The Computation of Purity Filtrations over Commutative Noetherian Rings of Operators and their Application to Behaviors

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***Abstract*—Due to the theoretical work and computer implementations of, for instance, Barakat, Quadrat and Robertz and their coauthors the theory of finitely generated (f.g.) modules over *non-commutative* regular noetherian rings of partial differential operators with variable coefficients like the Weyl algebras and over other similar rings has become constructive in recent years. In particular these authors compute the *purity or grade filtration* of a f.g. module by homological means and discuss its significance for the associated behavior. Pommaret and Quadrat noted this significance already in 1999. In this note it is shown that over an arbitrary *commutative* noetherian ring of operators the purity filtration can be easily derived from an appropriate primary decomposition, so that various methods and implementations to compute primary decompositions can also be used to compute the purity filtration, the actual implementation of this observation being left to the younger generation. In most books on *Constructive Commutative Algebra* the connection between purity filtrations and primary decompositions is implicitly stated for cyclic modules. For the standard *commutative* rings of operators and signal modules the latter are injective cogenerators. This implies, in particular, a one-one correspondence between filtrations of the module and the induced filtrations of its associated behavior and thus establishes the systems theoretic significance of the module filtrations. For *non-commutative* rings of operators the standard signal modules are in general neither injective nor cogenerators and therefore the connection of a module filtration with its associated behavior filtration is weaker. It is also shown by a counter-example that dimensional purity of the module or behavior does not imply dimensional purity of the initial conditions according to Riquier of the associated homogeneous Cauchy problem.**

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I. Introduction

The talk and this paper represent an excerpt without proofs of a paper with the same title which was submitted to *Multidimensional Systems and Signal Processing* in Spring 2013.

Due to the theoretical work and computer implementations of, for instance, Barakat [1], Quadrat [12], Robertz [15] and their coauthors the theory of finitely generated (f.g.) modules over *non-commutative* regular noetherian rings of partial differential operators with variable coefficients like the Weyl algebras and over other similar rings has become constructive in recent years. A larger part of this work develops constructive *Homological Algebra*. One interest of these computations lies in their application to the analysis and synthesis of multidimensional linear systems or behaviors. The papers [1] and [12] especially compute the *purity or grade filtration* of a f.g. module by homological means and discuss its systems theoretic significance. Björk introduced this filtration in his important book [2, Thm. 2.4.15] by means of spectral sequences and the Ext-functors and discussed its properties in [2, §2.7]. For commutative pure-dimensional regular rings the filtration goes back to Roos [16]. Pommaret and Quadrat [11, Def. 12, Thm. 3] observed the systems theoretic significance of Björk's results. In his paper [12] Quadrat uses, extends and simplifies Björk's theory and makes it constructive.

The present paper uses only arbitrary *commutative* noetherian rings A of operators and elaborates [9, Ex. III.3]. Its main goal is to show that in this commutative case different purity filtrations of a f.g. module M can be easily derived from the primary decomposition of the zero submodule of M (Section III). In the commutative case the various methods and implementations for the computation of primary decompositions [5], [6, ch.4], [7, pp. 315-320] can thus also serve for the computation of purity filtrations. The method from [5] also uses homological algebra and the Ext-functor. In [19, Thm. 7.1] the primary decomposition was used to construct a decomposition of a behavior as a sum of its controllable part and of (co)primary behaviors. For cyclic modules $A/\mathfrak{a}$ the purity filtration was already discussed in [18, §3.2] with a different terminology.

Over the standard *commutative* rings A of operators there are many natural signal A-modules $\mathcal{F}$ (with action $a \circ w$ for $a \in A$ and $w \in \mathcal{F}$) which are injective cogenerators. This signifies that the functor $M \mapsto \mathcal{B} := \mathrm{Hom}_A(M, \mathcal{F})$ from f.g. modules M to their associated behavior $\mathcal{B}$ preserves and reflects exact sequences and thus establishes a strong duality between modules and behaviors. Injectivity of the signal module $\mathcal{F}$ also signifies that any inhomogeneous linear system $R \circ w = u$ with $R \in A^{k\times \ell}$ and given $u \in \mathcal{F}^k$ has a solution $w \in \mathcal{F}^\ell$ if and only if u satisfies the obvious necessary compatibility condition $L \circ u = 0$ where L is a universal left annihilator of R. In particular, for such signal modules module filtrations of M and behavior filtrations of $\mathcal{B}$ are in one-one correspondence (Section IV). We describe various

ways to compute a behavior by means of its subbehaviors in the filtration, in particular in Thm. IV.1 from [13, Thm. 7,§5] and [12, Thm. 7, §4] with a different proof. The actual implementation of the remarks of the present note are left to the younger generation.
For the standard *non-commutative* multidimensional rings of operators the standard signal modules are neither injective nor cogenerators. Module filtrations give still rise to behavior filtrations, but their connection is weaker. Thm. IV.1 gives partial answers for non-injective signal modules. Quadrat performed relevant computations in [12, §4].
Already Riquier [14] solved the *initial value or Cauchy problem* for an *analytic* n-dimensional behavior $\mathcal{B} = \mathrm{Hom}_A(M,\mathcal{F})$ (in a different language, of course) and showed that every trajectory in $\mathcal{B}$ is uniquely determined by a finite family of initial functions $u_\sigma,\ \sigma\in\Sigma$, where u_σ depends on $n(\sigma)\le n$ independent variables. This suggests the natural question whether all initial functions of a pure d-dimensional behavior $\mathcal{B}$ depend on $n(\sigma)=d$ independent variables. Compare the respective indications in [12, Introduction, second paragraph]. We show in Section V that the answer to this question is negative.

II. Basic data

The following notions and results from *Commutative Algebra* are standard [3], [8]. Let A be a *commutative* noetherian ring, $\mathrm{spec}(A)$ resp. $\max(A)$ the sets of its prime resp. maximal ideals and Mod_A the category of A-modules. For an A-module M and a prime ideal $\mathfrak{p}\in\mathrm{spec}(A)$ we have to consider the local noetherian ring $A_\mathfrak{p}=\left\{\frac{a}{s};\ a\in A, s\in A\setminus\mathfrak{p}\right\}$ and the corresponding $A_\mathfrak{p}$-module $M_\mathfrak{p}$. The *support* resp. *associator* of M is

$$\mathrm{supp}(M):=\{\mathfrak{p}\in\mathrm{spec}(A);\ M_\mathfrak{p}\neq 0\}\supset$$
$$\mathrm{ass}(M):=\{\mathfrak{p}\in\mathrm{spec}(A);\ A/\mathfrak{p}\subseteq M \text{ (up to isomorphism)}\}.$$

Then

$$\mathrm{supp}(M)=\{\mathfrak{q}\in\mathrm{spec}(A);\ \exists\mathfrak{p}\in\mathrm{ass}(M)\text{ with }\mathfrak{p}\subseteq\mathfrak{q}\}. \quad (1)$$

If $M=A/\mathfrak{a}$ is a cyclic module then

$$\mathrm{supp}(A/\mathfrak{a})=V(\mathfrak{a}):=\{\mathfrak{p}\in\mathrm{spec}(A);\ \mathfrak{a}\subseteq\mathfrak{p}\}. \quad (2)$$

If M is finitely generated (f.g.) with *annihilator* ideal

$$\begin{array}{c}\mathrm{ann}_A(M):=\{a\in A;\ aM=0\}\ \text{ then}\\ \mathrm{supp}(M)=V(\mathrm{ann}_A(M))\end{array} \quad (3)$$

and the associator $\mathrm{ass}(M)$ is finite. The *(Krull) dimension* of the A-module M is

$$\begin{array}{c}\dim(M):=\\ \sup\{n\in\mathbb{N};\ \exists\mathfrak{p}_0\subsetneq\mathfrak{p}_1\subsetneq\cdots\subsetneq\mathfrak{p}_n,\ \mathfrak{p}_i\in\mathrm{supp}(M)\}\end{array} \quad (4)$$

especially for $\mathfrak{p}\in\mathrm{spec}(A)$:

$$\begin{array}{c}\dim(A/\mathfrak{p})=\sup\{n\in\mathbb{N};\ \exists\mathfrak{p}=\mathfrak{p}_0\subsetneq\mathfrak{p}_1\subsetneq\cdots\subsetneq\mathfrak{p}_n\},\\ \dim(M)\underset{(1)}{=}\sup_{\mathfrak{p}\in\mathrm{ass}(M)}\dim(A/\mathfrak{p}).\end{array} \quad (5)$$

The Krull dimension of the *ring* A is that of A as A-module with $\mathrm{supp}(A)=\mathrm{spec}(A)$ and may be infinite. The Krull dimension of the local noetherian ring $A_\mathfrak{p},\ \mathfrak{p}\in\mathrm{spec}(A)$ is

$$\dim(A_\mathfrak{p})=\sup\{n\in\mathbb{N};\ \exists\mathfrak{p}_0\subsetneq\mathfrak{p}_1\subsetneq\cdots\subsetneq\mathfrak{p}_n=\mathfrak{p}\}, \quad (6)$$

and finite. It is also called the *height* or *codimension* of the prime ideal $\mathfrak{p}$. The inequality $\dim(A/\mathfrak{p})+\dim(A_\mathfrak{p})\le\dim(A),\ \ \mathfrak{p}\in\mathrm{spec}(A)$, holds obviously. If $\mathfrak{p},\mathfrak{q}\in\mathrm{spec}(A),\ \mathfrak{p}\subseteq\mathfrak{q}$ then $\dim(A_\mathfrak{p})\le\dim(A_\mathfrak{q})$. The *codimension* of any nonzero module ${}_AM$ is defined as

$$\begin{array}{c}\mathrm{cd}(M):=\min_{\mathfrak{p}\in\mathrm{supp}(M)}\dim(A_\mathfrak{p})\underset{(1)}{=}\min_{\mathfrak{p}\in\mathrm{ass}(M)}\dim(A_\mathfrak{p}),\\ \text{hence } h(\mathfrak{p}):=\mathrm{cd}(A/\mathfrak{p})=\dim(A_\mathfrak{p})\text{ for }\mathfrak{p}\in\mathrm{spec}(A),\\ h(\mathfrak{a}):=\mathrm{cd}(A/\mathfrak{a})\text{ for }\mathfrak{a}\subsetneq A,\ h(\mathfrak{a})+\dim(A/\mathfrak{a})\le\dim(A),\\ \dim(M)+\mathrm{cd}(M)\le\dim(A).\end{array} \quad (7)$$

The codimension of the zero module is defined to be ∞. The number $h(\mathfrak{a})=\mathrm{cd}(A/\mathfrak{a})$ is also called the *height* of the ideal $\mathfrak{a}$.
A submodule Q of a f.g. A-module M is called $\mathfrak{p}$*-primary* if $\mathfrak{p}\in\mathrm{spec}(A)$ and $\mathrm{ass}(M/Q)=\{\mathfrak{p}\}$.

Lemma II.1. ([3, Thm. IV.2.1, Prop. IV.2.5]) *Every submodule N of a finitely generated (f.g.) module M admits a reduced primary decomposition $N=\bigcap_{\mathfrak{p}\in\mathrm{ass}(M/N)}Q(\mathfrak{p})$ with* $\mathrm{ass}(M/Q(\mathfrak{p}))=\{\mathfrak{p}\}$ *for all* $\mathfrak{p}\in\mathrm{ass}(M/N)$*. For a minimal* $\mathfrak{p}\in\mathrm{ass}(M/N)$ *the component $Q(\mathfrak{p})$ is uniquely determined by*

$$Q(\mathfrak{p})=\{x\in M;\ \exists s\in A\setminus\mathfrak{p}\text{ with }sx\in N\}. \quad (8)$$

A *Serre subcategory* $\mathfrak{C}$ of Mod_A is a class of modules which is closed under taking isomorphic copies, submodules, factor modules, extensions and direct sums; compare [17, §3] for a short introduction into this notion, its history and its basic properties. For a given Serre category $\mathfrak{C}$ every A-module has a largest submodule $\mathrm{Ra}_\mathfrak{C}(M)$ in $\mathfrak{C}$ which is called the $\mathfrak{C}$-*radical* of M. Serre categories are in one-one-correspondence with disjoint decompositions $\mathrm{spec}(A)=\mathfrak{P}_1\uplus\mathfrak{P}_2$ with the property that $\mathfrak{p},\mathfrak{q}\in\mathrm{spec}(A),\mathfrak{p}\subseteq\mathfrak{q}$ and $\mathfrak{p}\in\mathfrak{P}_1$ imply $\mathfrak{q}\in\mathfrak{P}_1$. The correspondence is given by

$$\begin{array}{c}\mathfrak{P}_1:=\{\mathfrak{p}\in\mathrm{spec}(A);\ A/\mathfrak{p}\in\mathfrak{C}\},\\ \mathfrak{C}=\{C\in\mathrm{Mod}_A;\ \mathrm{supp}(C)\subseteq\mathfrak{P}_1\}=\\ \{C\in\mathrm{Mod}_A;\ \mathrm{ass}(C)\subseteq\mathfrak{P}_1\}=\\ \{C\in\mathrm{Mod}_A;\ \forall\mathfrak{p}\in\mathfrak{P}_2:\ C_\mathfrak{p}=0\}.\end{array} \quad (9)$$

Then

$$\mathrm{Ra}_\mathfrak{C}(M)=0\iff\mathrm{ass}(M)\subseteq\mathfrak{P}_2. \quad (10)$$

If M is f.g. its radical $\mathrm{Ra}_\mathfrak{C}(M)$ can be computed via the primary decomposition of 0 in M from Lemma II.1 [17, Alg.

3.1]:

$$0=\bigcap_{\mathfrak{p}\in\operatorname{ass}(M)}Q(\mathfrak{p})=U_1\bigcap U_2 \text{ where}$$
$$U_i:=\bigcap\left\{Q(\mathfrak{p});\ \mathfrak{p}\in\operatorname{ass}(M)\bigcap\mathfrak{P}_i\right\}, \tag{11}$$
$$\operatorname{ass}(M/U_i)=\operatorname{ass}(M)\bigcap\mathfrak{P}_i. \text{ Then}$$
$$M/U_1\in\mathfrak{C},\ U_2=\operatorname{Ra}_{\mathfrak{C}}(M),\ \operatorname{Ra}_{\mathfrak{C}}\left(M/\operatorname{Ra}_{\mathfrak{C}}(M)\right)=0.$$

III. The purity filtration

The assumptions of the preceding section are in force. For an exact sequence of A-modules

$$0\to M'\to M\to M''\to 0 \text{ one concludes}$$
$$\operatorname{supp}(M)=\operatorname{supp}(M')\bigcup\operatorname{supp}(M''),\ \text{hence}$$
$$\operatorname{cd}(M)=\inf_{\mathfrak{p}\in\operatorname{supp}(M)}\dim(A_{\mathfrak{p}})=\min\left(\operatorname{cd}(M'),\operatorname{cd}(M'')\right). \tag{12}$$

If $M=\bigcup_{i\in I}M_i$ is a directed union of submodules one likewise infers

$$\operatorname{supp}(M)=\bigcup_{i\in I}\operatorname{supp}(M_i) \text{ and } \operatorname{cd}(M)=\inf_{i\in I}\operatorname{cd}(M_i). \tag{13}$$

These two equations imply that for every $d\in\mathbb{N}$ the class

$$\mathfrak{C}_d:=\{C\in\operatorname{Mod}_A;\ \operatorname{cd}(C)\geq d\},\ d\in\mathbb{N}, \tag{14}$$

is a Serre category. The corresponding decomposition (9) is

$$\operatorname{spec}(A)=\mathfrak{P}_{d,1}\biguplus\mathfrak{P}_{d,2}=$$
$$\{\mathfrak{p}\in\operatorname{spec}(A);h(\mathfrak{p})\geq d\}\biguplus\{\mathfrak{p}\in\operatorname{spec}(A);h(\mathfrak{p})<d\} \tag{15}$$

There are the obvious inclusions

$$\mathfrak{C}_0=\operatorname{Mod}_A\supseteq\mathfrak{C}_1\supseteq\cdots\supseteq\mathfrak{C}_d\supseteq\mathfrak{C}_{d+1}\supseteq\cdots$$
$$\operatorname{Ra}_0(M):=M\supseteq\cdots\supseteq\operatorname{Ra}_d(M):= \tag{16}$$
$$\operatorname{Ra}_{\mathfrak{C}_d}(M)\supseteq\operatorname{Ra}_{d+1}(M)=\operatorname{Ra}_{d+1}(\operatorname{Ra}_d(M))\supseteq\cdots.$$

In [11, Def. 12] $\operatorname{Ra}_d(M)$ is denoted by $\operatorname{tor}_{d-1}(M)$. From (11) we infer

$$\operatorname{Ra}_d(M)=\bigcap\left\{Q(\mathfrak{p});\ \mathfrak{p}\in\operatorname{ass}(M)\bigcap\mathfrak{P}_{d,2}\right\}=$$
$$\bigcap\{Q(\mathfrak{p});\ \mathfrak{p}\in\operatorname{ass}(M),\ h(\mathfrak{p})<d\},\ \operatorname{cd}\left(\operatorname{Ra}_d(M)\right)\geq d,$$
$$\operatorname{ass}(M/\operatorname{Ra}_d(M))=\operatorname{ass}(M)\bigcap\mathfrak{P}_{d,2}=$$
$$\{\mathfrak{p}\in\operatorname{ass}(M);\ h(\mathfrak{p})<d\},\ \operatorname{Ra}_{d+1}(M)=$$
$$\operatorname{Ra}_d(M)\bigcap\bigcap\{Q(\mathfrak{p});\ \mathfrak{p}\in\operatorname{ass}(M),\ h(\mathfrak{p})=d\}=$$
$$\bigcap\left\{\operatorname{Ra}_d(M)\bigcap Q(\mathfrak{p});\ \mathfrak{p}\in\operatorname{ass}(M),\ h(\mathfrak{p})=d\right\} \tag{17}$$

Corollary III.1. *Let M be nonzero and f.g. For the data from* (11) *and* (17) *let $m:=\operatorname{cd}(M)$, hence*

$$m=\min\nolimits_{\mathfrak{p}\in\operatorname{ass}(M)}h(\mathfrak{p})\leq p:=\max\nolimits_{\mathfrak{p}\in\operatorname{ass}(M)}h(\mathfrak{p}).$$
$$\textit{Then } M=\operatorname{Ra}_0(M)=\cdots=\operatorname{Ra}_m(M)\supsetneq$$
$$\operatorname{Ra}_{m+1}(M)\supseteq\cdots\supseteq\operatorname{Ra}_p(M)\supsetneq\operatorname{Ra}_{p+1}(M)=0.$$

Theorem and Definition III.2. *Let ${}_AM$ be finitely generated, $d\geq 0$ and*

$$0=\bigcap_{\mathfrak{p}\in\operatorname{ass}(M)}Q(\mathfrak{p}),\ \operatorname{ass}(M/Q(\mathfrak{p}))=\{\mathfrak{p}\} \tag{18}$$

a reduced primary decomposition of 0 *in* M. *Then*

1) *The intersection*
$$\operatorname{Ra}_d(M)=\bigcap\{Q(\mathfrak{p});\ \mathfrak{p}\in\operatorname{ass}(M),\ h(\mathfrak{p})<d\} \textit{ with}$$
$$\operatorname{ass}\left(M/\operatorname{Ra}_d(M)\right)=\{\mathfrak{p}\in\operatorname{ass}(M);\ h(\mathfrak{p})<d\} \tag{19}$$
is a reduced primary decomposition of $\operatorname{Ra}_d(M)$ *in* M.
2) *The intersection*
$$\operatorname{Ra}_{d+1}(M)=$$
$$\bigcap\left\{\operatorname{Ra}_d(M)\bigcap Q(\mathfrak{p});\ \mathfrak{p}\in\operatorname{ass}(M),\ h(\mathfrak{p})=d\right\} \textit{ with}$$
$$\operatorname{ass}\left(\operatorname{Ra}_d(M)/\operatorname{Ra}_{d+1}(M)\right)=\{\mathfrak{p}\in\operatorname{ass}(M);\ h(\mathfrak{p})=d\} \tag{20}$$
is a reduced primary decomposition of $\operatorname{Ra}_{d+1}(M)$ *in* $\operatorname{Ra}_d(M)$. *In particular,* $\operatorname{Ra}_d(M)/\operatorname{Ra}_{d+1}(M)$ *is pure d-codimensional, i.e., it is either zero or all its associated primes $\mathfrak{p}$ have the same codimension $d=\dim(A_{\mathfrak{p}})$ and are therefore minimal in* $\operatorname{ass}\left(\operatorname{Ra}_d(M)/\operatorname{Ra}_{d+1}(M)\right)$. *This implies*
$$\forall\mathfrak{p}\in\operatorname{ass}(M) \textit{ with } \dim(A_{\mathfrak{p}})=d:$$
$$\operatorname{Ra}_d(M)\bigcap Q(\mathfrak{p})=\operatorname{Ra}_d(Q(\mathfrak{p}))=$$
$$\{x\in\operatorname{Ra}_d(M);\ \exists s\in A\setminus\mathfrak{p} \textit{ with } sx\in\operatorname{Ra}_{d+1}(M)\}. \tag{21}$$
While the component $Q(\mathfrak{p})$, $\mathfrak{p}\in\operatorname{ass}(M)$, $h(\mathfrak{p})=d$, is not unique in general its radical $\operatorname{Ra}_d(Q(\mathfrak{p}))$ is by (21). *The filtration $M=\operatorname{Ra}_0(M)\supseteq\operatorname{Ra}_1(M)\supseteq\cdots$ is called the* purity filtration *of M.*
3) *In particular, if* $\operatorname{cd}(M)=m$ *then*
$$M=\operatorname{Ra}_m(M)\supsetneq$$
$$\operatorname{Ra}_{m+1}(M)=\bigcap\{Q(\mathfrak{p});\ h(\mathfrak{p})=m\} \tag{22}$$
and $M/\operatorname{Ra}_{m+1}(M)$ is reasonably called the pure $\operatorname{cd}(M)$-codimensional or equidimensional factor *of M. For its computation one has to compute the primary components $Q(\mathfrak{p})$ with $\operatorname{cd}(M/Q(\mathfrak{p}))=\operatorname{cd}(M)$ only. Since $\operatorname{Ra}_{d+1}(M)=\operatorname{Ra}_{d+1}\left(\operatorname{Ra}_d(M)\right)$ equation* (22) *enables the inductive computation of the purity filtration.*

Remark III.3. If $M=A/I$ is cyclic of codimension

$$m:=\operatorname{cd}(A/I)=\min\nolimits_{\mathfrak{p}\in\operatorname{ass}(A/I)}h(\mathfrak{p})\leq p:=\max\nolimits_{\mathfrak{p}\in\operatorname{ass}(A/I)}h(\mathfrak{p})$$
$$\text{and if } I=\bigcap_{\mathfrak{p}\in\operatorname{ass}(A/I)}I(\mathfrak{p}),\ Q(\mathfrak{p})=I(\mathfrak{p})/I$$

is a reduced primary decomposition then also

$$I=\bigcap_{j=m}^{p}I_j,\ I_j:=\bigcap\{I(\mathfrak{p});\ \mathfrak{p}\in\operatorname{ass}(A/I),\ h(\mathfrak{p})=j\}, \tag{23}$$

and Vasconcelos [18, Def. 3.37] calls I_j the *j-th equidimensional component* of I. The unique component $I_m:=$

$\bigcap\{I(\mathfrak{p});\ \mathfrak{p}\in\mathrm{ass}(A/I),\ h(\mathfrak{p})=m\}$ with $\mathrm{Ra}_{m+1}(A/I) = I_m/I$ is called the *equidimensional part* of I in [6, Def. 4.4.1].

Remark III.4. If $A=F[s]/I,\ F[s]=F[s_1,\cdots,s_n],\ I\subseteq F[s]$, is an arbitrary affine F-algebra over a field F it is preferable to define the purity filtration by means of dimensions instead of codimensions and to consider the increasing sequence of Serre categories

$$\begin{gathered}\mathfrak{C}'_{-1}=0\subseteq\mathfrak{C}'_0\subseteq\cdots\subseteq\mathfrak{C}'_d:=\\ \{C\in\mathrm{Mod}_A;\ \dim(C)\le d\}\subseteq\cdots\subseteq\mathfrak{C}'_{\dim(A)}=\mathrm{Mod}_A\end{gathered}\tag{24}$$

with their increasing sequence of radicals

$$\begin{gathered}\mathrm{Ra}'_{-1}(M)=0\subseteq\mathrm{Ra}'_0(M)\subseteq\cdots\subseteq\mathrm{Ra}'_d(M):=\\ \mathrm{Ra}_{\mathfrak{C}'_d}(M)\subseteq\cdots\subseteq\mathrm{Ra}'_{\dim(A)}(M)=M.\end{gathered}\tag{25}$$

The analogue of Theorem III.2 holds.

IV. Application to behaviors

The assumptions of Thm. III.2 are in force. In addition we assume an injective cogenerator ${}_A\mathcal{F}$. The module $\mathcal{F}$ is interpreted as a space of *signals* on which the ring A of *operators* acts via $a\circ w,\ a\in A, w\in\mathcal{F}$. The discrete resp. continuous standard cases are the following: The ring A is the polynomial algebra $A:=F[s]:=F[s_1,\cdots,s_n]$ over a field F. In the discrete case $\mathcal{F}$ is the space of n-variate sequences

$$\mathcal{F}:=F^{\mathbb{N}^n}:=\{w=(w(\mu))_{\mu\in\mathbb{N}^n}:\mathbb{N}^n\to F,\ \mu\mapsto w(\mu)\}\tag{26}$$

with the shift action $(s^\nu\circ w)(\mu) := w(\mu+\nu),\ \ \mu,\nu\in\mathbb{N}^n,\ w\in F^{\mathbb{N}^n}$. In the continuous case F is the field $\mathbb{R}$ of real resp. $\mathbb{C}$ of complex numbers and the signal space $\mathcal{F}$ is chosen as the space of smooth functions in $t=(t_1,\cdots,t_n)\in\mathbb{R}^n$ or of distributions, i.e.,

$$\mathcal{F}:=\mathrm{C}^\infty(\mathbb{R}^n,F)\text{ or }\mathcal{F}:=\mathcal{D}'(\mathbb{R}^n,F)\tag{27}$$

with the action by partial differentiation $s_i\circ w:=\partial_i w:=\partial w/\partial t_i$ for $w\in\mathcal{F}$ and $i=1,\cdots,n$. The action of a matrix $R\in A^{k\times\ell}$ on $w\in\mathcal{F}^\ell$(columns) is given by

$$R\circ w:=\left(\sum_{j=1}^{\ell}R_{ij}\circ w_j\right)_{i=1,\cdots,k}\in\mathcal{F}^k.\tag{28}$$

Consider an arbitrary f.g. A-module M with its associated behavior $\mathcal{B}$, viz.

$$\begin{gathered}R\in A^{k\times\ell},\ U:=A^{1\times k}R\subseteq A^{1\times\ell},\ M:=A^{1\times\ell}/U\\ \mathcal{B}:=U^\perp:=\left\{w\in\mathcal{F}^\ell;\ \forall\xi\in U:\ \xi\circ w=0\right\}=\\ \left\{w\in\mathcal{F}^\ell;\ R\circ w=0\right\}\underset{\text{Malgrange 1962}}{\cong}\mathrm{Hom}_A(M,\mathcal{F}),\ w\longleftrightarrow\varphi\\ \varphi(\delta_j+U)=w_j,\ \delta_j=(0,\cdots,0,\overset{j}{1},0,\cdots,0),\ j=1,\cdots,\ell.\end{gathered}\tag{29}$$

We will always identify $U^\perp=\mathrm{Hom}_A(A^{1\times\ell}/U,\mathcal{F})$. Since $\mathcal{F}$ is an injective cogenerator the functor $\mathrm{Hom}_A(-,\mathcal{F})$ preserves and reflects exact sequences, in particular $M=0$ if and only if $\mathrm{Hom}_A(M,\mathcal{F})=0$. For a submodule M' of M there result the two dual exact sequences with the canonical injection inj and surjection can:

$$\begin{gathered}0\to M'\xrightarrow{\text{inj}}M\xrightarrow{\text{can}}M'':=M/M'\to 0\\ 0\leftarrow\mathrm{Hom}_A(M',\mathcal{F})\overset{\mathrm{Hom}(\text{inj},\mathcal{F})}{\longleftarrow}\mathrm{Hom}_A(M,\mathcal{F})\overset{\mathrm{Hom}(\text{can},\mathcal{F})}{\longleftarrow}\\ \mathrm{Hom}_A(M'',\mathcal{F})\leftarrow 0,\end{gathered}\tag{30}$$

hence the identifications

$$\begin{gathered}\mathrm{Hom}_A(M'',\mathcal{F})=\{\varphi\in\mathrm{Hom}_A(M,\mathcal{F});\ \varphi(M')=0\},\\ \mathrm{Hom}_A(M,\mathcal{F})/\mathrm{Hom}_A(M/M',\mathcal{F})=\mathrm{Hom}_A(M',\mathcal{F}),\\ \varphi+\mathrm{Hom}_A(M/M',\mathcal{F})=\varphi|M'.\end{gathered}\tag{31}$$

Define

$$\begin{gathered}m:=\mathrm{cd}(M)=\min\nolimits_{\mathfrak{p}\in\mathrm{ass}(M)}\dim(A_{\mathfrak{p}})\text{ and}\\ p:=\max\nolimits_{\mathfrak{p}\in\mathrm{ass}(M)}\dim(A_{\mathfrak{p}}).\end{gathered}\tag{32}$$

The purity filtration $M=\mathrm{Ra}_m(M)\supset\cdots\supset\mathrm{Ra}_{p+1}(M)=0$ from Thm. III.2 and Cor. III.1 induces the filtration of subbehaviors

$$\begin{gathered}0=\mathcal{B}_0=\cdots\mathcal{B}_m\subsetneq\mathcal{B}_{m+1}\subseteq\cdots\\ \subseteq\mathcal{B}_d:=\mathrm{Hom}_A(M/\mathrm{Ra}_d(M),\mathcal{F})\subseteq\mathcal{B}_{d+1}\subseteq\cdots\\ \mathcal{B}_p\subsetneq\mathcal{B}_{p+1}=\mathcal{B}:=\mathrm{Hom}_A(M,\mathcal{F})\text{ with}\\ \mathcal{B}_{d+1}/\mathcal{B}_d\underset{\text{ident.}}{=}\mathrm{Hom}_A\left(\mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M),\mathcal{F}\right).\end{gathered}\tag{33}$$

The reduced primary decomposition (20) and application of the exact functor $\mathrm{Hom}_A(-,\mathcal{F})$ to the ensuing diagonal monomorphisms

$$M/\mathrm{Ra}_{d+1}(M)\to M/\mathrm{Ra}_d(M)\times\prod_{\mathfrak{p}\in\mathrm{ass}(M),\ h(\mathfrak{p})=d}M/Q(\mathfrak{p})\tag{34}$$

and

$$\begin{gathered}\mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M)\to\\ \prod_{\mathfrak{p}\in\mathrm{ass}(M),\ h(\mathfrak{p})=d}\mathrm{Ra}_d(M)/\left(\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p})\right)\end{gathered}\tag{35}$$

furnish the decompositions

$$\begin{gathered}\mathcal{B}_{d+1}=\mathcal{B}_d+\sum_{\mathfrak{p}\in\mathrm{ass}(M),\ h(\mathfrak{p})=d}\mathcal{B}_{\mathfrak{p}}\text{ with}\\ \mathcal{B}_{\mathfrak{p}}=\mathrm{Hom}_A(M/Q(\mathfrak{p}),\mathcal{F})\\ \mathcal{B}_{d+1}/\mathcal{B}_d=\sum_{\mathfrak{p}\in\mathrm{ass}(M),\ h(\mathfrak{p})=d}(\mathcal{B}_d+\mathcal{B}_{\mathfrak{p}})/\mathcal{B}_d,\\ (\mathcal{B}_d+\mathcal{B}_{\mathfrak{p}})/\mathcal{B}_d=\mathrm{Hom}_A(\mathrm{Ra}_d(M)/(\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p})),\mathcal{F}),\\ \mathcal{B}=\sum_{\mathfrak{p}\in\mathrm{ass}(M)}\mathcal{B}_{\mathfrak{p}}.\end{gathered}\tag{36}$$

The last sum representation coincides with that in [19, Thm. 7.1]. The preceding data have the following matrix descriptions in which all matrices can be computed in the standard cases:

$$\begin{aligned} &M=A^{1\times\ell}/U,\ U=A^{1\times k}R,\ R\in A^{k\times\ell},\\ &\mathcal{B}=\left\{w\in\mathcal{F}^{\ell};\ R\circ w=0\right\},\\ &\mathrm{Ra}_d(M)=A^{1\times k_d}R_d/U,\ R_d\in A^{k_d\times\ell},\ A^{1\times k_d}R_d\supseteq U,\\ &M/\mathrm{Ra}_d(M)\underset{\text{ident.}}{=}A^{1\times\ell}/A^{1\times k_d}R_d,\\ &\mathcal{B}_d=\left\{w\in\mathcal{F}^{\ell};\ R_d\circ w=0\right\},\\ &Q(\mathfrak{p})=A^{1\times k_\mathfrak{p}}R_\mathfrak{p}/U,\ R_\mathfrak{p}\in A^{k_\mathfrak{p}\times\ell},\ A^{1\times k_\mathfrak{p}}R_\mathfrak{p}\supseteq U,\\ &\mathcal{B}_\mathfrak{p}=\left\{w\in\mathcal{F}^{\ell};\ R_\mathfrak{p}\circ w=0\right\}. \end{aligned} \tag{37}$$

If in addition matrices $P_d\in A^{m_d\times k_d}$ and $P_\mathfrak{p}\in A^{m_\mathfrak{p}\times k_d}$ are computed such that the maps

$$\begin{aligned} &A^{1\times k_d}/A^{1\times m_d}P_d\cong A^{1\times k_d}R_d/A^{1\times k_{d+1}}R_{d+1}\cong\\ &\mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M),\\ &\xi+A^{1\times m_d}P_d\mapsto \xi R_d+A^{1\times k_{d+1}}R_{d+1}\mapsto\\ &(\xi R_d+U)+\mathrm{Ra}_{d+1}(M),\\ &A^{1\times k_d}/A^{1\times m_\mathfrak{p}}P_\mathfrak{p}\cong \mathrm{Ra}_d(M)/\left(\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p})\right),\\ &\xi+A^{1\times m_\mathfrak{p}}P_\mathfrak{p}\mapsto(\xi R_d+U)+\left(\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p})\right) \end{aligned} \tag{38}$$

are isomorphisms then also

$$\begin{aligned} &R_d\circ:\mathcal{B}_{d+1}/\mathcal{B}_d\cong\left\{v\in\mathcal{F}^{k_d};\ P_d\circ v=0\right\}=\\ &\sum_{\mathfrak{p}\in\mathrm{ass}(M),\ h(\mathfrak{p})=d}\left\{v\in\mathcal{F}^{k_d};\ P_\mathfrak{p}\circ v=0\right\},\ w+\mathcal{B}_d\mapsto R_d\circ w, \end{aligned} \tag{39}$$

is an isomorphism.

If in the situation of (30) the module $\mathcal{F}$ is not injective it is still possible, at least partially, to describe $\mathrm{Hom}_A(M,\mathcal{F})$ with data from M' and $M''=M/M'$. The following considerations are applicable to arbitrary filtrations and derive a result of Quadrat and Robertz [13, Thm. 7,§5], [12, Thm. 7, §4] with a different proof based on the seminal *Homological Algebra* textbook [4] instead of the Baer extensions in the quoted papers. Assume that free resolutions

$$\begin{aligned} &\cdots\to A^{1\times s(1)}\xrightarrow{R'}A^{1\times s(0)}\xrightarrow{\phi'}M'\to 0\\ &\cdots\to A^{1\times p(1)}\xrightarrow{R''}A^{1\times p(0)}\xrightarrow{\phi''}M''\to 0 \end{aligned} \tag{40}$$

are given. The $R',\ R''$ etc. are matrices of appropriate sizes and act on *row* vectors by multiplication on the *right*, ϕ' and ϕ'' are linear maps and the (infinite) sequences of modules in (40) are exact. In particular,

$$\begin{aligned} &\phi'_{\mathrm{ind}}:A^{1\times s(0)}/\ker(R')\cong M',\ \xi'+\ker(R')\mapsto\phi'(\xi'),\\ &\text{and }\phi''_{\mathrm{ind}}:A^{1\times p(0)}/\ker(R'')\cong M'' \end{aligned}$$

are explicit representations of M' resp. M''. In [4, Prop. V.2.2] a resolution of the first *short exact sequence* in (30) is *constructed*, i.e., a further free resolution

$$\begin{aligned} &\cdots\to A^{1\times(p(1)+s(1))}\xrightarrow{R}A^{1\times(p(0)+s(0))}\xrightarrow{\phi}M\to 0,\\ &\text{hence }\phi_{\mathrm{ind}}:A^{1\times(p(0)+s(0))}/\ker(R)\cong M, \end{aligned} \tag{41}$$

where especially the map ϕ and matrix R have the form

$$\begin{aligned} &\phi=\begin{pmatrix}\phi_p\\ \phi'\end{pmatrix},\ \mathrm{can}\,\phi_p=\phi'',\\ &R=\begin{pmatrix}R'' & R_{12}\\ 0 & R'\end{pmatrix}\in A^{(p(1)+s(1))\times(p(0)+s(0))}. \end{aligned} \tag{42}$$

The equation

$$\mathrm{can}\,\phi_p=\phi'':A^{1\times p_0}\xrightarrow{\phi_p}M\xrightarrow{\mathrm{can}}M'' \tag{43}$$

shows how to construct ϕ_p as a lifting of ϕ'' and $R_{12}\in A^{p(1)\times s(0)}$ is constructed similarly.

For any signal module $\mathcal{F}$ the functor $\mathrm{Hom}_A(-,\mathcal{F})$ is left exact. Its application to (41) furnishes the following

Theorem IV.1. ([13, Thm. 7,§5],[12, Thm. 7, §4]) *With the data from above the map* $\mathrm{Hom}(\phi,\mathcal{F})$ *induces the behavior isomorphism*

$$\begin{aligned} &\mathrm{Hom}(\phi,\mathcal{F}):\mathrm{Hom}_A(M,\mathcal{F})\cong\\ &\mathcal{B}:=\left\{w=\begin{pmatrix}w''\\ w'\end{pmatrix}\in\mathcal{F}^{p(0)+s(0)};\ R\circ w=0\right\}=\\ &\left\{w\in\mathcal{F}^{p(0)+s(0)};\ R''\circ w''+R_{12}\circ w'=0,\ R'\circ w'=0\right\}\\ &\textit{where }\mathcal{B}':=\mathrm{Hom}_A(M',\mathcal{F})=\left\{w'\in\mathcal{F}^{s(0)};\ R'\circ w'=0\right\} \end{aligned} \tag{44}$$

is the behavior of M'. *If* $\mathcal{F}$ *is injective the inhomogeneous equation*

$$R''\circ w''=-R_{12}\circ w' \tag{45}$$

is solvable for each $w'\in\mathcal{B}'$. *Thus the solution of* $R'\circ w'=0$ *and then of* (45) *furnishes all trajectories of* $\mathcal{B}$ *according to* (44) *and then also of* $\mathrm{Hom}_A(M,\mathcal{F})$ *via* $\mathrm{Hom}(\phi,\mathcal{F})$. *If* $\mathcal{F}$ *is not injective then this method does not work in general and the representation* (44) *has more limited significance.*

Like [4, Prop. V.2.2] and [13, Thm. 7,§5] this theorem holds for arbitrary non-commutative *noetherian rings of operators.*

V. Initial conditions according to Riquier

We consider the discrete case from equation (26). Consider an ideal $\mathfrak{a}$ of A and the dual behavior $\mathfrak{a}^\perp\subseteq\mathcal{F}$ according to (29). The componentwise partial order $\le_{\mathrm{cw}}$ on $\mathbb{N}^n$ is defined by $\mu\le_{\mathrm{cw}}\nu:\Longleftrightarrow\nu\in\mu+\mathbb{N}^n$. A subset $N\subseteq\mathbb{N}^n$ is called an *order ideal* if $N+\mathbb{N}^n=N$. These order ideals are in bijective correspondence with the monomial ideals $F[N]=\oplus_{\nu\in N}Fs^\nu\subseteq A=F[s]$ generated by the monomials $s^\nu,\ \nu\in N$. Each order ideal is of the form $N=D+\mathbb{N}^n$ where D is the finite set of minimal elements of N with respect to $\le_{cw}$. Let $\le$ be any *term (well)-order* on $\mathbb{N}^n$ and $\deg:F[s]=F[\mathbb{N}^n]\to\mathbb{N}^n\uplus\{-\infty\}$ the corresponding polynomial degree function.

Any ideal $\mathfrak{a}\subseteq A$ gives rise to its behavior $\mathfrak{a}^\perp:=\left\{w\in F^{\mathbb{N}^n};\ \mathfrak{a}\circ w=0\right\}$ and the degree set $N:=\deg(\mathfrak{a}):=\{\deg(f);\ 0\neq f\in\mathfrak{a}\}$ which is an order ideal. The complement $\Gamma:=\mathbb{N}^n\setminus\deg(\mathfrak{a})$ of the degree set is called the *initial region* of $\mathfrak{a}^\perp$ for the chosen term order. Then

$$\mathfrak{a}^\perp\xrightarrow{\cong}F^\Gamma,\ w\mapsto w|\Gamma:=(w(\mu))_{\mu\in\Gamma}, \tag{46}$$

is an isomorphism [10, Thm. 3.1]. In other words: For each *initial data* $u = (u(\mu))_{\mu\in\Gamma} \in F^\Gamma$ the *initial value or Cauchy problem*

$$\mathfrak{a} \circ w = 0,\ w|\Gamma = u, \tag{47}$$

has a unique solution w (in $\mathfrak{a}^\perp$).
According to Riquier [14] the initial region has an additional structure which was discussed in [10, §4]: For every subset $S \subseteq [n] := \{1, \cdots, n\}$ with complement $S' = [n] \setminus S$ we identify

$$\mathbb{N}^S := \{\mu \in \mathbb{N}^n;\ \forall i \in S' :\ \mu_i = 0\} \subset \mathbb{N}^n. \tag{48}$$

Then there are a finite subset $\Sigma \subset \Gamma$ and subsets $S(\sigma) \subseteq [n],\ \sigma \in \Sigma$, such that that the initial region Γ has the disjoint decomposition

$$\Gamma = \biguplus_{\sigma\in\Sigma} \left(\sigma + \mathbb{N}^{S(\sigma)}\right) \text{ with } \dim(A/\mathfrak{a}) = \max_{\sigma\in\Sigma} \sharp(S(\sigma)), \tag{49}$$

where $\sharp(S(\sigma))$ denotes the number of elements of $S(\sigma)$ [10, §4]. In *Commutative Algebra* Riquier's disjoint decomposition (49) of Γ is also called a *Stanley decomposition*. Combining (46) and (49) we obtain the isomorphism

$$\begin{aligned} &\mathfrak{a}^\perp \cong \prod_{\sigma\in\Sigma} F^{\mathbb{N}^{S(\sigma)}},\ w \mapsto (u_\sigma)_{\sigma\in\Sigma}, \\ &u_\sigma(\mu) := w(\sigma + \mu),\ \mu \in \mathbb{N}^{S(\sigma)}. \end{aligned} \tag{50}$$

The functions u_σ depend on the $\sharp(S(\sigma))$ *independent variables* $\mu_i \in \mathbb{N},\ i \in S(\sigma)$.
As mentioned in the Introduction the equations (49) and (50) suggest the following question: If $A/\mathfrak{a}$ is pure m-dimensional, i.e., if $\dim(A/\mathfrak{p}) = m$ for all $\mathfrak{p} \in \mathrm{ass}(A/\mathfrak{a})$ do then all sets $S(\sigma)$ have the same cardinality m? The following example gives a negative answer.

Example V.1. Consider the 4-dimensional polynomial algebra $A = F[s] = F[s_1, s_2, s_3, s_4]$ over a field F and the prime ideals $\mathfrak{p}_1 := As_1 + As_2$ and $\mathfrak{p}_2 := As_3 + As_4$. The ideal $\mathfrak{a} := \mathfrak{p}_1 \bigcap \mathfrak{p}_2$ has the following properties: It is pure 2-dimensional, i.e., $\dim(A/\mathfrak{p}) = 2$ for all $\mathfrak{p} \in \mathrm{ass}(A/\mathfrak{a})$. Since $\mathfrak{a}$ is monomial the degree set $\deg(\mathfrak{a})$ is unique and does not depend on the chosen term well-order. For every Riquier decomposition (49) $\Gamma = \biguplus_{\sigma\in\Sigma} \left(\sigma + \mathbb{N}^{S(\sigma)}\right)$ of the initial region $\Gamma := \mathbb{N}^n \setminus \deg(\mathfrak{a})$ of the associated behavior $\mathfrak{a}^\perp$ there is a set $S(\sigma)$ with less than 2 elements, i.e., there is at least one initial function $u_\sigma \in F^{\mathbb{N}^{S(\sigma)}}$ according to (50) which depends on at most one independent variable.

References

[1] M. Barakat, Purity Filtration and the Fine Structure of Autonomy, *Proceedings of the 19th International Symposium on the Mathematical Theory of Networks and Systems, MTNS 2010, Budapest*, 1657-1661
[2] J.-E. Björk, *Rings of Differential Operators*, North-Holland, Amsterdam, 1979
[3] N. Bourbaki, *Commutative Algebra*, Hermann, Paris, 1972
[4] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956
[5] D. Eisenbud, C. Huneke, W. Vasconcelos, Direct methods for primary decompostion, *Invent. Math.* 110 (1992), 207-235
[6] G.-M. Greuel, G. Pfister, *A* **Singular** *Introduction to Commutative Algebra*, Springer, Berlin, 2002
[7] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra 2*, Springer, Berlin, 2005
[8] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986
[9] U. Oberst, The significance of Gabriel localization for stability and stabilization of multidimensional input/output behaviors, *Proceedings of the 19th International Symposium on the Mathematical Theory of Networks and Systems MTNS 2010*, Budapest, 1651-1655
[10] U. Oberst, F. Pauer, Solving Systems of Linear Partial Difference and Differential Equations with Constant Coefficients Using Gröbner Bases, *Radon Series Comp. Appl. Math.* 2(2007), 23-41
[11] J.F. Pommaret, A. Quadrat, Algebraic analysis of linear multidimensional control systems, *IMA J. Math. Control and Inform.* 16(1999), 275-297
[12] A. Quadrat, Grade Filtration of Linear Functional Systems, *Acta Appl. Math.* DOI 10.1007/s10440-012-9791-2 (2013), pp.1-62
[13] A. Quadrat, D. Robertz, Baer's extension problem for multidimensional linears systems, *Proceedings of the 18th International Symposium on the Mathematical Theory of Networks and Systems MTNS 2008*, Blacksburg (VA), pp.1-12
[14] C. Riquier, *Les Systèmes d'Équations aux Dérivées Partielles*, Gautiers-Villars, Paris, 1910
[15] D. Robertz, Formal Algorithmic Elimination for PDEs, Habilitationsschrift, RWTH Aachen
[16] J.-E. Roos, Bidualité et structure des foncteurs dérivés de $\varprojlim$ dans la catégorie des modules sur un anneau régulier, *C.R. Acad. Sci. Paris* 254(1962), 1556-1558
[17] M. Scheicher, U. Oberst, Multidimensional stability by Serre categories and the construction and parametrization of observers via Gabriel localizations, pp.1-38, to appear in *SIAM J. Control and Optimization*
[18] W. Vasconcelos, *Computational Methods in Commutative Algebra and Algebraic Geometry*, Springer, Berlin, 1998
[19] J. Wood, U. Oberst, E. Rogers, D.H. Owens, A Behavioral Approach to the Pole Structure of One-Dimensional and Multidimensional Linear Systems, *SIAM J. Control Optim.* 38(2000), 627-661