb{N}})^{\ell_2};\ Q \circ u = 0\right\}. \end{aligned} \tag{192}$$

This implies that for $n \geq n_1$ the equation $P \circ w = u,\ u \in (\mathbb{C}^{n+\mathbb{N}})^{\ell_2}$, has a solution $w \in (\mathbb{C}^{n+\mathbb{N}})^{\ell_1}$ if and only if $Q \circ u = 0$ or $\eta \circ u = 0$ if $\eta P = 0$.
5. *Input/output structures*: Let $R \in \mathbf{A}^{k\times\ell} \subset \mathbf{K}^{k\times\ell}$ of rank $p := \mathrm{rank}(R)$ where

$\mathbf{K} := \mathrm{quot}(\mathbf{A})$ (cf. (80)-(82)). Then there are various choices of p columns of R that are a basis of the column space $R\mathbf{K}^\ell$ of R. After the standard column permutation one writes $R = (P, -Q) \in \mathbf{K}^{k\times(p+m)},\ \ell = p+m$, and obtains $\mathrm{rank}(R) = \mathrm{rank}(P) = p$ and the unique *transfer matrix* $H \in \mathbf{K}^{p\times m}$ with $PH = Q$. Define

$$U := \mathbf{A}^{1\times k}(P,-Q),\ U^0 := \mathbf{A}^{1\times k}P,\ M := \mathbf{A}^{1\times(p+m)}/U,\ M^0 := \mathbf{A}^{1\times p}/U^0,\\ \mathcal{B} := \mathcal{B}(U),\ \mathcal{B}^0 := \mathcal{B}(U^0). \tag{193}$$

Since $P \in \mathbf{A}^{k\times p}$ and $\mathrm{rank}(P) = p$ the module M^0 is torsion and $\mathcal{B}^0$ is autonomous. In analogy to the LTI-case the sequence of $\mathbf{A}$-modules

$$0 \to \mathbf{A}^{1\times m} \xrightarrow{(\circ(0,\mathrm{id}_m))_{\mathrm{ind}}} M = \mathbf{A}^{1\times(p+m)}/U \xrightarrow{\left(\circ\left(\begin{smallmatrix}\mathrm{id}_p\\ 0\end{smallmatrix}\right)\right)_{\mathrm{ind}}} M^0 = \mathbf{A}^{1\times p}/U^0 \to 0\\ (\circ(0,\mathrm{id}_m))_{\mathrm{ind}}:\ \eta \mapsto (0,\eta)+U,\ \left(\circ\left(\begin{smallmatrix}\mathrm{id}_p\\ 0\end{smallmatrix}\right)\right)_{\mathrm{ind}}:\ (\xi,\eta)+U \mapsto \xi + U^0 \tag{194}$$

is exact. Conversely, the exactness of this sequence and the torsion property of M^0 imply $\mathrm{rank}(P,-Q) = \mathrm{rank}(P) = p$. The decomposition $R = (P,-Q)$ is called an ***input/output (IO) decomposition or structure*** of R, M or $\mathcal{B}$. The description by the exactness of (194) shows that the structure depends on M, but not on the special choice of R. The exactness of the module sequence (194) implies that of the behavior sequence ($\mathcal{W} = \mathrm{cl}\left((\mathbb{C}^{n+\mathbb{N}})_{n\geq 0}\right)$)

$$\begin{array}{ccccccc} 0\to & \mathcal{B}^0 & \xrightarrow{\left(\begin{smallmatrix}\mathrm{id}_p\\ 0\end{smallmatrix}\right)\circ} & \mathcal{B} & \xrightarrow{(0,\mathrm{id}_m)\circ} & \mathcal{W}^m & \to 0\ . \\ & y & \mapsto & \left(\begin{smallmatrix}y\\ 0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}y\\ u\end{smallmatrix}\right) & \mapsto & u & \end{array} \tag{195}$$

For sufficiently large n_1 and $n \geq n_1$ this implies the exactness of

$$\begin{array}{ccccccc} 0\to & \mathcal{B}(P,n) & \xrightarrow{\left(\begin{smallmatrix}\mathrm{id}_p\\ 0\end{smallmatrix}\right)\circ} & \mathcal{B}((P,-Q),n) & \xrightarrow{(0,\mathrm{id}_m)\circ} & \left(\mathbb{C}^{n+\mathbb{N}}\right)^m & \to 0 \\ & y & \mapsto & \left(\begin{smallmatrix}y\\ 0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}y\\ u\end{smallmatrix}\right) & \mapsto & u & \end{array}\\ \text{where } \mathcal{B}(P,n) = \left\{y \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^p;\ P\circ y = 0\right\} \text{ and}\\ \mathcal{B}((P,-Q),n) = \left\{\left(\begin{smallmatrix}y\\ u\end{smallmatrix}\right) \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{p+m};\ P\circ y = Q\circ u\right\}. \tag{196}$$

Hence the component u of a trajectory $\left(\begin{smallmatrix}y\\ u\end{smallmatrix}\right)$ of $\mathcal{B}((P,-Q),n),\ n \geq n_1$, is free, i.e., can be freely chosen as input, but there is no larger component with this property. Up to the introduction of the initial time n_1 this is the standard LTI-result.

The e.s. of an IO-behavior $\mathcal{B}$ is defined by that of its autonomous part $\mathcal{B}^0$. Using this we finally prove Cor. 1.9 with the help of [11, §2.4]. With the data from (129) and (130) consider, for sufficiently large n, IO-behaviors

$$\widetilde{\mathcal{B}}_1(n) := \left\{\left(\begin{smallmatrix}w_1\\ u_1\end{smallmatrix}\right) \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_1+k}; R_1\circ w_1 + Q\circ u_1 = 0\right\},\\ \widetilde{\mathcal{B}}_3(n) := \left\{\left(\begin{smallmatrix}w_3\\ u_3\end{smallmatrix}\right) \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_3+\ell_1}; R_3\circ w_3 + R\circ u_3 = 0\right\} \tag{197}$$

where $(R_1, Q) \in \mathbf{A}^{\ell_1\times(\ell_1+k)}$. The conditions $R_i \in \mathbf{A}^{\ell_i\times\ell_i}$ and $\mathrm{rank}(R_i) = \ell_i$ for $i = 1,3$ imply that these $\widetilde{\mathcal{B}}_i$ are indeed IO-behaviors. The series interconnection of first $\widetilde{\mathcal{B}}_1$ and then $\widetilde{\mathcal{B}}_3$ is given by

$$\widetilde{\mathcal{B}}(n) := \left\{\left(\begin{smallmatrix}w_3\\ w_1\\ u_1\end{smallmatrix}\right) \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_3+\ell_1+k}; R_3\circ w_3 + R\circ w_1 = 0,\ R_1\circ w_1 + Q\circ u_1 = 0\right\} \tag{198}$$

and is itself an IO-behavior with input u_1. Its autonomous part is

$$\widetilde{\mathcal{B}}^0(n) := \left\{ \left(\begin{smallmatrix} w_3 \\ w_1 \end{smallmatrix}\right) \in \left(\mathbb{C}^{n+\mathbb{N}}\right)^{\ell_3+\ell_1};\, R_3 \circ w_3 + R \circ w_1 = 0,\ R_1 \circ w_1 = 0 \right\} \underset{(129)}{=} \mathcal{B}_2(n). \tag{199}$$

So Cor. 1.9 is indeed a simple consequence of Thm. 1.8.

6. *Controllability*: The behavior $\mathcal{B}(U),\ U \subseteq \mathbf{A}^{1\times\ell}$, is called *controllable* if its module $M = \mathbf{A}^{1\times\ell}/U$ is torsionfree and thus free, cf. (79). This is equivalent to the existence of an *image representation* (Willems) or *parametrization* (Pommaret), i.e., an epimorphism $P\circ : \mathcal{W}^m = \mathcal{B}(0)^m \to \mathcal{B}(U)$, i.e., of surjections $P\circ : W(n)^m \to \mathcal{B}(R,n)$ for sufficiently large $n \geq n_1$. This also means that trajectories of $\mathcal{B}(U)$ can be concatenated as usual, but only after the time instant n_1. If $\mathrm{tor}(M) = V/U$ then $\mathcal{B}_{\mathrm{cont}}(U) := \mathcal{B}(V)$ is the largest controllable subbehavior of $\mathcal{B}(U)$. The isomorphisms (72), (79) imply $\mathrm{tor}(M) \cong \prod_{i=1}^r \mathbf{A}/\mathbf{A}f_i$ and $M \cong \mathbf{A}^{\ell_2 - r} \times \mathrm{tor}(M)$ and thus the *controllable-autonomous decomposition*

$$\mathcal{B}(U) \cong \mathcal{W}^{\ell_2 - r} \times \left(\prod_{i=1}^r \mathcal{B}(\mathbf{A}f_i)\right). \tag{200}$$

7. *A larger coefficient field than* $\mathbb{C}(t)$, cf. [22, Ex. 1.2]: Consider $m \geq 1$ and the *locally convergent* Laurent series $a(z) = \sum_{i=k}^\infty a_i z^i$, $k \in \mathbb{Z}$, with $\sigma(a) := \limsup_{i\geq 0} \sqrt[i]{|a_i|} < \infty$. Standard complex variable theory shows that $a(z)$ is a holomorphic function in the annulus $\left\{z \in \mathbb{C};\ 0 < |z| < \sigma(a)^{-1}\right\}$. Therefore the function $f(t) := a(t^{-1/m})$ of the real variable t is a smooth function in the interval $(\sigma(a)^m, \infty)$, in particular it has no poles for $t > \sigma(a)^m$. Also there is a $\tau > \sigma(a)^m$ such that $f(t)$ has no zeros for $t \geq \tau$. Like in the case of rational functions we identify $f = (f(t))_{t\in\mathbb{N}, t>\sigma(a)^m}$. These sequences form a field F that has the properties (i)-(iv) of Remark 1.1,(a). The field F is isomorphic to the algebraic closure of the field $\mathbb{C}\langle\langle z\rangle\rangle$ of locally convergent Laurent series and contains $\mathbb{C}(t) = \mathbb{C}(t^{-1})$. Examples for such sequences are $\left(t\cos(t^{-1/2})\right)_{t\geq 1}$ or $(\exp(t^{-1}))_{t\geq 0}$, but not $(\cos(t))_{t\geq 0}$.
8. *Stabilization*: In [20] the method of this paper and that of [3] are used for the construction and (Kučera-Youla)-parametrization of all stabilizing compensators for tracking, disturbance rejection and model matching of a stabilizible LTV-differential system. The differential analogue of Thm. 1.8 turns out to be a decisive tool.
Acknowledgement: *We thank the three reviewers and the associate editor for their comprehensive reports, valuable criticism and various suggestions that essentially improved the paper's presentation.*

References

[1] N. Bourbaki, *Commutative Algebra*, Hermann, Paris, 1972

[2] H. Bourlès, B. Marinescu, *Linear Time-Varying Systems*, Springer, Berlin, 2011

[3] H. Bourlès, B. Marinescu, U. Oberst, 'The injectivity of the canonical signal module for multidimensional linear systems of difference equations with variable coefficients', pp. 1-29, *Multidim. Sys. Sign. Proc.* 2015, DOI 10.1007/s110451-015-0331-x

[4] H. Bourlès, B. Marinescu, U. Oberst, 'Weak exponential stability of linear time-varying differential behaviors', *Linear Algebra and its Applications* 486(2015), 523-571, http://dx.doi.org/10.1016/j.laa.2015.08.034

[5] C. Cárcamo, C. Vidal, 'The Chetaev Theorem for Ordinary Difference Equations', *Proyecciones Journal of Mathematics* 31(2012), 391-402

[6] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956

[7] F. Chyzak, A. Quadrat, D. Robertz, 'OreModules: A symbolic package for the study of multidimensional linear systems' in J. Chiasson, J.-J. Loiseau (Eds.), *Applications of Time-Delay Systems*, Lecture Notes in Control and Information Sciences 352, Springer, 2007, pp. 233-264.

[8] T. Cluzeau, A. Quadrat, 'Factoring and decomposing a class of linear functional systems', *Lin. Alg. Appl.* 428(2008), 324-381.

[9] M. Fliess, 'Some basic structural properties of generalized linear systems', *Systems and Control Letters* 15(1990), 391-396

[10] M. Fliess, 'Reversible Linear and Nonlinear Discrete-Time Dynamics', *IEEE Transactions on Automatic Control* 37(1992), 1144-1153

[11] Fliess M., Bourlès H., 'Discussing some examples of linear system interconnections', *Systems and Control Letters* 27(1996), 1-7

[12] S. Fröhler, U. Oberst, 'Continuous time-varying linear systems', *Systems and Control Letters* 35(1998), 97-110

[13] W. Hahn, 'Über die Anwendung der Methode von Ljapunov auf Differenzengleichungen', *Math. Ann.* 136(1958), 430-441

[14] A.T. Hill, A. Ilchman, 'Exponential stability of time-varying linear systems', *IMA J. Numer. Analysis* 31(2011), 865-885

[15] D. Hinrichsen, A.J. Pritchard, *Mathematical Systems Theory I*, Springer, Berlin, 2005

[16] A. Ilchmann, V. Mehrmann, 'A Behavioral Approach to Time-Varying Linear Systems: Part 1: General Theory', *SIAM J. Control Optim.* 44(2005), 1725-1747

[17] E.W. Kamen, P.P. Khargonekar, P.P. Poolla, 'A Transfer-Function Approach to Linear Time-Varying Discrete-Time Systems', *SIAM J. Control Optim.* 23(1985), 550-565

[18] P. Lancaster, L. Rodman, *Algebraic Riccati Equations*, Oxford University Press, Oxford, 1995

[19] J.C. McConnell, J.C. Robson, *Noncommutative Noetherian Rings*, John Wiley and Sons, Chichester, 1987

[20] U. Oberst, 'Stabilizing compensators for linear time-varying differential systems', pp. 1-30, to appear in *International J. of Control*, DOI: 10.1080/00207179.2015.1091949

[21] I-L. Popa, T. Ceauşu, M. Megan, 'On exponential stability for linear discrete-time systems in Banach spaces', arXiv: 1305.2036v1, 2013

[22] M. Van der Put, M.F. Singer, *Galois Theory of Difference Equations*, Springer, 1997

[23] A. Quadrat, 'Grade filtration of linear functional systems', *Acta Appl. Math.* 2013, 62 pages doi:10.1007/s10440-012-9791-2

[24] A. Quadrat, D. Robertz, 'Baer's extension problem for multidimensional linear systems', *Proceedings of MTNS 2008*, 12 pages

[25] W.J. Rugh, *Linear System Theory*, Prentice Hall, Upper Saddle River, NJ, 1996

[26] E. Zerz, ''An Algebraic Analysis Approach to Linear Time-varying Systems', *IMA J. Math. Control Inform.* 23(2006), 113-126

[27] I. Zidane, B. Marinescu, M. Abbas-Turki,'Linear time-varying control of the vibrations of flexible structures', *IET Control Theory and Applications*, doi: 10.1049/iet-cta.2013.1118