Multidimensional discrete stability by Serre categories and the construction and parametrization of observers via Gabriel localizations

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Martin Scheicher* and Ulrich Oberst
Institut für Mathematik, Universität Innsbruck
Technikerstrasse 13, A-6020 Innsbruck, Austria
martin.scheicher@uibk.ac.at, ulrich.oberst@uibk.ac.at

Abstract

Immediate predecessors of this work were a paper on two-dimensional dead-beat observers by Bisiacco and Valcher (*Multidim. Systems and Signal Processing* 19(2008), 287-306) and one on one-dimensional functional observers by Blumthaler (*Linear Algebra and its Applications* 432(2010), 1560-1577) (compare also Fuhrmann's comprehensive paper *Linear Algebra Appl.* 428(2008), 44-136). The present paper extends Blumthaler's results to continuous or discrete multidimensional behaviors, i.e., constructs and parametrizes all controllable observers of a given multidimensional behavior, and for this purpose also discusses the required multidimensional stability. Such an observer produces a signal that *approximates* or *estimates* a desired component of the behavior such that the signal difference is *negligible* in a suitable sense. This definition thus presupposes that of *negligible* or *stable* autonomous systems. In the standard one-dimensional case these are the asymptotically stable behaviors. We define and investigate the *characteristic variety* of an autonomous behavior in the needed generality of this paper and define *stability*, as in the one-dimensional case, by the *spectral condition* that the characteristic variety is contained in a preselected *stability region* of an appropriate multidimensional affine space. This stability is equivalent to the property that all *polynomial-exponential* trajectories in the behavior have frequencies in the stability region only. The stability region gives rise to a *Serre category* or class of modules over the relevant ring of operators that, by definition, is closed under isomorphisms, submodules, factor modules, extensions and direct sums and that determines the stability region. The *spectral condition* for stability is *equivalent* to the *algebraic condition* that the system module belongs to the associated Serre category. This category, in turn, gives rise to an associated *Gabriel localization* that is indispensable for the construction and parametrization of controllable observers.

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1 Introduction

The paper's main result Thm. 4.4 whose principal special case is exposed in Thm. 1.1 concerns the existence, construction and parametrization of all controllable functional observers of a given multidimensional discrete behavior. The *definition* of a multidimensional observer presupposes that of a suitable multidimensional *stability* as in dimension one where different stability notions lead, for instance, to exact, deadbeat, tracking and asymptotic observers [9]. We define the stability of an autonomous behavior by a *spectral condition* on its *characteristic variety* and establish the analytic significance of this condition. The algebraic counterpart of any chosen stability notion is the corresponding *Serre category* of modules; an autonomous behavior is stable if and only if its dual module belongs to the corresponding Serre category. The mathematical theory of multidimensional stability, its analytic significance and its associated Serre category under the general assumptions of this paper is developed in Section 5. The proof of our main theorem requires the *Gabriel localization functor* associated with the Serre category and hence with the chosen stability notion. Section 3 develops this localization theory as far as needed. In Section 2 we explain, without proofs, the stability theory with a multidimensional standard example and the standard one-dimensional theory.
The paper is an elaboration of [25]. Immediate predecessors of our work were the paper [3] on two-dimensional deadbeat observers by Bisiacco and Valcher and the paper [6] by Blumthaler on one-dimensional functional observers. These recent papers and the present one continue and extend the one-dimensional observer constructions of many prominent researchers, see [9] and the references of [6].
The goal of the following more precise description of the data introduced above is to enable the understanding of our main Thm. 1.1 on the existence, construction and parametrization of multidimensional observers without going into all details of the technical Sections 3 and 5. We also compare the multidimensional concepts with the standard one-dimensional ones from [7], [11], [24]. In the most important cases of this paper the *signal spaces* and corresponding rings of *operators* are the following: As base field F we choose the complex field $\mathbb{C}$ or the real field $\mathbb{R}$. The theory for the real field $\mathbb{R}$ is more complicated as shown in Section 5. Let $m = m_I + m_{II} \in \mathbb{N}$ be an additive decomposition. As discrete domain of the independent variables of the signals we use the sublattice of $\mathbb{Z}^m$

$$N := \mathbb{N}^{m_I} \times \mathbb{Z}^{m_{II}} \ni \mu = (\mu_1, \cdots, \mu_m) = (\mu_I, \mu_{II}),\ \mu_I = (\mu_1, \cdots, \mu_{m_I}). \tag{1}$$

The cases $N = \mathbb{N} \times \mathbb{Z}$ from [3], [4] and $N = \mathbb{Z}^2$ from [32] and [17] are special cases and motivated this generality. The lattice N gives rise to the *signal F-space*

$$\mathscr{F} := F^N := \{w : N \to F\} \ni w = (w(\mu))_{\mu \in N}. \tag{2}$$

Let $s = (s_1, \cdots, s_m) = (s_I, s_{II}),\ s_I = (s_1, \cdots, s_{m_I})$, be a list of indeterminates. The monoid algebra of the monoid or lattice N is the factorial F-affine integral domain

$$A := F[N] = \oplus_{\mu \in N} F s^\mu = F[s_I, s_{II}, s_{II}^{-1}],\ s_{II}^{-1} := (s_{m_I+1}^{-1}, \cdots, s_m^{-1}). \tag{3}$$

Obviously the ring A is a *mixed Laurent polynomial algebra*. The ring $A = F[N]$ acts on $\mathscr{F} = F^N$ by the usual *shift or translation action* $\circ$ defined by

$$(s^\mu \circ w)(\nu) := w(\mu + \nu),\ w \in F^N,\ \mu, \nu \in N, \tag{4}$$

and makes $\mathscr{F}$ an A-module and indeed a *large injective cogenerator*. Hence there is a strong *duality*

$$\begin{gathered} M \leftrightarrow \mathscr{B} := U^\perp := \left\{ w \in \mathscr{F}^\ell;\ R \circ w = 0 \right\} \underset{\text{Malgrange 1962}}{\cong} D(M) := \mathrm{Hom}_A(M, \mathscr{F}), \\ R \in A^{k\times\ell},\ U := A^{1\times k}R,\ M := A^{1\times\ell}/U \end{gathered} \tag{5}$$

between finitely generated A-modules M and their associated *behaviors* $\mathscr{B}$. The modules U resp. M are called the *equation module* resp. the *system module* of $\mathscr{B}$. The behavior is *autonomous* if and only if $\mathrm{rank}(R) = \ell$ or M is a torsion module. In the sequel we will often abbreviate the terms 'finitely generated' resp. 'finite-dimensional' by 'f.g.' resp. 'f.d.'. Stability and stabilization in the case $F = \mathbb{C}$, $N = \mathbb{N}^m$ and $A = \mathbb{C}[s]$ were first treated in [20, §5] with the technique of the present paper.
We construct observers for a multidimensional behavior $\mathscr{B} \subseteq \mathscr{F}^\ell$ with two additional matrices (operators) $P \in A^{m\times\ell}$ and $Q \in A^{q\times\ell}$. (Here the row dimension m of P is *not* the number of components of N, the correct interpretation of m follows from the context.) Often $P \circ w$ resp. $Q \circ w$ are called the *measurable part* resp. the *relevant part* of a trajectory w of $\mathscr{B}$ [3]. A *(functional) observer of $Q \circ w$ from $P \circ w$, $w \in \mathscr{B}$*, is an input/output (IO) behavior $\mathscr{B}_{\mathrm{obs}}$ with trajectories $\binom{y}{u} \in \mathscr{F}^{q+m}$ that accepts the image $P \circ w$ of a trajectory $w \in \mathscr{B}$ as input u and outputs an *approximation* y of $Q \circ w$. This signifies that $y - Q \circ w$ is *small* or *negligible* in a sense that has to be defined. In other words, the error behavior

$$\mathscr{B}_{\mathrm{err}} := \left\{ y - Q \circ w;\ w \in \mathscr{B},\ \binom{y}{P\circ w} \in \mathscr{B}_{\mathrm{obs}} \right\} \tag{6}$$

should be (autonomous and) small (negligible, stable), again in a sense that has to be defined in the multidimensional situation. The interconnection diagram of $\mathscr{B}$ and $\mathscr{B}_{\mathrm{obs}}$ is shown in Figure 1.

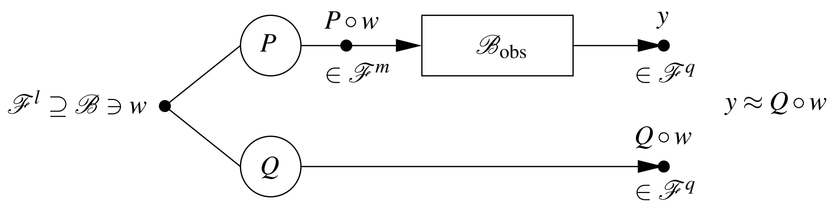

Figure 1: The interconnection diagram.

Already in [7, p.357] and [11, p.522] Luenberger's state observer from input and output is motivated by a special case of (6). Up to the more involved multidimensional stability the definition of a functional observer by (6) coincides with that of Fuhrmann [9] and Blumthaler [6] in dimension 1.
In the standard one-dimensional case the negligible autonomous behaviors are the asymptotically stable ones that are defined by *spectral conditions* on the *characteristic*

frequencies of the behaviors. The set of characteristic frequencies of a multidimensional autonomous behavior $\mathscr{B}$ is the *characteristic variety* $\operatorname{char}(\mathscr{B})$. This variety and its potential usefulness for multidimensional stability in the continuous case were already discussed in [18, pp.157-161] where it was quoted from [5, ch.8, §1.7], but only in the discrete case $F=\mathbb{C}$, $N=\mathbb{N}^m$ and $A=\mathbb{C}[s]$ and the standard continuous case. Shankar [28, §4] applied it to multidimensional continuous stability. Special instances of the characteristic variety appeared in [2] as *variety of rank singularities*, in [32, Prop. 3.2] for $N=\mathbb{Z}^2$ as *Laurent variety of maximal order minors*, in [4, p.3] for $N=\mathbb{N}\times\mathbb{Z}$ as *time/space(TS)-variety*, in [17, Introd.] for $N=\mathbb{Z}^2$ as the *set of zeros of the determinant of the square polynomial matrix describing the system* and in [1, Introd.] as *set of characteristic frequencies*. For its definition in the general situation of this paper we first define the global space

$$\Lambda_N := \mathbb{C}^{m_I}\times(\mathbb{C}\setminus\{0\})^{m_{II}} = \{\lambda=(\lambda_I,\lambda_{II})\in\mathbb{C}^m;\ \forall i=m_I+1,\cdots,m:\ \lambda_i\neq 0\}\subset\mathbb{C}^m \tag{7}$$

of all complex vectors λ that can be substituted into all Laurent polynomials $f\in A$, i.e., for which $f(\lambda)$ is defined. If the autonomous behavior $\mathscr{B}$ is given by a matrix $R\in A^{k\times\ell}$ as in (5) the *characteristic variety* of $\mathscr{B}$ or M is defined as

$$\operatorname{char}(\mathscr{B}) := \operatorname{char}(M) := \{\lambda\in\Lambda_N;\ \operatorname{rank}(R(\lambda))<\ell=\operatorname{rank}(R)\}. \tag{8}$$

It coincides with the *variety*, *vanishing set* or set of zeros

$$V_{\Lambda_N}(\mathfrak{a}) := \{\lambda\in\Lambda_N;\ \forall f\in\mathfrak{a}:\ f(\lambda)=0\},\ \mathfrak{a}\subseteq A, \tag{9}$$

of the *annihilator ideal*

$$\mathfrak{a} := \operatorname{ann}_A(M) := \{f\in A;\ fM=0\} \tag{10}$$

of the system module $M=A^{1\times\ell}/A^{1\times k}R$ of $\mathscr{B}$, see (86), (87). This implies in particular that $\operatorname{char}(\mathscr{B})$ depends on $\mathscr{B}$ only and not on the special choice of R. If in dimension $m=1$ with $N=\mathbb{N}$, $A=\mathbb{C}[s]$, $\mathscr{F}=\mathbb{C}^{\mathbb{N}}$ the autonomous behavior has the state space form

$$\mathscr{B}=\left\{w\in\mathscr{F}^\ell;\ s\circ w=Gw\right\}=\left\{w\in\mathscr{F}^\ell;\ R\circ w=0\right\},\ G\in F^{\ell\times\ell},\ R:=s\,\mathrm{id}_\ell-G,\ \text{then}$$
$$\operatorname{char}(\mathscr{B})=\{\lambda\in\mathbb{C};\ \operatorname{rank}(\lambda\,\mathrm{id}_\ell-G)<\ell\}=\{\lambda\in\mathbb{C};\ \det(\lambda\,\mathrm{id}_\ell-G)=0\} \tag{11}$$

is the *spectrum* or set of eigenvalues of G, i.e., the set of roots of its *characteristic polynomial* $\det(s\,\mathrm{id}_\ell-G)$, whence the term *characteristic variety*. In higher dimensions the characteristic variety replaces the spectrum of a complex matrix. A one-dimensional transfer matrix $H\in\mathbb{C}(s)^{p\times m}$ has a unique *controllable* or *irreducible* [7, Thm. 6-2], [11, p.574] input/output realization

$$\begin{gathered}\mathscr{B}:=\left\{\binom{y}{u}\in\mathscr{F}^{p+m};\ P\circ y=Q\circ u\right\},\ (P,-Q)\in A^{p\times(p+m)},\ \det(P)\neq 0,\ H=P^{-1}Q,\\ A^{1\times p}P=\left\{\xi\in A^{1\times p};\ \xi H\in A^{1\times m}\right\},\ Q:=PH,\\ \mathscr{B}^0:=\{y\in\mathscr{F}^p;\ P\circ y=0\},\ \operatorname{char}(\mathscr{B}^0)=\{\lambda\in\mathbb{C};\ \det(P(\lambda))=0\}.\end{gathered} \tag{12}$$

Then $\det(P)$ is called the characteristic polynomial [7, Def. 6-1'] and $\operatorname{char}(\mathscr{B}^0)$ the set of *(finite) poles* of H [7, p.443], [11, §6.5.3, §8.3.2].
Spectral conditions on $\mathscr{B}$ are conditions on $\operatorname{char}(\mathscr{B})$. To introduce these we choose a disjoint *stability decomposition*

$$\Lambda_N=\Lambda_1\uplus\Lambda_2 \text{ with } \Lambda_2\neq\emptyset \tag{13}$$

where Λ_1 resp. Λ_2 are called the *stable (stability) region* resp. the *unstable (instability) region*. In the real case $F=\mathbb{R}$ we assume as usual that the Λ_i are invariant under the complex conjugation $\lambda \mapsto \overline{\lambda} := (\overline{\lambda_1} \cdots, \overline{\lambda_m})$. In the one-dimensional discrete standard case Λ_1 is the interior of the unit disc. With these data Λ_1-*stability* or Λ_1-*negligibility* of the autonomous behavior $\mathscr{B}$ is defined by the *spectral condition* $\mathrm{char}(\mathscr{B}) \subseteq \Lambda_1$. *Analytically* this spectral condition signifies that the *polynomial-exponential* trajectories of $\mathscr{B}$ have frequencies in Λ_1 only (Thms. 5.8, 5.11). This explains the systems theoretic relevance of the spectral condition. Here a signal $w \in F^N$ is called *polynomial-exponential* or *finite* if the cyclic module $A \circ w$ is F-finite-dimensional. These finite signals are described in Results 5.7 and 5.10 that are quoted from [19]. In the simplest case $N=\mathbb{Z}^n$, $F=\mathbb{C}$ a signal is finite if and only if it is a finite $\mathbb{C}$-linear combination of signals $(p(\mu)\lambda^{\mu})_{\mu\in\mathbb{Z}^n}$ where p is a polynomial function and λ a frequency vector in $\Lambda_{\mathbb{Z}^n} = (\mathbb{C}\setminus\{0\})^n$. An ideal $\mathfrak{a}$ resp. an element f of A are called Λ_1-stable if the cyclic modules $A/\mathfrak{a}$ resp. A/Af have this property or, equivalently, if the varieties $V_{\Lambda_N}(\mathfrak{a})$ resp. $V_{\Lambda_N}(f) := V_{\Lambda_N}(Af)$ are contained in Λ_1. In the one-dimensional discrete standard case a polynomial is stable if its roots have absolute value less than 1. An element h in the quotient field $\mathrm{quot}(A) = F(s) = F(s_1, \cdots, s_n)$ of rational functions is called Λ_1-stable if it admits a representation $h = \frac{f}{g}$ with $f, g \in A$ and Λ_1-stable g. *Properness* of Λ_1-stable rational functions or matrices as in [33, Ch.2] is not discussed in this paper.

Serre categories appear if one looks for *algebraic* characterizations of f.g. A-modules M whose dual behaviors $\mathscr{B} \cong D(M)$ are Λ_1-stable. By definition, such a category is a class $\mathfrak{C}$ of A-modules that is closed under isomorphisms, submodules, factor modules, extensions and direct sums. These defining properties enable various constructions with and inside $\mathfrak{C}$ that we employ in connection with stability and our main theorem on observers. Especially every module M has a largest submodule $\mathrm{Ra}_{\mathfrak{C}}(M)$ in $\mathfrak{C}$, its $\mathfrak{C}$-*radical*. In [27, Ch. I] Serre introduced Serre categories of abelian groups under the name *classes* and already called the groups in such a class *negligible*. In Thms. 5.8 and 5.11 we construct such a category $\mathfrak{C}(\Lambda_1)$ for every stability decomposition (13) and show that Λ_1 is determined by $\mathfrak{C}(\Lambda_1)$ and that the *spectral condition* $\mathrm{char}(\mathscr{B}) \subseteq \Lambda_1$ for $\mathscr{B} \cong D(M)$ is indeed equivalent to the *algebraic condition* $M \in \mathfrak{C}(\Lambda_1)$. In the one-dimensional situation of (12) the f.g. modules in $\mathfrak{C}(\Lambda_1)$ occur as system modules $M^0 := \mathbb{C}[s]^{1\times p}/\mathbb{C}[s]^{1\times p}P$ of the autonomous parts $\mathscr{B}^0$ with Λ_1-stable determinant $\det(P)$. Most books on one-dimensional systems theory study and construct Λ_1-stable square matrices P instead of M^0, for instance for the design of stabilizing compensators [7, Thms. 9-6, 9-9], [11, §7.5]. That P can be chosen square follows from the Smith form of univariate polynomial matrices. For multivariate polynomials such a form does not exist and therefore the study of f.g. polynomial modules is often more natural and simpler than that of polynomial matrices. The f.g modules M in a Serre category $\mathfrak{C}$ and their dual behaviors $D(M)$ are suggestively called $\mathfrak{C}$-*small*, $\mathfrak{C}$-*negligible* or $\mathfrak{C}$-*stable*. We show that a behavior is $\mathfrak{C}$-negligible if and only if it itself belongs to $\mathfrak{C}$. Thus a behavior $\mathscr{B}$ is Λ_1-stable or Λ_1-negligible (spectral condition) if and only if it is $\mathfrak{C}(\Lambda_1)$-stable or -negligible (algebraic condition). There are Serre categories $\mathfrak{C}$ of systems theoretic interest that are not of the form $\mathfrak{C}(\Lambda_1)$, for instance the class $\mathfrak{C}_{\mathrm{fin}}$ (see (64)) of all A-modules M whose cyclic submodules Ax, $x \in M$, are F-finite-dimensional (f.d.). The corresponding $\mathfrak{C}_{\mathrm{fin}}$-negligible f.g. modules M or dual behaviors $D(M)$ are precisely the F-f.d. ones and were studied, in particular with respect to their negligibility, in [31] and [15]. Therefore we often use Serre categories for the definition of stability and derive the spectral characterization as special case.
Every Serre category $\mathfrak{C}$ gives rise to a specific *Gabriel localization* functor $\mathscr{Q}_{\mathfrak{C}}$ on A-

modules with the property that $\mathfrak{C}=\ker(\mathscr{Q}_{\mathfrak{C}}):=\{C;\ \mathscr{Q}_{\mathfrak{C}}(C)=0\}$. Gabriel localization arises naturally when one wants to study A-modules up to negligible ones. In the proof of our main Thm. 4.4 we apply $\mathscr{Q}_{\mathfrak{C}}$ to negligible trajectories and modules and thereby annihilate them. This simplifies all equations considerably. The most important special case of the theorem is

Theorem 1.1. (Main theorem) *For a given stability decomposition* (13) *consider the associated Serre category $\mathfrak{C}(\Lambda_1)$, three matrices $R\in A^{k\times\ell}$, $P\in A^{m\times\ell}$, $Q\in A^{q\times\ell}$ and the behavior $\mathscr{B}:=\left\{w\in\mathscr{F}^{\ell};\ R\circ w=0\right\}$. Compute a matrix $R'\in A^{k'\times\ell}$ by Algorithm 3.1 such that*

$$A^{1\times k'}R'\supseteq A^{1\times k}R \text{ and } \mathrm{Ra}_{\mathfrak{C}(\Lambda_1)}\left(A^{1\times\ell}/A^{1\times k}R\right)=A^{1\times k'}R'/A^{1\times k}R. \tag{14}$$

There is an input/output observer behavior $\mathscr{B}_{\mathrm{obs}}$ with Λ_1-stable error behavior $\mathscr{B}_{\mathrm{err}}$ as in (6) *if and only if there are Λ_1-stable rational matrices (with Λ_1-stable entries)*

$$X\in F(s)^{q\times k'} \text{ and } H_{\mathrm{obs}}\in F(s)^{q\times m} \text{ such that } Q=XR'+H_{\mathrm{obs}}P. \tag{15}$$

For each such equation the unique controllable realization $\mathscr{B}_{\mathrm{obs}}$ of the (transfer) matrix H_{obs}, i.e., the input/output behavior

$$\begin{gathered}\mathscr{B}_{\mathrm{obs}}:=\left\{\left(\begin{smallmatrix}y\\u\end{smallmatrix}\right)\in\mathscr{F}^{q+m};\ P_{\mathrm{obs}}\circ y=Q_{\mathrm{obs}}\circ u\right\},\ (P_{\mathrm{obs}},-Q_{\mathrm{obs}})\in A^{k_{\mathrm{obs}}\times(q+m)},\ \text{with}\\ A^{1\times k_{\mathrm{obs}}}P_{\mathrm{obs}}=\left\{\xi\in A^{1\times q};\ \xi H_{\mathrm{obs}}\in A^{1\times m}\right\},\ Q_{\mathrm{obs}}:=P_{\mathrm{obs}}H_{\mathrm{obs}},\end{gathered} \tag{16}$$

is a (controllable) observer of $Q\circ w$ from $P\circ w$, $w\in\mathscr{B}$. All controllable observers with Λ_1-stable error behavior (6) *are obtained in this fashion.*

Remark 1.2. The set Λ_2 is called *ideal-convex* [29] if each Λ_1-stable ideal contains a Λ_1-stable $f\in A$ or, equivalently, if the Gabriel localization $\mathscr{Q}_{\mathfrak{C}(\Lambda_1)}(M)$ of every module ${}_AM$ coincides with the standard quotient module M_T with respect to the multiplicatively closed set T of Λ_1-stable polynomials. If this property holds the matrix R' in (14) can be replaced by R, the proof of Thms. 4.4 and 1.1 can be simplified and the existence of an observer is equivalent to the usual *detectability* condition that the negligibility of $P\circ w$, $w\in\mathscr{B}$, implies that of $Q\circ w$. In dimension $m>2$ ideal convexity rarely holds and is hard to check.

The linear equation $Q=XR'+H_{\mathrm{obs}}P$ in Thm. 1.1 for observer constructions was stimulated by its one-dimensional predecessors (see [9] and [6] for one-dimensional observer results, their literature and principal contributors) and by the two-dimensional deadbeat observers of [3], see Example 4.6.
Section 3 furnishes a simpler and more comprehensive introduction to the Gabriel localization functor $\mathscr{Q}_{\mathfrak{C}}$ than that given in [20] and contains the indispensable technical preparations for the proof of the main theorem in Section 4. Its most important new results are Algorithm 3.1 for the computation of $\mathrm{Ra}_{\mathfrak{C}}(M)$ for a f.g. A-module M, Algorithm 3.9 for the computation of $\mathscr{Q}_{\mathfrak{C}}\left(A^{1\times k}R\right)$ for $R\in A^{k\times\ell}$ and Thm. 3.2 on the direct sum decomposition of the signal module into its *steady state* part and its negligible part. The algorithms make *Computer Algebra* applicable to the theorems of this paper as discussed in Section 7. Remark 3.10 on *Willems closures* is a side result of this paper and not used otherwise, but nevertheless interesting since, for instance, Shankar et al. [28], [16] and Sasane [26] have derived special results in this direction.

Several results of this paper, for instance those in Section 4 on the construction of observers, can be and are derived for any F-affine domain $A = F[s]/I$ of operators with a polynomial prime ideal I and any injective cogenerator signal module ${}_A\mathscr{F}$. Even arbitrary commutative noetherian domains A could be admitted. Already in [18, Chs. 2,7] multidimensional behavioral systems theory, in particular the module-behavior duality, were developed for signal spaces ${}_A\mathscr{F}$ of this generality. For Section 5 and the consideration of polynomial-exponential signals the large injective cogenerator

$$A^* := \mathrm{Hom}_F(A,F) \cong \left\{w \in F^{\mathbb{N}^m};\ I \circ w = 0\right\} \tag{17}$$

has to be used. The recent paper [1] applies the signal space (17) in a very interesting special case and mentions the significance of polynomial-exponential solutions and of the set of characteristic frequencies as developed in Section 5.

In Section 6 we consider an arbitrary finitely generated submonoid $N \subseteq \mathbb{Z}^n$ with $\mathbb{Z}^n = N - N$ w.l.o.g. as discrete domain of the independent variables and its monoid algebra $F[N]$ that acts on the large injective $F[N]$-cogenerator $F^N = F[N]^*$ by translation. In systems theory examples of such domains are $N := \mathscr{H}_0 := \left\{(\mu_1,\mu_2) \in \mathbb{Z}^2;\ \mu_1+\mu_2 \geq 0\right\}$ in [3, p.2] and the *cones* $\mathscr{C} \subset \mathbb{Z}^2$ of [14, Def. 5] that were used for causal input/output representations of two-dimensional behaviors. The connection of the algebraic properties of $F[N]$ with the combinatorial properties of N are investigated in the monograph [13]. The rings $F[N]$ are F-affine domains, but not factorial in general. This creates problems in the application of Gabriel localization in Section 4 and suggests the construction of Serre categories $\mathfrak{C}_N$ of $F[N]$-torsion modules that are induced from Serre categories $\mathfrak{C}_{\mathbb{Z}^n}$ of $F[\mathbb{Z}^n]$-torsion modules over the factorial Laurent polynomial algebra $F[\mathbb{Z}^n]$. The main results are Thms. 6.3 and 6.4. The least Serre category $\mathfrak{C}_N$ of this kind is that whose $\mathfrak{C}_N$-negligible trajectories resp. behaviors are the deadbeat resp. nilpotent ones from [3], [4], but here for general $N \subseteq \mathbb{Z}^n$ instead of $N = \mathbb{N}\times\mathbb{Z} \subset \mathbb{Z}^2$. That $F[N]$ is not factorial in general or, in other words, not a unique factorization domain was also observed in [14, Remark on p.1544] and created difficulties in the proof of [14, Thm. 7].

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2 A multidimensional example

We explain the stability notions of the Introduction for one important example, but refer to the following sections for the proofs. We use the complex base field for simplicity.
With the notations from the Introduction we consider the case

$$\begin{gathered} F := \mathbb{C},\ m>0,\ m_I := 1,\ m_{II} = m-1,\ N = \mathbb{N}\times\mathbb{Z}^{m-1},\ t := \mu_1,\ A = \mathbb{C}[s_1, s_{II}, s_{II}^{-1}], \\ \mathscr{F} = \mathbb{C}^{\mathbb{N}\times\mathbb{Z}^{m-1}} = \left(\mathbb{C}^{\mathbb{Z}^{m-1}}\right)^{\mathbb{N}} \ni w = (w(\mu))_{\mu\in N} = (w(0), w(1), \cdots) \text{ with} \\ w(t) \in \mathbb{C}^{\mathbb{Z}^{m-1}} \text{ and } w(t)(\mu_{II}) = w(t,\mu_{II}),\ t\in\mathbb{N},\ \mu_{II}\in\mathbb{Z}^{m-1} \\ \Lambda_N := \mathbb{C}\times(\mathbb{C}\setminus\{0\})^{m-1}. \end{gathered} \tag{18}$$

The number $t = \mu_1$ is interpreted as a discrete time instant and the signal w as a time-series of signals $w(t) \in \mathbb{C}^{\mathbb{Z}^{m-1}}$ at the time t. For $m = 1$, $m_{II} = 0$, these data are the

standard ones of one-dimensional discrete systems theory for the time axis $\mathbb{N}$.
For each $\lambda \in \Lambda_N$ we consider the *character* or *substitution homomorphism* χ_λ and its kernel $\mathfrak{m}(\lambda)$ defined by

$$\chi_\lambda : A \to \mathbb{C},\ f \mapsto f(\lambda),\ \mathfrak{m}(\lambda) := \ker(\chi_\lambda) = \{f \in A;\ f(\lambda) = 0\} = \sum_{i=1}^{m} A(s_i - \lambda_i). \tag{19}$$

Hilbert's Nullstellensatz implies that the $\mathfrak{m}(\lambda)$, $\lambda \in \Lambda_N$, are precisely all maximal ideals of A. A signal $w \in \mathscr{F}$ is *finite* or *polynomial-exponential* if the cyclic submodule $A \circ w \subset \mathscr{F}$ is $\mathbb{C}$-finite-dimensional (f.d.). The A-submodule $\mathscr{F}_{\text{fin}}$ of $\mathscr{F}$ of all finite signals admits a direct sum decomposition into A-submodules $\mathscr{F}(\lambda)$:

$$\begin{gathered}\mathscr{F}_{\text{fin}} = \oplus_{\lambda\in\Lambda_N}\mathscr{F}(\lambda) \text{ where } \mathscr{F}(\lambda) := \left\{w \in \mathscr{F};\ \exists k \in \mathbb{N} \text{ with } \mathfrak{m}(\lambda)^k \circ w = 0\right\} = \\ \oplus_{\alpha\in\mathbb{N}^m}\mathbb{C}e_{\lambda,\alpha},\ \alpha = (\alpha_1, \alpha_{II}) \in \mathbb{N}^m = \mathbb{N}\times\mathbb{N}^{m-1},\ \mu = (t, \mu_{II}) \in N = \mathbb{N}\times\mathbb{Z}^{m-1}, \\ e_{\lambda,\alpha}(t,\mu_{II}) := e_{\lambda_1,\alpha_1}(t)\binom{\mu_{II}}{\alpha_{II}}\lambda_{II}^{\mu_{II}},\ e_{\lambda_1,\alpha_1}(t) := \begin{cases}\binom{t}{\alpha_1}\lambda_1^t & \text{if } \lambda_1 \neq 0 \\ \delta_{\alpha_1,t} & \text{if } \lambda_1 = 0\end{cases}.\end{gathered} \tag{20}$$

As functions of t resp. μ_{II} the factors λ_1^t and $\lambda_{II}^{\mu_{II}}$ are powers (exponentials) whereas $\binom{t}{\alpha_1}$ and $\binom{\mu_{II}}{\alpha_{II}}$ are multinomial coefficients and thus polynomial functions. This explains the term *polynomial-exponential* for the signals in $\mathscr{F}_{\text{fin}}$. The growth of these signals is, of course, determined by their exponential factors. The decomposition (20) is a standard result for one-dimensional discrete systems theory ($m = 1$) (cf. [24, Thm. 3.2.5] in the continuous case). If in (5) M and $\mathscr{B}$ are $\mathbb{C}$-f.d., necessarily of the same dimension, then, of course, $\mathscr{B}$ contains finite trajectories only, i.e., $\mathscr{B} \subset \mathscr{F}_{\text{fin}}^{\ell}$. This holds for all autonomous systems in dimension $m = 1$. We choose the stability decomposition (cf. [17, Thm. 10] for $m = 2$)

$$\Lambda_N = \Lambda_1 \uplus \Lambda_2 \text{ with } \Lambda_2 := \{\lambda \in \Lambda_N;\ |\lambda_1| \geq 1,\ \forall i = 2, \cdots, m:\ |\lambda_i| = 1\}. \tag{21}$$

Consider the data from above in dimension $m = 1$:

$$N = \mathbb{N},\ s = s_1,\ A = \mathbb{C}[s],\ \mathscr{F} = \mathbb{C}^{\mathbb{N}},\ \Lambda_N = \mathbb{C},\ \Lambda_1 = \{\lambda \in \mathbb{C};\ |\lambda| < 1\}. \tag{22}$$

In dimension $m = 1$ the autonomous behavior $\mathscr{B}$ from (5) can always be described by a *square matrix* $R \in A^{\ell\times\ell}$ of $\text{rank}(R) = \ell$ (cf. [24, Thm. 2.5.23]) that is unique up to row equivalence and gives rise to the *characteristic polynomial* $\det(R) \in \mathbb{C}[s]$ of $\mathscr{B}$. The characteristic variety or (finite) set of characteristic frequencies is

$$\text{char}(\mathscr{B}) = \{\lambda \in \mathbb{C};\ \text{rank}(R(\lambda)) < \ell\} = \{\lambda \in \mathbb{C};\ \det(R(\lambda)) = 0\}. \tag{23}$$

The behavior admits the direct sum decomposition

$$\mathscr{B} = \oplus_{\lambda\in\text{char}(\mathscr{B})}\mathscr{B}(\lambda),\ \mathscr{B}(\lambda) := \mathscr{B}\bigcap\mathscr{F}(\lambda)^{\ell}, \tag{24}$$

(cf. [24, Thm. 3.2.16 and proof, pp.77-79] in the continuous case). For autonomous systems $\mathscr{B}$ in the state space form (11) the decomposition (24) is called the *modal decomposition* of $\mathscr{B}$ and the elements of $\mathscr{B}(\lambda)$ are called the *modes* of (complex) frequency λ [11, §2.5.2], [7, p.145]. The mathematical background of the modal decomposition is the *Jordan decomposition* of the matrix G. Moreover it is shown [7, Thm. 8-15], [11, p.176-177], [24, Thm. 7.2.2,(i) and proof, Ex. 7.8 on p.271] that the following conditions are equivalent:

$$\begin{gathered}\operatorname{char}(\mathscr{B}) \subset \Lambda_1, \text{ i.e., } \mathscr{B} \text{ is } \Lambda_1\text{-stable} \iff \mathscr{B} \subset \oplus_{\lambda\in\Lambda_1}\mathscr{F}(\lambda)^\ell \iff \\ \mathscr{B} \text{ is asymptotically or internally stable, i.e., } \forall w \in \mathscr{B}:\ \lim_{t\to\infty} w(t) = 0.\end{gathered} \tag{25}$$

The second equivalence follows from a simple analytic argument with geometric sequences. It is also customary (cf. [33, p.14] in the continuous case) to diminish Λ_1 to obtain better convergence properties and even to choose a finite set Λ_1 and thus to prescribe the characteristic frequencies of the behavior (cf. [7, Thm. 7-7, p.367], [11, p.511], [24, Thm. 10.3.1] in the continuous case).

All results of Section 5 are generalizations of the quoted theorems from [7], [11], [24], and of (23)-(25) to higher dimensions, additional fields and more general rings of operators. Their use in the papers [32], [4], [17] shows that they are significant in the context of multidimensional stability and not only for the observer definition and constructions of the present paper. For $\lambda \in \Lambda_N$ and $\mathfrak{m}(\lambda)$ from (19) one has to consider

$$\begin{gathered}\mathscr{B}_0(\lambda) := \mathscr{B}\bigcap \mathbb{C}^\ell e_{\lambda,0} \subseteq \mathscr{B}(\lambda) := \mathscr{B}\bigcap \mathscr{F}(\lambda)^\ell \subseteq \mathscr{F}_{\mathrm{fin}}^\ell \\ \text{and the quotient module } M_{\mathfrak{m}(\lambda)} = \left\{\frac{x}{f};\ x\in M,\ f\in A,\ f(\lambda)\neq 0\right\}\end{gathered} \tag{26}$$

over the local quotient ring $A_{\mathfrak{m}(\lambda)}$. In Section 5 it is then proven that

$$\begin{gathered}\mathscr{B}\bigcap \mathscr{F}_{\mathrm{fin}}^\ell = \oplus_{\lambda\in\operatorname{char}(\mathscr{B})}\mathscr{B}(\lambda) \text{ and} \\ \operatorname{char}(\mathscr{B}) = \{\lambda\in\Lambda_N;\ \mathscr{B}_0(\lambda)\neq 0\} = \{\lambda\in\Lambda_N;\ \mathscr{B}(\lambda)\neq 0\} = \left\{\lambda\in\Lambda_N;\ M_{\mathfrak{m}(\lambda)}\neq 0\right\}.\end{gathered} \tag{27}$$

The second equation shows again that $\operatorname{char}(\mathscr{B})$ is indeed independent of the special choice of R, the first establishes a direct sum decomposition of the module of polynomial-exponential trajectories in $\mathscr{B}$. But note that for $m \geq 2$ an autonomous behavior contains, in general, many trajectories that are not polynomial-exponential. Like in dimension 1 the first equation in (27) implies the equivalence

$$\operatorname{char}(\mathscr{B}) \subseteq \Lambda_1 \iff \mathscr{B}\bigcap \mathscr{F}_{\mathrm{fin}}^\ell \subseteq \oplus_{\lambda\in\Lambda_1}\mathscr{F}(\lambda)^\ell \tag{28}$$

and thus the description of Λ_1-stability of $\mathscr{B}$ by the equivalent property that all polynomial-exponential trajectories in $\mathscr{B}$ have frequencies in Λ_1 only. The second equation of (27) suggests to define the class of A-modules

$$\mathfrak{C}(\Lambda_1) := \left\{C;\ C\ A\text{-module},\ \forall\lambda\in\Lambda_2:\ C_{\mathfrak{m}(\lambda)} = 0\right\}. \tag{29}$$

Standard properties of the functors $C \mapsto C_{\mathfrak{m}(\lambda)}$ imply that this class $\mathfrak{C}(\Lambda_1)$ is a *Serre category*. For $\lambda, \lambda' \in \Lambda_N$ the equivalence $\left((A/\mathfrak{m}(\lambda))_{\mathfrak{m}(\lambda')} \neq 0 \iff \lambda = \lambda'\right)$ implies that $\Lambda_1 = \{\lambda \in \Lambda_N;\ A/\mathfrak{m}(\lambda) \in \mathfrak{C}(\Lambda_1)\}$ so that the associated Serre category $\mathfrak{C}(\Lambda_1)$ determines the stability region Λ_1. Equation (27) also implies the equivalence

$$\operatorname{char}(\mathscr{B}) \subseteq \Lambda_1 \iff M \in \mathfrak{C}(\Lambda_1) \quad (\text{where } \mathscr{B} \cong D(M)) \tag{30}$$

and therefore the equivalence of the spectral and the algebraic definition of Λ_1-stability.

Whether the analytic condition $\lim_{t\to\infty} w(t) = 0$ in (25) has also a multidimensional counterpart was discussed in [17, Thm. 10] for $m = 2$ for the special stability decomposition from (21), but Λ_1-stability is not equivalent to the appropriate analytic condition. Predecessors of [17] were [8], [2] and [32]. The paper [23] extends [17] to arbitrary dimensions m and more general autonomous behaviors and contains the following result (see [23] for the details): Consider the Hilbert space

$$\mathrm{L}^2(\mathbb{Z}^{m-1}) := \left\{ u \in \mathbb{C}^{\mathbb{Z}^{m-1}};\ \sum_{\nu\in\mathbb{Z}^{m-1}} |u(\nu)|^2 < \infty \right\} \tag{31}$$

of square-summable multisequences with its standard inner product. Assume that the autonomous behavior $\mathscr{B}$ is Λ_1-stable and that $\mathscr{B}$ is *time-autonomous* (ta) (=*time-relevant* in [17]) in the sense that there is a time instant $d \in \mathbb{N}$ such that the map

$$\mathscr{B} \to \left((\mathbb{C}^{\mathbb{Z}^{m-1}})^{\ell} \right)^d,\ w \mapsto (w(0), \cdots, w(d-1)), \tag{32}$$

is injective so that $w \in \mathscr{B}$ is fully determined by its d initial data $w(0), \cdots, w(d-1) \in \left(\mathbb{C}^{\mathbb{Z}^{m-1}}\right)^{\ell}$. Time-autonomy can be constructively checked by [22, Thm. 3.7 and Cor. 3.8]. If the initial data $w(0), \cdots, w(d-1)$ belong to $\mathrm{L}^2(\mathbb{Z}^{m-1})^{\ell}$ and if a weak additional condition is satisfied then all $w(t),\ t \in \mathbb{N}$, belong to $\mathrm{L}^2(\mathbb{Z}^{m-1})^{\ell}$ and $\lim_{t\to\infty} w(t) = 0$ in the Hilbert space topology. We call this analytic stability L^2-stability. We conjecture that the weak additional condition is superfluous, but have not yet proven this. An example in [23] shows that time-autonomy and L^2-stability do not imply Λ_1-stability. At present we know of no *analytic* condition that is *equivalent* to Λ_1-stability of a *time-autonomous* behavior.

The L^2-stability of the error behavior $\mathscr{B}_{\mathrm{err}}$ from (6) as a consequence of its Λ_1-stability and time-autonomy is, of course, very important for the usefulness of the corresponding observer $\mathscr{B}_{\mathrm{obs}}$. The algebraic *construction* and *parametrization* of the observers, however, proceed via the Serre category $\mathfrak{C}(\Lambda_1)$ and its associated Gabriel localization and the L^2-stability is useless for this purpose.

3 Serre categories and Gabriel localization

Gabriel developed Serre's ideas from [27] into a comprehensive theory of quotient categories, quotient modules and quotient rings in his thesis [10]. Gabriel localization as used here is well exposed in [30, ch.VII, IX,X,XI]. We use standard notions and results concerning commutative noetherian rings, especially on prime ideals and primary decomposition, that are exposed in [12, ch. 1-2], for instance.

In the whole paper let F be a field, A an F-affine integral domain of the form $A = F[s]/I$ with $s = (s_1, \cdots, s_m)$ and a prime ideal I. Let Mod_A be the category of A-modules and $\mathrm{spec}(A)$ resp. $\max(A)$ the set of prime resp. of maximal ideals of A. For $M \in \mathrm{Mod}_A$ and $\mathfrak{p} \in \mathrm{spec}(A)$ the quotient module $M_{\mathfrak{p}} := \left\{ \frac{x}{t};\ x \in M,\ t \in A \setminus \mathfrak{p} \right\}$ is a module over the local quotient ring $A_{\mathfrak{p}}$. Let $\mathrm{supp}(M) := \{\mathfrak{p} \in \mathrm{spec}(A);\ M_{\mathfrak{p}} \neq 0\}$ be the *support* of M and $\mathrm{ass}(M) := \{\mathfrak{p} \in \mathrm{spec}(A);\ A/\mathfrak{p} \subseteq M(\text{up to isomorphism})\}$ its *associator* or set of *associated prime ideals*. These sets are related via

$$\mathrm{supp}(M) = \{\mathfrak{q} \in \mathrm{spec}(A);\ \exists \mathfrak{p} \in \mathrm{ass}(M) \text{ with } \mathfrak{p} \subseteq \mathfrak{q}\}. \tag{33}$$

The support of a cyclic module $M = A/\mathfrak{a}$ is

$$V(\mathfrak{a}) := \mathrm{supp}(A/\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{spec}(A);\ \mathfrak{a} \subseteq \mathfrak{p}\}. \tag{34}$$

If ${}_AM$ is finitely generated with annihilator ideal

$$\mathfrak{a} := \mathrm{ann}_A(M) := \{f \in A;\ fM = 0\} \text{ then } \mathrm{supp}(M) = V(\mathfrak{a}). \tag{35}$$

We use an injective cogenerator signal module $\mathscr{F}$ that is large, i.e., satisfies $\mathrm{ass}(\mathscr{F}) = \mathrm{spec}(A)$, as in all standard cases [18, Thm. 2.54]. For a matrix $R \in A^{k\times\ell}$, its row module $U := A^{1\times k}R \subseteq A^{1\times\ell}$ and factor module $M := A^{1\times\ell}/U$ the dual behavior is (cf. (5))

$$\begin{gathered} D(M) := \mathrm{Hom}_A(M, \mathscr{F}) \cong \\ \mathscr{B} := U^{\perp} = \left\{w \in \mathscr{F}^{\ell};\ U \circ w = 0\right\} = \left\{w \in \mathscr{F}^{\ell};\ R \circ w = 0\right\} \end{gathered} \tag{36}$$

where $\mathscr{F}^{\ell}$ denotes *column* vectors with entries in $\mathscr{F}$.
Gabriel localization is associated with a given Serre subcategory $\mathfrak{C}$ of Mod_A. We assume $\mathfrak{C} \neq \mathrm{Mod}_A$ and therefore $\mathfrak{C}$ consists of torsion modules only. The largest submodule in $\mathfrak{C}$ of $M \in \mathrm{Mod}_A$ is the $\mathfrak{C}$-*radical* $\mathrm{Ra}_{\mathfrak{C}}(M)$. It consists of the elements $x \in M$ that are annihilated by some ideal $\mathfrak{a}$ with $A/\mathfrak{a} \in \mathfrak{C}$, i.e., $\mathfrak{a}x = 0$, and are called $\mathfrak{C}$-*negligible*. As defined in the Introduction modules in $\mathfrak{C}$ and their dual autonomous behaviors are also called $\mathfrak{C}$-small, $\mathfrak{C}$-negligible or $\mathfrak{C}$-stable. The closure under extensions of $\mathfrak{C}$ also implies that the radical $\mathrm{Ra}_{\mathfrak{C}}(M)$ is the least submodule $U \in \mathfrak{C}$ of M with $\mathrm{Ra}_{\mathfrak{C}}(M/U) = 0$. The Serre categories $\mathfrak{C} \neq \mathrm{Mod}_A$ are in one-one correspondence with disjoint decompositions $\mathrm{spec}(A) = \mathfrak{P}_1 \uplus \mathfrak{P}_2$, $\mathfrak{P}_2 \neq \emptyset$, with the property that $\mathfrak{p}, \mathfrak{q} \in \mathrm{spec}(A)$, $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in \mathfrak{P}_1$ imply $\mathfrak{q} \in \mathfrak{P}_1$:

$$\begin{gathered} \mathfrak{P}_1 := \{\mathfrak{p} \in \mathrm{spec}(A);\ A/\mathfrak{p} \in \mathfrak{C}\}, \quad \mathfrak{C} = \{C \in \mathrm{Mod}_A;\ \mathrm{ass}(C) \subseteq \mathfrak{P}_1\}, \\ \text{hence } \mathfrak{C} = \{C \in \mathrm{Mod}_A;\ \mathrm{supp}(C) \subseteq \mathfrak{P}_1\} = \{C \in \mathrm{Mod}_A;\ \forall \mathfrak{p} \in \mathfrak{P}_2 :\ C_{\mathfrak{p}} = 0\}. \end{gathered} \tag{37}$$

This connection between $\mathfrak{C}$ and the $\mathfrak{P}_i$ and the properties of $\mathfrak{C}$ also furnish the equivalence

$$\mathrm{Ra}_{\mathfrak{C}}(M) = 0 \iff \mathrm{ass}(M) \subseteq \mathfrak{P}_2 \tag{38}$$

for $M \in \mathrm{Mod}_A$. The set

$$\mathfrak{T}_{\mathfrak{C}} := \{\mathfrak{a} \subseteq A;\ \mathfrak{a} \text{ ideal}, A/\mathfrak{a} \in \mathfrak{C}\} = \{\mathfrak{a} \subseteq A;\ V(\mathfrak{a}) \subseteq \mathfrak{P}_1\} \tag{39}$$

is called the *Gabriel topology* induced from $\mathfrak{C}$. The properties of $\mathfrak{C}$ imply immediately that an A-module M belongs to $\mathfrak{C}$ if and only if each element of M is annihilated by some ideal in $\mathfrak{T}_{\mathfrak{C}}$ and therefore $\mathfrak{T}_{\mathfrak{C}}$ determines $\mathfrak{C}$ uniquely. Likewise, if M is f.g. with annihilator ideal

$$\mathfrak{a} := \mathrm{ann}_A(M) \text{ then } (M \in \mathfrak{C} \iff A/\mathfrak{a} \in \mathfrak{C} \iff V(\mathfrak{a}) \subseteq \mathscr{P}_1). \tag{40}$$

The next algorithm uses standard properties of the associator and of primary decompositions [12, p.41].

Algorithm 3.1. (Computation of the radical) *Let $M \in \mathrm{Mod}_A$ be finitely generated and let $0 = \bigcap_{\mathfrak{p}\in\mathrm{ass}(M)} U(\mathfrak{p}) \subseteq M$ be an irredundant primary decomposition of 0 in M. Define $U_i := \bigcap_{\mathfrak{p}\in\mathrm{ass}(M)\cap\mathfrak{P}_i} U(\mathfrak{p})$ for $i = 1, 2$. Then*

$$U_2 = \mathrm{Ra}_{\mathfrak{C}}(M),\ M/U_1 \in \mathfrak{C},\ \mathrm{Ra}_{\mathfrak{C}}(M/U_2) = 0,\ D(M) = D(M/U_1) + D(M/\mathrm{Ra}_{\mathfrak{C}}(M)). \tag{41}$$

Hence, if this primary decomposition can be computed and if the membership problem $\mathfrak{p} \in \mathfrak{P}_1$ or $\mathfrak{p} \in \mathfrak{P}_2$ can be decided for prime ideals $\mathfrak{p}$ of A then all modules in (40) *can be computed too.*

Proof. The irredundant primary decomposition is characterized by $\operatorname{ass}(M/U(\mathfrak{p})) = \{\mathfrak{p}\}$ for $\mathfrak{p} \in \operatorname{ass}(M)$. Then $0 = U_1 \bigcap U_2$ and the induced diagonal homomorphisms

$$M \to M/U_1 \times M/U_2, x \mapsto (x+U_1, x+U_2), \text{ and}$$
$$M/U_i \to \prod_{\mathfrak{p} \in \operatorname{ass}(M) \cap \mathfrak{P}_i} M/U(\mathfrak{p}),\ i = 1,2, \text{and also } U_2 \to M/U_1,\ x \mapsto x+U_1,$$

are injective which implies $\operatorname{ass}(U_2) \subseteq \operatorname{ass}(M/U_1)$ and

$$\begin{aligned}
\operatorname{ass}(M) &\subseteq \operatorname{ass}(M/U_1) \cup \operatorname{ass}(M/U_2)\\
&\subseteq \operatorname{ass}\left(\prod_{\mathfrak{p} \in \operatorname{ass}(M) \cap \mathfrak{P}_1} M/U(\mathfrak{p})\right) \bigcup \operatorname{ass}\left(\prod_{\mathfrak{p} \in \operatorname{ass}(M) \cap \mathfrak{P}_2} M/U(\mathfrak{p})\right)\\
&= \left(\bigcup_{\mathfrak{p} \in \operatorname{ass}(M) \cap \mathfrak{P}_1} \operatorname{ass}(M/U(\mathfrak{p}))\right) \bigcup \left(\bigcup_{\mathfrak{p} \in \operatorname{ass}(M) \cap \mathfrak{P}_2} \operatorname{ass}(M/U(\mathfrak{p}))\right)\\
&= (\operatorname{ass}(M) \cap \mathfrak{P}_1) \biguplus (\operatorname{ass}(M) \cap \mathfrak{P}_2) = \operatorname{ass}(M),
\end{aligned}$$

thus $\operatorname{ass}(M/U_i) = \operatorname{ass}(M) \cap \mathfrak{P}_i$. The inclusions $\operatorname{ass}(U_2) \subseteq \operatorname{ass}(M/U_1) \subseteq \mathfrak{P}_1$ and $\operatorname{ass}(M/U_2) \subseteq \mathfrak{P}_2$ and equations (37) and (38) imply

$$M/U_1 \in \mathfrak{C},\ U_2 \in \mathfrak{C}, \text{ hence } U_2 \subseteq \operatorname{Ra}_{\mathfrak{C}}(M) \text{ and } \operatorname{Ra}_{\mathfrak{C}}(M/U_2) = 0, \text{ thus } U_2 = \operatorname{Ra}_{\mathfrak{C}}(M).$$

By duality the diagonal monomorphism $M \to M/U_1 \times M/U_2$ induces the surjection

$$D(M/U_1) \times D(M/U_2) \xrightarrow{+} D(M), \text{ thus } D(M/U_1) + D(M/U_2) = D(M).$$

□

The following results are exposed in [30, §IX.1, §X.1-2]. For $M \in \operatorname{Mod}_A$ the maps

$$M \to \operatorname{Hom}_A(\mathfrak{a}, M),\ x \mapsto (a \mapsto ax),\ \mathfrak{a} \in \mathfrak{T}_{\mathfrak{C}}, \tag{42}$$

are injective if $\operatorname{Ra}_{\mathfrak{C}}(M) = 0$. If they are isomorphisms for all $\mathfrak{a} \in \mathfrak{T}_{\mathfrak{C}}$ the module M is called $\mathfrak{C}$-*closed*. The full subcategory $\operatorname{Mod}_{A,\mathfrak{C}}$ of all $\mathfrak{C}$-closed modules is closed under kernels, direct products and direct sums in Mod_A and abelian. In particular, the inclusion $\operatorname{inj}_{\mathfrak{C}} : \operatorname{Mod}_{A,\mathfrak{C}} \subset \operatorname{Mod}_A$ is left exact, but in general epimorphisms in $\operatorname{Mod}_{A,\mathfrak{C}}$ are not surjective. The functor $\operatorname{inj}_{\mathfrak{C}}$ has the left adjoint *Gabriel localization functor* $\mathscr{Q}_{\mathfrak{C}} : \operatorname{Mod}_A \to \operatorname{Mod}_{A,\mathfrak{C}}$ with its associated functorial morphism $\eta_M : M \to \mathscr{Q}_{\mathfrak{C}}(M)$, i.e., the map

$$\operatorname{Hom}_A(\mathscr{Q}_{\mathfrak{C}}(M), N) \to \operatorname{Hom}_A(M,N),\ g \mapsto g\eta_M,\ M \in \operatorname{Mod}_A, N \in \operatorname{Mod}_{A,\mathfrak{C}}, \tag{43}$$

is a functorial isomorphism. The functor $\mathscr{Q}_{\mathfrak{C}}$ is exact and moreover

$$\begin{gathered}
\ker(\eta_M) = \operatorname{Ra}_{\mathfrak{C}}(M),\ \operatorname{cok}(\eta_M) \in \mathfrak{C},\ \left(M \in \operatorname{Mod}_{A,\mathfrak{C}} \iff \eta_M : M \cong \mathscr{Q}_{\mathfrak{C}}(M)\right).\\
\text{Thus } M \overset{\text{ident.}}{\subseteq} \mathscr{Q}_{\mathfrak{C}}(M) \text{ if } \operatorname{Ra}_{\mathfrak{C}}(M) = 0.
\end{gathered} \tag{44}$$

If $V \subseteq N$ are $\mathfrak{C}$-closed modules their factor object in $\operatorname{Mod}_{A,\mathfrak{C}}$ is denoted by $N/_{\mathfrak{C}}V$. The exactness of $\mathscr{Q}_{\mathfrak{C}}$ and (44) imply $N/_{\mathfrak{C}}V = \mathscr{Q}_{\mathfrak{C}}(N/V)$ where N/V is the standard factor module in Mod_A.

The next theorem is a consequence of Matlis' theory of injective modules over commutative noetherian rings [12, pp.145-152] that was essentially used in [18] already. A direct sum of injective modules is injective, and each injective module admits a direct decomposition into (directly) indecomposable injectives. A submodule $M \subseteq E$ is called *large* or *essential* if for each nonzero submodule $U \subseteq E$ also $M \cap U$ is nonzero. This implies $\operatorname{ass}(M) = \operatorname{ass}(E)$. If in addition E is injective then it is called an *injective hull* of M. Each A-module M has an injective hull which is unique up to isomorphism and denoted by $E(M)$. The map

$$\mathfrak{p} \mapsto E(A/\mathfrak{p}) \text{ with } \operatorname{ass}(E(A/\mathfrak{p})) = \operatorname{ass}(A/\mathfrak{p}) = \{\mathfrak{p}\} \tag{45}$$

is a bijection from $\operatorname{spec}(A)$ onto the set of isomorphism classes of indecomposable injectives. For each $t \in A \setminus \mathfrak{p}$ the multiplication $t: E(A/\mathfrak{p}) \to E(A/\mathfrak{p})$ is bijective, i.e., $E(A/\mathfrak{p})$ is a module over the local ring $A_{\mathfrak{p}}$, and is indeed the least injective cogenerator over this ring.

Theorem and Definition 3.2. *Let $\mathfrak{C} \subsetneq \operatorname{Mod}_A$ be a Serre subcategory.*
(i) If E is an indecomposable injective with $\operatorname{ass}(E) = \{\mathfrak{p}\}$ then

$$\operatorname{Ra}_{\mathfrak{C}}(E) = \begin{cases} E & \text{if } \mathfrak{p} \in \mathfrak{P}_1 \\ 0 & \text{if } \mathfrak{p} \in \mathfrak{P}_2 \end{cases}.$$

(ii) The large (with $\operatorname{ass}(\mathscr{F}) = \operatorname{spec}(A)$) injective cogenerator $\mathscr{F}$ admits a non-unique direct decomposition

$$\mathscr{F} = \operatorname{Ra}_{\mathfrak{C}}(\mathscr{F}) \oplus \mathscr{F}_2 \text{ with } \operatorname{Ra}_{\mathfrak{C}}(\mathscr{F}_2) = 0,\ \operatorname{ass}(\operatorname{Ra}_{\mathfrak{C}}(\mathscr{F})) = \mathfrak{P}_1,\ \operatorname{ass}(\mathscr{F}_2) = \mathfrak{P}_2. \tag{46}$$

In particular, $\operatorname{Ra}_{\mathfrak{C}}(\mathscr{F})$ and $\mathscr{F}_2$ are injective as direct summands of $\mathscr{F}$. If $w = w_1 + w_2 \in \mathscr{F} = \operatorname{Ra}_{\mathfrak{C}}(\mathscr{F}) \oplus \mathscr{F}_2$ then w_1 resp. w_2 are suggestively called the $\mathfrak{C}$-negligible part *resp. the* $\mathfrak{C}$-steady state *of the trajectory w.*
(iii) For $C \in \operatorname{Mod}_A:\ C \in \mathfrak{C} \Longleftrightarrow \operatorname{Hom}_A(C, \mathscr{F}_2) = 0$.
(iv) The module $\mathscr{F}_2$ is $\mathfrak{C}$-closed and $\mathscr{F}_2 \cong \mathscr{F}/\operatorname{Ra}_{\mathfrak{C}}(\mathscr{F}) \cong \mathscr{Q}_{\mathfrak{C}}(\mathscr{F})$, hence

$$\operatorname{Hom}_A(M, \mathscr{F}_2) \cong \operatorname{Hom}_A(\mathscr{Q}_{\mathfrak{C}}(M), \mathscr{F}_2) \cong \operatorname{Hom}_A(\mathscr{Q}_{\mathfrak{C}}(M), \mathscr{Q}_{\mathfrak{C}}(\mathscr{F})).$$

(v) The module $\mathscr{F}_2 \cong \mathscr{Q}_{\mathfrak{C}}(\mathscr{F})$ is an injective cogenerator in $\operatorname{Mod}_{A,\mathfrak{C}}$.
(vi) The radical $\operatorname{Ra}_{\mathfrak{C}}(\mathscr{F})$ is an injective cogenerator in the abelian category $\mathfrak{C}$ and thus induces the behavioral duality $C \mapsto \operatorname{Hom}_A(C, \operatorname{Ra}_{\mathfrak{C}}(\mathscr{F})) = \operatorname{Hom}_A(C, \mathscr{F})$ between f.g. $\mathfrak{C}$-negligible modules and behaviors.

Proof. (i) This follows directly from equations (37) and (38).
(ii) The module $\mathscr{F}$ admits a direct decomposition $\mathscr{F} = \oplus_{i \in I} E_i$ into indecomposable injectives E_i with $\operatorname{ass}(E_i) = \{\mathfrak{p}_i\}$ and $\operatorname{spec}(A) = \operatorname{ass}(\mathscr{F}) = \{\mathfrak{p}_i;\ i \in I\}$ because $\mathscr{F}$ is a large injective cogenerator, Therefore, by (i),

$$\operatorname{Ra}_{\mathfrak{C}}(E_i) = \begin{cases} E_i & \text{if } \mathfrak{p}_i \in \mathfrak{P}_1 \\ 0 & \text{if } \mathfrak{p}_i \in \mathfrak{P}_2 \end{cases} \text{ and}$$

$$\operatorname{Ra}_{\mathfrak{C}}(\mathscr{F}) = \operatorname{Ra}_{\mathfrak{C}}(\oplus_{i \in I} E_i) = \oplus_{i \in I} \operatorname{Ra}_{\mathfrak{C}}(E_i) = \oplus_{i \in I, \mathfrak{p}_i \in \mathfrak{P}_1} E_i,$$

$$\mathscr{F} = \operatorname{Ra}_{\mathfrak{C}}(\mathscr{F}) \oplus \mathscr{F}_2 \text{ with } \mathscr{F}_2 := \oplus_{i \in I, \mathfrak{p}_i \in \mathfrak{P}_2} E_i,\ \operatorname{Ra}_{\mathfrak{C}}(\mathscr{F}_2) = \operatorname{Ra}_{\mathfrak{C}}(\mathscr{F}) \cap \mathscr{F}_2 = 0.$$

As direct summands of $\mathscr{F}$ the submodules $\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})$ and $\mathscr{F}_2$ are injective. By equations (37) and (38) their associators satisfy

$$\begin{gathered}\operatorname{ass}(\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}))\subseteq\mathfrak{P}_1 \text{ and } \operatorname{ass}(\mathscr{F}_2)\subseteq\mathfrak{P}_2. \text{ But}\\ \operatorname{spec}(A)=\operatorname{ass}(\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})\oplus\mathscr{F}_2)=\operatorname{ass}(\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}))\cup\operatorname{ass}(\mathscr{F}_2)\subseteq\mathfrak{P}_1\uplus\mathfrak{P}_2=\operatorname{spec}(A),\\ \text{hence } \operatorname{ass}(\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}))=\mathfrak{P}_1 \text{ and } \operatorname{ass}(\mathscr{F}_2)=\mathfrak{P}_2.\end{gathered}$$

(iii) Let $C\in\mathrm{Mod}_A$. If $C\in\mathfrak{C}$ any linear map $f:C\to\mathscr{F}_2$ maps $C=\mathrm{Ra}_{\mathfrak{C}}(C)$ into $\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}_2)=0$, hence $\mathrm{Hom}_A(C,\mathscr{F}_2)=0$.
Assume, conversely, $\mathrm{Hom}_A(C,\mathscr{F}_2)=0$. For $\mathfrak{p}\in\operatorname{spec}(A)$ the module $E(A/\mathfrak{p})$ is the least injective cogenerator over $A_{\mathfrak{p}}$, hence

$$\begin{gathered}\text{for } C\in\mathrm{Mod}_A:\ \mathrm{Hom}_A(C,E(A/\mathfrak{p}))\cong\mathrm{Hom}_{A_{\mathfrak{p}}}(C_{\mathfrak{p}},E(A/\mathfrak{p})) \text{ and}\\ (C_{\mathfrak{p}}=0\iff\mathrm{Hom}_A(C,E(A/\mathfrak{p}))=0)\,.\end{gathered}$$

By (ii) we have $\mathscr{F}_2=\oplus_{\mathfrak{p}_i\in\mathfrak{P}_2}E_i,\ E_i\cong E(A/\mathfrak{p}_i)$, and $\operatorname{ass}(\mathscr{F}_2)=\{\mathfrak{p}_i;\ i\in I,\mathfrak{p}_i\in\mathfrak{P}_2\}=\mathfrak{P}_2$. If in addition C is finitely generated this implies

$$\begin{gathered}0=\mathrm{Hom}_A(C,\mathscr{F}_2)=\mathrm{Hom}_A(C,\oplus_{\mathfrak{p}_i\in\mathfrak{P}_2}E_i)\underset{C\text{ f.g}}{\cong}\oplus_{\mathfrak{p}_i\in\mathfrak{P}_2}\mathrm{Hom}_A(C,E_i)\Longrightarrow\\ \forall\mathfrak{p}_i\in\mathfrak{P}_2:\ \mathrm{Hom}_A(C,E(A/\mathfrak{p}_i))=0\Longrightarrow\forall\mathfrak{p}\in\mathfrak{P}_2:\ C_{\mathfrak{p}}=0\underset{(37)}{\iff}C\in\mathfrak{C}.\end{gathered}$$

In general, a f.g. submodule C' of C and the injectivity of $\mathscr{F}_2$ induce the surjection

$$\begin{gathered}0=\mathrm{Hom}_A(C,\mathscr{F}_2)\to\mathrm{Hom}_A(C',\mathscr{F}_2),\ f\mapsto f|C',\text{ hence}\\ \forall C'\subseteq C,\ C'\text{ f.g.}:\ \mathrm{Hom}_A(C',\mathscr{F}_2)=0\Longrightarrow\forall C'\subseteq C,\ C'\text{ f.g.}:\ C'\in\mathfrak{C}\iff C\in\mathfrak{C}.\end{gathered}$$

(iv) The maps (42) (with $M=\mathscr{F}_2$) are injective since $\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}_2)=0$ and surjective since $\mathscr{F}_2$ is injective, hence $\mathscr{F}_2\in\mathrm{Mod}_{A,\mathfrak{C}}$. With (44) this implies $\mathscr{Q}_{\mathfrak{C}}(\mathscr{F}_2)\cong\mathscr{F}_2$ and

$$\mathscr{Q}_{\mathfrak{C}}(\mathscr{F})=\mathscr{Q}_{\mathfrak{C}}(\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})\oplus\mathscr{F}_2)\cong\mathscr{Q}_{\mathfrak{C}}(\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}))\oplus\mathscr{Q}_{\mathfrak{C}}(\mathscr{F}_2)\cong 0\oplus\mathscr{F}_2=\mathscr{F}_2.$$

(v) Since monomorphisms in $\mathrm{Mod}_{A,\mathfrak{C}}$ coincide with those in Mod_A and since $\mathscr{F}_2$ is injective in Mod_A it is also injective in $\mathrm{Mod}_{A,\mathfrak{C}}$. Further assume $N\in\mathrm{Mod}_{A,\mathfrak{C}}$ and $\mathrm{Hom}_A(N,\mathscr{F}_2)=0$. From (iii) we infer $N\in\mathfrak{C}$, hence $N\in\mathfrak{C}\cap\mathrm{Mod}_{A,\mathfrak{C}}=0$ and $N=0$. This is the cogenerator property of the injective object $\mathscr{F}_2\in\mathrm{Mod}_{A,\mathfrak{C}}$. The proof of (v) also follows from [30, Prop. X.1.9].
(vi) Since $\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})$ is injective in Mod_A and contained in $\mathfrak{C}$ it is also injective in $\mathfrak{C}$. The identity $\mathrm{Hom}_A(C,\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}))=\mathrm{Hom}_A(C,\mathscr{F})$ for $C=\mathrm{Ra}_{\mathfrak{C}}(C)\in\mathfrak{C}$ and the injective cogenerator property of $\mathscr{F}$, i.e., $(C=0\iff\mathrm{Hom}_A(C,\mathscr{F})=0)$, imply the same property for $\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})\in\mathfrak{C}$. □

Corollary 3.3. *If $\mathscr{L}$ is any A-module, for instance a submodule of the signal module $\mathscr{F}$, then the Serre subcategory $\mathfrak{C}$ with $\mathfrak{P}_1:=\operatorname{supp}(\mathscr{L})$ (see* (33), (37)*) is the least one that contains $\mathscr{L}$. Behavioral duality is then valid according to Thm. 3.2,(vi).*

Proof. If $\mathfrak{p}\subseteq\mathfrak{q}$ are prime ideals and M an A-module then $M_{\mathfrak{p}}=(M_{\mathfrak{q}})_{\mathfrak{p}}$, hence $M_{\mathfrak{p}}\neq 0$ implies $M_{\mathfrak{q}}\neq 0$ and $\mathfrak{P}_1:=\operatorname{supp}(\mathscr{L})$ satisfies the condition for $\mathfrak{P}_1$ from (37) that also implies that $\mathfrak{C}$ is the least Serre subcategory with $\mathscr{L}\in\mathfrak{C}$. □

The determination of $\mathrm{supp}(\mathscr{L})$ is difficult in general.
For modules and behaviors as in (36) there are the canonical isomorphisms

$$\mathscr{B}\bigcap\mathscr{F}_2^\ell=\left\{w\in\mathscr{F}_2^\ell;\ R\circ w=0\right\}\cong\mathrm{Hom}_A(M,\mathscr{F}_2)\cong\mathrm{Hom}_A(\mathscr{Q}_{\mathfrak{C}}(M),\mathscr{F}_2)\tag{47}$$

and the decomposition (46) induces the behavior decomposition

$$\mathscr{B}=\left(\mathscr{B}\bigcap\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})^\ell\right)\bigoplus\left(\mathscr{B}\bigcap\mathscr{F}_2^\ell\right)=\mathrm{Ra}_{\mathfrak{C}}(\mathscr{B})\bigoplus\left(\mathscr{B}\bigcap\mathscr{F}_2^\ell\right).\tag{48}$$

A multiplicatively closed set $T\subseteq A\setminus\{0\}$ with the standard quotient ring A_T and exact quotient module functor $M\mapsto M_T$ gives rise to a Serre subcategory [30, Ex.2 on p.200]

$$\begin{gathered}\mathfrak{C}(T):=\{C\in\mathrm{Mod}_A;\ C_T=0\}\ \text{ with }\mathfrak{T}(T):=\mathfrak{T}_{\mathfrak{C}(T)}=\{\mathfrak{a}\subseteq A;\ \mathfrak{a}\cap T\neq\emptyset\},\\ \mathfrak{P}_1(T):=\{\mathfrak{p}\in\mathrm{spec}(A);\ \mathfrak{p}\cap T\neq\emptyset\},\ \mathfrak{P}_2(T):=\{\mathfrak{p}\in\mathrm{spec}(A);\ \mathfrak{p}\cap T=\emptyset\},\\ \mathrm{Ra}_T(M):=\mathrm{Ra}_{\mathfrak{C}(T)}(M)=\{x\in M;\ \exists t\in T:\ tx=0\}\ \text{ for }M\in\mathrm{Mod}_A,\\ \mathrm{Mod}_{A,\mathfrak{C}(T)}=\mathrm{Mod}_{A_T},\ \mathscr{Q}_{\mathfrak{C}(T)}(M)=M_T.\end{gathered}\tag{49}$$

Conversely, any $(\mathfrak{C},\mathfrak{P}_i)$ from (37) gives rise to the multiplicatively closed set

$$T(\mathfrak{C}):=\bigcap_{\mathfrak{p}\in\mathfrak{P}_2}(A\setminus\mathfrak{p})=\{t\in A;\ A/At\in\mathfrak{C}\}\ \text{ with }\mathfrak{C}(T(\mathfrak{C}))\subseteq\mathfrak{C}.\tag{50}$$

In general, the last inclusion is not an equality and $\mathscr{Q}_{\mathfrak{C}}(M)\neq M_{T(\mathfrak{C})}$, but the isomorphism (42) and (50) imply that each module in $\mathrm{Mod}_{A,\mathfrak{C}}$ is an $A_{T(\mathfrak{C})}$-module.
In the sequel we fix a Serre subcategory with $\mathrm{spec}(A)=\mathfrak{P}_1\uplus\mathfrak{P}_2$ from (37) and use the notations

$$\mathrm{Ra}:=\mathrm{Ra}_{\mathfrak{C}},\ \mathscr{Q}:=\mathscr{Q}_{\mathfrak{C}},\ T:=T(\mathfrak{C}).\tag{51}$$

Result 3.4. ([21, Thm. 2.4]) *If M is a submodule of a $\mathfrak{C}$-closed module N then*

$$M\subseteq\mathscr{Q}(M)\subseteq N\ \text{and}\ \mathscr{Q}(M)/M=\mathrm{Ra}\,(N/M).$$

The proof in the quoted paper was given for a special $\mathfrak{C}$ only, but holds for general $\mathfrak{C}$.

Corollary 3.5. ([20, Lemma 3.4]) *The quotient field $K:=\mathrm{quot}(A)$ of A is $\mathfrak{C}$-closed and $\mathscr{Q}(A)=\bigcap_{\mathfrak{p}\in\mathfrak{P}_2}A_{\mathfrak{p}}$.*

Proof. That the maps (42) are bijective for $M=K$ is easy to see, hence $\mathscr{Q}(A)/A=\mathrm{Ra}(K/A)$ by the preceding result. The local quotient rings $A_{\mathfrak{p}},\ \mathfrak{p}\in\mathfrak{P}_2$, are contained in K. Let $U:=\bigcap_{\mathfrak{p}\in\mathfrak{P}_2}A_{\mathfrak{p}}$, hence $A\subseteq U\subseteq K$. For all $\mathfrak{p}\in\mathfrak{P}_2$ we conclude

$$(U/A)_{\mathfrak{p}}=U_{\mathfrak{p}}/A_{\mathfrak{p}}\subseteq(A_{\mathfrak{p}})_{\mathfrak{p}}/A_{\mathfrak{p}}=A_{\mathfrak{p}}/A_{\mathfrak{p}}=0,$$

hence $U/A\in\mathfrak{C}$ by (37) and $U/A\subseteq\mathrm{Ra}(K/U)=\mathscr{Q}(A)/A$ or $U\subseteq\mathscr{Q}(A)$. Conversely, we get for all $\mathfrak{p}\in\mathfrak{P}_2$:

$$0=(\mathrm{Ra}(K/A))_{\mathfrak{p}}=(\mathscr{Q}(A)/A)_{\mathfrak{p}}=\mathscr{Q}(A)_{\mathfrak{p}}/A_{\mathfrak{p}}=0\ \text{or}\ \mathscr{Q}(A)_{\mathfrak{p}}=A_{\mathfrak{p}}.$$

This implies $\mathscr{Q}(A)\subseteq\bigcap_{\mathfrak{p}\in\mathfrak{P}_2}\mathscr{Q}(A)_{\mathfrak{p}}=\bigcap_{\mathfrak{p}\in\mathfrak{P}_2}A_{\mathfrak{p}}=U.$ □

Since A is torsionfree, hence $\mathrm{Ra}_{\mathfrak{C}}(A)=0$, the inclusions $A\subseteq A_T\subseteq\mathscr{Q}(A)=\bigcap_{\mathfrak{p}\in\mathfrak{P}_2}A_{\mathfrak{p}}$ hold, but in general the equality $A_T=\mathscr{Q}(A)$ is not valid. For constructive and other purposes this equality is, however, important. Therefore we make the

Assumption 3.6. The affine domain A and the Serre subcategory $\mathfrak{C}$ satisfy

$$\mathscr{Q}(A) := A_T, \text{ i.e., } \bigcap_{\mathfrak{p}\in\mathfrak{P}_2} A_{\mathfrak{p}} = A_T,\ T := \bigcap_{\mathfrak{p}\in\mathfrak{P}_2} (A\setminus\mathfrak{p}).$$

For factorial A the assumption holds [20, Lemma 3.2]. In Section 6 non-factorial A with $\mathscr{Q}(A) = A_T$ play an important part.

Due to this assumption any A- or A_T-submodule V of $A_T^{1\times\ell}$ induces

$$\begin{aligned} V &\subseteq V_T = A_T V \subseteq \mathscr{Q}(V) = \mathscr{Q}(V_T) \subseteq \mathscr{Q}(A)^{1\times\ell} = A_T^{1\times\ell}, \text{ and} \\ V^{\perp_2} &:= \left\{w\in\mathscr{F}_2^\ell;\ V\circ w = 0\right\} \cong \operatorname{Hom}_A\left(A_T^{1\times\ell}/V, \mathscr{F}_2\right) \\ &\cong \operatorname{Hom}_A\left(\mathscr{Q}\left(A_T^{1\times\ell}/V\right), \mathscr{F}_2\right) \cong \operatorname{Hom}_A\left(A_T^{1\times\ell}/_{\mathfrak{C}}\mathscr{Q}(V), \mathscr{F}_2\right) \cong \mathscr{Q}(V)^{\perp_2}. \end{aligned} \tag{52}$$

Thus $V^{\perp_2}$ is an $\mathscr{F}_2$-behavior and orthogonal to the $\mathfrak{C}$-closed submodule $\mathscr{Q}(V)$ of $A_T^{1\times\ell}$. Here we used that $\mathscr{F}_2$ is $\mathfrak{C}$-closed. If

$$V = A_T^{1\times k'}R' \subseteq A_T^{1\times\ell},\ R'\in A_T^{k'\times\ell}, \text{ then } V^{\perp_2} = \left\{w\in\mathscr{F}_2^\ell;\ R'\circ w = 0\right\}. \tag{53}$$

Notice that $R'\circ w$ is defined since $\mathscr{F}_2$ is an A_T-module. For the special case of an A-submodule $U\subseteq A^{1\times\ell}$ we get

$$U^{\perp_2} = \mathscr{Q}(U)^{\perp_2} = U^\perp\cap\mathscr{F}_2^\ell,\ U\subseteq A^{1\times\ell}. \tag{54}$$

Since $\mathscr{F}_2$ is an injective cogenerator in the abelian category $\operatorname{Mod}_{A,\mathfrak{C}}$ standard arguments imply the

Corollary 3.7. *For A- or A_T-submodules $V_i\subseteq A_T^{1\times\ell}$, $i=1,2$, with $\mathscr{Q}(V_i) = A_T^{1\times k_i'}R_i'$, $R_i'\in A_T^{k_i'\times\ell}$ the following equivalences hold:*

$$V_1^{\perp_2}\subseteq V_2^{\perp_2} \iff \mathscr{Q}(V_1)\supseteq\mathscr{Q}(V_2) \iff \exists X\in A_T^{k_2'\times k_1'} \text{ with } R_2' = XR_1'.$$

Consider especially an input/output (IO) behavior

$$\begin{aligned} &\mathscr{B} := \left\{\left(\begin{smallmatrix} y\\ u\end{smallmatrix}\right)\in\mathscr{F}^{p+m};\ P\circ y = Q\circ u\right\} \text{ with} \\ &(P,-Q)\in A^{k\times(p+m)},\ \operatorname{rank}(P,-Q) = \operatorname{rank}(P) = p, \\ &\text{and transfer matrix } H\in\operatorname{quot}(A)^{p\times m} \text{ with } PH = Q. \end{aligned} \tag{55}$$

The IO property signifies that $\mathscr{B}^0 := \{y\in\mathscr{F}^p;\ P\circ y = 0\}$ is autonomous and that for every input $u\in\mathscr{F}^m$ there is an output $y\in\mathscr{F}^p$ such that $\left(\begin{smallmatrix} y\\ u\end{smallmatrix}\right)\in\mathscr{B}$. The IO behavior is called $\mathfrak{C}$-*stable* [20, Thm. and Def. 4.2] if its autonomous part $\mathscr{B}^0$ is $\mathfrak{C}$-negligible, i.e., belongs to $\mathfrak{C}$. This is equivalent to

$$\mathscr{B}^0\cap\mathscr{F}_2^p = \ker(P\circ : \mathscr{F}_2^p\to\mathscr{F}_2^k) = 0 \text{ and implies } H\in A_T^{p\times m}. \tag{56}$$

The last implication was shown in [20, Thm. and Def. 4.2] for factorial A only, but the given proof remains valid if $\bigcap_{\mathfrak{p}\in\mathfrak{P}_2} A_{\mathfrak{p}} = A_T$ (Assumption 3.6).

Corollary 3.8. *If the IO behavior from* (55) *is $\mathfrak{C}$-stable then*

$$\mathscr{B}\cap\mathscr{F}_2^{p+m} = \left\{\left(\begin{smallmatrix} y\\ u\end{smallmatrix}\right)\in\mathscr{F}_2^{p+m};\ y = H\circ u\right\}.$$

Proof. This follows from (56). The rational matrix $H \in A_T^{p\times m}$ gives rise to the operator $H\circ : \mathscr{F}_2^m \to \mathscr{F}_2^p$. The equation $PH = Q$ implies that for all

$$u \in \mathscr{F}_2^m:\ P\circ H\circ u = Q\circ u,\ \text{hence}\ \binom{H\circ u}{u} \in \mathscr{B}\cap\mathscr{F}_2^{p+m}.$$

Assume, conversely,

$$\binom{y}{u} \in \mathscr{B}\cap\mathscr{F}_2^{p+m} \Longrightarrow P\circ y = Q\circ u = (PH)\circ u = P\circ H\circ u \Longrightarrow$$
$$P\circ(y - H\circ u) = 0 \Longrightarrow y - H\circ u \in \mathscr{B}^0\cap\mathscr{F}_2^p \underset{(56)}{=} 0.$$

□

The second algorithm of this section computes the module $\mathscr{Q}(U)$ in the situation of (36) under Assumption 3.6 and is applicable to arbitrary f.g. torsionfree modules U since these are submodules of f.g. free modules. It extends and improves [21, Alg. 4.1] with Algorithm 3.1 as essential tool.

Algorithm 3.9. (Computation of $\mathscr{Q}(U)$) *Under Assumption 3.6 let*

$$R \in A^{k\times\ell},\ U := A^{1\times k}R \subseteq A^{1\times\ell},\ \textit{hence}\ \mathscr{Q}(U) = \mathscr{Q}(U_T) \subseteq \mathscr{Q}(A)^{1\times\ell} = A_T^{1\times\ell}.$$

By means of Algorithm 3.1 compute $R' \in A^{k'\times\ell}$ *with* $\mathrm{Ra}(A^{1\times\ell}/U) = (A^{1\times k'}R')/U$. *Then* $\mathscr{Q}(U) = A_T^{1\times k'}R'$.

Proof. All modules in the preceding equation are considered as A-modules and not as A_T-modules. Due to Assumption 3.6 the modules $A_T = \mathscr{Q}(A)$ and $A_T^{1\times\ell}$ are $\mathfrak{C}$-closed. From Result 3.4 we infer

$$\mathscr{Q}(U)/U_T = \mathscr{Q}(U_T)/U_T \underset{\text{Result 3.4}}{=} \mathrm{Ra}\left(A_T^{1\times\ell}/U_T\right) = \left(\mathrm{Ra}\left(A^{1\times\ell}/U\right)\right)_T =$$
$$\left((A^{1\times k'}R')/U\right)_T = (A_T^{1\times k'}R')/U_T,\ \text{hence}\ \mathscr{Q}(U) = A_T^{1\times k'}R'.$$

Here we used the simple identity $\mathrm{Ra}(M_T) = \mathrm{Ra}(M)_T$. □

Remark 3.10. (*Willems closures*) Serre categories and their associated radical and especially Algorithm 3.1 can also be fruitfully applied to the computation of *Willems closures* with respect to arbitrary injective modules $\mathscr{G} \cong \oplus_{\mathfrak{r}\in\mathrm{ass}(\mathscr{G})}E(A/\mathfrak{r})^{(\alpha(\mathfrak{r}))}$ where $\alpha(\mathfrak{r})$ is a nonzero cardinal number and $E(A/\mathfrak{r})^{(\alpha(\mathfrak{r}))}$ a direct sum of $\alpha(\mathfrak{r})$ copies of $E(A/\mathfrak{r})$. Consider the Serre subcategory

$$\begin{gathered}\mathfrak{C} := \{C \in \mathrm{Mod}_A;\ \forall\mathfrak{r}\in\mathrm{ass}(\mathscr{G}):\ C_\mathfrak{r} = 0\}\ \text{with}\\ \mathfrak{P}_1 := \{\mathfrak{p}\in\mathrm{spec}(A);\ \forall\mathfrak{r}\in\mathrm{ass}(\mathscr{G}) : (A/\mathfrak{p})_\mathfrak{r} = 0\ \text{or}\ \mathfrak{p}\cap(A\setminus\mathfrak{r}) \neq \emptyset\}\\ \mathfrak{P}_2 := \{\mathfrak{p}\in\mathrm{spec}(A);\ \exists\mathfrak{r}\in\mathrm{ass}(\mathscr{G}) : (A/\mathfrak{p})_\mathfrak{r} \neq 0\ \text{or}\ \mathfrak{p}\subseteq\mathfrak{r}\},\ \text{hence}\\ \mathrm{ass}(\mathscr{G}) \subseteq \mathfrak{P}_2\ \text{and}\ \mathfrak{C} = \{C;\ \mathrm{supp}(C)\subseteq\mathfrak{P}_1\}.\end{gathered} \tag{57}$$

For $M \in \mathrm{Mod}_A$ consider the Gelfand map

$$\rho_M : M \to \mathscr{G}^{\mathrm{Hom}_A(M,\mathscr{G})},\ x \mapsto (g(x))_{g\in\mathrm{Hom}_A(M,\mathscr{G})}, \tag{58}$$

and its kernel $\ker(\rho_M) = \bigcap_{g\in\mathrm{Hom}_A(M,\mathscr{G})}\ker(g)$. If $M = A^{1\times\ell}/U$ and if we identify $\mathrm{Hom}_A(M,\mathscr{G})$ with its $\mathscr{G}$-behavior $U^{\perp_{\mathscr{G}}} = \left\{w \in \mathscr{G}^{\ell};\ U\circ w = 0\right\}$ as usual then

$$\begin{aligned}\ker(\rho_M) = \bigcap_{g\in\mathrm{Hom}_A(M,\mathscr{G})}\ker(g) = U^{\perp_{\mathscr{G}}\perp}/U \subseteq M = A^{1\times\ell}/U \text{ with}\\ U^{\perp_{\mathscr{G}}\perp} := \left\{\xi \in A^{1\times\ell};\ \xi\circ U^{\perp_{\mathscr{G}}} = 0\right\}.\end{aligned} \tag{59}$$

Shankar introduced the term *Willems closure* [28, Def. on p.1821] of U in $A^{1\times\ell}$ with respect to $\mathscr{G}$ for $U^{\perp_{\mathscr{G}}\perp}$ and computed it in various cases [28, Thm. 2.1], [16]. Another contribution is [26]. Let $\mathscr{G}$ be any injective A-module with the induced data from (57) - (59). Then

1. $\ker(\rho_M) = \mathrm{Ra}_{\mathfrak{C}}(M)$.
2. If $M = A^{1\times\ell}/U$ is finitely generated and $U = \bigcap_{\mathfrak{p}\in\mathrm{ass}(M)} U(\mathfrak{p})$ is an irredundant primary decomposition of $U \subseteq A^{1\times\ell}$ then the Willems closure of U with respect to $\mathscr{G}$ is $U^{\perp_{\mathscr{G}}\perp} = \bigcap_{\mathfrak{p}\in\mathrm{ass}(M)\cap\mathfrak{P}_2} U(\mathfrak{p})$.

The application of 1. and 2. requires the knowledge of $\mathrm{ass}(\mathscr{G})$. Such computations are contained in the quoted papers of Shankar et al.
The assertion 2. follows immediately from 1. and Algorithm 3.1. For the proof of 1. we show first that $C_1 := \ker(\rho_M) \supseteq C_2 := \mathrm{Ra}_{\mathfrak{C}}(M)$: Let $x \in C_2$, hence $Ax \in \mathfrak{C}$. Then

$$\mathrm{Hom}_A(Ax,\mathscr{G}) = \mathrm{Hom}_A\left(Ax, \bigoplus_{\mathfrak{r}\in\mathrm{ass}(\mathscr{G})} E(A/\mathfrak{r})^{(\alpha(\mathfrak{r}))}\right) \cong \bigoplus_{\mathfrak{r}\in\mathrm{ass}(\mathscr{G})} \mathrm{Hom}_A\left(Ax, E(A/\mathfrak{r})\right)^{(\alpha(\mathfrak{r}))}.$$

$$\text{But } \forall \mathfrak{r}\in\mathrm{ass}(\mathscr{G}) \subseteq \mathfrak{P}_2:\ \mathrm{Ra}_{\mathfrak{C}}(E(A/\mathfrak{r})) \underset{\text{Thm. 3.2,(i)}}{=} 0 \underset{Ax\in\mathfrak{C}}{\Longrightarrow}$$
$$\forall \mathfrak{r}\in\mathrm{ass}(\mathscr{G}) \subseteq \mathfrak{P}_2:\ \mathrm{Hom}_A(Ax, E(A/\mathfrak{r})) = 0 \Longrightarrow$$
$$\mathrm{Hom}_A(Ax,\mathscr{G}) = 0 \Longrightarrow \forall g\in\mathrm{Hom}_A(M,\mathscr{G}):\ g(x) = 0 \Longrightarrow x\in C_1 \Longrightarrow C_2 \subseteq C_1.$$

For the reverse inclusion $C_1 \subseteq \mathrm{Ra}_{\mathfrak{C}}(M) = C_2$ we show $C_1 \in \mathfrak{C}$: But

$$C_1 \in \mathfrak{C} \Longleftrightarrow \forall \mathfrak{r}\in\mathrm{ass}(\mathscr{G}):\ (C_1)_{\mathfrak{r}} = 0 \underset{A_{\mathfrak{r}}E(A/\mathfrak{r})\text{ inj. cog.}}{\Longleftrightarrow}$$
$$\forall \mathfrak{r}\in\mathrm{ass}(\mathscr{G}):\ 0 = \mathrm{Hom}_{A_{\mathfrak{r}}}((C_1)_{\mathfrak{r}}, E(A/\mathfrak{r})) \cong \mathrm{Hom}_A(C_1, E(A/\mathfrak{r})).$$

But any $f : C_1 \to E(A/\mathfrak{r}) \subseteq \mathscr{G}$ can be extended to $g \in \mathrm{Hom}_A(M,\mathscr{G})$, hence $f = g|C_1 = g|\ker(\rho_M) = 0$ and $\mathrm{Hom}_A(C_1, E(A/\mathfrak{r})) = 0$.

4 Multidimensional observers

The general assumptions of Section 3 are in force, i.e., F is a field, $A = F[s]/I$ with $s = (s_1,\cdots,s_m)$ and $I \in \mathrm{spec}(F[s])$ is an F-affine integral domain and ${}_A\mathscr{F}$ is a large injective cogenerator signal module with $\mathrm{ass}(\mathscr{F}) = \mathrm{spec}(\mathscr{F})$. Moreover $\mathfrak{C} \subsetneq \mathrm{Mod}_A$ is a Serre subcategory with $\mathscr{Q} := \mathscr{Q}_{\mathfrak{C}}$, $\mathrm{Ra} := \mathrm{Ra}_{\mathfrak{C}}$ and $T := T(\mathfrak{C})$ that satisfies Assumption 3.6, viz. $\mathscr{Q}(A) = A_T$. The decomposition $\mathscr{F} = \mathrm{Ra}(\mathscr{F}) \oplus \mathscr{F}_2$ holds according to Thm. and Def. 3.2. As explained in the Introduction we consider a behavior

$$\mathscr{B} = U^{\perp} \subseteq \mathscr{F}^{\ell},\ U = A^{1\times k}R,\ R\in A^{k\times l}, \text{ with } \mathscr{Q}(U) = A_T^{1\times k'}R',\ R'\in A^{k'\times l}, \tag{60}$$

and two additional matrices $P \in A^{m\times l}$ and $Q \in A^{q\times l}$. The matrix R' is computed by means of Algorithm 3.9.

Definition 4.1. Consider an input/output behavior (compare (55))

$$\begin{gathered}\mathscr{B}_{\rm obs}=\left\{\left(\begin{smallmatrix}y\\ u\end{smallmatrix}\right)\in\mathscr{F}^{q+m};\ P_{\rm obs}\circ y=Q_{\rm obs}\circ u\right\}\ \text{with}\\ (P_{\rm obs},-Q_{\rm obs})\in A^{k_{\rm obs}\times(q+m)},\ \text{rank}\,(P_{\rm obs},-Q_{\rm obs})=\text{rank}(P_{\rm obs})=q\\ \text{and transfer matrix } H_{\rm obs}\in\text{quot}(A)^{q\times m},\ P_{\rm obs}H_{\rm obs}=Q_{\rm obs}.\end{gathered}\tag{61}$$

Then $\mathscr{B}_{\rm obs}$ is called a *$\mathfrak{C}$-observer of $Q\circ w$ from $P\circ w$, $w\in\mathscr{B}$,* if the associated *error behavior*

$$\mathscr{B}_{\rm err}:=\left\{y-Q\circ w\in\mathscr{F}^q;\ w\in\mathscr{B},\ \left(\begin{smallmatrix}y\\ P\circ w\end{smallmatrix}\right)\in\mathscr{B}_{\rm obs}\right\}\tag{62}$$

is $\mathfrak{C}$-negligible, i.e., $\mathscr{B}_{\rm err}\in\mathfrak{C}$.

Example 4.2. In [3] the signal space $\mathscr{F}^\ell$ and thus every trajectory $w\in\mathscr{B}$ are decomposed into three components $\mathscr{F}^l=\mathscr{F}^{\ell_r}\times\mathscr{F}^{\ell_m}\times\mathscr{F}^{\ell_i}\ni w=(w_r,w_m,w_i)^\top$ where w_r is the relevant component that one wants to estimate, w_m is the measurable one and w_i the irrelevant one that one neither knows nor is interested in. With

$$Q:=(\mathrm{id}_{\ell_r},0,0),\ Q\circ w=w_r,\ \text{and}\ P:=(0,\mathrm{id}_{\ell_m},0),\ P\circ w=w_m,$$

this situation is included in the preceding setting.

Lemma 4.3. *A $\mathfrak{C}$-observer $\mathscr{B}_{\rm obs}$ of $\mathscr{B}$ is $\mathfrak{C}$-stable, i.e. $\mathscr{B}^0_{\rm obs}\in\mathfrak{C}$, hence by Cor. 3.8*

$$\mathscr{B}_{\rm obs}\cap\mathscr{F}_2^{q+m}=\left\{\left(\begin{smallmatrix}y\\ u\end{smallmatrix}\right)\in\mathscr{F}_2^{q+m};\ H_{\rm obs}\circ u=y\right\}.$$

Proof. A subbehavior of a $\mathfrak{C}$-negligible one is again such and indeed

$$\mathscr{B}^0_{\rm obs}=\{y\in\mathscr{F}^q;\ P_{\rm obs}\circ y=0\}=\left\{y-Q\circ 0;\ 0\in\mathscr{B},\ \left(\begin{smallmatrix}y\\ P\circ 0\end{smallmatrix}\right)\in\mathscr{B}_{\rm obs}\right\}\subseteq\mathscr{B}_{\rm err}.$$

□

The following theorem characterizes the existence of a $\mathfrak{C}$-observer and parametrizes all controllable ones.

Theorem 4.4 (cf. [3, Thm. 5.2] and [6, Thm. 2.7])**.** *Under the assumptions stated at the beginning of this section, in particular*

$$\mathscr{Q}(A)=A_T\ \text{and}\ U=A^{1\times k}R\subseteq A_T^{1\times k}R\subseteq\mathscr{Q}(U)=A_T^{1\times k'}R',\ R'\in A^{k'\times l},$$

the following statements are equivalent:

1. *There exists a $\mathfrak{C}$-observer of $Q\circ w$ from $P\circ w$, $w\in\mathscr{B}$.*
2. *$Q=XR'+H_{\rm obs}P$ for some $X\in A_T^{q\times k'}$ and $H_{\rm obs}\in A_T^{q\times m}$.*
3. *Under the additional assumption that $\mathscr{Q}(-)=(-)_T$, hence w.l.o.g. $R=R'$: If $w\in\mathscr{B}$ and $P\circ w$ is $\mathfrak{C}$-negligible then so is $Q\circ w$. (This is the standard* detectability condition*. The implication* 1., 2. $\Longrightarrow$ 3. *is always true.)*

For each equation $Q=XR'+H_{\rm obs}P$ as in 2. the unique controllable realization of the (transfer) matrix $H_{\rm obs}$, i.e., the input/output behavior

$$\begin{gathered}\mathscr{B}_{\rm obs}:=\left\{\left(\begin{smallmatrix}y\\ u\end{smallmatrix}\right)\in\mathscr{F}^{q+m};\ P_{\rm obs}\circ y=Q_{\rm obs}\circ u\right\},\ (P_{\rm obs},-Q_{\rm obs})\in A^{k_{\rm obs}\times(q+m)},\ \textit{with}\\ A^{1\times k_{\rm obs}}P_{\rm obs}=\left\{\xi\in A^{1\times q};\ \xi H_{\rm obs}\in A^{1\times m}\right\},\ Q_{\rm obs}:=P_{\rm obs}H_{\rm obs},\end{gathered}$$

is a controllable $\mathfrak{C}$-observer of $Q\circ w$ from $P\circ w$, $w\in\mathscr{B}$. Thus the matrices $H_{\rm obs}$ parametrize the set of all possible controllable $\mathfrak{C}$-observers.

Proof. 1. $\Longrightarrow$ 2.: Let $\mathscr{B}_{\mathrm{obs}}$ from (61) be a $\mathfrak{C}$-observer of $\mathscr{B}$. From Lemma 4.3 we infer that $\mathscr{B}_{\mathrm{obs}}$ is $\mathfrak{C}$-stable and that

$$\mathscr{B}_{\mathrm{obs}} \cap \mathscr{F}_2^{q+m} = \left\{ \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathscr{F}_2^{q+m};\ y = H_{\mathrm{obs}} \circ u \right\},\ H_{\mathrm{obs}} \in A_T^{q\times m}.$$

That $\mathscr{B}_{\mathrm{err}}$ is $\mathfrak{C}$-negligible signifies that

$$\begin{gathered} \mathscr{B}_{\mathrm{err}} \cap \mathscr{F}_2^q = \left\{ y - Q \circ w;\ w \in \mathscr{B} \bigcap \mathscr{F}_2^\ell,\ y = H_{\mathrm{obs}} P \circ w \right\} = 0, \text{ i.e.,} \\ \text{if } w \in \mathscr{B} \cap \mathscr{F}_2^\ell \text{ and } y := H_{\mathrm{obs}} P \circ w \text{ then } y - Q \circ w = (H_{\mathrm{obs}} P - Q) \circ w = 0. \end{gathered} \tag{63}$$

From (54) we conclude $\mathscr{B} \cap \mathscr{F}_2^\ell = U^{\perp_2} = \mathscr{Q}(U)^{\perp_2}$. Define $V := A_T^{1\times q}(H_{\mathrm{obs}} P - Q) \subseteq A_T^{1\times \ell}$. Equation (63) implies $U^{\perp_2} \subseteq V^{\perp_2}$ and then, by Cor. 3.7,

$$\begin{gathered} A_T^{1\times k'} R' = \mathscr{Q}(U) \supseteq \mathscr{Q}(V) \supseteq A_T^{1\times q}(H_{\mathrm{obs}} P - Q) \Longrightarrow \\ \exists X \in A_T^{q\times k'} \text{ with } H_{\mathrm{obs}} P - Q = -XR' \Longrightarrow Q = XR' + H_{\mathrm{obs}} P. \end{gathered}$$

2. $\Longrightarrow$ 1.: All matrices in $Q = XR' + H_{\mathrm{obs}} P$ have entries in A_T and hence act as operators on spaces $\mathscr{F}_2^\bullet$. Let $\mathscr{B}_{\mathrm{obs}}$ be the unique controllable realization of H_{obs} as in the statement of the theorem. From the definition of P_{obs} we conclude

$$\begin{gathered} A^{1\times k_{\mathrm{obs}}} P_{\mathrm{obs}} = \left\{ \xi \in A^{1\times q};\ \xi H_{\mathrm{obs}} \in A^{1\times m} \right\} \Longrightarrow \\ A_T^{1\times k_{\mathrm{obs}}} P_{\mathrm{obs}} = \left\{ \xi \in A_T^{1\times q};\ \xi H_{\mathrm{obs}} \in A_T^{1\times m} \right\} \underset{H_{\mathrm{obs}} \in A_T^{q\times m}}{\Longrightarrow} \\ A_T^{1\times k_{\mathrm{obs}}} P_{\mathrm{obs}} = A_T^{1\times q} \Longrightarrow \exists Y \in A_T^{q\times k_{\mathrm{obs}}} \text{ with } Y P_{\mathrm{obs}} = \mathrm{id}_q . \end{gathered}$$

Again we use $\mathscr{B} \cap \mathscr{F}_2^\ell = \mathscr{Q}(U)^{\perp_2} = \left\{ w \in \mathscr{F}_2^\ell;\ R' \circ w = 0 \right\}$. We have to show that $\mathscr{B}_{\mathrm{err}} \cap \mathscr{F}_2^q = 0$. But let $w \in \mathscr{B} \cap \mathscr{F}_2^\ell$ and $\left(\begin{smallmatrix} y \\ P\circ w \end{smallmatrix}\right) \in \mathscr{B}_{\mathrm{obs}} \cap \mathscr{F}_2^{q+m}$. Then

$$\begin{gathered} P_{\mathrm{obs}} \circ y = Q_{\mathrm{obs}} \circ P \circ w = P_{\mathrm{obs}} \circ (H_{\mathrm{obs}} P) \circ w = P_{\mathrm{obs}} \circ (Q - XR') \circ w \Longrightarrow \\ P_{\mathrm{obs}} \circ \big(y - Q \circ w + X \circ R' \circ w \big) = 0 \underset{Y P_{\mathrm{obs}} = \mathrm{id}_q,\ R' \circ w = 0}{\Longrightarrow} \\ y - Q \circ w = 0 \Longrightarrow \mathscr{B}_{\mathrm{err}} \cap \mathscr{F}_2^q = 0. \end{gathered}$$

In this fashion every solution $(X, H_{\mathrm{obs}}) \in A_T^{q\times (k'+m)}$ of the inhomogeneous linear equation $Q = XR' + H_{\mathrm{obs}} P = (X, H_{\mathrm{obs}}) \left(\begin{smallmatrix} R' \\ P \end{smallmatrix}\right)$ furnishes a controllable $\mathfrak{C}$-observer of $\mathscr{B}$ with transfer matrix H_{obs} or, in other words, these H_{obs} parametrize the set of all *controllable* $\mathfrak{C}$-*observers*. For fixed H_{obs} the matrix X is unique up to a left multiple of a universal left annihilator of R'.

2. $\Longleftrightarrow$ 3.: By assumption we have $\mathscr{Q}(M) = M_T$ for all ${}_AM$ and may and do choose $R' = R$. The A-submodules $V_1 := A^{1\times (k+m)} \left(\begin{smallmatrix} R \\ P \end{smallmatrix}\right)$ and $V_2 := A^{1\times q} Q$ of $A^{1\times \ell}$ give rise to

$$\begin{gathered} \mathscr{Q}(V_1) = A_T^{1\times (k+m)} \left(\begin{smallmatrix} R \\ P \end{smallmatrix}\right),\ \mathscr{Q}(V_2) = A_T^{1\times q} Q \text{ and} \\ V_1^{\perp_2} = \left\{ w \in \mathscr{B} \cap \mathscr{F}_2^\ell;\ P \circ w = 0 \right\},\ V_2^{\perp_2} = \left\{ w \in \mathscr{F}_2^\ell;\ Q \circ w = 0 \right\}. \end{gathered}$$

The condition of 3. signifies that $V_1^{\perp_2} \subseteq V_2^{\perp_2}$ or, equivalently by Cor. 3.7, that

$$\begin{gathered} A_T^{1\times (k+m)} \left(\begin{smallmatrix} R \\ P \end{smallmatrix}\right) = \mathscr{Q}(V_1) \supseteq \mathscr{Q}(V_2) = A_T^{1\times q} Q \Longleftrightarrow \\ \exists (X, H_{\mathrm{obs}}) \in A_T^{q\times (k+m)} \text{ with } Q = (X, H_{\mathrm{obs}}) \left(\begin{smallmatrix} R \\ P \end{smallmatrix}\right) = XR + H_{\mathrm{obs}} P \end{gathered}$$

which is the condition of 2. $\square$

Remark 4.5. If in Thm. 4.4 the matrix R' can be computed by Algorithm 3.9 and if inhomogeneous linear systems over A_T like in 2. can be solved, then the condition in 2. can be checked, all matrices H_{obs} can be computed and the unique controllable $\mathfrak{C}$-observers with transfer matrix H_{obs} can be constructed.

Example 4.6. (*Deadbeat trajectories*) 1. The multidimensional generalization of the two-dimensional deadbeat observers [3] is obtained for the signals and operators from (1), (2) and (3): The set $T := s^N := \{s^\mu;\ \mu \in N = \mathbb{N}^{m_I} \times \mathbb{Z}^{m_{II}}\}$ of monomials is multiplicatively closed and gives rise to the Serre subcategory $\mathfrak{C}(T) := \{C \in \text{Mod}_A;\ C_T = 0\}$ from (49) with $\mathscr{Q}_{\mathfrak{C}(T)}(M) = M_T$. A signal $w \in F^N$ is $\mathfrak{C}(T)$-negligible or a *deadbeat signal* if and only if $s^\mu \circ w = 0$ for some $\mu \in N$. The $\mathfrak{C}(T)$-negligible behaviors are called *nilpotent* [3].
2. In the situation of Example 4.2 the matrices of Thm. 4.4 have the form

$$
\begin{gathered}
w = (w_r, w_m, w_i)^\top \in \mathscr{F}^{\ell_r+\ell_m+\ell_i},\ R' = (R'_r, R'_m, R'_i) \in A^{k' \times (\ell_r+\ell_m+\ell_i)},\\
P = (0, \text{id}_{\ell_m}, 0) \in A^{\ell_m \times (\ell_r+\ell_m+\ell_i)},\ Q = (\text{id}_{\ell_r}, 0, 0) \in A^{\ell_r \times (\ell_r+\ell_m+\ell_i)},\\
X \in A_T^{\ell_r \times k'},\ H_{\text{obs}} \in A_T^{\ell_r \times \ell_m}.
\end{gathered}
$$

The equation 2. in Thm. 4.4 obtains the form

$$
\begin{gathered}
Q = (\text{id}_{\ell_r}, 0, 0) = XR' + H_{\text{obs}}P = X(R'_r, R'_m, R'_i) + H_{\text{obs}}(0, \text{id}_{\ell_m}, 0) \text{ or}\\
\text{id}_{\ell_r} = XR'_r,\ XR'_i = 0,\ H_{\text{obs}} = -XR'_m.
\end{gathered}
$$

Hence for any chosen stability notion an observer for w_r from w_m exists if and only if the matrix R'_r has a left inverse $X \in A_T^{\ell_r \times k'}$ with $XR_i = 0$. The transfer matrix of the controllable observer is $H_{\text{obs}} = -XR_m$.
For deadbeat trajectories as in 1. one may choose $R' = R$. In dimension 2 this is the result [3, Thm. 5.2, (iii)].

5 Characteristic variety, stability and Serre categories

In this section we construct and characterize the Serre categories $\mathfrak{C}(\Lambda_1)$ from the Introduction and Section 2 and derive their connection with Λ_1-stability. The main results are Thms. 5.8, 5.11 and 5.14. We repeat that the case of the real base field $\mathbb{R}$ requires more difficult considerations than that of the complex base field $\mathbb{C}$.
For the proofs we need and therefore recall the results of [19] and present them in a simplified form. In the beginning we assume an arbitrary field F, an F-affine integral domain $A = F[s]/I$ and an arbitrary injective cogenerator ${}_A\mathscr{F}$ or, equivalently, an injective module ${}_A\mathscr{F}$ with $\text{ass}(\mathscr{F}) \supseteq \max(A)$. We use standard results from *Commutative Algebra* [12, §5]. Consider the Serre subcategory (cf. [21])

$$
\begin{gathered}
\mathfrak{C}_{\text{fin}} := \{C \in \text{Mod}_A;\ \forall x \in C : \dim_F(Ax) < \infty\} \text{ with}\\
\mathfrak{P}_{1,\text{fin}} = \{\mathfrak{p} \in \text{spec}(A);\ A/\mathfrak{p} \in \mathfrak{C}_{\text{fin}}\} = \max(A) \text{ and } \text{Ra}_{\text{fin}} := \text{Ra}_{\mathfrak{C}_{\text{fin}}},\ \mathscr{F}_{\text{fin}} := \text{Ra}_{\text{fin}}(\mathscr{F}).
\end{gathered} \tag{64}
$$

That $\dim_F(A/\mathfrak{m}) < \infty$ for $\mathfrak{m} \in \max(A)$ follows from Hilbert's Nullstellensatz. An element $x \in M$, $M \in \text{Mod}_A$, is $\mathfrak{C}_{\text{fin}}$-negligible or *finite* if the cyclic module Ax is F-finite dimensional (f.d.) whereas the modules in $\mathfrak{C}_{\text{fin}}$ are also called *locally finite*. The Gabriel topology $\mathfrak{T}_{\text{fin}} := \mathfrak{T}_{\mathfrak{C}_{\text{fin}}}$ consists of the ideals $\mathfrak{a}$ with $\dim_F(A/\mathfrak{a}) < \infty$ or $\dim(A/\mathfrak{a}) = 0$ where dim denotes the Krull dimension.

Result 5.1. ([19, Thm. 1.14, (2.12-13)]) *The radical* $\mathscr{F}_{\text{fin}} = \text{Ra}_{\text{fin}}(\mathscr{F})$ *admits the direct decomposition*

$$\begin{gathered}\mathscr{F}_{\text{fin}} = \oplus_{\mathfrak{m}\in\max(A)}\mathscr{F}(\mathfrak{m}) \text{ where} \\ \mathscr{F}(\mathfrak{m}) = \bigcup_{k=0}^{\infty} \text{ann}_{\mathscr{F}}(\mathfrak{m}^k), \quad \text{ann}_{\mathscr{F}}(\mathfrak{m}^k) := \left(\mathfrak{m}^k\right)^{\perp} = \left\{w \in \mathscr{F};\ \mathfrak{m}^k \circ w = 0\right\}\end{gathered} \tag{65}$$

and is itself an injective cogenerator with $\text{ass}\,(\mathscr{F}_{\text{fin}}) = \max(A)$. *This decomposition induces the decomposition*

$$\mathscr{F}^{\ell}_{\text{fin}} = \text{Ra}_{\text{fin}}\left(\mathscr{F}^{\ell}\right) = \oplus_{\mathfrak{m}\in\max(A)}\mathscr{F}(\mathfrak{m})^{\ell}.$$

For $t \in A\setminus\mathfrak{m}$ *the map* $t\circ:\ \mathscr{F}(\mathfrak{m}) \to \mathscr{F}(\mathfrak{m})$ *is bijective and therefore* $\mathscr{F}(\mathfrak{m})$ *is a module over the local ring* $A_{\mathfrak{m}} = \left\{\frac{a}{t};\ a \in A,\ t \in A\setminus\mathfrak{m}\right\}$, *and indeed an injective cogenerator of* $\text{Mod}_{A_{\mathfrak{m}}}$. *These properties can also be proven as in Thm. 3.2.*

For an A-module M we define its *maximal support* as

$$\begin{gathered}\text{supp}_{\max}(M) := \text{supp}(M)\cap\max(A) = \{\mathfrak{m}\in\max(A);\ M_{\mathfrak{m}}\neq 0\}\,,\ \text{especially} \\ V_{\max}(\mathfrak{a}) := \text{supp}_{\max}(A/\mathfrak{a}) = V(\mathfrak{a})\cap\max(A) = \{\mathfrak{m}\in\max(A);\ \mathfrak{a}\subseteq\mathfrak{m}\}\,.\ \text{Then} \\ \text{supp}_{\max}(M) \underset{(35)}{=} V_{\max}(\mathfrak{a}) \text{ if } M \text{ f.g., } \mathfrak{a} := \text{ann}_A(M).\end{gathered} \tag{66}$$

Corollary 5.2. *Consider modules and the associated behavior as in* (36)*, i.e.,*

$$\begin{gathered}R\in A^{k\times\ell},\ U := A^{1\times k}R,\ M := A^{1\times\ell}/U,\ \mathfrak{a} := \text{ann}_A(M) \text{ and } \mathscr{B} := U^{\perp} \cong \text{Hom}_A(M,\mathscr{F}). \\ \text{Then } \text{Ra}_{\text{fin}}(\mathscr{B}) = \mathscr{B}\cap\mathscr{F}^{\ell}_{\text{fin}} = \bigoplus_{\mathfrak{m}\in\max(A)} \mathscr{B}(\mathfrak{m}) \text{ with} \\ \mathscr{B}(\mathfrak{m}) := \mathscr{B}\cap\mathscr{F}(\mathfrak{m})^{\ell} = \left\{w\in\mathscr{F}(\mathfrak{m})^{\ell};\ R\circ w = 0\right\} \cong \\ \text{Hom}_A(M,\mathscr{F}(\mathfrak{m})) \cong \text{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}},\mathscr{F}(\mathfrak{m})).\end{gathered}$$

This implies

$$\begin{gathered}\text{supp}_{\max}(M) = V_{\max}(\mathfrak{a}) = \{\mathfrak{m}\in\max(A);\ \mathscr{B}(\mathfrak{m})\neq 0\} \text{ and} \\ \mathscr{B}\bigcap\mathscr{F}^{\ell}_{\text{fin}} = \oplus_{\mathfrak{m}\in\text{supp}_{\max}(M)}\mathscr{B}(\mathfrak{m}).\end{gathered}$$

Proof. The equality $\text{Ra}_{\text{fin}}(\mathscr{B}) = \oplus_{\mathfrak{m}}\mathscr{B}(\mathfrak{m})$ follows directly from $\mathscr{F}^{\ell}_{\text{fin}} = \oplus_{\mathfrak{m}}\mathscr{F}(\mathfrak{m})^{\ell}$ and $\mathscr{B} = \left\{w\in\mathscr{F}^{\ell};\ R\circ w = 0\right\}$. The last isomorphism comes from the universal property of the quotient module since $\mathscr{F}(\mathfrak{m})$ is an $A_{\mathfrak{m}}$-module according to Result 5.1. Since ${}_{A_{\mathfrak{m}}}\mathscr{F}(\mathfrak{m})$ is an injective cogenerator the equivalence

$$M_{\mathfrak{m}} = 0 \iff \mathscr{B}(\mathfrak{m}) \cong \text{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}},\mathscr{F}(\mathfrak{m})) = 0$$

holds and hence $\text{supp}_{\max}(M) = \{\mathfrak{m}\in\max(A);\ \mathscr{B}(\mathfrak{m})\neq 0\}$. □

Since $\mathscr{F}_{\text{fin}}$ is an injective cogenerator $\mathscr{B}\cap\mathscr{F}^{\ell}_{\text{fin}}$ is a "big" submodule of $\mathscr{B}$ and determines $\mathscr{B}$ which, however, contains many non-finite trajectories in general.
The following theorem is a simple, but important consequence of Cor. 5.2 and characterizes, for the constructed $\mathfrak{C}$, the $\mathfrak{C}$-negligible behaviors by properties of their finite trajectories.

Theorem 5.3. (Compare [20, Sect. 3]) *Let F be a field, A an F-affine domain, ${}_A\mathscr{F}$ an injective cogenerator (with $\operatorname{ass}(\mathscr{F}) \supseteq \max(A)$) and $\mathscr{B} \cong \operatorname{Hom}_A(M,\mathscr{F})$ the behavior from Cor. 5.2. Choose an arbitrary disjoint* stability decomposition $\max(A) = \mathfrak{M}_1 \uplus \mathfrak{M}_2$ *into a* stable set $\mathfrak{M}_1$ *and a non-empty* unstable set $\mathfrak{M}_2$. *Then*

$$\mathfrak{C} := \{C \in \operatorname{Mod}_A;\ \forall \mathfrak{m} \in \mathfrak{M}_2 :\ C_{\mathfrak{m}} = 0\} = \{C \in \operatorname{Mod}_A;\ \operatorname{supp}_{\max}(C) \subseteq \mathfrak{M}_1\}$$

is a Serre subcategory of A-torsion modules and the following properties are equivalent:

1. *The module M belongs to $\mathfrak{C}$ or $\mathscr{B}$ is $\mathfrak{C}$-negligible.*
2. *The rank of $R \in A^{k\times\ell}$ is ℓ, i.e., M is an A-torsion module, and $V_{\max}(\mathfrak{a}) = \operatorname{supp}_{\max}(M)$ is contained in $\mathfrak{M}_1$.*
3. *The $\mathscr{F}$-behavior $\mathscr{B}$ is autonomous and $\operatorname{Ra}_{\mathrm{fin}}(\mathfrak{B}) = \mathscr{B} \bigcap \mathscr{F}^{\ell}_{\mathrm{fin}} = \oplus_{\mathfrak{m}\in\mathfrak{M}_1}\mathscr{B}(\mathfrak{m})$, i.e., the finite trajectories of $\mathscr{B}$ have components in $\mathscr{F}(\mathfrak{m})^{\ell}$ for $\mathfrak{m} \in \mathfrak{M}_1$ only.*

With the notations from (37) *this implies*

$$\begin{aligned}\mathfrak{P}_{2,\mathfrak{C}} &:= \{\mathfrak{q} \in \operatorname{spec}(A);\ \exists \mathfrak{m}_2 \in \mathfrak{M}_2 \text{ with } \mathfrak{q} \subseteq \mathfrak{m}_2\},\ \mathfrak{M}_i = \mathfrak{P}_{i,\mathfrak{C}} \bigcap \max(A),\ i = 1,2,\\ \mathfrak{T}_{\mathfrak{C}} &= \{\mathfrak{a} \subseteq A;\ V_{\max}(\mathfrak{a}) \subseteq \mathfrak{M}_1\},\ \mathfrak{P}_1 = \{\mathfrak{p} \in \operatorname{spec}(A);\ V_{\max}(\mathfrak{p}) \subseteq \mathfrak{M}_1\}.\end{aligned} \tag{67}$$

Proof. The functors $\operatorname{Mod}_A \to \operatorname{Mod}_{A_{\mathfrak{m}}}$, $C \mapsto C_{\mathfrak{m}}$, are exact and preserve direct sums. This implies immediately that $\mathfrak{C}$ satisfies the defining closure properties of a Serre subcategory. All three properties imply that M is an A-torsion module or that $\mathscr{B}$ is an autonomous $\mathscr{F}$-behavior, hence we assume this.
1. $\Longleftrightarrow$ 2.: $\mathfrak{C} = \{C \in \operatorname{Mod}_A;\ \operatorname{supp}_{\max}(C) \subseteq \mathfrak{M}_1\}$ and $V_{\max}(\mathfrak{a}) = \operatorname{supp}_{\max}(M)$ since M is finitely generated.
2. $\Longleftrightarrow$ 3.: From Cor. 5.2 we infer

$$\begin{gathered}\operatorname{supp}_{\max}(M) = \{\mathfrak{m} \in \max(A);\ \mathscr{B}(\mathfrak{m}) \neq 0\} \text{ and } \operatorname{Ra}_{\mathrm{fin}}(\mathscr{B}) = \oplus_{\mathfrak{m}\in\operatorname{supp}_{\max}(M)}\mathscr{B}(\mathfrak{m}),\\ \text{hence } \left(\operatorname{Ra}_{\mathrm{fin}}(\mathscr{B}) = \oplus_{\mathfrak{m}\in\mathfrak{M}_1}\mathscr{B}(\mathfrak{m}) \Longleftrightarrow \operatorname{supp}_{\max}(M) \subseteq \mathfrak{M}_1\right).\end{gathered}$$

Equation (67) is an immediate consequence. ☐

Remark 5.4. (cf. [21, Rem. 5.1]) There are Serre subcategories of Mod_A that do not arise according to Thm. 5.3. For $k \in \mathbb{N}$ consider the Serre subcategory associated to (see (37))

$$\begin{gathered}\operatorname{spec}(A) = \mathfrak{P}_{1,k} \uplus \mathfrak{P}_{2,k} \text{ with } \mathfrak{P}_{1,k} := \{\mathfrak{p} \in \operatorname{spec}(A);\ \dim(A/\mathfrak{p}) \leq k\},\\ \mathfrak{P}_{2,k} := \{\mathfrak{p} \in \operatorname{spec}(A);\ \dim(A/\mathfrak{p}) > k\} \text{ and } \mathfrak{C}_k = \{M \in \operatorname{Mod}_A;\ \dim(M) \leq k\}\end{gathered}$$

where dim denotes the Krull dimension. Since an ideal $\mathfrak{p}$ is maximal if and only if $\dim(A/\mathfrak{p}) = 0$ we infer $\mathfrak{P}_{2,k} \cap \max(A) = \emptyset$. Therefore, by (67), $\mathfrak{C}$ does not arise according to Thm. 5.3. For $k = 0$ one obtains $\mathfrak{P}_{1,0} = \mathfrak{P}_{1,\mathrm{fin}} = \max(A)$ and $\mathfrak{C}_0 = \mathfrak{C}_{\mathrm{fin}}$.

5.1 Discrete behaviors

We now specialize $\mathscr{F}$ to get more analytic information on the finite trajectories of the last theorem. For $A = \mathbb{C}[s]$ and the continuous and discrete standard $\mathbb{C}[s]$-signal modules the subsequent theory follows from [19] and [20].

For $M \in \mathrm{Mod}_A$ let $M^* := \mathrm{Hom}_F(M,F)$ denote the dual vector space. It is an A-module via

$$(a \circ \varphi)(x) := \varphi(ax),\ a \in A,\ \varphi \in M^*,\ x \in M. \tag{68}$$

Result 5.5. *([18, Thm. 3.15], [19, Thm. 1.14]) The dual space $\mathscr{F} := A^*$ with the action from* (68) *is a large injective cogenerator, i.e., satisfies* $\mathrm{ass}(A^*) = \mathrm{spec}(A)$*. Its injective submodule $\mathscr{F}_{\mathrm{fin}}$ from Result 5.1 is the least injective A-cogenerator. For each $\mathfrak{m} \in \max(A)$ the direct summand $\mathscr{F}(\mathfrak{m})$, $\mathfrak{m} \in \max(A)$, is indecomposable, hence $\mathscr{F}(\mathfrak{m}) \cong E(A/\mathfrak{m})$ (compare* (45)*) and $\mathscr{F}(\mathfrak{m})$ is the least injective $A_{\mathfrak{m}}$-cogenerator.*

The standard multidimensional discrete signal spaces are of this form. In the following we use this large injective cogenerator signal module A^*.
For a finitely generated A-module $M = A^{1\times\ell}/U$ with A^*-behavior $U^\perp \subseteq (A^*)^\ell$ there are the canonical A-isomorphisms

$$\begin{gathered} M^* = \mathrm{Hom}_F(M,F) \cong \mathrm{Hom}_A(M,A^*) \cong U^\perp, \quad \varphi \mapsto \phi \underset{(36)}{\mapsto} w, \\ \phi(x)(a) = \varphi(ax),\ w_j = \phi(\delta_j + U), \quad \delta_j = (0,\cdots,0,\overset{j}{1},0,\cdots,0), \\ \text{hence the frequent identification } M^* = \mathrm{Hom}_A(M,A^*) = U^\perp \subseteq (A^*)^\ell. \end{gathered} \tag{69}$$

Example 5.6. (*Monoid algebras [13, Ch. 7]*) We consider the elements of the free abelian group $\mathbb{Z}^n$ as *row vectors*. Consider a matrix $\Theta \in \mathbb{Z}^{m\times n}$ and the finitely generated additive monoid

$$N := \mathbb{N}^{1\times m}\Theta = \sum_{i=1}^m \mathbb{N}\Theta_{i-} \subseteq \mathbb{Z}^n. \tag{70}$$

Let $s = (s_1,\cdots,s_m)$ and $\sigma = (\sigma_1,\cdots,\sigma_n)$ be two lists of indeterminates. The group algebra of $\mathbb{Z}^n$ over F is the Laurent polynomial algebra

$$F[\mathbb{Z}^n] = F[\sigma,\sigma^{-1}] = \oplus_{\nu\in\mathbb{Z}^n} F\sigma^\nu,\ \sigma^{-1} := (\sigma_1^{-1},\cdots,\sigma_n^{-1}), \tag{71}$$

where we identify $\nu \in \mathbb{Z}^n$ with the monomial σ^ν as usual. The monoid algebra of N then has the form

$$F[N] := \oplus_{\nu\in N} F\sigma^\nu \subseteq F[\mathbb{Z}^n] = F[\sigma,\sigma^{-1}]. \tag{72}$$

The monoid epimorphism $\circ\Theta : \mathbb{N}^m \to N, \mu \mapsto \mu\Theta$, induces the algebra epimorphism

$$\begin{gathered} \varphi : F[s] \to F[N],\ s_i \mapsto \sigma^{\Theta_{i-}} = \prod_{j=1}^n \sigma_j^{\Theta_{ij}},\ s^\mu \mapsto \sigma^{\mu\Theta},\ \text{with} \\ I_N := \ker(\varphi) \in \mathrm{spec}(F[s]) \text{ and } F[s]/I_N \cong F[N], s^\mu + I_N \mapsto \varphi(s^\mu) = \sigma^{\mu\Theta}. \end{gathered} \tag{73}$$

We often identify $F[s]/I_N = F[N]$, $s^\mu + I_N = \sigma^{\mu\Theta}$. The ideal I_N is called the *lattice ideal* of N and has the form [13, Thm. 7.3]

$$I_N = \sum \left\{ F\left(s^\mu - s^{\mu'}\right);\ \mu,\mu' \in \mathbb{N}^m,\ \mu\Theta = \mu'\Theta \right\}. \tag{74}$$

The algebra $F[N]$ acts on $F^N := \{w : N \to F,\ \nu \mapsto w(\nu)\}$ by shifts or translation. It also acts on $F[N]^*$ via (68) and the map

$$F[N]^* \to F^N,\ \varphi \mapsto w,\ \varphi(\sigma^\nu) = w(\nu), \tag{75}$$

is an $F[N]$-isomorphism. Therefore we may and do identify $F[N]^* = F^N$, i.e., $\varphi = w$ and $w(\sigma^\nu) = w(\nu)$. According to Result 5.5 the module ${}_{F[N]}F^N$ is a large injective cogenerator with all the additional properties and gives rise to a corresponding behavioral systems theory.

5.2 Operator rings $F[\mathbb{N}^{m_I} \times \mathbb{Z}^{m_{II}}]$

We introduce some more notations for the lattices and their monoid algebras from (1)-(3). For any disjoint decomposition

$$\begin{aligned}\{1,\cdots,m\} = S \uplus S' \text{ and any set } X \text{ we identify } X^m = X^{S\uplus S'} = X^S \times X^{S'}\\ x = (x_j)_{j=1,\cdots,m} = (x_S, x_{S'}),\ x_S := (x_j)_{j\in S},\ x_{S'} := (x_j)_{j\in S'}.\end{aligned} \tag{76}$$

Let $m = m_I + m_{II}$, $m_I, m_{II} \in \mathbb{N}$, and pose

$$\begin{gathered}\{1,\cdots,m\} = S_I \uplus S_{II} \text{ with}\\ S_I := \{1,\cdots,m_I\},\ S_{II} := \{m_I+1,\cdots,m = m_I+m_{II}\},\ \text{and}\\ \mathbb{Z}^m = \mathbb{Z}^{S_I} \times \mathbb{Z}^{S_{II}} = \mathbb{Z}^{m_I} \times \mathbb{Z}^{m_{II}} \ni \mu = (\mu_{S_I}, \mu_{S_{II}}) =: (\mu_I, \mu_{II}),\\ N := \mathbb{N}^{m_I} \times \mathbb{Z}^{m_{II}} \subseteq \mathbb{Z}^m = \mathbb{Z}^{m_I} \times \mathbb{Z}^{m_{II}},\ s = (s_{S_I}, s_{S_{II}}) =: (s_I, s_{II}),\\ A := F[N] = F[s_I, s_{II}, s_{II}^{-1}] \ni s^\mu = s_I^{\mu_I} s_{II}^{\mu_{II}},\ \mu_I \in \mathbb{N}^{m_I},\ \mu_{II} \in \mathbb{Z}^{m_{II}}.\end{gathered} \tag{77}$$

So $F[N]$ is the *mixed Laurent polynomial algebra* from (3). In the polynomial algebra $F[s]$ the set $T := \{s_{II}^{\mu_{II}};\ \mu_{II} \in \mathbb{N}^{m_{II}}\}$ is multiplicatively closed and $F[N] = F[s_I, s_{II}, s_{II}^{-1}] = F[s]_T$ is the quotient ring of $F[s]$ with respect to T. This implies the factoriality of $F[N]$ and thus the validity of Assumption 3.6. Moreover there is the bijection [12, Thm. 4.1]

$$\max(F[N]) \cong \{\mathfrak{m} \in \max(F[s]);\ \mathfrak{m} \cap T = \emptyset\},\ \mathfrak{n} = \mathfrak{m}_T = F[N]\mathfrak{m} \leftrightarrow \mathfrak{m} = \mathfrak{n} \cap F[s]. \tag{78}$$

For the Laurent polynomial algebras and notations from above over any algebraically closed field F and over the real field $\mathbb{R}$ we recall the description of the finite trajectories in F^N from [19].

5.2.1 Algebraically closed base fields F

Assume first that F is algebraically closed and define the total space (cf. (7))

$$\Lambda_N := F^{S_I} \times (F\setminus\{0\})^{S_{II}} = F^{m_I} \times (F\setminus\{0\})^{m_{II}} \subset F^m = F^{m_I} \times F^{m_{II}} \tag{79}$$

of vectors $\lambda \in F^m$ that can be substituted into Laurent polynomials $f \in A$. Hilbert's Nullstellensatz for polynomial ideals implies the bijection

$$F^m \cong \max(F[s]),\ \lambda \mapsto \mathfrak{m}(\lambda) := \sum_{i=1}^m F[s](s_i - \lambda_i) = \{f \in F[s];\ f(\lambda) = 0\}, \tag{80}$$

that, together with (78), furnishes an analogue for $F[N]$:

$$\begin{gathered}\Lambda_N \cong \max(F[N]) = \max(F[s_I, s_{II}, s_{II}^{-1}]),\\ \lambda \mapsto \mathfrak{m}_N(\lambda) := \sum_{i=1}^m F[s_I, s_{II}, s_{II}^{-1}](s_i - \lambda_i) = \{f \in F[s_I, s_{II}, s_{II}^{-1}];\ f(\lambda) = 0\}.\end{gathered} \tag{81}$$

The *vanishing set* or *variety* of an ideal $\mathfrak{a} \subseteq F[N]$, compare (9), is

$$\begin{gathered}V(\mathfrak{a}) := V_{\Lambda_N}(\mathfrak{a}) := \{\lambda \in \Lambda_N;\ \forall f \in \mathfrak{a}:\ f(\lambda) = 0\}.\ \text{Then}\\ V_{\Lambda_N}(\mathfrak{a}) \cong V_{\max}(\mathfrak{a}),\ \lambda \mapsto \mathfrak{m}_N(\lambda),\end{gathered} \tag{82}$$

is the canonical bijection induced from (81).
According to Result 5.5 we define

$$F^N(\lambda) := F^N(\mathfrak{m}_N(\lambda)) = \left\{w \in F^N;\ \exists k \in \mathbb{N} : \mathfrak{m}_N(\lambda)^k \circ w = 0\right\} \text{ for}$$
$$\lambda \in \Lambda_N = F^{m_I} \times (F \setminus \{0\})^{m_{II}} \text{ and obtain } \mathrm{Ra}_{\mathrm{fin}}(F^N) = \oplus_{\lambda\in\Lambda_N} F^N(\lambda). \tag{83}$$

We construct an F-basis of $F^N(\lambda)$. Let $S := \mathrm{supp}(\lambda) := \left\{j;\ \lambda_j \neq 0\right\}$, hence $S \supseteq S_{II}$ and $\{1,\cdots,m\} = S' \uplus S$, and consider the derived data

$$\lambda_S := (\lambda_j)_{j\in S},\ \alpha := (\alpha_{S'},\alpha_S),\ t = (t_{S'},t_S) \in \mathbb{Z}^m = \mathbb{Z}^{S'} \times \mathbb{Z}^S.$$
$$\text{Define } e_{\lambda,\alpha} = \left(e_{\lambda,\alpha}(t)\right)_{t\in N} \in F^N,\ \lambda \in \Lambda_N,\ \alpha \in \mathbb{N}^m,$$
$$\text{by } e_{\lambda,\alpha}(t) = \delta_{\alpha_{S'},t_{S'}} \binom{t_S}{\alpha_S} \lambda_S^{t_S-\alpha_S} \text{ with } \binom{t_S}{\alpha_S} := \prod_{j\in S} \binom{t_j}{\alpha_j}. \tag{84}$$
$$\text{Then } (s-\lambda)^\beta \circ e_{\lambda,\alpha} = \begin{cases} e_{\lambda,\alpha-\beta} & \text{if } \alpha \in \beta + \mathbb{N}^m \\ 0 & \text{otherwise} \end{cases}.$$

In characteristic zero, but not in positive characteristic the multinomial coefficients $\binom{t_S}{\alpha_S}$ are polynomial functions of $t \in N$.

Result 5.7. ([19, Thm. 1.25, Cor. 1.26, Thm. 4.23]) *For algebraically closed F, the data from* (76)-(84) *and $\lambda \in \Lambda_N$ one has*

$$F^N(\lambda) = \oplus_{\alpha\in\mathbb{N}^m} F e_{\lambda,\alpha} \textit{ with } (s-\lambda)^\beta \circ e_{\lambda,\alpha} = \begin{cases} e_{\lambda,\alpha-\beta} & \textit{if } \alpha \in \beta + \mathbb{N}^m \\ 0 & \textit{otherwise} \end{cases}.$$

Due to (84) *and* $\mathrm{Ra}_{\mathrm{fin}}(F^N) = \oplus_{\lambda\in\Lambda_N} F^N(\lambda)$ *the finite signals in* $\mathrm{Ra}_{\mathrm{fin}}(F^N)$ *are called* polynomial-exponential *in characteristic zero. For $F = \mathbb{C}$ the growth of $e_{\lambda,\alpha}(t)$ as function of t is determined by its factor $\lambda_S^{t_S}$.*

For the adaption of Thm. 5.3 we introduce the characteristic variety, cf. (8), (27). For a f.g. $F[N]$- torsion module and its dual autonomous behavior, viz.

$$M = F[N]^{1\times\ell}/F[N]^{1\times k}R,\ R \in F[N]^{k\times\ell},\ \mathrm{rank}(R) = \ell,$$
$$\mathscr{B} = \mathrm{Hom}_{F[N]}(M,F^N) \underset{\text{ident.}}{=} \left\{w \in \left(F^N\right)^\ell;\ R \circ w = 0\right\} \tag{85}$$

and for $\lambda \in \Lambda_N$ there are the canonical isomorphisms $F[N]/\mathfrak{m}_N(\lambda) \cong F$ and

$$M/\mathfrak{m}_N(\lambda)M \cong M_{\mathfrak{m}_N(\lambda)}/\mathfrak{m}_N(\lambda)_{\mathfrak{m}_N(\lambda)} M_{\mathfrak{m}_N(\lambda)} \cong F^{1\times\ell}/F^{1\times k}R(\lambda).$$

From Krull's Lemma we infer

$$\mathfrak{m}_N(\lambda) \in \mathrm{supp}_{\max}(M) \iff M_{\mathfrak{m}_N(\lambda)} \neq 0 \underset{\text{Krull}}{\iff}$$
$$M/\mathfrak{m}_N(\lambda)M \cong M_{\mathfrak{m}_N(\lambda)}/\mathfrak{m}_N(\lambda)_{\mathfrak{m}_N(\lambda)} M_{\mathfrak{m}_N(\lambda)} \neq 0 \iff \mathrm{rank}(R(\lambda)) < \ell.$$
$$\text{Moreover } D(M/\mathfrak{m}_N(\lambda)M) = D(M) \bigcap D(F[N]/\mathfrak{m}_N(\lambda))^\ell = \mathscr{B} \bigcap F^{1\times\ell} e_{\lambda,0}.$$

Thus (81) and Cor. 5.2 imply the bijection

$$\mathrm{char}(\mathscr{B}) := \mathrm{char}(M) := \left\{\lambda \in \Lambda_N;\ \mathscr{B} \cap F^\ell e_{\lambda,0} \neq \emptyset\right\} = \left\{\lambda \in \Lambda_N;\ \mathscr{B} \cap F^N(\lambda)^\ell \neq \emptyset\right\} =$$
$$\{\lambda \in \Lambda_N;\ \mathrm{rank}(R(\lambda)) < \mathrm{rank}(R)\} \cong \mathrm{supp}_{\max}(M) \text{ and}$$
$$\mathscr{B} \bigcap \mathrm{Ra}_{\mathrm{fin}}(F^N)^\ell = \oplus_{\lambda\in\mathrm{char}(\mathscr{B})} \mathscr{B}(\lambda),\ \mathscr{B}(\lambda) := \mathscr{B} \cap F^N(\lambda)^\ell. \tag{86}$$

The set $\mathrm{char}(\mathscr{B})$ is the *characteristic variety* from (8). The bijection $V_{\Lambda_N}(\mathfrak{a}) \cong V_{\max}(\mathfrak{a})$ from (82) for the annihilator ideal $\mathfrak{a} := \mathrm{ann}_A(M)$ of M and the equation $V_{\max}(\mathfrak{a}) = \mathrm{supp}_{\max}(M)$ from (66) imply equation (9):

$$\mathrm{char}(\mathscr{B}) = V_{\Lambda_N}(\mathrm{ann}_A(M)). \tag{87}$$

For the preceding more special situation Theorem 5.3 furnishes

Theorem 5.8. *Assumptions as in Result 5.7, especially $F[N] = F[s_I, s_{II}, s_{II}^{-1}]$ over an algebraically closed field F. Consider a f.g. $F[N]$-torsion module and its dual autonomous behavior as in* (85). *Choose an arbitrary disjoint stability decomposition $\Lambda_N = F^{m_I} \times (F \setminus \{0\})^{m_{II}} = \Lambda_1 \uplus \Lambda_2$ into a* stable region *Λ_1 and an* unstable region *$\Lambda_2 \neq \emptyset$. Then*

$$\mathscr{B} \bigcap \mathrm{Ra}_{\mathrm{fin}}(F^N)^\ell = \oplus_{\lambda \in \mathrm{char}(\mathscr{B})} \mathscr{B}(\lambda),\ \mathscr{B}(\lambda) := \mathscr{B} \cap \left(F^N(\lambda)\right)^\ell,\ F^N(\lambda) = \oplus_{\alpha \in \mathbb{N}^m} F e_{\lambda,\alpha}.$$

Moreover $\mathfrak{C}(\Lambda_1) := \left\{C \in \mathrm{Mod}_{F[N]};\ \forall \lambda \in \Lambda_2 : C_{\mathfrak{m}_N(\lambda)} = 0\right\}$

is a Serre subcategory of $\mathrm{Mod}_{F[N]}$*, and the module M and $\mathscr{B}$ are $\mathfrak{C}(\Lambda_1)$-negligible if and only* $\mathrm{char}(M) = \mathrm{char}(\mathscr{B}) = V_{\Lambda_N}(\mathrm{ann}_A(M)) \subseteq \Lambda_1$ *or if and only if* $\mathscr{B} \bigcap \mathrm{Ra}_{\mathrm{fin}}(F^N)^\ell = \oplus_{\lambda \in \Lambda_1} \mathscr{B}(\lambda)$.

For $N = \mathbb{N}^m$, $F[N] = F[s]$ and $\Lambda_N = F^m$ Thm. 5.7 permits a simplification. Consider the F-algebra automorphism $\varphi_\lambda : F[s] \to F[s],\ f \mapsto f(s - \lambda)$, with its inverse $\varphi_\lambda^{-1} = \varphi_{-\lambda}$.

Corollary 5.9. *Let F be algebraically closed and I any ideal of $F[s]$. The decomposition* $\mathrm{Ra}_{\mathrm{fin}}(F^{\mathbb{N}^m}) = \oplus_{\lambda \in F^m} F^{\mathbb{N}^m}(\lambda)$ *and $e_{0,\alpha} = (\delta_{\alpha,t})_{t \in \mathbb{N}^m} =: \delta_\alpha$ hold.*
1. With $\mathrm{supp}(w) := \{\mu \in \mathbb{N}^m;\ w(\mu) \neq 0\}$ *for $w \in F^{\mathbb{N}^m}$ one gets*

$$F^{\mathbb{N}^m}(0) = \oplus_{\alpha \in \mathbb{N}^m} F \delta_\alpha = F^{(\mathbb{N}^m)} := \left\{w \in F^{\mathbb{N}^m};\ \mathrm{supp}(w)\ \textit{finite}\right\}.$$

2. For fixed $\lambda \in F^m$ and $I_{-\lambda} := \varphi_{-\lambda}(I) = \{f(s+\lambda);\ f \in I\}$ the map

$$\phi_\lambda : F^{(\mathbb{N}^m)} = \oplus_{\alpha \in \mathbb{N}^m} F \delta_\alpha \cong F^{\mathbb{N}^m}(\lambda),\ y = (y(\alpha))_{\alpha \in \mathbb{N}^m} \mapsto \sum_{\alpha \in \mathbb{N}^m} y(\alpha) e_{\lambda,\alpha}, \tag{88}$$

is a φ_λ-semilinear isomorphism, i.e., is an F-isomorphism and satisfies $\phi_\lambda(f \circ y) = f(s - \lambda) \circ \phi_\lambda(y)$ for $f \in F[s]$ and $y \in F^{(\mathbb{N}^m)}$. It induces the isomorphism

$$I_{-\lambda}^\perp \cap F^{(\mathbb{N}^m)} = \left\{y \in F^{(\mathbb{N}^m)};\ I_{-\lambda} \circ y = 0\right\} \cong I^\perp \cap F^{\mathbb{N}^m}(\lambda), \tag{89}$$

This reduces computations in $F^{\mathbb{N}^m}(\lambda)$ to computations in $F^{(\mathbb{N}^m)}$.

Proof. 2. The semi-linearity follows from the last equation in (84) which holds for $e_{\lambda,\alpha}$ and especially for $\delta_\alpha = e_{0,\alpha}$. □

5.2.2 The real case $F = \mathbb{R}$

The analogue of Result 5.7 for the real algebra $\mathbb{R}[N]$ and its large injective cogenerator ${}_{\mathbb{R}[N]}\mathbb{R}^N$ is derived from the complex case. Let $\Gamma := \mathrm{Aut}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}_{\mathbb{C}}, \gamma\}$ denote the Galois group of $\mathbb{C}$ over $\mathbb{R}$ where $\gamma : \mathbb{C} \to \mathbb{C},\ z \mapsto \overline{z}$, is the complex conjugation. Its action on $\mathbb{C}$ is extended componentwise to a semi-linear action on any function space

$$\mathbb{C}^J := \{w : J \to \mathbb{C}\} \text{ by } (\gamma w)(j) := \overline{w}(j) := \overline{w(j)},\ \gamma(zw) = \overline{z}(\gamma w) \text{ for } z \in \mathbb{C}. \tag{90}$$

This action induces an analogous action on any Γ-invariant subset $V \subseteq \mathbb{C}^J$, i.e., with $\Gamma V = V$, and then the fixed set ${}^{\Gamma}V := \{v \in V;\ \Gamma v = \{v\}\}$ and the orbit space $\Gamma\backslash V := \{\Gamma v;\ v \in V\}$. If V is a Γ-invariant $\mathbb{C}$-subspace of $\mathbb{C}^J$ then ${}^{\Gamma}V$ is an $\mathbb{R}$-subspace of V and gives rise to the direct decomposition

$$V = {}^{\Gamma}V \oplus i\left({}^{\Gamma}V\right) \ni v = \frac{v+\gamma v}{2} + i\frac{v-\gamma v}{2i} =: \Re(v) + i\Im(v). \tag{91}$$

where $\Re(v)$ resp. $\Im(v)$ are called the *real* resp. *imaginary* part of v. In particular, Γ acts on $\mathbb{C}^m$ with its Γ-invariant subset

$$\begin{gathered}\Lambda_{N,\mathbb{C}} := \mathbb{C}^{m_I} \times (\mathbb{C}\setminus\{0\})^{m_{II}} \text{ and orbits } \Gamma\lambda = \left\{\lambda, \overline{\lambda}\right\} \in \Gamma\backslash\Lambda_{N,\mathbb{C}}. \text{ Moreover}\\ \lambda \in \Lambda_{N,\mathbb{R}} := \mathbb{R}^{m_I} \times (\mathbb{R}\setminus\{0\})^{m_{II}} \iff \Gamma\lambda = \{\lambda\}.\end{gathered} \tag{92}$$

For $\lambda \in \Lambda_{N,\mathbb{C}}$ let $\mathfrak{m}_{N,\mathbb{C}}(\lambda) := \sum_{j=1}^{m} \mathbb{C}[N](s_j - \lambda_j)$ (cf. (81)) and

$$\mathfrak{m}_{N,\mathbb{R}} := \mathbb{R}[N] \cap \mathfrak{m}_{N,\mathbb{C}}(\lambda) = \{f \in \mathbb{R}[N];\ f(\lambda) = 0\} \in \max(\mathbb{R}[N]). \tag{93}$$

Again the Nullstellensatz implies the bijection [19, Lemma 5.5]

$$\Gamma\backslash\Lambda_{N,\mathbb{C}} \cong \max(\mathbb{R}[s]),\ \Gamma\lambda \mapsto \mathfrak{m}_{N,\mathbb{R}}(\lambda). \text{ Define } \mathbb{R}^N(\lambda) := \mathbb{R}^N(\mathfrak{m}_{N,\mathbb{R}}(\lambda)). \tag{94}$$

If $\lambda \in \Lambda_{N,\mathbb{R}}$ then $e_{\lambda,\alpha} \in \mathbb{R}^N$. For $\lambda \in \Lambda_{N,\mathbb{C}} \setminus \Lambda_{N,\mathbb{R}}$ we have $\overline{e_{\lambda,\alpha}} = e_{\overline{\lambda},\alpha}$ in (84) and define $c_{\lambda,\alpha} := \Re(e_{\lambda,\alpha})$ and $s_{\lambda,\alpha} := \Im(e_{\lambda,\alpha})$. For $\lambda_j \neq 0$ we use the polar representation $\lambda_j = |\lambda_j| e^{i\omega_j}$, $\omega_j \in \mathbb{R}$,. For $S := \mathrm{supp}(\lambda)$, $t \in N$ and $\alpha \in \mathbb{N}^m$ we obtain $\lambda_S^{t_S} = |\lambda_S|^{t_S} e^{it_S \bullet \omega_S}$ where $|\lambda_S| := (|\lambda_j|)_{j\in S}$, $\omega_S := (\omega_j)_{j\in S}$ and $t_S \bullet \omega_S := \sum_{j\in S} t_j \omega_j$ and

$$\begin{aligned} e_{\lambda,\alpha} &= \delta_{\alpha_{S'},t_{S'}} \binom{t_S}{\alpha_S} |\lambda_S|^{t_S-\alpha_S} e^{i(t_S-\alpha_S)\bullet\omega_S}\\ c_{\lambda,\alpha}(t) &= \delta_{\alpha_{S'},t_{S'}} \binom{t_S}{\alpha_S} |\lambda_S|^{t_S-\alpha_S} \cos\left((t_S-\alpha_S)\bullet\omega_S\right)\\ s_{\lambda,\alpha}(t) &= \delta_{\alpha_{S'},t_{S'}} \binom{t_S}{\alpha_S} |\lambda_S|^{t_S-\alpha_S} \sin\left((t_S-\alpha_S)\bullet\omega_S\right). \end{aligned} \tag{95}$$

Result 5.10. ([19, Ex. 5.27, Thm. 4.23]) *The indecomposable injective $\mathbb{R}[N]$-module $\mathbb{R}^N(\lambda)$ has the form*

$$\mathbb{R}^N(\lambda) = \begin{cases} \oplus_{\alpha\in\mathbb{N}^m} \mathbb{R} e_{\lambda,\alpha} & \text{if } \lambda \in \Lambda_{N,\mathbb{R}}\\ {}^{\Gamma}\!\left(\mathbb{C}^N(\lambda) \oplus \mathbb{C}^N(\overline{\lambda})\right) = \oplus_{\alpha\in\mathbb{N}^m}\left(\mathbb{R}c_{\lambda,\alpha} \oplus \mathbb{R}s_{\lambda,\alpha}\right) & \text{if } \lambda \in \Lambda_{N,\mathbb{C}} \setminus \Lambda_{N,\mathbb{R}} \end{cases}.$$

With these preparations the real specialization of Theorem 5.3 is

Theorem 5.11. *Let $F := \mathbb{R}$. Data from* (76)-(78)*, especially $\mathbb{R}[N] = \mathbb{R}[s_I, s_{II}, s_{II}^{-1}]$ and from* (92)-(95)*. Consider a f.g. $\mathbb{R}[N]$-torsion module and its dual autonomous behavior as in* (85)*. Choose an arbitrary disjoint stability decomposition $\Lambda_{N,\mathbb{C}} = \mathbb{C}^{m_I} \times (\mathbb{C}\setminus\{0\})^{m_{II}} = \Lambda_1 \uplus \Lambda_2$ into Γ-invariant regions Λ_1 and $\Lambda_2 \neq \emptyset$. With $\mathbb{R}^N(\lambda)$ from* (94) *and Result 5.10 one obtains*

$$\begin{gathered}\mathscr{B} \bigcap \mathrm{Ra}_{\mathrm{fin}}(\mathbb{R}^N)^\ell = \oplus_{\Gamma\lambda\in\Gamma\backslash\mathrm{char}(\mathscr{B})} \mathscr{B}(\lambda) \text{ with } \mathscr{B}(\lambda) := \mathscr{B} \cap \left(\mathbb{R}^N(\lambda)\right)^\ell. \text{ Moreover}\\ \mathfrak{C}_{\mathbb{R}}(\Lambda_1) := \left\{C \in \mathrm{Mod}_{\mathbb{R}[N]};\ \forall \lambda \in \Lambda_2 : C_{\mathfrak{m}_{N,\mathbb{R}}(\lambda)} = 0\right\}\end{gathered}$$

is a Serre subcategory of $\mathrm{Mod}_{\mathbb{R}[N]}$*, and the module* M *or* $\mathscr{B}$ *are* $\mathfrak{C}_{\mathbb{R}}(\Lambda_1)$*-negligible if and only* $\mathrm{char}(\mathscr{B}) \subseteq \Lambda_1$ *or if and only if* $\mathscr{B} \bigcap \mathrm{Ra}_{\mathrm{fin}}(\mathbb{R}^N)^\ell = \oplus_{\Gamma\lambda \in \Gamma\setminus\Lambda_1} \mathscr{B}(\lambda)$.
As in the complex case the characteristic variety is

$$\mathrm{char}(M) = \mathrm{char}(\mathscr{B}) = V_{\Lambda_N}(\mathrm{ann}_{\mathbb{R}[N]}(M)) = \left\{\lambda \in \Lambda_{N,\mathbb{C}};\ \mathrm{rank}(R(\lambda)) < \ell\right\} \text{ and}$$
$$\Gamma\setminus\mathrm{char}(M) \cong \mathrm{supp}_{\max}(M) = V_{\max}(\mathrm{ann}_{\mathbb{R}[N]}(M)), \Gamma\lambda \mapsto \mathfrak{m}_{N,\mathbb{R}}(\lambda)$$

is bijective.

Remark 5.12. The theorems 5.8 and 5.11 can be applied to the observer constructions of Thm. 4.4 in all discrete standard cases when F is the complex or real field, the domain of the independent discrete variables has the form $N = \mathbb{N}^{m_I} \times \mathbb{Z}^{m_{II}}$ and the signal space is F^N. In Section 6 we extend the theory to more general lattices.

5.3 Affine integral domains as operator rings

Finally we extend the preceding results to arbitrary F-affine integral domains $A = F[s]/I$. For this purpose we interpret ${}_AA^*$-behaviors as special ${}_{F[s]}F^{\mathbb{N}^m}$-behaviors: The isomorphism (69) applied to $F[s]$ instead of A yields the isomorphisms

$$\begin{gathered} A^* = \mathrm{Hom}_F(F[s]/I, F) \cong \mathrm{Hom}_{F[s]}(F[s]/I, F^{\mathbb{N}^m}) \cong I^\perp \subseteq F[s]^* \underset{\text{ident.}}{=} F^{\mathbb{N}^m}, \\ \varphi \mapsto \phi \mapsto w = \phi(1+I),\ \varphi(s^\mu + I) = w(\mu). \\ \text{We identify } A^* = I^\perp \subseteq F^{\mathbb{N}^m}, \text{ i.e., } w = \varphi,\ w(\mu) = w(s^\mu + I), \end{gathered} \tag{96}$$

and thus interpret A^* as the subbehavior $I^\perp \subseteq F^{\mathbb{N}^m}$ [18, Thm. 2.99]. The behavior $I^\perp$ is a large injective A-cogenerator, but, in general, not an injective $F[s]$-module. In [1] $I^\perp$ is called the *reduced signal space* for the prime ideal I. An $F[s]$-module M with $IM = 0$ is the same as an A-module with the action $\overline{f}x := fx$ for $f \in F[s]$, $\overline{f} = f + I \in A$ and $x \in M$. If $M = F[s]^{1\times\ell}/U$ is f.g. the condition $IM = 0$ is equivalent to the inclusion $IF[s]^{1\times\ell} \subseteq U$. Each such U has the form

$$U = F[s]^{1\times k}R + IF[s]^{1\times\ell},\ R \in F[s]^{k\times\ell}, \text{ with } M := F[s]^{1\times\ell}/U \underset{\text{ident.}}{=} A^{1\times\ell}/A^{1\times k}\overline{R} \tag{97}$$

where $\overline{R}_{ij} := R_{ij} + I \in A = F[s]/I$ and where the last identification comes from the isomorphism theorem. Notice that M is an $F[s]$-torsion module since $IM = 0$. It is also an A-torsion module if and only if $\mathrm{rank}(\overline{R}) = \ell$. By behavioral duality the module U contains $IF[s]^{1\times\ell}$ if and only if

$$\begin{gathered} U^\perp = \left\{w \in \left(F^{\mathbb{N}^m}\right)^\ell;\ U \circ w = 0\right\} \subseteq \left(IF[s]^{1\times\ell}\right)^\perp = (I^\perp)^\ell, \text{ hence} \\ U^\perp = \left\{w \in (I^\perp)^\ell;\ R \circ w = 0\right\} = \left\{w \in \left(F^{\mathbb{N}^m}\right)^\ell;\ I \circ w = 0,\ R \circ w = 0\right\}. \end{gathered} \tag{98}$$

On the other hand, this f.g. A-module M gives rise to the dual A^*-behavior $\mathrm{Hom}_A(M, A^*)$. Since $IM = 0$ and $I \circ I^\perp = 0$ the isomorphism (69) also implies the A-isomorphisms

$$\begin{gathered} \mathrm{Hom}_F(M, F) \cong \mathrm{Hom}_A(M, A^*) = \mathrm{Hom}_A(M, I^\perp) = \mathrm{Hom}_{F[s]}(M, F^{\mathbb{N}^m}) \cong \\ \mathscr{B} := U^\perp = \left\{w \in (I^\perp)^\ell;\ R \circ w = 0\right\} = \left\{w \in \left(F^{\mathbb{N}^m}\right)^\ell;\ I \circ w = 0,\ R \circ w = 0\right\}. \end{gathered} \tag{99}$$

We thus obtain the following

Corollary 5.13. *(Compare [18, Thm. 2.99]) Consider any field F, an F-affine integral domain $A = F[s]/I$ with $s = (s_1, \cdots, s_m)$ and $I \in \operatorname{spec}(F[s])$, the large injective A-cogenerator $A^* = \operatorname{Hom}_F(A, F) \underset{\text{ident.}}{=} I^\perp$ and U and M from* (97) *and the dual ${}_AI^\perp$-behavior $\mathscr{B} \underset{\text{ident.}}{=} \operatorname{Hom}_A(M, I^\perp)$.*

1. *The A-isomorphisms* (99) *hold, i.e., every ${}_AI^\perp$-behavior $\mathscr{B}$ is the same as an an ${}_{F[s]}F^{\mathbb{N}^m}$-behavior that is contained in $(I^\perp)^\ell$. For F-finite-dimensional M the isomorphisms* (99) *imply* $\dim_F(U^\perp) = \dim_F(M)$.
2. $\max(A) = \{\mathfrak{m}/I;\ \mathfrak{m} \in \max(F[s]),\ \mathfrak{m} \supseteq I\}$ *and for* $\mathfrak{m} \supseteq I$*:*
$$\mathscr{B}(\mathfrak{m}/I) = \mathscr{B}(\mathfrak{m}) = \left\{ w \in \left(F^{\mathbb{N}^m}(\mathfrak{m}) \right)^\ell;\ R \circ w = 0,\ I \circ w = 0 \right\}.$$
Cor. 5.2 implies the decomposition
$$\mathscr{B} \bigcap \operatorname{Ra}_{\text{fin}}(I^\perp)^\ell = \mathscr{B} \bigcap \operatorname{Ra}_{\text{fin}}(F^{\mathbb{N}^m})^\ell = \oplus_{\mathfrak{m}/I \in \max(A)} \mathscr{B}(\mathfrak{m}).$$

An important special case of Cor. 5.13 was already used in Zerz' thesis [35].
We finally connect Cor. 5.13 with Thms. 5.8 and 5.11. For algebraically closed F the variety of the prime ideal $I \subset F[s]$ is $V_F(I) := \{\lambda \in F^m = \Lambda_{\mathbb{N}^m};\ \forall f \in I:\ f(\lambda) = 0\}$ and for $\lambda \in F^m$ the ideal $\mathfrak{m}_F(\lambda) := \{f \in F[s]; f(\lambda) = 0\}$ is maximal (see (19)). Then

$$(\lambda \in V_F(I) \iff \mathfrak{m}_F(\lambda) \supseteq I) \text{ and } V_F(I) \cong \max\left(F[s]/I\right),\ \lambda \mapsto \mathfrak{m}_F(\lambda)/I. \tag{100}$$

For M from (97) the equivalence

$$\lambda \notin V_F(I) \iff I \cap (F[s] \setminus \mathfrak{m}_F(\lambda)) \neq \emptyset \iff I_{\mathfrak{m}_F(\lambda)} = F[s]_{\mathfrak{m}_F(\lambda)}$$

and $IM = 0$ imply $M_{\mathfrak{m}_F(\lambda)} = 0$ for $\lambda \notin V_F(I)$. For $\lambda \in V_F(I)$ the equation $IM = 0$ implies $M_{\mathfrak{m}_F(\lambda)} = M_{\mathfrak{m}_F(\lambda)/I}$. We infer

$$\begin{aligned} V_F(I) \supseteq \operatorname{char}(M) = \operatorname{char}(\mathscr{B}) = \left\{\lambda \in F^m = \Lambda_{\mathbb{N}^m};\ M_{\mathfrak{m}_F(\lambda)} \neq 0\right\} = \\ \{\lambda \in V_F(I);\ \operatorname{rank}(R(\lambda)) < \ell\} \cong \operatorname{supp}_{\max}({}_AM),\ \lambda \mapsto \mathfrak{m}_F(\lambda)/I. \end{aligned} \tag{101}$$

For the real case, i.e., $F = \mathbb{R}$ and $I \subseteq \mathbb{R}[s]$, one considers the complex variety $V_{\mathbb{C}}(I) = V_{\mathbb{C}}(\mathbb{C}I) \subset \mathbb{C}^m$ which is Γ-invariant. The bijections (94) and (101) then induce the bijections

$$\begin{aligned} \Gamma \backslash V_{\mathbb{C}}(I) \cong \max(\mathbb{R}[s]/I),\ \Gamma\lambda \mapsto \mathfrak{m}_{\mathbb{R}}(\lambda)/I,\ \mathfrak{m}_{\mathbb{R}}(\lambda) := \mathbb{R}[s] \cap \mathfrak{m}_{\mathbb{C}}(\lambda), \\ \text{and } \Gamma \backslash \operatorname{char}(M) \cong \operatorname{supp}_{\max}({}_AM) \text{ where} \\ \operatorname{char}(M) = \{\lambda \in V_{\mathbb{C}}(I);\ \operatorname{rank}(R(\lambda)) < \ell\} \subseteq V_{\mathbb{C}}(I). \end{aligned} \tag{102}$$

Due to Cor. 5.13, (101) and (102) the following theorem is a special case of Thms. 5.3, 5.8 and 5.11.

Theorem 5.14. *Consider an algebraically closed field F resp. the real field $F = \mathbb{R}$, an F-affine integral domain $A = F[s]/I$ and the large injective A-cogenerator $A^* \underset{\text{ident.}}{=} I^\perp \subseteq F^{\mathbb{N}^m}$. Choose any stability decomposition $V_F(I) = \Lambda_1 \uplus \Lambda_2$, $\Lambda_2 \neq \emptyset$, resp. $V_{\mathbb{C}}(I) = \Lambda_1 \uplus \Lambda_2$, $\Lambda_2 \neq \emptyset$, with Γ-invariant Λ_i. Consider the A-module M from* (97) *and the associated ${}_AI^\perp$- or ${}_{F[s]}F^{\mathbb{N}^m}$-behavior $\mathscr{B} = \left\{ w \in \left(F^{\mathbb{N}^m}\right)^\ell;\ I \circ w = 0,\ R \circ w = 0 \right\}$.*

1. *There are the direct decompositions*

$$\begin{gathered}\mathrm{Ra}_{\mathrm{fin}}(\mathscr{B})=\mathscr{B}\bigcap\mathrm{Ra}_{\mathrm{fin}}(F^{\mathbb{N}^m})^\ell=\oplus_{\lambda\in\mathrm{char}(\mathscr{B})}\mathscr{B}(\lambda)\ \textit{resp.}\\ \mathrm{Ra}_{\mathrm{fin}}(\mathscr{B})=\mathscr{B}\bigcap\mathrm{Ra}_{\mathrm{fin}}(\mathbb{R}^{\mathbb{N}^m})^\ell=\oplus_{\Gamma\lambda\in\Gamma\backslash\mathrm{char}(\mathscr{B})}\mathscr{B}(\lambda)\ \textit{with}\\ \mathscr{B}(\lambda)=\left\{w\in\mathscr{B};\ \exists k\in\mathbb{N}\ \textit{with}\ \mathfrak{m}_F(\lambda)^k\circ w=0\right\}=\\ \left\{w\in\left(F^{\mathbb{N}^m}(\lambda)\right)^\ell;\ I\circ w=0,\ R\circ w=0\right\},\end{gathered}$$

where the $F^{\mathbb{N}^m}(\lambda)$ are described analytically in Results 5.7 resp. 5.10.

2. *The category $\mathfrak{C}(\Lambda_1):=\left\{M\in\mathrm{Mod}_A;\ \forall\lambda\in\Lambda_2:\ M_{\mathfrak{m}_F(\lambda)}=M_{\mathfrak{m}_F(\lambda)/I}=0\right\}$ is a Serre subcategory of torsion modules of* Mod_A*, and an A-torsion module M belongs to $\mathfrak{C}(\Lambda_1)$ if and only* $\mathrm{char}(M)=\mathrm{char}(\mathscr{B})\subseteq\Lambda_1$ *or if and only if*

$$\mathrm{Ra}_{\mathrm{fin}}(\mathscr{B})=\oplus_{\lambda\in\Lambda_1}\mathscr{B}(\lambda)\ \textit{resp.}\ \mathrm{Ra}_{\mathrm{fin}}(\mathscr{B})=\oplus_{\Gamma\lambda\in\Gamma\backslash\Lambda_1}\mathscr{B}(\lambda).$$

6 Serre categories for general lattices

We assume the data of Example 5.6, i.e., a finitely generated submonoid $N=\mathbb{N}^{1\times m}\Theta\subseteq Z:=\mathbb{Z}^n$ as domain of the independent variables of the signals $w\in F^N$. W.l.o.g. we assume $Z=\mathbb{Z}^n=\mathbb{Z}N=N-N$. The base field F is arbitrary. From Ex. 5.6 we use two lists $s=(s_1,\cdots,s_m)$ and $\sigma=(\sigma_1,\cdots,\sigma_n)$ of indeterminates, the polynomial algebra $F[s]$, the Laurent polynomial algebra $F[Z]$ and its subalgebra $F[N]$:

$$\begin{gathered}F[Z]=F[\sigma,\sigma^{-1}]=\oplus_{\nu\in Z}F\sigma^\nu\supsetneq F[N]=\oplus_{\nu\in N}F\sigma^\nu\ \text{with}\\ \varphi_{\mathrm{ind}}:F[s]/I_N\cong F[N],\ s^\mu+I_N\mapsto\sigma^{\mu\Theta},\ \text{where}\\ I_N=\sum\left\{F(s^\mu-s^{\mu'});\ \mu,\mu'\in\mathbb{N}^m,\ \mu\Theta=\mu'\Theta\right\}\subset F[s].\end{gathered}\tag{103}$$

The relation of our data to those discussed in [14] for the construction of causal input/output representations of two-dimensional behaviors is given by the following dictionary:

$$\begin{gathered}N=\mathscr{C}\subset\mathbb{Z}^2\ [14,\text{Def. }5]\ ,\ F[N]=F[\sigma_1,\sigma_2,\sigma_1^{-1},\sigma_2^{-1}]_{\mathscr{C}}\ [14,\ (2)\text{ on p.1542 }],\\ F^N=(F)^{\mathbb{Z}^2}_{\mathscr{C}},\ \left(F^N\right)^\ell=(F^\ell)^{\mathbb{Z}^2}_{\mathscr{C}}\ [14,\text{ p.1547}]\ .\end{gathered}\tag{104}$$

Due to the given form of I_N the behavior $I_N^\perp\subseteq F^{\mathbb{N}^m}$ is the invariant set

$$I_N^\perp=\left\{u\in F^{\mathbb{N}^m};\ \forall\mu,\mu'\in\mathbb{N}^m\ \text{with}\ \mu\Theta=\mu'\Theta:\ u(\mu)=u(\mu')\right\}.\tag{105}$$

The adjoint isomorphism φ_{ind}^* of φ_{ind} is the isomorphism

$$\begin{gathered}\varphi_{\mathrm{ind}}^*:F[N]^*=F^N\cong(F[s]/I_N)^*=I_N^\perp:w\mapsto u,\ w(\mu\Theta)=u(\mu).\\ \text{We identify }F^N=I_N^\perp,\ w=u,\ w(\mu)=w(\mu\Theta),w(\mu)=w(\mu')\text{ if }\mu\Theta=\mu'\Theta.\end{gathered}\tag{106}$$

and also $F[s]/I_N=F[N],\ s^\mu+I_N=\sigma^{\mu\Theta}$. Thus Cor. 5.13 can be applied to $F[N]$, i.e., finitely generated $F[N]$-modules M and their dual ${}_{F[N]}F^N$-behaviors

$$\mathscr{B}=\mathrm{Hom}_{F[N]}(M,F^N)\underset{(99)}{=}\mathrm{Hom}_{F[s]}(M,F^{\mathbb{N}^m})$$

can also be considered as $F[s]$-modules resp. ${}_{F[s]}F^{\mathbb{N}^m}$-behaviors which is significant for constructive purposes.
In general, the ring $F[N]$ is not factorial [13, p.136] and Assumption 3.6 is not satisfied for an arbitrary Serre subcategory of $\mathrm{Mod}_{F[N]}$ and then Section 4 is not applicable. In contrast the Laurent polynomial algebra $F[Z]=F[\sigma,\sigma^{-1}]$ is factorial and Assumption 3.6 is satisfied for every Serre subcategory of $\mathrm{Mod}_{F[Z]}$. This suggests the following procedure to construct suitable Serre subcategories of $\mathrm{Mod}_{F[N]}$.
The set $\sigma^N:=\{\sigma^\nu;\ \nu\in N\}$ is multiplicatively closed in $F[N]$. Due to $Z=N-N$ the quotient ring $F[N]_{\sigma^N}$ coincides with $F[Z]$. The quotient module functor

$$\mathrm{Mod}_{F[N]}\to\mathrm{Mod}_{F[Z]},\quad M\mapsto M_{\sigma^N}=F[Z]\otimes_{F[N]}M, \tag{107}$$

is exact. There is the bijection [12, Thm. 4.1]

$$\begin{gathered}\mathrm{spec}_{\mathrm{e}}(F[N]):=\left\{\mathfrak{p}\in\mathrm{spec}(F[N]);\ \mathfrak{p}\cap\sigma^N=\emptyset\right\}\cong\mathrm{spec}(F[Z]),\\ \mathfrak{p}=\mathfrak{q}\cap F[N]\leftrightarrow\mathfrak{q}=\mathfrak{p}_{\sigma^N}=F[Z]\mathfrak{p}.\\ \text{Moreover for }\mathfrak{p}\in\mathrm{spec}(F[N]):\ \mathfrak{p}\cap\sigma^N\neq\emptyset\iff\mathfrak{p}_{\sigma^N}=F[Z],\text{ hence}\\ \mathrm{spec}(F[N])=\mathrm{spec}_{\mathrm{e}}(F[N])\uplus\mathrm{spec}_{\mathrm{ne}}(F[N])\text{ with}\\ \mathrm{spec}_{\mathrm{ne}}(F[N]):=\{\mathfrak{p}\in\mathrm{spec}(F[N]);\ \mathfrak{p}_{\sigma^N}=F[Z]\}.\end{gathered}\tag{108}$$

Lemma 6.1. *Let $\mathfrak{C}_Z$ be any Serre subcategory of $\mathrm{Mod}_{F[Z]}$, for instance one from Thms. 5.8 or 5.11, with the associated decomposition* (37) $\mathrm{spec}(F[Z])=\mathfrak{P}_{Z,1}\uplus\mathfrak{P}_{Z,2}$, *the radical* $\mathrm{Ra}_Z:=\mathrm{Ra}_{\mathfrak{C}_Z}$, *the Gabriel topology* (39) $\mathfrak{T}_Z:=\mathfrak{T}_{\mathfrak{C}_Z}$ *the localization functor* $\mathscr{Q}_Z:=\mathscr{Q}_{\mathfrak{C}_Z}:\mathrm{Mod}_{F[Z]}\to\mathrm{Mod}_{F[Z],\mathfrak{C}_Z}$ *and* $T_Z:=T(\mathfrak{C}_Z)$ *from* (50) *according to Section 3. Define*

$$\mathfrak{C}_N:=\left\{C\in\mathrm{Mod}_{F[N]};\ C_{\sigma^N}\in\mathfrak{C}_Z\text{ or }\mathscr{Q}_Z(C_{\sigma^N})=0\right\}. \tag{109}$$

Then $\mathfrak{C}_N$ is a Serre subcategory of $\mathrm{Mod}_{F[N]}$ *with its associated data* $\mathrm{spec}(F[N])=\mathfrak{P}_{N,1}\uplus\mathfrak{P}_{N,2}$, $\mathrm{Ra}_N:=\mathrm{Ra}_{\mathfrak{C}_N}$, $\mathfrak{T}_N:=\mathfrak{T}_{\mathfrak{C}_N}$, $\mathscr{Q}_N:=\mathscr{Q}_{\mathfrak{C}_N}$, $T_N:=T(\mathfrak{C}_N)$. *With*

$$\begin{gathered}\mathfrak{P}_{N,1,\mathrm{e}}:=\mathfrak{P}_{N,1}\cap\mathrm{spec}_{\mathrm{e}}(F[N])=\left\{\mathfrak{p}\in\mathrm{spec}(F[N]);\ F[N]/\mathfrak{p}\in\mathfrak{C}_N,\ \mathfrak{p}\cap\sigma^N=\emptyset\right\},\\ \mathfrak{M}_{N,1}:=\max(F[N])\cap\mathfrak{P}_{N,1},\quad\mathfrak{M}_{N,2}:=\max(F[N])\cap\mathfrak{P}_{N,2}\\ \mathfrak{M}_{N,1,\mathrm{e}}:=\mathfrak{M}_{N,1}\cap\mathrm{spec}_{\mathrm{e}}(F[N]),\\ \mathfrak{M}_{Z,1}:=\max(F[Z])\cap\mathfrak{P}_{Z,1},\quad\mathfrak{M}_{Z,2}:=\max(F[Z])\cap\mathfrak{P}_{Z,2}\end{gathered}$$

we obtain

$$\begin{gathered}\mathfrak{T}_N=\{\mathfrak{a}\subseteq F[N];\ \mathfrak{a}_{\sigma^N}=F[Z]\mathfrak{a}\in\mathfrak{T}_Z\},\ \text{hence }\sigma^N\subseteq T_N,\\ \mathfrak{P}_{N,1}=\mathfrak{P}_{N,1,\mathrm{e}}\uplus\mathrm{spec}_{\mathrm{ne}}(F[N])\text{ and }\mathfrak{M}_{N,1}=\mathfrak{M}_{N,1,\mathrm{e}}\uplus(\mathfrak{M}_{N,1}\cap\mathrm{spec}_{\mathrm{ne}}(F[N])).\\ \text{For an ideal }\mathfrak{b}\subseteq F[Z]:\ \mathfrak{b}\in\mathfrak{T}_Z\iff F[N]\cap\mathfrak{b}\in\mathfrak{T}_N.\end{gathered}\tag{110}$$

Moreover the bijection $\mathrm{spec}_{\mathrm{e}}(F[N])\cong\mathrm{spec}(F[Z])$ *from* (108) *induces the bijections*

$$\mathfrak{P}_{N,1,\mathrm{e}}\cong\mathfrak{P}_{Z,1},\ \mathfrak{P}_{N,2}\cong\mathfrak{P}_{Z,2},\ \mathfrak{M}_{N,1,\mathrm{e}}\cong\mathfrak{M}_{Z,1},\ \mathfrak{M}_{N,2}\cong\mathfrak{M}_{Z,2} \tag{111}$$

Proof. 1. Since $(-)_{\sigma^N}$ is exact and preserves direct sums the closure properties of $\mathfrak{C}_N$ follow immediately from those of $\mathfrak{C}_Z$, so $\mathfrak{C}_N$ is a Serre subcategory. The exactness also implies the first equation of (110). Since each σ^ν, $\nu\in N$, is a unit in $F[Z]$ we infer $F[Z]\sigma^\nu=F[Z]$ and thus $\sigma^\nu\in T_N$ by (50). The last equivalence in (110) follows from $(F[N]\cap\mathfrak{b})_{\sigma^n}=\mathfrak{b}$.
2. *Ad* (110), (111): The prime ideals $\mathfrak{p}\in\mathrm{spec}_{\mathrm{ne}}(F[N])$ with $\mathfrak{p}_{\sigma^N}=F[Z]\mathfrak{p}=F[Z]$ belong to $\mathfrak{P}_{N,1}=\mathrm{spec}(F[N])\cap\mathfrak{T}_N$, but do not generate prime ideals in $F[Z]$. The remaining assertions follow from the bijection in (108). ∎

Example 6.2. (cf. [3] for $N=\mathbb{N}\times\mathbb{Z}$) For $\mathfrak{C}_Z=\{0\}$ one obtains the least category $\mathfrak{C}_N$, viz. $\mathfrak{C}_N=\left\{C\in\mathrm{Mod}_{F[N]};\ C_{\sigma^N}=0\right\}$ of the special type (49) induced from $T_N=\sigma^N$. A signal $w\in F^N$ is $\mathfrak{C}_N$-negligible if and only if $\sigma^\nu\circ w=0$ for some $\nu\in N$. These signals are called *deadbeat signals* and already appeared in Ex. 4.6 for the case $N=\mathbb{N}^{m_I}\times\mathbb{Z}^{m_{II}}$.

Theorem 6.3. *For the data from Lemma 6.1 the following assertions hold:*

1. *An $F[Z]$-module is $\mathfrak{C}_N$-closed if and only if it is $\mathfrak{C}_Z$-closed, and indeed*

$$\mathrm{Mod}_{F[N],\mathfrak{C}_N}=\mathrm{Mod}_{F[Z],\mathfrak{C}_Z}.$$

2. *$\mathscr{Q}_N(M)=\mathscr{Q}_Z(M_{\sigma_N})$ for $M\in\mathrm{Mod}_{F[N]}$.*
3. *$T_N=F[N]\cap T_Z$, $T_Z=(T_N)_{\sigma^N}$ and $F[N]_{T_N}=F[Z]_{T_Z}=\mathscr{Q}_Z(F[Z])=\mathscr{Q}_N(F[N])$, hence Assumption 3.6 is satisfied for $F[N]$.*
4. *If $\mathfrak{C}_Z$ is given as in Thm. 5.3, i.e., $\max(F[Z])=\mathfrak{M}_{Z,1}\uplus\mathfrak{M}_{Z,2}$ with $\mathfrak{M}_{Z,2}\neq\emptyset$ and*

$$\mathfrak{C}_Z:=\left\{C'\in\mathrm{Mod}_{F[Z]};\ \forall\mathfrak{n}_2\in\mathfrak{M}_{Z,2}:\ C'_{\mathfrak{n}_2}=0\right\}$$

then so is $\mathfrak{C}_N$ with $\max(F[N])=\mathfrak{M}_{N,1}\uplus\mathfrak{M}_{N,2}$ where

$$\mathfrak{M}_{N,2}:=\left\{F[N]\bigcap\mathfrak{n}_2;\ \mathfrak{n}_2\in\mathfrak{M}_{Z,2}\right\}\cong\mathfrak{M}_{Z,2},\ \mathfrak{m}_2=\mathfrak{n}_2\bigcap F[N]\leftrightarrow\mathfrak{n}_2=\mathfrak{m}_{2,\sigma^N}.$$

Proof. 1. For an $F[Z]$-module M, an ideal $\mathfrak{a}\in\mathfrak{T}_N$ and $\mathfrak{b}=\mathfrak{a}_{\sigma^N}\in\mathfrak{T}_Z$ consider the map from (42):

$$M\to\mathrm{Hom}_{F[N]}(\mathfrak{a},M)\cong\mathrm{Hom}_{F[N]_{\sigma^N}}(\mathfrak{a}_{\sigma^N},M)=\mathrm{Hom}_{F[Z]}(\mathfrak{b},M).\tag{112}$$

If M is $\mathfrak{C}_Z$- closed then this is an isomorphism for all $\mathfrak{b}\in\mathfrak{T}_Z$, hence also for all $\mathfrak{a}\in\mathfrak{T}_N$ and thus M is $\mathfrak{C}_N$-closed. If M is $\mathfrak{C}_N$-closed choose any $\mathfrak{b}\in\mathfrak{T}_Z$. Then $\mathfrak{a}:=F[N]\cap\mathfrak{b}$ satisfies $\mathfrak{a}_{\sigma^N}=\mathfrak{b}$ and $\mathfrak{a}\in\mathfrak{T}_N$, hence (112) is again an isomorphism for all $\mathfrak{b}$ and M is $\mathfrak{C}_Z$-closed.
Since a $\mathfrak{C}_N$-closed module X is an $F[N]_{T_N}$-module and since $\sigma^N\subseteq T_N$ by (110) we conclude that X is also an $F[Z]=F[N]_{\sigma^N}$-module. By 1. X is a $\mathfrak{C}_Z$-closed module.
2. For any $F[N]$-module M and $\mathfrak{C}_N$-closed module X item 1. and (43) imply the functorial isomorphisms

$$\mathrm{Hom}_{F[N]}(M,X)\cong\mathrm{Hom}_{F[Z]}(M_{\sigma^n},X)\cong\mathrm{Hom}_{F[Z]}\left(\mathscr{Q}_Z(M_{\sigma^N}),X\right).$$

This signifies that $M\mapsto\mathscr{Q}_Z(M_{\sigma^N})$ is the left adjoint functor of the inclusion of $\mathrm{Mod}_{F[N],\mathfrak{C}_N}$ into $\mathrm{Mod}_{F[N]}$ and therefore coincides with $\mathscr{Q}_N$. Notice that left adjoint functors are unique(ly defined) up to functorial isomorphism only.
3. The bijection (111) $\mathfrak{P}_{N,2}\cong\mathfrak{P}_{Z,2}$, $\mathfrak{p}=F[N]\cap\mathfrak{q}\leftrightarrow\mathfrak{q}=\mathfrak{p}_{\sigma^N}$, and (50) imply

$$F[N]\bigcap T_Z=F[N]\cap\left(\bigcap_{\mathfrak{q}\in\mathfrak{P}_{Z,2}}(F[Z]\setminus\mathfrak{q})\right)=$$
$$\bigcap_{\mathfrak{q}\in\mathfrak{P}_{Z,2}}(F[N]\setminus(F[N]\cap\mathfrak{q}))=\bigcap_{\mathfrak{p}\in\mathfrak{P}_{N,2}}(F[N]\setminus\mathfrak{p})=T_N,$$

hence also $(T_N)_{\sigma^N} \subseteq T_Z$ since σ^N consists of units of $F[Z]$. For $t \in T_Z$ there is a denominator σ^ν, $\nu \in N$, with $\sigma^\nu t \in F[N] \cap T_Z = T_N$, hence $t = \sigma^{-\nu}(\sigma^\nu t) \in (T_N)_{\sigma^N}$ and $(T_N)_{\sigma^N} = T_Z$. Since $\sigma^N \subseteq T_N$ and $F[N]_{\sigma^N} = F[Z]$ we conclude $F[N]_{T_N} = (F[N]_{\sigma^N})_{T_N} = F[Z]_{T_Z}$. The factoriality of $F[Z]$ implies $F[Z]_{T_Z} = \mathcal{Q}_Z(F[Z])$, compare Ass. 3.6. Summing up we obtain $F[N]_{T_N} = F[Z]_{T_Z} = \mathcal{Q}_Z(F[Z]) = \mathcal{Q}_Z(F[N]_{\sigma^N}) \underset{2.}{=} \mathcal{Q}_N(F[N])$.

4. Define $\mathfrak{C}'_N := \{C \in \mathrm{Mod}_{F[N]};\ \forall \mathfrak{m}_2 \in \mathfrak{M}_{N,2} :\ C_{\mathfrak{m}_2} = 0\}$. We have to show $\mathfrak{C}_N = \mathfrak{C}'_N$. Consider maximal ideals $\mathfrak{n}_2 \in \mathfrak{M}_{Z,2}$ and $\mathfrak{m}_2 := F[N] \bigcap \mathfrak{n}_2 \in \mathfrak{M}_{N,2}$, hence $\mathfrak{n}_2 = \mathfrak{m}_{2,\sigma^N}$ and $\emptyset = \sigma^N \bigcap \mathfrak{m}_2$. Let first $C \in \mathfrak{C}'_N$. By definition we get

$$\forall \mathfrak{m}_2 \in \mathfrak{M}_{N,2} :\ C_{\mathfrak{m}_2} = 0 \Longrightarrow \forall \mathfrak{m}_2 \in \mathfrak{M}_{N,2} :\ 0 = (C_{\mathfrak{m}_2})_{\sigma^N} = (C_{\sigma^N})_{\mathfrak{m}_{2,\sigma^N}} = (C_{\sigma^N})_{\mathfrak{n}_2}$$
$$\Longrightarrow \forall \mathfrak{n}_2 \in \mathfrak{M}_{Z,2} :\ (C_{\sigma^N})_{\mathfrak{n}_2} = 0 \Longrightarrow C_{\sigma^N} \in \mathfrak{C}_Z \Longrightarrow C \in \mathfrak{C}_N.$$

Let, conversely, $C \in \mathfrak{C}_N$ and assume w.l.o.g. that C is f.g. Then

$$(C_{\sigma^N})_{\mathfrak{n}_2} = 0 \Longrightarrow \exists t' \in F[Z] \setminus \mathfrak{n}_2 \text{ with } t'C_{\sigma^N} = 0 \underset{\mathfrak{n}_2 = \mathfrak{m}_{2,\sigma^N}}{\Longrightarrow}$$
$$\exists t \in F[N] \setminus \mathfrak{m}_2 \text{ with } (tC)_{\sigma^N} = 0 \Longrightarrow \exists t \notin \mathfrak{m}_2, \exists \mu \in N \text{ with } (\sigma^\mu t)C = 0 \underset{\sigma^N \subseteq F[N]\setminus \mathfrak{m}_2}{\Longrightarrow}$$
$$\exists t_1 \in F[N] \setminus \mathfrak{m}_2 :\ t_1 C = 0 \Longrightarrow C_{\mathfrak{m}_2} = 0 \underset{\forall \mathfrak{m}_2}{\Longrightarrow} C \in \mathfrak{C}'_N.$$

□

The Serre categories of the type $\mathfrak{C}_N$ are characterized in the following

Theorem 6.4. *For a Serre subcategory $\mathfrak{C} \subsetneq \mathrm{Mod}_{F[N]}$ with $\mathrm{spec}(F[N]) = \mathfrak{P}_1 \uplus \mathfrak{P}_2$ according to (37) the following assertions are equivalent:*

- *(i) The category $\mathfrak{C}$ is of the form $\mathfrak{C} = \mathfrak{C}_N$ for some $\mathfrak{C}_Z$. The category $\mathfrak{C}_Z$ is then uniquely determined.*
- *(ii) $\mathfrak{P}_1 \supseteq \mathrm{spec}_{\mathrm{ne}}(F[N]) := \{\mathfrak{p} \in \mathrm{spec}(F[N]);\ \mathfrak{p} \bigcap \sigma^N \neq \emptyset\}$.*
- *(iii) $\sigma^N \subseteq T(\mathfrak{C})$.*
- *(iv) $F[Z] \subseteq F[N]_{T(\mathfrak{C})}$.*
- *(v) All deadbeat signals w, i.e., those with $\sigma^\nu \circ w = 0$ for some $\nu \in N$, are $\mathfrak{C}$-negligible.*

Proof. $(i) \Longrightarrow (ii), (iii), (iv), (v)$: Lemma 6.1. Moreover the bijection $\mathfrak{P}_{N,1,\mathrm{e}} \cong \mathfrak{P}_{Z,1}$, $\mathfrak{p} \mapsto \mathfrak{p}_{\sigma^N}$, from Lemma 6.1 shows that $\mathfrak{P}_{Z,1}$ and hence $\mathfrak{C}_Z$ can be reconstructed from $\mathfrak{P}_{N,1}$. This suggests how to construct $\mathfrak{C}_Z$.
$(ii) \Longrightarrow (i)$: Assume $\mathfrak{P}_1 \supseteq \mathrm{spec}_{\mathrm{ne}}(F[N])$ and define

$$\mathfrak{P}_{1,\mathrm{e}} := \mathfrak{P}_1 \cap \mathrm{spec}_{\mathrm{e}}(F[N]), \text{ hence } \mathfrak{P}_1 = \mathfrak{P}_{1,\mathrm{e}} \uplus \mathrm{spec}_{\mathrm{ne}}(F[N]), \text{ and } \mathfrak{P}_{Z,1} \text{ by}$$
$$\mathfrak{P}_{1,\mathrm{e}} \cong \mathfrak{P}_{Z,1} := \{\mathfrak{p}_{\sigma^N} = F[Z]\mathfrak{p};\ \mathfrak{p} \in \mathfrak{P}_{1,\mathrm{e}}\} \subset \mathrm{spec}(F[Z]) \text{ with } \mathfrak{p} = F[N] \cap \mathfrak{p}_{\sigma^N}.$$

From $\mathfrak{P}_1$ the new set $\mathfrak{P}_{Z,1}$ inherits the property that $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ and $\mathfrak{q}_1 \in \mathfrak{P}_{Z,1}$ imply $\mathfrak{q}_2 \in \mathfrak{P}_{Z,1}$. Therefore $\mathfrak{P}_{Z,1}$ gives rise to the Serre categories $\mathfrak{C}_Z$ and then $\mathfrak{C}_N$ with

$$\mathfrak{P}_{N,1} = \mathfrak{P}_{N,1,\mathrm{e}} \uplus \mathrm{spec}_{\mathrm{ne}}(F[N]),\ \mathfrak{P}_{N,1,\mathrm{e}} \cong \mathfrak{P}_{Z,1}. \text{ Since}$$
$$\mathrm{spec}_{\mathrm{e}}(F[N]) \cong \mathrm{spec}(F[Z]),\ \mathfrak{p} \mapsto \mathfrak{p}_{\sigma^N}, \text{ and } \mathfrak{P}_{1,\mathrm{e}} \cong \mathfrak{P}_{Z,1}$$

we conclude

$$\mathfrak{P}_{N,1,\mathrm{e}} = \mathfrak{P}_{1,\mathrm{e}} \text{ and } \mathfrak{P}_1 = \mathfrak{P}_{1,\mathrm{e}} \uplus \operatorname{spec}_{\mathrm{ne}}(F[N]) = \mathfrak{P}_{N,1,\mathrm{e}} \uplus \operatorname{spec}_{\mathrm{ne}}(F[N]) = \mathfrak{P}_{N,1}.$$

The equality $\mathfrak{P}_1 = \mathfrak{P}_{N,1}$ implies $\mathfrak{C} = \mathfrak{C}_N$.
$(iii) \Longrightarrow (ii)$: $\sigma^N \subseteq T(\mathfrak{C})$ and (50) imply $\mathfrak{p} \cap \sigma^N = \emptyset$ for all $\mathfrak{p} \in \mathfrak{P}_2$. In other words, $\mathfrak{p} \cap \sigma^N \neq \emptyset$ implies $\mathfrak{p} \in \mathfrak{P}_1$, hence $\operatorname{spec}_{\mathrm{ne}}(F[N]) \subseteq \mathfrak{P}_1$.
$(iv) \Longrightarrow (iii)$: We apply that $T(\mathfrak{C})$ is saturated, i.e., $(t_1 t_2 \in T(\mathfrak{C}) \Longrightarrow t_i \in \mathfrak{C})$.

$$\text{For } \nu \in N: \ \sigma^{-\nu} \in F[Z] \subseteq F[N]_{T(\mathfrak{C})} \Longrightarrow$$
$$\sigma^{-\nu} = at^{-1},\ a \in F[N],\ t \in T(\mathfrak{C}) \Longrightarrow t = \sigma^{\nu} a \Longrightarrow \sigma^{\nu} \in T(\mathfrak{C}).$$

$(v) \Longrightarrow (iii)$: Consider the behavior $\mathscr{B} := (F[N]\sigma^{\nu})^{\perp} \subseteq F^N$, $\sigma^{\nu} \in \sigma^N$. All signals in $\mathscr{B}$ are annihilated by σ^{ν}, hence are $\mathfrak{C}$-negligible by (v). This, however, implies that $F[N]/F[N]\sigma^{\nu} \in \mathfrak{C}$ and $\sigma^{\nu} \in T(\mathfrak{C})$ by (50).

□

7 Constructiveness of the algorithms

Let $F = \mathbb{R}$ or $F = \mathbb{C}$ and assume the situation of Thm. 5.14, viz. an F-affine integral domain $A = F[s]/I$, a stability decomposition $V_{\mathbb{C}}(I) = \Lambda_1 \uplus \Lambda_2$ and the associated Serre subcategory $\mathfrak{C}$ with the derived data

$$\mathfrak{P}_1 = \{\mathfrak{p}/I \in \operatorname{spec}(A);\ I \subseteq \mathfrak{p} \in \operatorname{spec}(F[s]),\ V_{\mathbb{C}}(\mathfrak{p}) \subseteq \Lambda_1\},$$
$$\mathfrak{T} := \mathfrak{T}_{\mathfrak{C}} = \{\mathfrak{a} = \mathfrak{b}/I \subseteq A;\ I \subseteq \mathfrak{b} \subseteq F[s],\ V_{\mathbb{C}}(\mathfrak{b}) \subseteq \Lambda_1\},$$
$$\mathscr{Q} := \mathscr{Q}_{\mathfrak{C}},\ T := T(\mathfrak{C})$$

from (51). We identify $\mathbb{C}^m = \mathbb{R}^{2m}$ and $\mathbb{C}[s] \subseteq \mathbb{R}[x,y]^2$ via

$$\mathbb{C}[s] \xrightarrow{\subseteq} \mathbb{R}[x,y]^2 = \mathbb{R}[x_1,\dots,x_m,y_1,\dots,y_m]^2,\ f \longmapsto (\Re(f(x+iy)), \Im(f(x+iy))).$$

Assume that $\Lambda_1 \subseteq \mathbb{R}^{2m}$ is semi-algebraic, i.e., the solution set of finitely many polynomial equalities and inequalities with polynomials in $\mathbb{R}[x,y]$. Then for an ideal $\mathfrak{a} = \mathfrak{b}/I \subseteq A$, $I \subseteq \mathfrak{b} = \sum_{k=1}^{\ell} F[s] f_k$, the relation $V_{\mathbb{C}}(\mathfrak{b}) \subseteq \Lambda_1$ and thus the inclusion $\mathfrak{a} \in \mathfrak{T}$ can be checked algorithmically via finding a quantifier-free formulation of the formula

$$\forall (x,y) \in \mathbb{R}^{2m}: \ \left(\left(\bigwedge_{k=1}^{\ell} \Re(f_k(x+iy)) = 0 \wedge \Im(f_k(x+iy)) = 0 \right) \Longrightarrow (x,y) \in \Lambda_1 \right),$$

that amounts to "true", i.e., $\mathfrak{a} \in \mathfrak{T}$, or "false", i.e., $\mathfrak{a} \notin \mathfrak{T}$. These quantifier eliminating algorithms are based on the theorem of Tarski-Seidenberg and have been implemented e.g. in QEPCAD [1] and Redlog [2].

In the computer algebra system SINGULAR [3] algorithms for the computation of a primary decomposition of a submodule $U \subseteq F[s]^{1\times \ell}$ of a free module of a polynomial ring are included that can be extended to compute primary decompositions of the finitely generated modules $F[s]^{1\times\ell}/U \cong A^{1\times\ell}/A^{1\times k}\overline{R}$ from (97). This implementation

[1] http://www.usna.edu/cs/~qepcad/B/QEPCAD.html
[2] http://redlog.dolzmann.de
[3] http://www.singular.uni-kl.de

works for many base fields F. Summarizing, in the situation described above one can compute the $\mathfrak{C}$-radical $\mathrm{Ra}_{\mathfrak{C}}(M)$ of a finitely generated A-module M using Algorithm 3.1 and the Gabriel localization $\mathscr{Q}(U)$ of a submodule $U \subseteq A^{1\times\ell}$ using Algorithm 3.9. In a more special setting one can take advantage of the quantifier elimination algorithms to solve systems of inhomogeneous linear equations over A_T which is crucial for the construction of observers in Theorem 4.4. In addition to the assumptions above let Λ_2 be *ideal convex* in the sense that $\mathscr{Q}(M) = M_T$ for all $M \in \mathrm{Mod}_A$ or

$$V_{\mathbb{C}}(\mathfrak{b}) \subseteq \Lambda_1 \iff \mathfrak{a} \in \mathfrak{T} \iff \mathfrak{a} \cap T \neq \emptyset \quad \text{for } \mathfrak{a} = \mathfrak{b}/I \subseteq A. \tag{113}$$

Consider a system of inhomogeneous linear equations $Yz = y$ over A_T with $Y \in A_T^{k\times\ell}$ and $y \in A_T^k$. By multiplying the rows with elements in T (that are invertible) we can assume w.l.o.g. that the matrices Y and y have entries in A. Solving the homogeneous system $Yz = 0$ over A_T is simple since a generating system of the solution module over A is also one over A_T. To find a solution of the inhomogeneous system we solve the homogeneous system $(Y, -y)\binom{x}{t} = 0$ over A first. Let $\binom{z_1}{t_1}, \dots, \binom{z_p}{t_p}$ be a generating system of its solution module and let $\mathfrak{a} := \sum_{j=1}^p A t_j \subseteq A$. There exists a solution of $Yz = y$ over A_T if and only if there exists an element $t \in \mathfrak{a} \cap T$, and this can be checked using Equation (113): If $t = \sum_{j=1}^p c_j t_j \in \mathfrak{a} \cap T$ with $c_j \in A$ then $\frac{1}{t}\sum_{j=1}^p c_j z_j \in A_T^\ell$ solves $Yz = y$.

The only problem left is actually finding $t \in \mathfrak{a} \cap T$. If we know that such t exist, e.g. using Equation (113), we make an ansatz $t = \sum_{j=1}^p c_j t_j$ with indeterminate $c_j = \sum_{\mu\in\mathbb{N}^m} d_{j\mu} s^\mu \in F[s]$ with $d_{j\mu} \in F$ and bounded total degree, say $\mathrm{tdeg}(c_j) \leqslant q \in \mathbb{N}$, and check if under these assumptions there exists such a t via finding a quantifier free formulation of

$$\exists d_{j\mu} \in F \ (\text{where } j = 1, \dots p,\ \mu \in \mathbb{N}^m \text{ with } \mu_1 + \cdots + \mu_m \leqslant q) : \\ V_{\mathbb{C}}\left(\sum_{j=1}^p \sum_{\mu} d_{j\mu} s^\mu t_j\right) \subseteq \Lambda_1 .$$

If the result is not "false" then it comprises a parametrization of possible t from which we can choose one. If the result is "false" then we enlarge q. Since we know that $\mathfrak{a} \cap T$ is not empty the algorithm stops after finitely many iterations. It should be noted, however, that the computation times for the quantifier elimination algorithms increase rapidly with the number of variables and the degrees of the polynomials involved, thus with today's computers these algorithms, especially the last one, are not suited for large problems.

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