

*Canonical State Representations and  
Hilbert Functions of Multidimensional  
Systems*

Talk at Gröbner-Semester  
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after a submitted paper of the same title,  
with a discussion of remarks of Professor J.F. Pommaret after my talk and  
several historical comments.

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**1 Talk**

*CONTENTS*

1. Modules and behaviors
2. Riquier's decomposition
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## *History*

1. Minimal state representations of one-dimensional proper transfer matrices:  
*R. Kalman* 1960s
2. First order representations of multidimensional systems: *E. Zerz* 2000 et.al.
3. The modified Spencer form for multidimensional systems: *J.F. Pommaret* 2004

# DATA

base field  $\mathbb{C}$  (in this talk)

ring of operators

$$\mathcal{D} := \mathbb{C}[s_1, \dots, s_r] \ni f = \sum_{\mu \in \mathbb{N}^r} f_{\mu} s^{\mu}$$

function space of  
formal power series

$$\mathcal{F} := \mathbb{C}^{\mathbb{N}^r} = \mathbb{C}[[z_1, \dots, z_r]] \ni$$
$$y = (y(\mu))_{\mu \in \mathbb{N}^r} = \sum_{\mu \in \mathbb{N}^r} y(\mu) z^\mu$$

*Other function spaces:* (locally) convergent power series, entire functions of exponential type,  $C^\infty$ -functions, distributions

Actions of  $\mathcal{D}$  on  $\mathcal{F}$  or  $\mathcal{D}$ -module structure  $f \circ y$  of  $\mathcal{F}$ :

1. *Discrete case of partial difference equations*

by left shifts:

$$(s^\mu \circ y)(\nu) := y(\nu + \mu)$$

2. *Continuous case of partial differential equations*

by partial differentiation:

$$s_\rho \circ y := \partial y / \partial z_\rho$$

## *BEHAVIORS*

matrices

$I$  finite index set, eg.  $I = \{1, \dots, q\}$

matrix  $R \in \mathcal{D}^{k \times I}$  or  $R \in \mathcal{D}^{k \times q}$

$\mathcal{D}^{1 \times I}$  = module of rows,  $\delta_i$  standard basis

$\mathcal{F}^I$  = module of columns

polynomial modules

$U := \mathcal{D}^{1 \times k} R \subset \mathcal{D}^{1 \times I}$  row module of  $R$

$M := \mathcal{D}^{1 \times I} / U$  factor module

behavior, system or solution space

$$\mathcal{B} := \text{sol}(M) := \{w \in \mathcal{F}^I; R \circ w = 0\}$$

$$\mathcal{B} \cong \text{Hom}_{\mathcal{D}}(M, \mathcal{F})$$

$$w = (w_i)_{i \in I} \leftrightarrow (\bar{\delta}_i \mapsto w_i)$$

Malgrange 1962

*categorical duality*

finitely generated module  ${}_{\mathcal{D}}M \leftrightarrow$

behavior  $\mathcal{B} \cong \text{Hom}_{\mathcal{D}}(M, \mathcal{F})$



Gröbner basis data

index set

$$I \times \mathbb{N}^r \ni \alpha = (i, \mu) = (i(\alpha), \mu(\alpha))$$

action

$$\begin{aligned} \mathbb{N}^r \times (I \times \mathbb{N}^r) &\rightarrow I \times \mathbb{N}^r : \\ (\lambda, \alpha = (i, \mu)) &\mapsto \lambda + \alpha := (i, \lambda + \mu) \end{aligned}$$

term order  $<$  on  $I \times \mathbb{N}^r$

order submodule of  $I \times \mathbb{N}^r$

$$\deg(U) = \mathbb{N}^r + \deg(U) = \uplus_{i \in I} \{i\} \times \deg(U)_i$$

computation by Buchberger's algorithm

region *below the stairs*

$$\Gamma := (I \times \mathbb{N}^r) \setminus \deg(U)$$

**Theorem 1.1.** (C. Riquier 1910,  
Oberst/Pauer 2001)

*Algorithmic canonical disjoint decomposition of  $\Gamma$  and  $M$ :*

$$\Gamma = \uplus_{\alpha \in \Delta} (\alpha + \mathbb{N}^{S(\alpha)}) \Rightarrow$$

$$M = \oplus_{\alpha \in \Delta} \mathbb{C} [(s_\rho)_{\rho \in S(\alpha)}] s^{\mu(\alpha)} \overline{\delta_{i(\alpha)}}$$

*with finite subset*

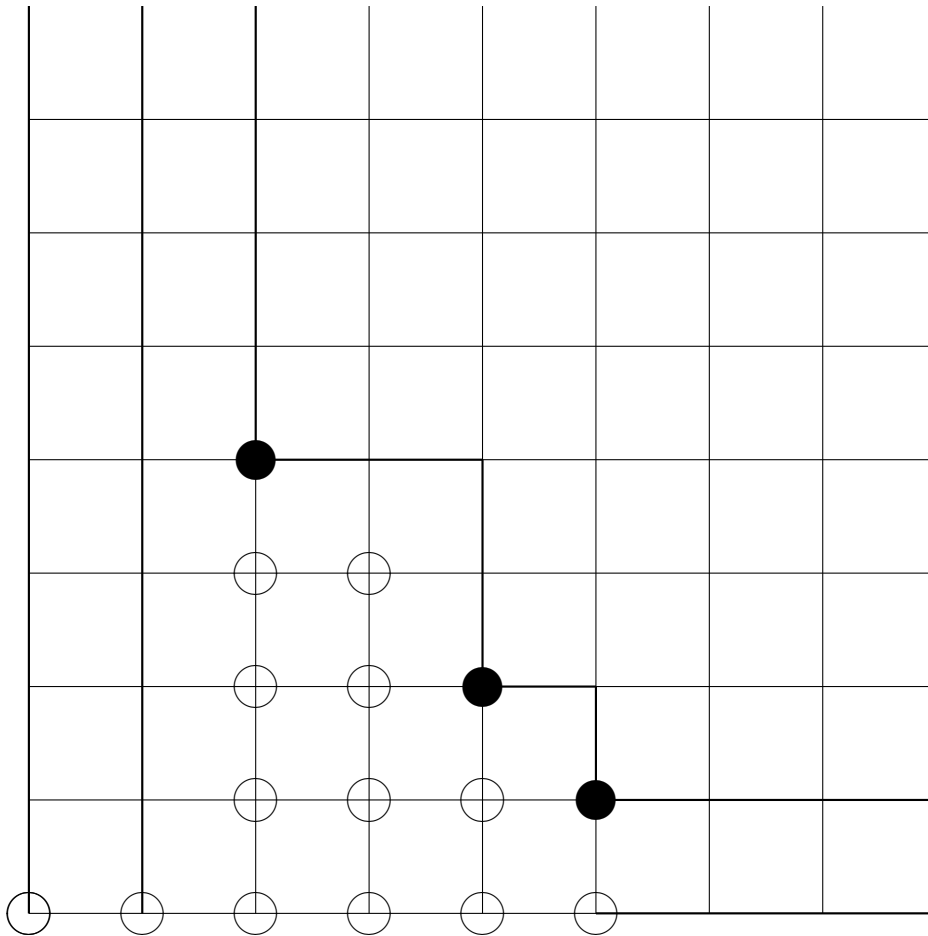
$$\Delta = \uplus_{i \in I} \{i\} \times \Delta_i \subset \Gamma$$

$$\forall \alpha \in \Delta : S(\alpha) \subset \{1, \dots, r\}$$

$$\mathbb{N}^{S(\alpha)} \subset \mathbb{N}^r \text{ (extension by zero).}$$

*Typical picture*

●  $\in \text{deg}(U)_i$ , ○  $\in \Delta_i$



## HILBERT FUNCTION

$$m \in \mathbb{N}$$

$$\mathcal{D}_{\leq m}^{1 \times I} := \{f \in \mathcal{D}^{1 \times I}; \text{ total degree of } f \leq m\}$$

$$U_{\leq m} := U \cap \mathcal{D}_{\leq m}^{1 \times I}$$

Hilbert function  $HF(m) :=$

$$\dim_{\mathbb{C}} (\mathcal{D}_{\leq m}^{1 \times I} / U_{\leq m})$$

$$\mu \in \mathbb{N}^r : |\mu| := \mu_1 + \cdots + \mu_r$$

$$S \subset \{1, \cdots, r\}$$

$$|S| := \text{number of elements of } S$$

**Theorem 1.2.** *For a graded term order:*

*Krull dimension*

$$\dim(M) = \max_{\alpha \in \Delta} |S(\alpha)|$$

*Hilbert function*  $HF(m) :=$

$$\sum_{\alpha \in \Delta, |\mu(\alpha)| \leq m} \binom{m - |\mu(\alpha)| + |S(\alpha)|}{|S(\alpha)|}$$

*Hilbert polynomial*  $HP(m) =$

$$\sum_{\alpha \in \Delta} \binom{m - |\mu(\alpha)| + |S(\alpha)|}{|S(\alpha)|}$$

*Hilbert series*  $HS := \sum_{m=0}^{\infty} HF(m)t^m =$

$$\sum_{\alpha \in \Delta} t^{|\mu(\alpha)|} (1-t)^{-|S(\alpha)|-1}.$$

*STATE BEHAVIOR*

$\mathcal{G} :=$  unique reduced  
Gröbner basis of  $U \subset \mathcal{D}^{1 \times I}$   
 $\epsilon_\rho := (0, \dots, 0, \overset{\rho}{1}, 0, \dots, 0) \in \mathbb{N}^r$

### Theorem and Definition 1.3.

$\mathcal{B}$  = state behavior or in state form  
with respect to the chosen term order:  $\Leftrightarrow$

$$\forall g \in \mathcal{G} : \text{lt}(g) = s_\rho \delta_i \text{ or } \deg(g) = (i, \epsilon_\rho) \Rightarrow$$

$S(i) := \{\rho; 1 \leq \rho \leq r, (i, \epsilon_\rho) \notin \deg(U)\}$ . Then

$$\Delta = \{(i, 0); i \in I\}, S(i, 0) = S(i).$$

$$\Gamma = \uplus_{i \in I} ((i, 0) + \mathbb{N}^{S(i)})$$

$$M = \oplus_{i \in I} \mathbb{C} [(s_\rho)_{\rho \in S(i)}] \overline{\delta}_i$$

$$\begin{aligned}\mathcal{F} &= \mathbb{C} [[z_1, \dots, z_r]] \\ \mathcal{F}(S(i)) &:= \mathbb{C} [[(z_\rho)_{\rho \in S(i)}]] \\ \text{proj} : \mathcal{F} &\rightarrow \mathcal{F}(S(i)), \quad a \mapsto a((z_\rho)_{\rho \in S(i)}, 0)\end{aligned}$$

*Assumptions for the next theorem:*

(i)  $\mathcal{B}$  in state form.

(ii)  $\mathcal{F}$  =space of power series (*analytic case*).

(ii) For the *continuous* case and *locally* convergent power series: The term order is *graded* (already [C. Riquier 1910]).



**Theorem 1.4.**

$$\mathcal{B} \cong \prod_{i \in I} \mathcal{F}(S(i))$$

$$w \mapsto (w_i((z_\rho)_{\rho \in S(i)}, 0))_{i \in I}$$

$$\begin{cases} |S(i)| = r \\ |S(i)| < r \end{cases} \Leftrightarrow w_i((z_\rho)_{\rho \in S(i)}, 0) := \begin{cases} \text{free component or input} \\ \text{initial condition or local state} \end{cases}$$

$w :=$  unique solution of the  
canonical Cauchy problem

*constructive solution*  
*via Buchberger's algorithm*  
*Oberst 1990, Oberst/Pauer 2001*

History: *Cauchy/Kovalevskaya, C. Riquier*  
*1910 (also so-called orthonormic non-*  
*linear systems)*

Standard example  
*Kalman system*

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$w = \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{F}^{n+m}$$

$x$  = state ,  $u$  = input

$x(0)$  = initial condition or local state

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}u(\tau)d\tau$$

*Caution:* In general, the preceding theorem is *false* in the *non-analytic* case of  $C^\infty$ –functions or distributions without additional *growth* conditions on the functions, compare [Gelfand-Shilov III 1967].

Standard counter-example  
*characteristic heat equation*

$$y = y(t, x) \in C^\infty(\mathbb{R}^2)$$

$$(s_1 - s_2^2) \circ y = 0 \text{ or}$$

$$\partial y / \partial t - \partial^2 y / \partial x^2 = 0$$

*lexicographic* term order  $s_1 > s_2$   
 $\text{lt}(s_1 - s_2^2) = s_1$ , hence *state form*. But

$$\exists \text{ solution } y \text{ with}$$

$$\text{support}(y) = \{(t, x); t \leq 0\} \Rightarrow$$

$$y \neq 0, \quad y(0, x) = 0$$

## *STATE REPRESENTATION*

*arbitrary*

$\mathcal{D}$  – module  $\mathcal{F}$ , term order  $<$  and  
behavior  $\mathcal{B} = \{w \in \mathcal{F}^I; R \circ w = 0\}$

Riquier's decomposition for

$$M := \mathcal{D}^{1 \times I} / \mathcal{D}^{1 \times k} R :$$

$$\Gamma := \uplus_{\alpha \in \Delta} (\alpha + \mathbb{N}^{S(\alpha)})$$

Without loss of generality:

$\mathcal{B}$  has only *essential components*, ie.

$$\forall i \in I : \deg(U)_i \neq \mathbb{N}^r \text{ or } (i, 0) \in \Delta$$

$$M = \bigoplus_{\alpha \in \Delta} \mathbb{C} [(s_\rho)_{\rho \in S(\alpha)}] s^{\mu(\alpha)} \overline{\delta_{i(\alpha)}}$$

$$\forall \alpha \in \Delta \quad \forall \rho \notin S(\alpha) :$$

$$s_\rho s^{\mu(\alpha)} \overline{\delta_{i(\alpha)}} = \sum_{\beta \in \Delta} f_{\alpha, \rho, \beta} s^{\mu(\beta)} \overline{\delta_{i(\beta)}}$$

$$f_{\alpha, \rho, \beta} \in \mathbb{C} [(s_\rho)_{\rho \in S(\beta)}]$$

Computation by Buchberger's algorithm

new behavior in  $\mathcal{F}^\Delta$

$$\mathcal{B}^s := \{x = (x_\alpha)_{\alpha \in \Delta} \in \mathcal{F}^\Delta;$$

$$\forall \alpha \in \Delta \forall \rho \notin S(\alpha) :$$

$$s_\rho \circ x_\alpha = \sum_{\beta \in \Delta} f_{\alpha, \rho, \beta} \circ x_\beta$$

$\mathcal{B}^s$  *not* of the first order in general, ie.

total degree of  $f_{\alpha, \rho, \beta} > 1$  in general



**Theorem 1.5.** 1. *New term order on  $\Delta \times \mathbb{N}^r$  (with MAPLE!):*

$$\begin{aligned}
 & (\beta, \nu) < (\alpha, \mu) : \Leftrightarrow \\
 & \beta + \nu < \alpha + \mu \text{ in } I \times \mathbb{N}^r \text{ or} \\
 & \beta + \nu = \alpha + \mu \text{ and } \mu(\alpha) <_{\text{lex}} \mu(\beta) \text{ in } \mathbb{N}^r.
 \end{aligned}$$

2.  $\mathcal{B}^s$  is in state form with respect to the term order from 1.

3. *Mutually inverse system isomorphisms*

$$\begin{aligned}
 \mathcal{B}^s & \cong \mathcal{B}, \quad x = (x_\alpha)_{\alpha \in \Delta} \leftrightarrow w = (w_i)_{i \in I} \\
 w_i & = x_{(i,0)}, \quad x_\alpha = s^{\mu(\alpha)} \circ w_{i(\alpha)}
 \end{aligned}$$

$w \mapsto x =:$  state map

after [Rapisarda/Willems 1997]

4.

$\mathcal{B}$  controllable  $\Leftrightarrow \mathcal{B}^s$  controllable.

## 2 The Hilbert data, excerpt from [30]

The following material is taken from my paper [30] and contains the proof of the theorem on the Hilbert data on page 13 of the present talk. It is based only on the Riquier decomposition  $\Gamma = \uplus_{\alpha \in \Delta} (\alpha + \mathbb{N}^{S(\alpha)})$  (see page 11) and is quoted from [31, Section 2.2] which, in turn, is a translation of [43, pp.143-168] into modern language. In the present talk I took the base field  $F = \mathbb{C}$  for simplicity.

**Corollary 2.1.** *The preceding considerations and data imply the direct sum decompositions*

$$\begin{aligned} \mathcal{D}^{1 \times I} &= \oplus_{\gamma \in \Gamma} F s^{\mu(\gamma)} \delta_{i(\gamma)} \oplus U = \\ &\oplus_{\alpha \in \Delta, \mu \in \mathbb{N}^{S(\alpha)}} F s^{\mu(\alpha) + \mu} \delta_{i(\alpha)} \oplus U = \oplus_{\alpha \in \Delta} F[s^{(\alpha)}] s^{\mu(\alpha)} \delta_{i(\alpha)} \oplus U \text{ and} \\ M &= \mathcal{D}^{1 \times I} / U = \oplus_{\alpha \in \Delta, \mu \in \mathbb{N}^{S(\alpha)}} F s^{\mu(\alpha) + \mu} \overline{\delta_{i(\alpha)}} = \oplus_{\alpha \in \Delta} F[s^{(\alpha)}] s^{\mu(\alpha)} \overline{\delta_{i(\alpha)}}. \end{aligned}$$

In particular, each  $f \in \mathcal{D}^{1 \times I}$  admits a unique representation

$$\begin{aligned} f &= f_{\text{nf}} + f_U \text{ with the normal form} \\ f_{\text{nf}} &= \sum_{\alpha \in \Delta} f_{\alpha} s^{\mu(\alpha)} \delta_{i(\alpha)} = \sum_{\alpha \in \Delta, \mu \in \mathbb{N}^{S(\alpha)}} f_{\alpha, \mu} s^{\mu(\alpha) + \mu} \delta_{i(\alpha)} \text{ where} \\ &f_{\alpha} \in F[s^{(\alpha)}] \text{ and } f_{\alpha, \mu} \in F. \end{aligned}$$

The normal form of  $f \in \mathcal{D}^{1 \times I}$  depends on  $U$  and on the chosen term order on  $I \times \mathbb{N}^r$ .

The construction of a basis of  $M$  according to corollary 2.1 is due F.S. Macaulay [51, Th. 1.1.1], [22, Th. 1.5.7]. The direct sum decomposition into  $F[s^{(\alpha)}]$ -modules is called a *Stanley decomposition* [51, Def. 1.4.1]. Its construction is a generalization or variant of [47, Th. 5.13], [51, Prop. 1.4.3] and [2]. The decomposition of  $M$  also permits to compute a suitable *Hilbert function, polynomial and series* in the following fashion. For  $\mu \in \mathbb{N}^r$  let  $|\mu| := \mu_1 + \dots + \mu_r$  and for each  $m \in \mathbb{N}$  define

$$\begin{aligned} \mathcal{D}_m^{1 \times I} &:= \oplus_{(i, \mu)} \{F s^{\mu} \delta_i; |\mu| = m\} \subseteq \mathcal{D}_{\leq m}^{1 \times I} := \oplus_{(i, \mu)} \{F s^{\mu} \delta_i; |\mu| \leq m\} = \oplus_{l=0}^m \mathcal{D}_l^{1 \times I} \\ U_{\leq m} &:= U \cap \mathcal{D}_{\leq m}^{1 \times I}, \widehat{M}_m := \mathcal{D}_{\leq m}^{1 \times I} / U_{\leq m} \ni [f]_m := f + U_{\leq m}, f \in \mathcal{D}_{\leq m}^{1 \times I} \\ \widehat{M} &:= \oplus_{m=0}^{\infty} \widehat{M}_m. \end{aligned}$$

The space  $\mathcal{D}_{\leq m}^{1 \times I}$  is that of polynomial vectors of total degree at most  $m$ . Let  $s_0$  be an additional indeterminate. The module  $\widehat{M}$  is a graded  $F[s_0, s] = F[s_0, s_1, \dots, s_r]$ -module with the scalar multiplication

$$s_{\rho} [f]_m := \begin{cases} [s_{\rho} f]_{m+1} & \text{if } 1 \leq \rho \leq r \\ [f]_{m+1} & \text{if } \rho = 0 \end{cases}, f \in \mathcal{D}_{\leq m}^{1 \times I}.$$

For  $|\mu| \leq m$  we obtain  $[s^{\mu} \delta_i]_m = s_0^{m-|\mu|} s^{\mu} [\delta_i]_0$ . Hence  $\widehat{M}$  is a finitely generated  $F[s_0, s]$ -module and therefore its *Hilbert function*

$$HF_{\widehat{M}} : \mathbb{N} \rightarrow \mathbb{N}, m \mapsto \dim_F(\widehat{M}_m) = \dim_F(\mathcal{D}_{\leq m}^{1 \times I} / U_{\leq m}),$$

is a polynomial in  $m$  for large  $m$ , ie. there is the unique *Hilbert polynomial*  $HP_{\widehat{M}}$  of  $\widehat{M}$  such that  $HF_{\widehat{M}}(m) = HP_{\widehat{M}}(m)$  for almost all  $m \in \mathbb{N}$ . Let  $d(M)$  denote its degree. The module  $\widehat{M}$  coincides with the  $s_0$ -extension of  $M$  in the sense of [54, p.4-5, Th. 4.2], up to isomorphism, and the numbers  $HF_{\widehat{M}}(m)$  are called the *complexity indices* of the behavior  $\mathcal{B} = U^\perp$  in [54, Def. 4.1, Th. 4.2]. The map  $M \mapsto \widehat{M}$  corresponds to projectivization in geometry.

**Result 2.2.** *The number  $d(M)$  is the Krull dimension  $\dim(M)$  of  $M$ .*

It is surprising that this basic result on the Krull dimension of a polynomial module is not *explicitly* contained in any of the standard references on Commutative Algebra. It is, however, a consequence of [54, results 7.4-7.6] and the literature quoted there. The graded  $F[s_0, s]$ -module  $\widehat{M}$  gives also rise to its *Hilbert power series*  $HS_{\widehat{M}} := \sum_{m=0}^{\infty} HF_{\widehat{M}}(m)t^m \in \mathbb{Z}[[t]] \cap \mathbb{Q}(t)$ .

**Theorem 2.3.** *Data as before.*

1. *The module  $M$  is a torsion module if and only if*

$$\max_{\alpha \in \Delta} |S(\alpha)| < r = \dim(F[s_1, \dots, s_r]).$$

2. *If the term order is graded then*

$$\begin{aligned} \dim(M) &= \max_{\alpha \in \Delta} |S(\alpha)| \\ HF_{\widehat{M}}(m) &= \sum_{\alpha} \left\{ \binom{m - |\mu(\alpha)| + |S(\alpha)|}{|S(\alpha)|}; \alpha = (i(\alpha), \mu(\alpha)) \in \Delta, |\mu(\alpha)| \leq m \right\} \\ HP_{\widehat{M}}(m) &= \sum_{\alpha \in \Delta} \binom{m - |\mu(\alpha)| + |S(\alpha)|}{|S(\alpha)|} \\ HS_{\widehat{M}} &= \sum_{\alpha \in \Delta} t^{|\mu(\alpha)|} (1-t)^{-|S(\alpha)|-1}. \end{aligned}$$

Here  $|S|$  is the number of elements of the finite set  $S$ . The term order is graded if  $|\mu| < |\nu|$  implies  $(i, \mu) < (j, \nu)$  for all  $i, j$ .

**Remark 2.4.** J. Wood, P.Rocha et al. [54] observed that the Hilbert function, polynomial and series of the module  $\widehat{M}$  have *system theoretic* significance for the behavior  $\mathcal{B} = U^\perp$ . Such a connection was, however, already established by F.S. Macaulay [24]. Indeed, for an *ideal*  $U \subset F[s]$  and the injective cogenerator  $\mathcal{F} := F[[z]]$  the behavior  $\mathcal{B} := U^\perp$  is exactly the *inverse system* of  $U$  and

$$HF_{\widehat{M}}(m) = \dim_F(\mathcal{D}_{\leq m}^{1 \times I} / U_{\leq m}) = \dim_F(\text{Hom}_F(\mathcal{D}_{\leq m}^{1 \times I} / U_{\leq m}, F))$$

is the number of *independent modular equations* of  $U$  for degree  $m$  according to [24, Ch.IV, pp.64-65]. The preceding computation of the Hilbert series is a variant or generalization of such calculations in the literature, see for instance [24, Th. 58, p.65], [51, prop.1.4.2], [1] and and the Groebner package of MAPLE.

*Proof.* 1. [54, Th. 7.6].

2. If  $f \in \mathcal{D}^{1 \times I}$  is a non-zero vector of degree  $\deg(f) = \alpha = (i, \mu)$  and  $|\mu| \leq m$  then  $f \in \mathcal{D}_{\leq m}^{1 \times I}$ . This follows from the assumed gradedness of the term order. Consider the decomposition

$$\begin{aligned} \mathcal{D}^{1 \times I} &= V \oplus U, \quad V := \bigoplus_{\gamma=(i,\mu) \in \Gamma} F s^\mu \delta_i, \quad \text{in particular} \\ s^\nu \delta_j &= f + g, \quad f := \text{nf}(s^\nu \delta_j) \text{ where } s^\nu \delta_j \in \mathcal{D}_{\leq m}^{1 \times I}, \quad |\nu| \leq m, \end{aligned} \quad (1)$$

is any basis vector of  $\mathcal{D}_{\leq m}^{1 \times I}$ . Since  $\deg(f) \in \Gamma \uplus \{-\infty\}$  and  $\deg(g) \in \deg(U) \uplus \{-\infty\}$  these degrees are distinct and hence

$$\begin{aligned} \alpha &:= (j, \nu) = \deg(s^\nu \delta_j) = \max(\deg(f), \deg(g)), \\ \text{say } \alpha &= \deg(f) > \deg(g) \text{ and hence } f, g \in \mathcal{D}_{\leq m}^{1 \times I} \\ \mathcal{D}_{\leq m}^{1 \times I} &= V_{\leq m} \oplus U_{\leq m} \text{ with } V_{\leq m} := V \cap \mathcal{D}_{\leq m}^{1 \times I} = \bigoplus_{\gamma=(i,\mu) \in \Gamma, |\mu| \leq m} F s^\mu \delta_i \\ V_{\leq m} &\cong \mathcal{D}_{\leq m}^{1 \times I} / U_{\leq m}, \quad HF_{\widehat{M}}(m) = \dim_F(V_{\leq m}). \end{aligned}$$

From

$$\begin{aligned} V &= \bigoplus_{(i,\mu) \in \Gamma} F s^\mu \delta_i = \bigoplus_{\alpha=(i(\alpha), \mu(\alpha)) \in \Delta} F [s^{(\alpha)}] s^{\mu(\alpha)} \delta_{i(\alpha)} \text{ we infer} \\ V_{\leq m} &= \bigoplus_{\alpha \in \Delta, |\mu(\alpha)| \leq m, \bigoplus_{\mu \in \mathbb{N}^{S(\alpha)}} \{F s^{\mu(\alpha) + \mu} \delta_{i(\alpha)}; |\mu| \leq m - |\mu(\alpha)|\}} \text{ and} \\ HF_{\widehat{M}}(m) &= \sum_{\alpha \in \Delta, |\mu(\alpha)| \leq m} |\{\mu \in \mathbb{N}^{S(\alpha)}; |\mu| \leq m - |\mu(\alpha)|\}|. \end{aligned}$$

For  $s, k \in \mathbb{N}$  the standard formulas

$$|\{\mu \in \mathbb{N}^s; |\mu| \leq k\}| = \binom{k+s}{s} \text{ and } t^k (1-t)^{-s-1} = \sum_{m \geq k} \binom{m-k+s}{s} t^m$$

hold. For  $|\mu(\alpha)| \leq m$  these imply

$$\begin{aligned} |\{\mu \in \mathbb{N}^{S(\alpha)}; |\mu| \leq m - |\mu(\alpha)|\}| &= \binom{m - |\mu(\alpha)| + |S(\alpha)|}{|S(\alpha)|} \text{ and} \\ \sum_{|\mu(\alpha)| \leq m} |\{\mu \in \mathbb{N}^{S(\alpha)}; |\mu| \leq m - |\mu(\alpha)|\}| t^m &= t^{|\mu(\alpha)|} (1-t)^{-|S(\alpha)|-1} \end{aligned}$$

and thus the expressions for the Hilbert function, polynomial and series. Since  $\binom{k+s}{s}$  is a polynomial of degree  $s$  in  $k$  the Hilbert polynomial has the degree  $\dim(M) = d(M) = \max_{\alpha \in \Delta} |S(\alpha)|$ .  $\square$

### 3 The Cauchy problem, excerpt from [30]

The following material is taken from the paper [30] where  $F$  denotes an arbitrary field. It is assumed that the system is in state form as in theorem 1.4 of this talk where the set  $\Delta$  of this section is denoted by  $I$ . The subset  $\Delta^0$  consists of those indices  $\alpha$  for which the support set  $S(\alpha)$  is not all of  $\{1, \dots, r\}$  and  $\Delta^{\text{free}}$  is its complement.

In the following theorem we use the space

$$F^{\mathbb{N}^r} = F[[z]] = F[[z_1, \dots, z_r]] \ni y = (y_\mu)_{\mu \in \mathbb{N}^r} = \sum_{\mu \in \mathbb{N}^r} y_\mu z^\mu.$$

of multi-sequences or formal power series. For  $\alpha \in \Delta$  we denote  $z^{(\alpha)} = (z_\rho)_{\rho \in S(\alpha)}$  and identify  $F[[z^{(\alpha)}]]$  as  $F[s^{(\alpha)}]$ -submodule of  $F[[z]]$ . For  $y \in F[[z]]$  and  $\alpha \in \Delta$  we define  $y(z^{(\alpha)}, 0) := \sum_{\mu \in \mathbb{N}^{S(\alpha)}} y_\mu z^\mu$ . If  $F = \mathbb{R}$  or  $F = \mathbb{C}$  and if  $y$  is a convergent power series and therefore an analytic function on  $\mathbb{R}^r$  resp.  $\mathbb{C}^r$  in the neighborhood of zero the power series  $y(z^{(\alpha)}, 0)$  is exactly the function on  $\mathbb{R}^{S(\alpha)}$  resp.  $\mathbb{C}^{S(\alpha)}$  where all variables  $z_\rho$ ,  $\rho \notin S(\alpha)$ , are set to zero.

**Assumption 3.1.** We assume that the function  $\mathcal{D}$ -module  $\mathcal{F}$  is a subspace of  $F[[z]]$  on which  $\mathcal{D} = F[s]$  acts either by left shifts, ie.  $(s_\rho \circ y)_\mu = y_{\mu + \epsilon_\rho}$ , or by partial differentiation, ie.  $s_\rho \circ y = \partial y / \partial z_\rho$ , and consider the following five cases which were treated in [31].

1.  $F$  is a field of arbitrary characteristic and  $\mathcal{F} := F[[z]]$  with the action by left shifts [31, th.15]
2.  $F$  is a field of characteristic zero and  $\mathcal{F} = F[[z]]$  with the action by partial differentiation [31, th.16]
3.  $F$  is the field of real or complex numbers and  $\mathcal{F} = F\langle z \rangle$  is the algebra of (locally) convergent power series with the action by left shifts [31, th.24]. A power series  $y = \sum_{\mu \in \mathbb{N}^r} y_\mu z^\mu$  is convergent if and only if its coefficients satisfy a growth condition

$$|y_\mu| \leq C a_1^{\mu_1} * \cdots * a_r^{\mu_r} \text{ for all } \mu \in \mathbb{N}^r$$

where  $C$  and the  $a_\rho$ ,  $\rho = 1, \dots, r$ , are positive real numbers.

4.  $F$  is the field of real or complex numbers and  $\mathcal{F} = F\langle z \rangle$  is the algebra of convergent power series with the action by partial differentiation [31, th.29] and the term order is graded (compare theorem 2.3). The solution of the Cauchy problem in this case is the main result in C. Riquier's book [43], even for certain, so-called orthonomic non-linear systems. The Cauchy-Kovalevskaya theorem was a predecessor of Riquier's results.
5.  $F := \mathbb{C}$  is the field of complex numbers and  $\mathcal{F} := O(\mathbb{C}^r, \exp) \in \mathbb{C}\langle z \rangle$  is the algebra of entire holomorphic functions of exponential type with the action by partial differentiation [31, th.26]. An entire holomorphic function  $y$  is called of exponential type if it satisfies a growth condition

$$|y(z)| \leq C \exp(a_1 |z_1| + \cdots + a_r |z_r|) \text{ for all } z \in \mathbb{C}^r$$

where  $C$  and the  $a_\rho$ ,  $\rho = 1, \dots, r$ , are positive real numbers.

If  $S$  is a subset of  $[r]$  and if  $\mathcal{F}$  is any of the function spaces of the preceding assumption let  $\mathcal{F}(S) = F[[z_\rho]_{\rho \in S}]$  denote the corresponding function space in the indeterminates or variables  $z_\rho$ ,  $\rho \in S$ .

**Result 3.2.** (The Cauchy problem [31]) *Assume that  $\mathcal{F}$  is one of the function spaces from assumption 3.1. Then for arbitrary functions  $v_\alpha \in \mathcal{F}(S(\alpha))$ ,  $\alpha \in \Delta$ , there is a unique trajectory  $w \in \mathcal{B}$  such that  $w_\alpha(z^{(\alpha)}, 0) = v_\alpha$  for all  $\alpha \in \Delta$ . In other terms, the map*

$$\mathcal{B} \rightarrow \prod_{\alpha \in \Delta^0} \mathcal{F}(S(\alpha)) \times \mathcal{F}^{\Delta^{\text{free}}}, w = \begin{pmatrix} x \\ u \end{pmatrix} \mapsto \begin{pmatrix} x_\alpha(z^{(\alpha)}, 0)_{\alpha \in \Delta^0} \\ u \end{pmatrix}, \quad (2)$$

is an isomorphism.

The component  $v_\alpha$  is called an input resp. an initial condition if  $\alpha \in \Delta$  belongs to  $\Delta^{\text{free}}$  resp. to  $\Delta^0$ . The vectors  $x = (w_\alpha)_{\alpha \in \Delta^0}$  resp.  $(x_\alpha(z^{(\alpha)}, 0))_{\alpha \in \Delta^0}$  are called the global resp. the local state of the system  $\mathcal{B}$ . Hence the input and the initial condition or local state give rise to a unique trajectory of the system.

*Proof.* The proof of this theorem is a special case of the theorems in [31], the exact references being given in assumption 3.1.  $\square$

**Remarks 3.3.** 1. The preceding state definition generalizes the state property of  $x$  in the Kalman system  $\dot{x} = Ax + Bu$  as explained in the introduction.

2. The following remarks are taken from the literature and concern the Cauchy problem for linear systems of partial differential equations with constant coefficients for function spaces like  $C^\infty$ -functions or distributions which do not consist of formal or convergent power series as assumed in result 3.2. The general behaviors of the present paper have not been treated in this context. The preceding theorem does not hold in general, but requires additional conditions on the system or the initial data. However, the theorem is applicable to entire functions of exponential type and to convergent power series with a graded term order and therefore suggests the formulation of the Cauchy problem also for more general function spaces. One has to distinguish between *uniqueness* and *correctness* results [9, ch. II, ch. III]. Correctness or well-posedness of the Cauchy problem signifies that it is uniquely solvable and that the solution depends continuously on the initial data. In general, uniqueness results require initial data with bounded growth at  $\infty$  [9, Th.1, p.42], [32, ch.VI, §6.2]. If the function space is an injective cogenerator it suffices to solve the Cauchy problem for autonomous systems as for instance in [9].
3. V. Palamodov has proven a quite general uniqueness result for weakly hypoelliptic autonomous systems  $R \circ w = 0$  [32, ch.VI, §6.2]. In this case the sets  $S(\alpha)$  do not depend on  $\alpha \in \Delta^0$ .
4. Most systems treated in the literature [9, ch.II, §3, (1)], [49, (15.1)] are of or can be easily reduced to the form

$$\begin{aligned} (\text{id}_n \partial / \partial z_1 + P_{II}(i\partial / \partial z_{II}))x &= 0, \quad P_{II} \in \mathbb{C}[s_{II}]^{n \times n}, \quad x \in \mathcal{F}^n, \\ s_{II} &= (s_2, \dots, s_r), \quad z_{II} = (z_2, \dots, z_r). \end{aligned} \quad (3)$$

The distinguished variable  $z_1$  is usually interpreted as time. The constant  $i := \sqrt{-1}$  in front of  $\partial / \partial z_{II}$  is just a convention in connection with the Fourier transform in the  $z_{II}$ -space. For the lexicographic order  $s_1 > s_2 > \dots > s_r$  the matrix  $P := s_1 \text{id}_n + P_{II}(is_{II})$  is a Gröbner matrix since the S-polynomials of the rows

$$P_{j-} = s_1 \delta_j + P_{II,j-}(is_{II}) \in \mathbb{C}[s]^{1 \times n}$$

are zero by definition. In particular, the autonomous system  $P \circ x = 0$  and its inhomogeneous counter-part  $P \circ x = u$  are in state space form and

the isomorphism

$$\left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{F}^{n+n}; P \circ x = u \right\} \cong \mathcal{F}(z_{II})^n \times \mathcal{F}^n, \quad \begin{pmatrix} x \\ u \end{pmatrix} \mapsto \begin{pmatrix} x(0, z_{II}) \\ u \end{pmatrix},$$

holds for  $\mathcal{F} = O(\mathbb{C}^r, \text{exp})$ . In general, the isomorphism does not hold for *locally* convergent power series since the pure lexicographic term order is not graded. A counter-example [5, ch.1, §2.2] is given by the heat equation

$$y_{z_1} - y_{z_2 z_2} = 0, \quad y(0, z_2) = (1 - z_2)^{-1}$$

whose unique formal power series solution  $y$  satisfies  $y(z_1, 0) = \sum_{k=0}^{\infty} (2k)! z_1^k$  and is not convergent.

5. For the heat equation  $(s_1 - s_2^2) \circ y = y_{z_1} - y_{z_2 z_2} = 0$ ,  $\mathcal{F} := C^\infty(\mathbb{R}^2)$  and the lexicographic term order  $s_1 > s_2$  the map (2)  $y \mapsto y(0, z_2)$  is not injective. Indeed, there is a solution  $y$  whose support is exactly the half-space  $\{z \in \mathbb{R}^2; z_1 = z \bullet \delta_1 \leq 0\}$  [16, Th. 8.6.7]. The reason is that  $\{z_1 = z \bullet \delta_1 = 0\}$  is a *characteristic* hyper-plane (line). For  $\mathcal{F} := C^\infty(\mathbb{R}^2)$  and indeed any  $\mathbb{C}[s]$ -module the system isomorphism

$$\begin{aligned} \{y \in \mathcal{F}; (s_1 - s_2^2) \circ y = 0\} &\cong \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{F}^2; \begin{pmatrix} s_2 & -1 \\ s_1 & -s_2 \end{pmatrix} \circ x = 0 \right\} \\ y = x_1 &\leftrightarrow x = \begin{pmatrix} y \\ s_2 \circ y \end{pmatrix} \end{aligned}$$

holds. The preceding non-uniqueness also applies to this first order autonomous system with a square-matrix. These examples underline the necessity of growth conditions on the function spaces for uniqueness results as mentioned in item 2.

6. As usual, the *order* of a polynomial matrix and of the behavior which it defines is the maximum of the total degrees of its entries.
7. We specialize the systems of (3) to first order ones, ie. we consider systems [49, ch.15, (15.1)]

$$P \circ x = u \text{ with } P := s_1 \text{id}_n - \sum_{\rho=2}^r s_\rho A_\rho - A_0, \quad A_\sigma \in \mathbb{C}^{n \times n}, \quad x, u \in \mathcal{F}^n$$

$$\mathcal{B} := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{F}^{n+n}; P \circ x = u \right\}, \quad \mathcal{B}^0 := \{x \in \mathcal{F}^n; P \circ x = 0\}. \quad (4)$$

The determinant of  $P$  is the characteristic polynomial of  $\sum_{\rho=2}^r s_\rho A_\rho + A_0$  and of the form

$$\chi = \chi_n + \cdots + \chi_0, \quad \chi_n = s_1^n + \cdots$$

where  $\chi_k$  is the homogeneous component of degree  $k$  of  $\chi$ . Any  $x \in \mathcal{B}^0$  also satisfies  $\chi \circ x = 0$ . The vector  $\delta_1 := (1, 0, \dots, 0)$  is non-characteristic for  $\chi$ , ie.  $\chi_n(\delta_1) = 1^n \neq 0$ . Holmgren's theorem and its consequence [16, Cor. 8.6.9] imply that any distributional solution of  $\chi \circ y = 0$  whose support lies in the right half-space  $\{z \in \mathbb{R}^r; z_1 = z \bullet \delta_1 \geq 0\}$  is indeed

zero. If  $x$  is a  $C^1$ -solution of  $P \circ x = 0$  with initial condition  $x(0, z_{II}) = 0$  the continuous function  $x_+$  defined by

$$x_+(z) := \begin{cases} x(z) & \text{if } z_1 \geq 0 \\ 0 & \text{if } z_1 \leq 0 \end{cases} \text{ satisfies } P \circ x_+ = 0 \text{ in } \mathcal{D}'(\mathbb{R}^r)$$

and is zero by the preceding remarks. This argument uses the first order property of  $P$ ; I learned it from my colleague Peter Wagner. Hence the map

$$\{x \in C^1(\mathbb{R}^r)^n; P \circ x = 0\} \rightarrow C^1(\mathbb{R}^{r-1})^n, x \mapsto x(0, z_{II}),$$

is injective [5, Ch.1, pp.34-36]. This argument can be extended to distributional solutions in  $C^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{r-1})) \subset \mathcal{D}'(\mathbb{R}^r)$ .

Correctness of the Cauchy problem in the present context characterizes *hyperbolicity*. Indeed, according to [49, ch.15, Def. 15.1, Th. 15.1, Th. 15.2] the following assertions are equivalent:

- (a) For  $\mathcal{F} := C^\infty(\mathbb{R}^r)$  the map (2)

$$\mathcal{B} \rightarrow C^\infty(\mathbb{R}^{r-1})^n \times \mathcal{F}^n, w = \begin{pmatrix} x \\ u \end{pmatrix} \mapsto \begin{pmatrix} x(0, z_{II}) \\ u \end{pmatrix}$$

is an isomorphism.

- (b) The system (4) is *hyperbolic*. This signifies that there is constant  $C > 0$  such that for every matrix  $A := A_2 \xi_2 + \cdots + A_r \xi_r - iA_0$ ,  $\xi \in \mathbb{R}^r$ ,  $i := \sqrt{-1}$ , and every eigenvalue  $\lambda$  of  $A$  the inequality  $|\operatorname{im}(\lambda)| \leq C$  holds.

Hyperbolicity for single equations is thoroughly discussed in [17, ch.XII]. Hyperbolic systems of the form (3) are defined and investigated in [9, Ch. III, §3]. To my knowledge hyperbolicity has not been defined for the general behaviors of the present paper.

**Remark 3.4.** (*First order representations*) In [55, section 6.1] E. Zerz proves that any polynomial matrix  $R \in F[s_1, \dots, s_r]^{k \times l}$  admits an LFT- representation (*linear fractional transformation*)

$$\begin{aligned} R &= D + C\Delta(s)(\operatorname{id}_n - A\Delta(s))^{-1}B \text{ where} \\ n &:= n_1 + \cdots + n_r, \Delta(s) = \operatorname{diag}(s_1 \operatorname{id}_{n_1}, \dots, s_r \operatorname{id}_{n_r}) \\ A &\in F^{n \times n}, B \in F^{n \times l}, C \in F^{k \times n}, D \in F^{k \times l} \\ \det(\operatorname{id}_n - A\Delta(s)) &= 1. \end{aligned} \tag{5}$$

The latter condition is automatically satisfied if  $A$  is a strictly lower triangular matrix, and hence representations (5) can be randomly generated for experimental purposes. The polynomial matrix  $\operatorname{id}_n - A\Delta(s)$  is invertible and gives



rise to the mutually inverse system isomorphisms

$$\begin{aligned} \mathcal{B}_{\text{lin}} &:= \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathcal{F}^{n+l}; R_{\text{lin}} \circ \begin{pmatrix} x \\ w \end{pmatrix} = 0 \right\} \cong \{w \in \mathcal{F}^l; R \circ w = 0\} \\ &\begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} (\text{id}_n - A\Delta(s))^{-1} \circ Bw \\ w \end{pmatrix} \leftrightarrow w \\ \text{where } R_{\text{lin}} &:= \begin{pmatrix} \text{id}_n - A\Delta(s) & -B \\ C\Delta(s) & D \end{pmatrix} = (MR_0) \text{ with} \\ M &= \begin{pmatrix} \text{id}_n - A\Delta(s) \\ C\Delta(s) \end{pmatrix}, R_0 := \begin{pmatrix} -B \\ D \end{pmatrix}, \\ \text{hence } \mathcal{B}_{\text{lin}} &= \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathcal{F}^{n+l}; M \circ x + R_0 w = 0 \right\}. \end{aligned} \quad (6)$$

The system  $\mathcal{B}_{\text{lin}}$  is a first order system and a state system with state  $x$  and manifest variables  $w$  in the sense of J.C. Willems [52, problem 1.12,p.56], but not in the sense of the present paper. No analogue of result 3.2 holds in this situation. For the solution of the Cauchy problem of  $R \circ w = 0$  the isomorphism  $\mathcal{B}_{\text{lin}} \cong \mathcal{B}$  is of little use only since, in general, the *canonical* state representation of  $\mathcal{B}_{\text{lin}}$  is not of the first order although the matrix  $R_{\text{lin}}$  is.

## 4 Discussion of Professor Pommaret's remarks and historical comments

1. In the very short discussion after my talk Professor J.-F. Pommaret from the École Normale des Ponts et Chaussées, Champs sur Marne, stated, first in public and then privately, that my talk and my paper [30] were *prehistoric mathematics*, were *superseded* by Janet, Spencer or Pommaret and altogether *very bad*. Of course, I am of a different opinion, and I object against the use of *such vague arguments* in a negative judgement of a mathematical paper. Defense against such *general* derogatory statements is difficult. Since I was not the only, but certainly the oldest participant of the conferences D2 and D3 who suffered from this type of attack I nevertheless try to refute Pommaret's assertions on the basis of some details which he was so friendly to provide me orally.

2. I never heard of prehistoric mathematics before since I believe that people had other problems in these times. But from Pommaret's utterances in this direction I conclude that non-prehistoric mathematics is that which was either produced or quoted by him. I am grateful that some of my papers belong to this group.

I object against Pommaret's repeated statements, in particular at the D2 or D3 conferences and in context with my talk and submitted paper, that everything essential has already been done by Janet, Spencer or Pommaret, and other people have only copied their good ideas. I will address specific points below.

Around ten years ago in Innsbruck Pommaret gave us the valuable hint at Riquier's fundamental book [43] from 1910 and recommended it warmly so that we invited his then PhD-student Quadrat to give us a mini-course with an introduction to it which took place in 1998. We are grateful to both for this enrichment of our knowledge. It is incomprehensible why these outstanding mathematics should suddenly be prehistoric.

Below I will expose my views of the history of the subject which often deviate from those of Pommaret, and these may, of course, also be erroneous since I am not a historian of mathematics. I apologize in advance for any mistakes. The last items concern the particular results of my paper [30].

3. The following remark is taken from the introduction of Pommaret's first book [35] from 1978 which was dedicated to Professor Janet on the occasion of his 88th birthday. We quote some lines from pages 2 and 3 with slight changes of the quotations and omissions to adapt them to the present situation.

**Remark 4.1.** (Begin of the quotation of [35, p.2,3])

1. p.2, lines 6<sup>-</sup>ff: *The methods of Riquier [43], used again by Janet [18], a student of Hilbert, in 1920 and modernized by Thomas and Ritt are quite different and give an operational process that can be accomplished in a finite number of steps in order to study any linear or nonlinear system of PDE.*
2. p.3, lines 2<sup>+</sup>ff.: *[...] by differentiating as many times as necessary a given finite number of PDE. This gives a way of computing certain derivatives called principal as functions of the variables and of the other variables called parametric. The method can be used both in the differentiable and analytic cases but we have to suppose that certain regularity conditions are fulfilled.*  
*The problem is thus to know what are the principal and the parametric terms in the Taylor formal expansion of any solution in a neighbourhood of a given point.*
3. p.3, lines 10<sup>-</sup>ff: *Janet used a total ordering of the derivatives by means of sets of integers, called cotes. Then, to any principal derivative, he associated in a non-intrinsic way depending on the coordinate system, certain of the variables called multiplicative variables (the others being called non-multiplicative).*
4. p.3, lines 17<sup>+</sup>ff.: *If this was done with respect to the non-multiplicative variables, the elimination of the principal derivative obtained twice, was used in order to start the elimination process and look for the compatibility of the system. If two such computations done by any method gave the same expression for any given principal derivative, the system was said to be passive.*
5. p.3, lines 20<sup>-</sup>ff.: *Moreover, in the case of inhomogeneous linear systems with second members, Janet employed a very important construction [18]. He proved in fact that, if the system was passive with zero second member, then it was also passive with nonzero second member, if and only if the second member itself satisfied certain differential conditions called integrability conditions that could be considered as a new passive linear system of PDE, and so on.*

**Remark 4.2.** 1. Item 1. of the preceding remark shows that at the time of writing [35] Pommaret still considered Riquier as the principal source of the ideas. Pommaret's attribution to Janet of the important terms *principal*, *parametric*, *total ordering*, *passive*, *integrability condition* is erroneous.

All these terms were introduced and essentially used already by Riquier. In a survey talk in Amsterdam in 2000 [29] I discussed Riquier's terminology in section 9. I use the notations of the present talk to explain the connection. In this context the introduction of [43] is also enlightening. However, anybody who reads [43] will notice that a translation of this fundamental work into modern language is a substantial task.

2. *total ordering, cotes*: [43, p. XI, §102/103, p.195/196, §104, p.201], [31, Assumption 28 on p.287], [30, Remark 3.2 on p.13]. In the Gröbner terminology Riquier uses the most general graded term order.
3. *passive*: [43, §101, p.193]. In the Gröbner basis terminology and with the notations of the present talk (pp.7-10) the system is called passive if the rows of  $R_{i-}$ ,  $i = 1, \dots, k$  are a reduced Gröbner basis of  $U$ . I also interpret Pommaret's Remark 4.1, (4), in this fashion.
4. *principal, parametric*: [43, p.169], [29, section 9]. For a passive system and in the Gröbner basis terminology and with the notations of this talk (pp.7-10) the principal resp. parametric derivatives correspond to the lattice points in  $\deg(U)$  resp. in  $\Gamma$ . Riquier attributes the terminology to Meray (1880). The term *parametric* is motivated, for instance, by theorem 1.4 or its more general form in [31] since the Taylor coefficients (=free parameters) for the unique solution  $w$  of the Cauchy problem can be freely prescribed for the lattice points in  $\Gamma$ .
5. *dérivée cardinale*: [43, §92, p.174]. In the Gröbner basis terminology the cardinal derivatives are the  $S$ -polynomials.
6. *compatibility, integrability condition*: [43, Introduction, p.XIX, line 11+, p.195].

4. In [35] Pommaret referred to [43], [18], the thesis of Quillen (1964) and the long paper by Spencer (1965) on overdetermined linear systems of partial differential equations. Thus the revival of Riquier's and Janet's work is obviously due to Pommaret. Another important contribution to this renaissance was the paper [46] by Schwarz. In the mean-time many colleagues have written papers which translate Riquier's and Janet's work into today's language and elaborate the algorithmic aspects of their work. In particular, this was done by Pommaret and by myself.

What Pommaret does not mention at all are the *preceding fundamental papers* on linear systems of partial differential equations with *constant* coefficients by L. Ehrenpreis [7], B. Malgrange [25] and V. Palamodov [32] which were written in the beginning sixties, the cited *books* followed later. Pommaret always, especially in [41, p.5, line 8<sup>+</sup>], shifts Palamodov's important work into the seventies. Hörmander [15] and Björk [3] exposed essential parts of this work with partially simpler proofs. In my opinion Björk's book is still the best source in this field and much easier to understand than all of Pommaret's books (compare MR 0549189).

These papers laid the foundations of *true algebraic analysis* in the sense that much algebra was used for the solution of analytic problems. Of course, much earlier Riquier and Janet and also Gröbner [13] worked in this area and their work was not mentioned by the just cited authors.

The term *algebraic analysis* was introduced, to my knowledge, by M. Kashiwara et.al. in [21] where they exposed Sato's hyperfunctions (analysis!!) and their application to linear systems of partial differential equations with variable coefficients. That hyperfunctions cannot be avoided in this context was also shown for ordinary differential equations and one-dimensional systems theory in [8]. In contrast, *algebraic analysis* or the *formal theory of partial differential equations* in the sense of Pommaret are purely algebraic and are usually called  $\mathcal{D}$ -module theory [10] in the linear case. His books and papers contain few, if any, results on the existence and uniqueness of solutions of systems of partial differential equations in reasonable function spaces (compare Remark 3.3, item 2.)

5. The most important results on linear systems of partial differential equations with *constant* coefficients for  $C^\infty$ -functions or distributions by Ehrenpreis et.al. are the so-called *integral representation theorem*, ie. the representation of solutions as integrals over polynomial-exponential solutions, and the so-called *fundamental principle* or *injectivity* of these function modules over the ring of differential operators with constant coefficients. The fundamental principle signifies that a non-homogeneous linear system of partial differential equations admits a solution if, in Riquier's, Janet's or Pommaret's language, the necessary *compatibility or integrability conditions* are satisfied (compare Remark 4.1, item (5)). These compatibility conditions can be computed via Janet's algorithm [18]. In 1989 I used Buchberger's algorithm for this purpose [27]. At this time I was in the good company of many mathematicians who had no idea of the work of Riquier and Janet.

The fundamental principle fails completely for variable coefficients and formal power series contrary to Pommaret's assertion, for instance in [41, p.4, lines 4<sup>-</sup> – 1<sup>-</sup>], and therefore such a result was certainly not proven by Janet or applied to generalize theorem 1.4. To see this consider one linear partial differential equation  $P(z, \partial/\partial z)(y(z)) = u(z)$  where  $u, y$  are formal power series and  $P$  is a non-zero operator in the  $r$ -dimensional Weyl algebra. Since the latter is an integral domain there are no compatibility conditions, and therefore the fundamental principle for power series would imply that there is always a solution  $y$  for given  $u$ . The simple ordinary differential equation  $z_1 y'(z_1) = 1$  with the solution  $\log(z_1) \notin \mathbb{C}[[z_1]]$  shows that this is false.

Of course, if the system is analytic and if according to Remark 4.1, item (2), *certain regularity conditions* are satisfied and if, in particular, the singular points are omitted then local solutions exist and have already been constructed by Riquier. For the *formal* power series of [41, p.4, lines 4<sup>-</sup> – 1<sup>-</sup>] such an omission makes no sense because, in general, a *formal* power series has no Taylor development at a point different from zero. Also *universal* differential fields are injective modules over the corresponding rings of differential operators, so that the fundamental principle is valid in them. But solutions in them are uninteresting according to [36, Rem. 3.27, p.228], especially for engineers.

The introduction of Malgrange's new article [26, Introduction, p.1, line 2<sup>+</sup>, p.3, lines 20<sup>+</sup> ff.] also contains an interesting historical survey. Its second line says "...sur l'involutivité *générique* des systèmes différentiels analytiques" or, in other terms, generically, such a system has a solution if the compatibility conditions are satisfied. In contrast to, for instance, theorem 1.4 *generic* signifies that solutions exist in most cases, but not in all, and that, in particular, the behavior of solutions in singular points is not discussed.

The fundamental principle is valid for ordinary linear differential equations and

hyperfunctions, but not for distributions [8].

At the conference D2 several colleagues talked on the solution of one inhomogeneous linear partial differential or difference equation with polynomial or power series coefficients in various function spaces of polynomials, formal or convergent power series, hypergeometric functions etc. or, equivalently, on the divisibility of these function spaces over various rings of differential or difference operators, among these S. Abramov, C. Christopher, F. Castro, S. Gann, H. Hauser, M. Petkovsek, G. Reid, F. Schwarz, N. Takayama, S. Tsarev, J.M. Ucha. Janet, Spencer or Pommaret did definitely not have any algorithm for the solution of their problems.

6. *Pommaret insinuates that theorem 1.2 on the Hilbert function, series and polynomial (=theorem 2.3 above or [30, Th. 3.5]) is contained in his books. This is false.*

The context of the theorem and in particular its relation to other work is described in detail and with various references in section 2 where also its one page proof from [30] is reproduced. Pommaret criticizes in particular that I do not mention the *characters*, ie. the coefficients of his Hilbert polynomial which were actually introduced by Hilbert himself [Math. Annalen 36 (1890)]. I learned them and their history in 1972 from Gröbner's excellent description in [14, pp.159-167, in particular (1.25 a-d) ].

In the first book [35] Pommaret defines the characters  $\bar{\alpha}_q^i$  [35, p.95, line 12<sup>-</sup>, Def. 3.2.5] and refers [35, Rem. 2.3 on p.94] to [18] in this context. The further text uses  $\delta$ -regular coordinate systems [35, Def. 3.2.7 on p.96] and equations of *class i*. The pages pp.98 also describe, for  $\delta$ -regular coordinates, a method of computing the characters by means of other numbers  $\beta_q^i$ , principal and parametric derivatives, multiplicative and non-multiplicative variables etc. The Hilbert polynomial and characters are also treated in in [36, pp.251] and [38, p.277] without algorithms, in [39, Cor. 2.35 on p.330] with essentially the same derivation as the one quoted from [35] and shortly also in [41, section I.3.3 and after Th. 6.14] where again the classes, multiplicative variables etc. are used. Moreover additional *elementary combinatorics* is needed for the computation of the Hilbert polynomial [39, p.330, line 6<sup>-</sup>]. Also the exact assumptions of [39, Cor. III.2.35] are very hard to extract from [39] if at all. Pommaret's descriptions are not algorithms in today's (non-prehistoric) understanding and their derivation can moreover be understood only by reading and understanding the pages [39, pp.298-330]. There he lays the foundations for the theory of *non-linear systems of partial differential equations on differentiable manifolds* in a language which is absolutely unnecessary for the Hilbert function of polynomial modules and in particular for my paper. Although almost thirty years have passed since the publication of [35] and although in the mean-time various people know about this and other books of the author apparently nobody has ever used his presentation for actually computing the Hilbert polynomial. Why? This is all the more surprising since today's young computer algebraists are always keen on learning new algorithms and on implementing them as, in particular, the conferences D2 and D3 have shown.

In contrast, theorem 1.2 gives a *closed formula* for *all* values  $HF(m)$ ,  $m \in \mathbb{N}$ , of the Hilbert *function* and thus for the Hilbert *series* and not only for the Hilbert *polynomial*, of course based on Riquier's decomposition  $\Gamma = \uplus_{\alpha \in \Delta} (\alpha + \mathbb{N}^{S(\alpha)})$  from theorem 1.1, (compare section 2) which is obtained by Buchberger's algorithm and elementary combinatorics [31, section 2.2] and which also represents

a (non-trivial) translation of Riquier's work into modern language. The complete proof of theorem 1.2 including all the preparatory and historical remarks requires three pages, compare section 2 above. I agree that my closed formulas and Pommaret's derivations described above have something in common. This is not surprising at all since I rely on the Riquier decomposition of  $\Gamma$  and Pommaret refers to the work of Janet who used and knew Riquier's work according to Pommaret's remarks 4.1, item (1). But my proof uses only the set  $\Gamma$  of *parametric derivatives*, its Riquier decomposition and the corresponding basis decomposition of the module  $M$  whereas the multiplicative and non-multiplicative variables used in Pommaret's derivation refer to the set of *principal derivatives* according to Pommaret's remarks 4.1, item (3). So Pommaret's and my derivations do certainly not coincide. D. Robertz in his talk and software presentation during the conference D2 described an implementation of (a variant of) Riquier's decomposition and this can be used to compute the Hilbert data [33], [44]. The paper [33] appeared one year ago when I submitted [30]. The authors knew [31] and present a survey of Janet's and not of Riquier's work.

7. *Pommaret insinuates that theorem 1.4 (=theorem 3.2 in section 3) on the Cauchy problem was shown by Janet, Pommaret etc. This is false.*

Its context is described in section 3 and in particular in items 1-7 of Remarks 3.3 with many references to the literature on partial differential equations, in particular to the treatises of the outstanding analysts Gelfand and Shilov [9], Hörmander [16], [17], Palamodov [32] and Egorov and Shubin [5], [6]. The theorem was quoted from [31] where we improved and extended the corresponding work in [27].

The proof of the unique existence theorem [43, Théorème d'Existence, p.254] for (passive and orthonomic non-linear) systems of partial differential equations and locally convergent power series is due to Riquier and not to Janet. Ten years later [18] Janet gave a new proof of Riquier's result and in particular a simpler algorithm after he had studied more algebra with Hilbert. Compare Pommaret's remarks 4.1, item (1), above.

In [31, th.29] we also gave a new proof of Riquier's theorem, only for *linear* systems of partial differential equations with *constant* coefficients, but in modern language and with a much shorter proof, and we acknowledged our debt to Riquier in several papers and in particular in [31] and my talk. In [27, pp.98-99] we proved the unique solvability of the Cauchy problem for linear systems of partial *difference* equations with constant coefficients for *formal* power series and described a solution algorithm which used Buchberger's algorithm. The paper [31, th.24] contains a better solution algorithm which was presented by Pauer at the conference D2 and the same result for locally convergent power series. Difference equations were not treated by Riquier, Janet, Spencer or Pommaret, but are as important for multidimensional systems and signal processing as their differential counter-part. By looking at those mathematical or engineering books on difference equations which also treat *partial* difference equations and multivariate functions, sequences or signals everybody can convince themselves that the results of [27] and [31] have added quite a bit in this important field of engineering mathematics.

8. *My paper [30] on Canonical State Representations presents the definition and construction of these. Pommaret considers these inappropriate and rejects them since they are not of the first order.*

To start the discussion on this matter I again quote from Pommaret's publica-

tions in italics and give my comment right away.

**Remark 4.3.** 1. (i) [37, p.310, line 1<sup>-</sup>–]: *We deal first with the state in order to convince the reader that the state representation must be avoided by any means.* (ii) [40, text before prop.3]: *Having understood that the concept of state is just a way to bring formally integrable (involutive) systems to formally integrable (involutive) first order systems with no zero order equation [...]* (iii) [41, p.26, line 9<sup>-</sup>]: *We shall prove that the Kalman form is nothing else but a particular case of the Spencer form [...]. No reference to control theory is needed [...]. It is not so well known that such a method, which allows to bring a system  $(R_q)$  of order  $q$  to a system of order 1, is only truly useful when  $R_q$  is formally integrable. Indeed, in this case and only in this case,  $R_{q+1}$  can be considered as a first order system over  $R_q$  without equations of order zero.*

Should the state be considered or should it not, that is the question.

2. [37, p.311, line 1<sup>+</sup>–]: *Indeed state is what must be given in order to integrate the control system whenever the input is given.*

This is exactly what theorem 1.4 says for signals in various spaces of power series. In contrast and to my knowledge Pommaret's books do not contain any results in this direction since integration of a system or, in other terms, construction of solutions in reasonable signal spaces is not addressed at all.

3. Pommaret's *modified Spencer form* as a generalization of the *Kalman form* [40, Def.4] is a first order system and therefore obviously different from my canonical state representation which, in general, is not of the first order. As quoted in [30] and in section 3 (compare equations (5) and (6)) Zerz [55] and Willems [52], [45] also consider and construct first order representations of linear systems with constant coefficients. But neither Pommaret nor these authors show an analogue of theorem 1.4 for their first order systems.

4. In Theorem 1.5 and its original [30, Th. 5.12, Th. 5.14] I show that every multidimensional system is isomorphic to a system in state form to which Theorem 1.4 is applicable. In my opinion, but also according to Pommaret's remark (2), this property is a decisive property for a state representation, and I prefer it to the first-order property of the system. Example [30, 5.16] shows that first-order systems may hide non-apparent complications.

5. My state systems apply to partial differential, partial difference and also to delay-partial differential linear systems with constant coefficients. They are generalizations of those non-first-order systems considered in [9] as explained in remark 3.3, item 4. Also the books [16], [17], [5], [6] show that leading analysts do not solve partial differential equations by first reducing them to first order. The constant coefficient linear systems which I consider and the even variable coefficient linear systems which Pommaret investigates have not yet been studied in so much detail, and the future will show which reductions to state form are preferable. Everybody is

invited to study and understand these different approaches and to make their choice.

9. *Janet's paper* [18]: In a very recent message to me F. Castro, Sevilla, points out that Riquier's decomposition is also explicitly contained in Janet's paper. Therefore I add some remarks on it although I have never used Janet's results in my papers and therefore did not read [18] in detail. As the foundation of my algorithms I primarily used Buchberger's Gröbner basis theory [27] and later also Riquier's work [31]. Colleagues like Apel, Blinkov, Castro, Gerdt, Plesken, Pommaret et.al. could discuss Janet's contributions in more detail. In the introduction of [18] Janet quotes early papers of Riquier and the book [43] and writes:

(i) p.66, line 11<sup>+</sup> ff.: *Cette forme canonique générale est due à M. Riquier, qui a le premier, en 1892, démontré l'existence des solutions d'un système différentiel quelconque.*

(ii) p.66, lines 11<sup>-</sup>, 12<sup>-</sup>: *Le présent travail a pour objet principal l'exposition simple des résultats de M. Riquier. Cette exposition nous conduira naturellement à certains résultats de nature algébrique qui complètent la théorie des formes donnée par M. Hilbert.*

Surprisingly and as far as I can see, Janet does not distinguish between Riquier's and his own original contributions in his paper, in particular he does not refer to the precise sections of [43] from which he extracted the ideas for his simplification of Riquier's work. An exception is the reference to the cotes [18, p.102, footnote].

Below I compare some of Janet's results from [18] with those of the present talk with the notations from pages 9 ff. Of course, I use the Gröbner basis theory whereas Janet had to develop its analogue on the basis of Riquier's work. The colleagues just mentioned have, in particular, translated Janet's work into modern language, but may have missed at various instances that Riquier is the original source of many of Janet's results.

(a) In Chapter I of [18] and also in Chapter II Janet discusses the combinatorial theory of the monoid  $\mathbb{N}^r$ , of its ideals  $N = N + \mathbb{N}^r$ , called modules in [18, Section 3, p.70], and, more generally, of order submodules

$$N := N + \mathbb{N}^r \subset I \times \mathbb{N}^r \text{ and their complement } \Gamma := (I \times \mathbb{N}^r) \setminus N, \quad (7)$$

the principal application being that to  $N := \deg(U)$ . In our generalization and translation [31, Section 2.2, pp.269-274] of Riquier's results from [43, pp.143-168] the main result is the canonical disjoint decomposition

$$I \times \mathbb{N}^r = \Gamma \uplus N, \quad \Gamma := \uplus_{\alpha \in \Delta_\Gamma} (\alpha + \mathbb{N}^{S(\alpha)}), \quad N := \uplus_{\alpha \in \Delta_N} (\alpha + \mathbb{N}^{S(\alpha)}), \quad (8)$$

$$\Delta_N \uplus \Delta_\Gamma = \text{finite subset of } I \times \mathbb{N}^r.$$

In the quoted pages Riquier treats only the decomposition of  $\Gamma$  which is sufficient for the purposes of the present paper, but not for the derivation of the Janet algorithm for the ring of linear differential operators. Our proof proceeds by induction as does Janet's derivation discussed in the following lines.

In [18, §6, §7, §8 on pp.74-79, §13 on pp.88-91] Janet uses a finite subset  $\mathcal{M} \subset \mathbb{N}^r$  and the order ideal  $N := \mathcal{M} + \mathbb{N}^r$  generated by  $\mathcal{M}$ , essentially proves the disjoint decomposition (8) if  $\mathcal{M}$  is *complete* [18, p.79, line 7<sup>-</sup>], and describes a completion algorithm [18, Section 9, pp.80-83]. With the notations from (8) the sets

$$C(\alpha) := \{s^{\alpha+\mu}; \mu \in \mathbb{N}^{S(\alpha)}\}, \quad \alpha \in \Delta_N \uplus \Delta_\Gamma \subset \mathbb{N}^r \quad (9)$$



are called *classes* of size  $m$  if  $S(\alpha)$  has  $m$  elements. The indeterminates

$$s_i, \begin{cases} i \in S(\alpha) \\ i \notin S(\alpha) \end{cases} \text{ are called } \begin{cases} \textit{multiplicative} \\ \textit{non-multiplicative} \end{cases} \text{ with respect to } C(\alpha). \quad (10)$$

However, Janet introduces this terminology also for a finite set  $s^\mu$ ,  $\mu \in \mathcal{M}$ , of monomials which is *not complete*. Compare [18, p.75, line 1<sup>-</sup>, p.89, line 8<sup>+</sup>] for *multiplicative* and [18, p.76, line 8<sup>-</sup>, p.89, line 11<sup>+</sup>] for *class*. This is the reason that the details of Janet's derivation of (8) differ quite a bit from our presentation in [31, Section 2.2]. I emphasize again that both Janet's derivations and also [31] are only translations of Riquier's work into the language of the time. I believe that our presentation in [31] based on the Gröbner basis theory is easier to understand for today's readership than the important, but also difficult paper [18].

Contrary to Pommaret's remark 4.1, item (3), the preceding discussion shows that Janet used the *multiplicative variables* not only for  $N = \deg(U)$  or the *principal derivatives*, but also for  $\Gamma$  or the *parametric derivatives*.

(b) The following quotations are taken from [18, Chapter II]:

(i) p.98, line 8<sup>+</sup>: *Afin de simplifier le langage, nous les supposerons linéaires; mais tout ce que nous dirons pourra s'étendre à des équations quelconques, sous réserve des conditions de régularité habituelles..*

(ii) p.98, line 11<sup>+</sup> ff.: *Les combinaisons d'équations que nous considérons seront toujours de combinaisons linéaires homogènes (résultats obtenues en additionnant ces équations membre à membre après les avoir éventuellement multipliées par des fonctions des variables indépendantes.*

(iii) p.98, line 13<sup>-</sup> ff.: *Il ne s'agira dans cette étude, comme l'exige la généralité du problème, que de fonctions développables en série de Taylor.*

In my interpretation and today's language this signifies that the solutions and the coefficients of the linear differential operators belong to the space  $\mathbb{C}\{z_1, \dots, z_r\}$  of (locally) convergent power series or of analytic functions in a neighborhood of some point in  $\mathbb{C}^r$ .

p.107, line 3<sup>+</sup> ff.: *Si, par dérivations et combinaisons, on ne peut tirer de (C) aucune relation entre les seules dérivées paramétriques (et variables indépendantes), on dira que le système est complètement intégrable.*

This notion is also due to Riquier. For linear systems with constant coefficients complete integrability signifies that the rows of  $R$  are a Gröbner basis of  $U = \mathbb{C}[s]^{1 \times k} R$  and that the necessary compatibility conditions are satisfied [29, Cor.16].

p.116, footnote: *Dans ce genre de raisonnements, il est sous-entendu que l'on se place au voisinage d'un système de valeurs (des variables indépendantes et des dérivées des premières classes) pour lequel toutes les résolutions successives supposées dans le texte sont possibles conformément à la théorie générale des fonctions implicites.*

In my interpretation this sentence signifies that Janet's reduction of a system to completely integrable form is only generically true. This agrees with the results of Malgrange [26], but contradicts Pommaret's repeated statements that the linear systems of partial differential equations with variable coefficients are always solvable in formal power series if the algebraic compatibility conditions are satisfied. Compare the simple counter-example in paragraph 5 above. The reason is that the used resolutions of linear equations with variable coefficients

for the highest derivative require a division in general, and the resulting meromorphic coefficients are analytic only in the complement of an analytic set. The implicit function theorem is used if a *non-linear* equation is solved for the highest derivative.

(c) In [18, Ch. II, §13] Janet applies his theory to the polynomial algebra  $(\mathbb{C}[s_1, \dots, s_r])$  in my talk) or to linear differential operators with *constant* coefficients in a short discussion. Like Hilbert he considers forms or homogeneous polynomials and ideals only and not arbitrary finitely generated polynomial modules as in Theorem 1.2(=Th. 2.3). The main result is Hilbert's syzygy theorem.

The Hilbert polynomial (compare [18, p.121, line 1<sup>+</sup>]), but not the whole Hilbert function as in Th. 1.2 is considered in [18, Ch.I, §14 on p.91 and Ch.II, pp.121-122], but no closed formulas as in Th. 1.2 are exhibited. Paragraph §14 on p.121 describes the highest term of the Hilbert polynomial of the homogeneous ideal. With the notations of the more general theorem 1.2 it is shown that

$$HP_M = \frac{\mu}{d!} t^d + \dots, \quad d = \dim(M) = \max_{\alpha \in \Delta} |S(\alpha)|, \quad (11)$$

$$\mu = \text{number of } S(\alpha) \text{ with } |S(\alpha)| = d.$$

The natural numbers  $d$  resp.  $\mu$  are interpreted as the *dimension* of the associated variety resp. its multiplicity. It is again surprising that Macaulay's treatise [24] is not mentioned in context with these considerations.

## References

- [1] J. Apel, *The Theory of Involutive Divisions and an Application to Hilbert Function Computations*, J.Symb.Computation 25(1998), 683-704
- [2] J. Apel, *On a conjecture of R.P. Stanley II: Quotients modulo monomial ideals*, J.Algebr.Comb. 17(2003), 57-74
- [3] J.-E. Björk, *Rings of Differential Operators*, North-Holland, Amsterdam, 1979
- [4] V.D. Blondel, A. Megretski (Editors), *Unsolved Problems in Mathematical Systems and Control Theory*, Princeton University Press, 2004
- [5] Y.V.Egorov, M.A. Shubin (Y.V.Egorov, M.A. Shubin Editors), *Partial Differential Equations I*, Springer-Verlag, Berlin, 1992
- [6] Y.V.Egorov, M.A. Shubin (Editors), *Partial Differential Equations III*, Springer-Verlag, Berlin, 1991
- [7] L Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Wiley-Interscience, New York, 1970
- [8] S. Fröhler, U. Oberst, *Continuous time-varying linear systems*, Systems and Control Letters 35 (1998), 97-110
- [9] I.M. Gelfand, G.E. Shilov, *Generalized Functions III. Theory of Differential Equations*, Academic Press, New York, 1967

- [10] S.I. Gelfand, Y.I. Manin (A.I. Kostrikin, I.R. Shafarevic Editors) *Homological Algebra*, Springer, Berlin, 1994
- [11] S.G. Gindikin, A.N. Belozersky, *The Cauchy Problem*, 1-86 in [6]
- [12] H. Gluesing-Luerssen, *Linear Delay-Differential Systems with Commensurate Delays: An Algebraic Approach*, Lecture Notes in Mathematics 1770, Springer-Verlag, Berlin, 2002
- [13] W. Gröbner, *Über die algebraischen Eigenschaften der Integrale von linearen Differentialgleichungen mit konstanten Koeffizienten*, Monatsh. Math. Phys. 47 (1939), 247-284
- [14] W. Gröbner, *Algebraische Geometrie II*, BI-Taschenbuchverlag, Mannheim 1970
- [15] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, North-Holland, Amsterdam, 1966
- [16] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin, 1983
- [17] L. Hörmander, *The Analysis of Linear Partial Differential Operators II*, Springer-Verlag, Berlin, 1983
- [18] M. Janet, *Sur les Systèmes d'Équations aux Dérivées Partielles*, J. de Mathématiques Pures et Appliquées 8(1920), 65-151
- [19] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980
- [20] R.E. Kalman, *Mathematical description of linear dynamical systems*, SIAM Journal on Control 1(1963), 152-192
- [21] M. Kashiwara, T. Kawai, T. Kimura, *Foundations of Algebraic Analysis*, Princeton University Press, Princeton, 1986
- [22] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra 1*, Springer-Verlag, Berlin, 2000
- [23] P. Lu, M. Liu, U. Oberst, *Linear Recurring Arrays, Linear Systems and Multidimensional Cyclic Codes over Quasi-Frobenius Rings*, Acta Appl. Math. 80(2004), 175-198
- [24] F.S. Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge University Press, Cambridge, 1916
- [25] B. Malgrange, *Systèmes différentiels à coefficients constants*, Séminaire Bourbaki 1962/1963, 246.01-246.11
- [26] B. Malgrange, *Systèmes Différentiels Involutifs*, Prépublication 636, Institut Fourier Grenoble, 2004
- [27] U. Oberst, *Multidimensional Constant Linear Systems*, Acta Appl. Math. 20(1990), 1 - 175

- [28] U. Oberst, *Variations on the Fundamental Principle for Linear Systems of Partial Differential and Difference Equations with Constant Coefficients*, AAECC 6(1995), 211-243
- [29] U. Oberst, *The Constructive Solution of Linear Systems of Partial Difference and Differential Equations with Constant Coefficients*, 32 pages, Proc. of the IMA Conference in Amsterdam in November 2000, (surprisingly) to appear soon
- [30] U. Oberst, *Canonical State Representations and Hilbert Functions of Multidimensional Systems*, submitted to Acta Appl. Math., June 2005
- [31] U. Oberst, F. Pauer, *The Constructive Solution of Linear Systems of Partial Difference and Differential Equations with Constant Coefficients*, Multidimensional Systems and Signal Processing 12(2001), 253-308
- [32] V.P. Palamodov, *Linear Differential Operators*, Springer-Verlag, Berlin, 1970
- [33] W. Plesken, D. Robertz, *Janet's approach to presentations and resolutions for polynomials and linear PDEs*, Archiv der Mathematik 84(2005), 22-37
- [34] J.W. Polderman, J.C. Willems, *Introduction to Mathematical Systems Theory*, Springer-Verlag, New York, 1998
- [35] J.-F. Pommaret, *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon and Breach, New York, 1978
- [36] J.-F. Pommaret, *Differential Galois Theory*, Kluwer, Dordrecht, 1983
- [37] J.-F. Pommaret, *New perspectives in control theory for partial differential equations*, IMA J. Math. Control and Information 9(1992), 305-330
- [38] J.-F. Pommaret, *Partial Differential Equations and Group Theory*, Kluwer, Dordrecht, 1994
- [39] J.-F. Pommaret, *Partial Differential Control Theory, Volume I: Mathematical Tools, Volume II: Control Systems*, Kluwer Academic Publishers, Dordrecht, 2001
- [40] J.-F. Pommaret, *Localization and Transfer Matrix Computation for Linear Multidimensional Control Systems*, Proceedings MTNS 2004
- [41] J.-F. Pommaret, Lecture Notes of the mini-course on *Algebraic Analysis of Control Systems defined by Partial Differential Equations*, pp.1-45, Gröbner Semester Linz, Sections D2 and D3, RISC, Hagenberg, and RICAM, Linz, May 2006
- [42] P. Rapisarda, J.C. Willems, *State Maps for Linear Systems*, SIAM J. Control Optim. 35(1997), 1053-1091
- [43] C. Riquier, *Les Systèmes d'Équations aux Dérivées Partielles*, Gauthiers - Villars, Paris, 1910
- [44] D. Robertz, *Implementation of Involutive Bases in MAPLE and C++*, homepage of Gröbner-Semester, Conference D2, Linz, 2006

- [45] P. Rocha, J.C. Willems, *Markov properties for systems described by PDEs and first-order representations*, Systems and Control Letters 55 (2006), 538-542
- [46] F. Schwarz, *The Riquier-Janet Theory and its Application to Nonlinear Evolution Equations*, Physica 11D (1984), 243-251
- [47] R.P. Stanley, *Combinatorics and Commutative Algebra*, 2.Edition, Birkhäuser, Boston, 1996
- [48] B. Sturmfels, *Solving Systems of Polynomial Equations*, American Math.Soc., 2002
- [49] F. Trèves, *Basic Linear Partial Differential Equations*, Academic Press, New York, 1975
- [50] A.I.G. Vardulakis, *Linear Multivariable Control*, John Wiley & Sons, Chichester, 1991
- [51] W.V. Vasconcelos, *Computational Methods in Commutative Algebra and Algebraic Geometry*, Springer-Verlag, Berlin, 1998
- [52] J.C. Willems, *State and First Order Representations*, [4, p.54-57]
- [53] W.A. Wolovich, *Linear Multivariable Systems*, Springer-Verlag, New York, 1974
- [54] J. Wood, P. Rocha, E. Rogers, and D.H. Owens, *Structure Indices for Multidimensional Systems*, IMA J.Math. Control Inform. 17(2000), 227-256
- [55] E. Zerz, *Topics in Multidimensional Linear Systems Theory*, Lecture Notes in Control and Information Sciences 256, Springer-Verlag, London, 2000