Time-autonomy and time-controllability of discrete multidimensional behaviors

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Abstract

Predecessors and starting point of the present paper on multidimensional discrete time-autonomy and time-controllability were three papers from 2002-2004 by Sasane, Controneo, Thomas and Willems on the continuous case, a paper from 2011 by Napp, Rapisarda and Rocha on time-relevant 2D-behaviors and a submitted paper by Bisiacco and Valcher on 2D-time-controllability and dead-beat control. We also acknowledge interesting talks on this subject by Bisiacco and Valcher in Innsbruck. We develop a general framework for discrete multidimensional behaviors in which the discrete time variable in $\mathbb{N}$ plays a special part among all independent variables. Following the models of our predecessors we define time-autonomy and time-controllability of a behavior by properties of its trajectories. Goal of the paper is the characterization of these properties by constructive algebraic conditions which is fully achieved for time-autonomy and for time-controllability in dimension two and partially for time-controllability in higher dimensions than two.

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1 Introduction

Predecessors and starting point of this paper on multidimensional discrete time-autonomy (ta) and time-controllability (tc) were the papers [18], [17] and [16] by Sasane, Controneo, Thomas and Willems on the continuous case, the very recent paper [12] by Napp, Rapisarda and Rocha on time-relevant 2D-behaviors and the submitted paper [2] by Bisiacco and Valcher on discrete 2D time-controllability and dead-beat control. We especially acknowledge inspiring talks on their resuls by Bisiacco and Valcher in Innsbruck.
We use the following general framework where the discrete independent time-variable in $\mathbb{N} = \{0, 1, \cdots\}$ plays a special part whereas all other independent variables are incorporated into an arbitrary affine integral domain A over a field F as ring of operators which acts on an injective cogenerator signal module $({}_AW, \circ)$, for instance

$W = A^* = \mathrm{Hom}_F(A,F)$ in all standard cases. Let Mod_A denote the category of A-modules. The polynomial algebra $B := A[s_0]$ in one indeterminate s_0 acts on the sequence space

$$W^{\mathbb{N}} = \left\{w = (w(t))_{t\in\mathbb{N}};\ w(t) \in W\right\}$$

via the given action of ${}_AW$ and by s_0 as left shift, i.e., $(s_0 \circ w)(t) = w(t+1)$. Thus s_0 is the time operator. The module ${}_BW^{\mathbb{N}}$ is again an injective cogenerator and gives rise to the standard duality between modules and behaviors. A matrix $R \in B^{k\times\ell}$ induces the *equation module* $U := B^{1\times k}R$, the finitely generated (f.g.) *system module* $M := B^{1\times\ell}/U$ and the *behavior*

$$\begin{gathered} D(M) := \mathrm{Hom}_B(M,W^{\mathbb{N}}) \cong \mathrm{Hom}_A(M,W) \cong \mathscr{B} := U^{\perp} := \\ \left\{w \in \left(W^{\mathbb{N}}\right)^{\ell};\ U \circ w = 0\right\} = \left\{w \in \left(W^{\mathbb{N}}\right)^{\ell};\ R \circ w = 0\right\}. \end{gathered} \tag{1}$$

In the standard case of an m-variate polynomial algebra A one has

$$\begin{gathered} A = F[s'] := F[s_1,\cdots,s_m] \subset B = A[s_0] = F[s] = F[s_0,\cdots,s_m] \\ F[s']^* \cong W := F^{\mathbb{N}^m},\ B^* \cong W^{\mathbb{N}} = \left(F^{\mathbb{N}^m}\right)^{\mathbb{N}} = F^{\mathbb{N}\times\mathbb{N}^m} = F^{\mathbb{N}^{1+m}}. \end{gathered} \tag{2}$$

The framework of [1], [3] is obtained by

$$\begin{gathered} A := F[s_1,s_1^{-1}] \subset B = A[s_0] = F[s_0,s_1,s_1^{-1}], \\ A^* \cong W := F^{\mathbb{Z}},\ B^* \cong W^{\mathbb{N}} = F^{\mathbb{N}\times\mathbb{Z}}. \end{gathered} \tag{3}$$

The same theory can be developed if the discrete time axis is $\mathbb{Z}$ instead of $\mathbb{N}$ to include the setting of [12].

A behavior $\mathscr{B}$ is called *time-autonomous* (ta) if there is a time instant $d \in \mathbb{N}$ such that each trajectory $w \in \mathscr{B}$ is uniquely determined by the d initial data $w(0),\cdots,$ $w(d-1)$. In the main Thm. 3.7 of Section 3 on ta we show that for $W = A^*$ a behavior $\mathscr{B}$ is ta if and only if there is a *monic* polynomial $f = s_0^d + f_{d-1}s_0^{d-1} + \cdots + f_0 \in B = A[s_0]$ with $f \circ \mathscr{B} = 0$, i.e.,

$$(f \circ w)(t) = w(t+d) + f_{d-1} \circ w(t+d-1) + \cdots + f_0 \circ w(t) = 0 \text{ for } w \in \mathscr{B},\ t \in \mathbb{N}.$$

A continuous analogue of this result is Theorem 3.4 of [18], which was, however, derived for scalar behaviors $\{w;\ f \circ w = 0\}$, $f \in B$, only. In [12] the authors discuss time-autonomy as *time-relevance* for discrete 2D-behaviors on $\mathbb{Z}^2$, i.e., for $A = F[s_1,s_1^{-1},s_2,s_2^{-1}]$ and $A^* \cong F^{\mathbb{Z}^2}$. Proposition 6 of [loc.cit.] characterizes time-relevant square autonomous behaviors in this situation and can be deduced from Thm. 3.7 for the time-axis $\mathbb{Z}$ instead of $\mathbb{N}$. For the *finite operator rings* $A = \mathbb{Z}/\mathbb{Z}n$, $n > 1$, and $W = A$ Kuijper et al. [7, Prop. 22,23] call time-autonomous behaviors just autonomous and prove the analogue of Thm. 3.7. Zerz [26, Thm. 10] calls time-autonomous behaviors *past-determined* and extends the result of [7] to arbitrary quasi-Frobenius rings A which are, by definition, self-injective and then artinian and of Krull dimension zero.

After the models in [18] and [3] we call a behavior $\mathscr{B}$ *time-controllable* if there is a time lag $L \geq 0$ such that for any time instant $p \in \mathbb{N}$ any two trajectories $w_1, w_2 \in \mathscr{B}$ can be concatenated in the time interval $\{p,\cdots,p+L-1\}$ to another trajectory

$$w = \left(w_1(0),\cdots,w_1(p-1),w(p),\cdots,w(p+L-1),\overset{p+L}{w_2(0)},w_2(1),\cdots\right) \in \mathscr{B}.$$

The goal of Section 4 is to describe time-controllability by constructive algebraic conditions and has been partially achieved. For the standard controllability notion the concatenability of trajectories, suitably defined, is equivalent with the torsionfreeness of the system module, i.e., that it is a submodule of a free module. This was originally observed by Willems [23] in one-dimensional behavioral systems theory. In the multidimensional discrete resp. continuous cases the equivalence of these notions was proven by Wood and Zerz [24] resp. by Pillai and Shankar [15].

Thm. 4.12 gives a, presently still non-constructive, algebraic characterization of tc. In Thm. 4.10 we construct a large class of tc behaviors $\mathscr{B}$, viz. behaviors of the form $\mathscr{B} := D(M)$ where M is a submodule of a f.g. so-called *extended module* $B\otimes_A M'$ [9, p.5]. So the extended modules play a similar part for time-controllability which the free modules $B^{1\times\ell} = B\otimes_A A^{1\times\ell}$ do for standard controllability. These $\mathscr{B}$ admit constructive sum decompositions $\mathscr{B} = \mathscr{B}_{\mathrm{cont}} + \mathscr{B}_{\mathrm{aut,tc}}$ where $\mathscr{B}_{\mathrm{cont}}$ is the controllable part of $\mathscr{B}$ and $\mathscr{B}_{\mathrm{aut,tc}}$ is autonomous and tc. We prove *constructive necessary* conditions for tc: Theorem 4.8 shows that a tc $D(M)$ contains no nonzero observables, i.e., elements of M, which are annihilated by *monic* polynomials; it is the discrete analogue of Theorem 4.3 of [18]. Theorem 4.15 furnishes further constructive necessary conditions for tc and shows especially that all associated prime ideals $\mathfrak{q}$ of a module ${}_BM$ with tc $D(M)$ are extended from A, i.e., of the form $\mathfrak{q} = (\mathfrak{q}\cap A)[\sigma]$. In Section 5 we show that for a principal ideal domain A and $B = A[\sigma]$ a behavior $D(M)$ is tc if and only if M is a submodule of an extended module and if and only if each associated prime of M is extended (Thm. 5.3). This is a slight generalization and sharpening of Thm. 5.1 of [3] with a different, shorter and *constructive* module theoretic proof, in particular tc can be algorithmically decided in dimension two by means of Computer Algebra. Such an algorithm is lacking in higher dimensions than two and therefore we have presently no counter-example to this theorem in higher dimensions. We also give a constructive proof of the two-dimensional matrix decompositions (Thms. 5.9, 5.10) which are at the basis of the two-dimensional results in [21], [22], [12], [3]. The constructive Quillen-Suslin Theorem [6] enables the constructiveness. These results are applicable to the category $\mathrm{Mod}_{\mathbb{Z}[\sigma]}$ with its injective cogenerator $W^{\mathbb{N}}$, $W := \mathbb{Q}/\mathbb{Z}$ or $W = \mathbb{R}/\mathbb{Z}$, and to its full subcategory $\mathrm{Mod}_{(\mathbb{Z}/\mathbb{Z}n)[\sigma]}$, $n > 1$, with its injective cogenerator $\left((\mathbb{Z}n)^{\perp}\right)^{\mathbb{N}}$ where $(\mathbb{Z}n)^{\perp} = \mathbb{Z}\frac{1}{n}/\mathbb{Z} \subset W$, hence $(\mathbb{Z}n)^{\perp} \cong \mathbb{Z}/\mathbb{Z}n$ [7],[25].

In Remark 5.11 we point out other notions of multidimensional controllability which seem more suitable than time-controllability for higher (Krull) dimension of B than two and which we are going to use for multidimensional controller design in arbitrary dimensions with a technique different from that in [2], [3].

Section 2 adds some details and examples to the meanwhile widely known module theoretic language of discrete multidimensional behavioral systems theory from [13] and thus prepares the proofs in Sections 3 and 4. Answering questions of the reviewers we make several remarks on the constructiveness of our results and have quoted Result 2.8 from our paper [20] which enables many constructive computations in the present paper.

The following problems are unsolved in general: (i) whether the module M of a tc behavior $\mathscr{B} := D(M)$ is a submodule of an extended module, i.e., whether the class of behaviors of Thm. 4.10 consists of *all* tc ones or whether Thm. 5.3 can be extended to higher dimensions. (ii) to check in higher dimensions *constructively* whether $\mathscr{B}$ is tc and whether M is a submodule of an extended module. According to the (computer) algebraists T.Y. Lam, G. Pfister, and R. Rao also the last task seems to be difficult.

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2 Multidimensional discrete time systems

In this section we introduce a multidimensional behavioral framework in which special attention is paid to the independent discrete time variable in $\mathbb{N} := \{0,1,\cdots\}$*, and first recall the standard module theoretic language for this purpose*.

Let F be a field. For a commutative noetherian F-algebra A let Mod_A, $\mathrm{spec}(A)$ and $\max(A)$ denote the category of A-modules, the set of prime and of maximal ideals of A respectively. If B is another such algebra and $\rho : A \to B$ an algebra homomorphism each B-module N is also an A-module via $ay := \rho(a)y,\ a \in A,\ y \in N$. This gives rise to the *forgetful functor*

$$\rho_* : \mathrm{Mod}_B \to \mathrm{Mod}_A,\ N \mapsto N. \tag{4}$$

Conversely, if ${}_AM$ is an A-module the module $\mathrm{Hom}_A(B,M)$ is a B-module via

$$(b_1\alpha)(b_2) := \alpha(b_1b_2),\ b_i \in B,\ \alpha \in \mathrm{Hom}_A(B,M), \tag{5}$$

and this gives rise to the *induced module functor*

$$\mathrm{Hom}_A(B,-) : \mathrm{Mod}_A \to \mathrm{Mod}_B. \tag{6}$$

The functor $\mathrm{Hom}_A(B,-)$ is right adjoint to the forgetful functor via the functorial isomorphism

$$\mathrm{Hom}_A(N,M) \cong \mathrm{Hom}_B(N,\mathrm{Hom}_A(B,M)),\ \varphi \mapsto \phi,\ \phi(y)(b) = \varphi(by). \tag{7}$$

An A-module W is *injective* if the functor $\mathrm{Hom}_A(-,W) : \mathrm{Mod}_A \to \mathrm{Mod}_A$ is *exact*, i.e., *preserves* exact sequences, and even an *injective cogenerator* if it also *reflects* such sequences. In systems theory such modules appear as *modules of signals* on which a *ring A of operators* acts. Let such an ${}_AW$ with action $a \circ w,\ a \in, w \in W$, be given. This action is extended to matrix multiplications

$$A^{p\times q} \times W^{q\times r} \to W^{p\times r},\ (P,X) \mapsto P \circ X,\ (P \circ X)_{ij} := \sum_{k=1}^{q} P_{ik} \circ X_{kj}. \tag{8}$$

Especially the free *row module* $A^{1\times\ell}$ with its standard basis

$$\delta_i := (0,\cdots,0,\overset{i}{1},0,\cdots,0) \in A^{1\times\ell},\ i = 1,\cdots,\ell \tag{9}$$

acts on the *column module* $W^\ell := W^{\ell\times 1}$. A finitely generated (f.g.) A-module

$$M = A^{1\times\ell}/U,\ U = A^{1\times k}R,\ R \in A^{k\times\ell}, \tag{10}$$

gives rise to its *dual module resp. solution module or* W*-behavior*

$$\begin{gathered} D(M) := D_W(M) := \mathrm{Hom}_A(M,W) \cong U^\perp := \\ \left\{ w \in W^\ell;\ \forall x \in U :\ x \circ w = x_1 \circ w_1 + \cdots + x_\ell \circ w_\ell = 0 \right\} =: \\ \left\{ w \in W^\ell;\ U \circ w = 0 \right\} = \left\{ w \in W^\ell;\ R \circ w = 0 \right\} \\ \varphi \mapsto w,\ \varphi(\delta_i + U) = w_i,\ i = 1,\cdots,\ell. \end{gathered} \tag{11}$$

In most situations we are going to identify

$$D(M) = U^{\perp},\ \varphi = w,\ w(\delta_i + U) = w_i, \tag{12}$$

and call M resp. U the module resp. the equation module of the behavior $U^{\perp}$.

Result 2.1. *If ${}_AW$ is an injective cogenerator and $\rho : A \to B$ a homomorphism then the induced module* (5) $\mathrm{Hom}_A(B,W)$ *is an injective cogenerator B-module [5, §1.8, Prop.2].*

This follows easily from the functorial isomorphism (7) and is an essential ingredient in discrete multidimensional systems theory [13, Cor. 3.12].
In the present paper all rings of operators A are *affine integral domains*, i.e., f.g. F-algebras without zero divisors. Let

$$F[s] := F[s_1, \cdots, s_m] = \oplus_{\mu \in \mathbb{N}^m} F s^{\mu},\ s^{\mu} = s_1^{\mu_1} * \cdots * s_m^{\mu_m}, \tag{13}$$

denote the m-variate polynomial algebra. Then any affine integral domain A is of the form

$$\begin{gathered} A = F[s]/I = F[\overline{s}],\ m \geq 1,\ I \in \mathrm{spec}(F[s]), \\ \overline{s} = (\overline{s_1}, \cdots, \overline{s_m}),\ \overline{s_i} := s_i + I \in A. \end{gathered} \tag{14}$$

Example 2.2. ([13, Cor. 3.12]) If A is any affine integral domain then $A^* := \mathrm{Hom}_F(A,F)$ is an injective A-cogenerator since ${}_FF$ is trivially an injective F-cogenerator. A f.g. A-module M gives rise to its dual space $D_F(M) := \mathrm{Hom}_F(M,F)$ and its dual module $D_{A^*}(M) = \mathrm{Hom}_A(M,A^*)$ which are identified by the canonical A-isomorphism (7)

$$\begin{gathered} D_F(M) := \mathrm{Hom}_F(M,F) \cong D_{A^*}(M) = \mathrm{Hom}_A(M,A^*), \\ \varphi \mapsto \phi,\ \phi(x)(a) = \varphi(ax),\ a \in A,\ x \in M, \end{gathered}$$

which holds in addition to the identification of (11).

Example 2.3. The polynomial algebra $F[s]$ acts on the multisequence space $F^{\mathbb{N}^m} := \{w : \mathbb{N}^m \to F,\ \mu \mapsto w(\mu)\}$ via left shifts, $(s^{\mu} \circ w)(\nu) = w(\mu + \nu)$. The canonical F-isomorphism

$$F[s]^* = \mathrm{Hom}_F(F[s],F) \cong F^{\mathbb{N}^m},\ \alpha \mapsto w,\ \alpha(s^{\mu}) = w(\mu),$$

is even $F[s]$-linear. We always identify $\mathrm{Hom}_F(F[s],F) = F^{\mathbb{N}^m}$, $w(s^{\mu}) = w(\mu)$, and thus obtain the standard injective cogenerator signal module for multidimensional discrete systems theory.
In the same manner the *mixed Laurent polynomial algebra*

$$A := F[s_0, s_1, s_1^{-1}, \cdots, s_m, s_m^{-1}] = \oplus_{\mu = (\mu_0, \cdots, \mu_m) \in \mathbb{N} \times \mathbb{Z}^m} F s^{\mu}$$

gives rise to the injective cogenerator signal A-module $F^{\mathbb{N} \times \mathbb{Z}^m}$ with its left shift action. For $m = 1$ Bisiacco and Valcher use this signal module and operator algebra for their 2D discrete observer and controller theory [1], [3] with time set $\mathbb{N}$.

Example 2.4. (compare [13, Thm. 2.99 on p.46]) For an affine integral domain A' and factor domain $A := A'/I$ with $I \in \mathrm{spec}(A')$ the category Mod_A is the full subcategory of $\mathrm{Mod}_{A'}$ of all A'-modules M' with $IM' = 0$ and the canonical map $\mathrm{can} : A' \to A$ induces the forgetful inclusion functor $\mathrm{can}_* : \mathrm{Mod}_A \subset \mathrm{Mod}_{A'}$ according to (4) with its right adjoint $\mathrm{Hom}_{A'}(A,-)$. From (7) and (11) we infer the canonical isomorphisms

$$\begin{gathered} A^* = D_F(A'/I) \cong D_{A'^*}(A'/I) = I^\perp (\subseteq A'^*),\ \alpha \mapsto \alpha\,\mathrm{can},\ \alpha \in A^*, \\ \text{and for a f.g. module } M \in \mathrm{Mod}_A \subset \mathrm{Mod}_{A'} \\ D_F(M) = M^* = \mathrm{Hom}_F(M,F) \cong D_{A'^*}(M) \cong D_{A^*}(M) \end{gathered} \tag{15}$$

by which we often identify all these modules. In particular, $A^* \cong I^\perp$ is an injective cogenerator signal A-module, and the A^*-behavior $D_{A^*}(M)$ is also an A'^*-behavior. We extend $\mathrm{can} : A' \to A$ to the linear map $\mathrm{can} := \mathrm{can}^{1\times\ell} : A'^{1\times\ell} \to A^{1\times\ell}$ with kernel $I^{1\times\ell} = IA'^{1\times\ell}$ and assume that $M = A^{1\times\ell}/U$. Then $U' := \mathrm{can}^{-1}(U)$ contains this kernel, and the homomorphism theorem induces the canonical A- and A'-isomorphisms

$$A'^{1\times\ell}/U' \cong A^{1\times\ell}/U \text{ and } U'^\perp = \left(A'^{1\times\ell}/U'\right)^* \cong U^\perp = \left(A^{1\times\ell}/U\right)^*. \tag{16}$$

Thus, up to isomorphism, an A^*-behavior $U^\perp \subseteq (A^*)^\ell$ is the same as an A'^*-behavior $U'^\perp \subseteq (A'^*)^\ell$ with the additional conditions

$$U'^\perp \subseteq (IA'^{1\times\ell})^\perp = (I^\perp)^\ell \iff U' \supseteq IA'^{1\times\ell} \iff I\left(A'^{1\times\ell}/U'\right) = 0.$$

Assume especially $A' = F[s] = F[s_1,\cdots,s_m]$ and $A'^* = F^{\mathbb{N}^m}$ as in Example 2.3. The A- and $F[s]$-isomorphism (15) then gets the form

$$A^* \cong I^\perp \subseteq F^{\mathbb{N}^m},\ \alpha \mapsto w,\ w(\mu) = \alpha(s^\mu + I). \tag{17}$$

Example 2.5. (Compare [11, Ch.7]). We consider the vectors in $\mathbb{N}^m$, $\mathbb{Z}^n$ as *row vectors*. Let $N = \mathbb{N}^m\Theta \subseteq \mathbb{Z}^n$, $\Theta \in \mathbb{Z}^{m\times n}$, be any f.g. submonoid of the group $\mathbb{Z}^n$ such that the rows of Θ generate N. The group algebra $F[\mathbb{Z}^n]$ of $\mathbb{Z}^n$ is the Laurent polynomial algebra in n indeterminates via

$$F[\mathbb{Z}^n] = F[s,s^{-1}] = \oplus_{\nu\in\mathbb{Z}^n} F s^\nu,\ s = (s_1,\cdots,s_n),\ s^{-1} := (s_1^{-1},\cdots,s_n^{-1}) \tag{18}$$

where $\nu \in \mathbb{Z}^n$ is identified with the monomial s^ν. The monoid F-algebra of N is

$$A := F[N] = \oplus_{\nu\in N} F s^\nu \subseteq F[\mathbb{Z}^n] = F[s,s^{-1}]. \tag{19}$$

Of course, $F[N]$ is an affine domain. Again we also identify

$$A^* = F[N]^* = F^N,\ w(s^\mu) = w(\mu), \tag{20}$$

and obtain the injective cogenerator $F[N]$-module F^N with its left shift action. We also consider the polynomial algebra

$$A' := F[\mathbb{N}^m] = F[\sigma],\ \sigma = (\sigma_1,\cdots,\sigma_m), \text{ with } A'^* = F[\sigma]^* = F^{\mathbb{N}^m} \tag{21}$$

and the left shift action of $F[\sigma]$ on $F^{\mathbb{N}^m}$. There are the monoid resp. algebra epimorphisms

$$\begin{gathered} \rho : \mathbb{N}^m \to N,\ \mu \mapsto \mu\Theta, \text{ resp.} \\ \rho : F[\sigma] \to F[N],\ \sigma^\mu \mapsto s^{\mu\Theta} \text{ with kernel } I_N \in \mathrm{spec}(F[\sigma]), \text{ hence} \\ F[\sigma]/I_N \underset{\text{ident.}}{=} F[N],\ \sigma^\mu + I_N = s^{\mu\Theta}. \end{gathered} \tag{22}$$

The prime ideal I_N is called the *toric or lattice ideal* of the monoid N [11, Def. 7.2, Thm. 7.4, p.148]. It is generated by the binomials $\sigma^\mu - \sigma^{\mu'}$ with $\mu\Theta = \mu'\Theta$ [11, Thm. 7.3]. The isomorphism $A^* \cong I^\perp \subseteq A'^*$ from (15) gets the form

$$\begin{gathered} F^N \cong I_N^\perp = \left\{ w' \in F^{\mathbb{N}^m};\ \forall \mu, \mu' \in \mathbb{N}^m :\ \mu\Theta = \mu'\Theta \Longrightarrow w'(\mu) = w'(\mu') \right\} \\ w \mapsto w',\ w'(\mu) = w(\mu\Theta). \end{gathered} \tag{23}$$

The paper [27] developed the first systems theoretic application of this situation for

$$\begin{gathered} N = \mathbb{Z}^n,\ m = 2n, \Theta : \mathbb{N}^{2n} = \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{Z}^n,\ (\mu, \mu') \mapsto \mu - \mu', \\ F[\sigma_1, \cdots, \sigma_n, \sigma_1', \cdots, \sigma_n']/I_{\mathbb{Z}^n} = F[\mathbb{Z}^n] = F[s, s^{-1}], \\ I_N = \sum_{i=1}^n F[\sigma, \sigma'](\sigma_i \sigma_i' - 1),\ \sigma_i + I_N = s_i,\ \sigma_i' + I_N = s_i^{-1} \end{gathered}$$

in order to reduce the *Cauchy or initial value problem* for $F^{\mathbb{Z}^n}$-behaviors to that for $F^{\mathbb{N}^{2n}}$-behaviors which was solved in [13]. Our characterization of time-autonomous behaviors below also depends on such a reduction.

Assumption 2.6. In the sequel we assume that A is an F-affine integral domain and W any injective cogenerator signal A-module with the action $a \circ w,\ a \in A,\ w \in W$, for instance among the signal modules of the preceding examples. One could also take the space $W := \mathrm{C}^\infty(\mathbb{R}^n, \mathbb{C})$ of smooth functions on which the polynomial algebra $A := \mathbb{C}[s_1, \cdots, s_n]$ acts by differentiation, $(s_i \circ w)(t) := \partial w / \partial t_i,\ t = (t_1, \cdots, t_n) \in \mathbb{R}^n$.

We use the signal A-module $W^{\mathbb{N}}$ of sequences $w = (w(0), w(1), \cdots)$ which are considered as time series of signals whose value at the discrete time $t \in \mathbb{N}$ is the signal $w(t)$. Let $B := A[\sigma] = \oplus_{t \in \mathbb{N}} A\sigma^t$ denote the univariate polynomial algebra which is also an F-affine integral domain. It acts on the A-module $W^{\mathbb{N}}$ of sequences via

$$\left((\sum_{k=0}^\infty f_k \sigma^k) \circ w \right)(t) := \sum_{k=0}^\infty f_k \circ w(t+k). \tag{24}$$

Standard one-dimensional discrete systems are obtained in this framework when $A = W = F$. As in Example 2.3 we obtain the identification of B-modules

$$\mathrm{Hom}_A(B, W) = W^{\mathbb{N}},\ w(\sigma^t) = w(t),\ t \in \mathbb{N}. \tag{25}$$

By Assumption 2.6 ${}_AW$ is an injective A-cogenerator and hence Result 2.1 furnishes that $W^{\mathbb{N}}$ is an injective B-cogenerator. In the sequel we investigate $W^{\mathbb{N}}$-behaviors whose modules are f.g. B-modules.

For *constructivity considerations* we quote a result from [20] which furnishes constructive computations. In this we use an arbitrary commutative noetherian integral domain B of operators and an arbitrary injective cogenerator signal B-module ${}_B\mathscr{F}$. If $M = B^{1\times\ell}/B^{1\times k}R$ is a f.g. B-module its associated behavior is

$$D(M) := \mathrm{Hom}_B(M, \mathscr{F}) \cong \left\{ w \in \mathscr{F}^\ell;\ R \circ w = 0 \right\}. \tag{26}$$

A subclass (subcategory) $\mathfrak{C} \subseteq \mathrm{Mod}_B$ is called a *Serre subcategory* if it is closed under isomorphisms, submodules, factor modules, extensions and direct sums. The following corollary is an immediate consequence.

Corollary and Definition 2.7. *Each B-module M has a largest submodule* $\mathrm{Ra}_{\mathfrak{C}}(M)$ *in* $\mathfrak{C}$ *which is also the unique submodule U of M with $U \in \mathfrak{C}$ and* $\mathrm{Ra}_{\mathfrak{C}}(M/U) = 0$ *and is called the* $\mathfrak{C}$-radical *of M.*

Compare [4, §IV.1-2] for the next remarks. A prime ideal $\mathfrak{q} \in \mathrm{spec}(B)$ is called associated to ${}_BM$ if $B/\mathfrak{q}$ is a submodule of M (up to isomorphism) or, equivalently, if $\mathfrak{q} = \mathrm{ann}_B(x) := \{b \in B;\ bx = 0\}$ for some $x \in M$. The *associator* of M is the set of all associated prime ideals of M and denoted by $\mathrm{ass}(M)$. It is empty if and only if $M = 0$ and finite if M is f.g.. The *support* of M is $\mathrm{supp}(M) := \{\mathfrak{q} \in \mathrm{spec}(B);\ M_{\mathfrak{q}} \neq 0\}$ and is related to $\mathrm{ass}(M)$ via

$$\mathrm{supp}(M) = \{\mathfrak{q} \in \mathrm{spec}(B);\ \exists \mathfrak{p} \in \mathrm{ass}(M) \text{ with } \mathfrak{p} \subseteq \mathfrak{q}\}. \tag{27}$$

The zero submodule of a f.g. B-module M admits an irredundant primary decomposition [4, Thm. IV.2.1]

$$0 = \bigcap_{\mathfrak{q} \in \mathrm{ass}(M)} U(\mathfrak{q}) \text{ with } \mathrm{ass}(M/U(\mathfrak{q})) = \{\mathfrak{q}\}. \tag{28}$$

The modules $U(\mathfrak{q})$ for the minimal $\mathfrak{q} \in \mathrm{ass}(M)$ are unique, the so-called embedded components $U(\mathfrak{q})$ with non-minimal $\mathfrak{q} \in \mathrm{ass}(M)$ are not.

Result 2.8. *([20, §2]) Let B be commutative noetherian domain, ${}_B\mathscr{F}$ an injective cogenerator and $\mathfrak{C} \subseteq \mathrm{Mod}_B$ a Serre subcategory of torsion modules. Define*

$$\begin{gathered}\mathfrak{T} := \mathfrak{T}(\mathfrak{C}) := \{\mathfrak{b} \subseteq B;\ \mathfrak{b} \textit{ ideal},\ B/\mathfrak{b} \in \mathfrak{C}\},\\ \mathfrak{P}_1 := \mathfrak{P}_1(\mathfrak{C}) := \mathfrak{T} \bigcap \mathrm{spec}(B),\ \mathfrak{P}_2 := \mathfrak{P}_2(\mathfrak{C}) := \mathrm{spec}(B) \setminus \mathfrak{P}_1(\mathfrak{C}).\end{gathered} \tag{29}$$

The set $\mathfrak{T}$ is called the Gabriel topology *of $\mathfrak{C}$.*
1. If M is any B-module then

$$\begin{gathered}M \in \mathfrak{C} \iff \forall x \in M:\ \mathrm{ann}_B(x) \in \mathfrak{T} \iff \mathrm{ass}(M) \subseteq \mathfrak{P}_1 \implies \mathrm{supp}(M) \subseteq \mathfrak{P}_1\\ \textit{and } \mathrm{Ra}_{\mathfrak{C}}(M) = 0 \iff \mathrm{ass}(M) \subseteq \mathfrak{P}_2.\end{gathered}$$

If $M = \sum_{i=1}^r Bx_i$ is f.g. then $\mathrm{ann}_B(M) := \{b \in B;\ bM = 0\} = \bigcap_{i=1}^r \mathrm{ann}_B(x_i)$ *and hence $M \in \mathfrak{C}$ if and only if* $\mathrm{ann}_B(M) \in \mathfrak{T}$.
2. Let ${}_BM$ be f.g. with the primary decomposition (28)*. Define*
$U_i := \bigcap_{\mathfrak{q} \in \mathrm{ass}(M) \cap \mathfrak{P}_i} U(\mathfrak{q})$ *for $i = 1,2$, hence*

$$U_1 \bigcap U_2 = 0 \textit{ and } M \xrightarrow{\Delta} M/U_1 \times M/U_2,\ x \mapsto (x+U_1, x+U_2),\ \textit{is injective}. \tag{30}$$

Then

$$\mathrm{ass}(M/U_i) = \mathrm{ass}(M) \bigcap \mathfrak{P}_i \subseteq \mathfrak{P}_i,\ M/U_1 \in \mathfrak{C},\ U_2 = \mathrm{Ra}_{\mathfrak{C}}(M). \tag{31}$$

So U_2 and $D(M/U_2)$ are uniquely determined by M and $\mathfrak{C}$, but U_1 and $D(M/U_1)$ are not in general. By duality the injection (30) *induces the surjection*

$$D(M/U_1) \times D(M/U_2) \xrightarrow{D(\Delta)=+} D(M),\ \textit{i.e.},\ D(M) = D(M/U_1) + D(M/U_2). \tag{32}$$

If the primary decomposition in (28) *can be computed which is the case for the typical commutative operator domains B of systems theory and if $\mathfrak{q} \in \mathfrak{P}_i$ can be decided then all modules of this Result can be computed. Gröbner bases and Computer Algebra are used for such computations.*

Corollary 2.9. *Any decomposition $D(M) = D(M/V_1) + D(M/V_2)$ with $M/V_1 \in \mathfrak{C}$ and $\mathrm{Ra}_{\mathfrak{C}}(M/V_2) = 0$ implies $V_2 = \mathrm{Ra}_{\mathfrak{C}}(M)$.*

Proof. The decomposition implies $0 = V_1 \bigcap V_2$ and the injection $V_2 \to M/V_1,\ x \mapsto x + V_1$. Thus V_2 is a submodule of $M/V_1 \in \mathfrak{C}$ which implies $V_2 \in \mathfrak{C}$. Together with $\mathrm{Ra}_{\mathfrak{C}}(M/V_2) = 0$ this implies $V_2 = \mathrm{Ra}_{\mathfrak{C}}(M)$ by Cor. and Def. 2.7 □

In the present paper we apply Result 2.8 in the following three cases.

Example 2.10. 1. Let $\mathfrak{C} \subset \mathrm{Mod}_B$ be the category of all torsion modules. Then $\mathfrak{P}_1(\mathfrak{C}) = \{\mathfrak{q} \in \mathrm{spec}(B);\ \mathfrak{q} \neq 0\}$ and $\mathrm{Ra}_{\mathfrak{C}}(M) = \mathrm{tor}(M)$ is the torsion submodule of M. Equations (31) and (32) get the form

$$\begin{gathered} U_2 = \mathrm{tor}(M), M/U_1 = \mathrm{tor}(M/U_1), \text{ and} \\ \mathscr{B} := D(M) = D(M/U_1) + D(M/\mathrm{tor}(M)) = \mathscr{B}_{\mathrm{aut}} + \mathscr{B}_{\mathrm{cont}} \end{gathered} \tag{33}$$

is a sum decomposition of $\mathscr{B}$ into its largest controllable subbehavior $\mathscr{B}_{\mathrm{cont}}$ [13, Thm. 7.21] and an autonomous subbehavior $\mathscr{B}_{\mathrm{aut}}$. Such decompositions were independently derived by various colleagues. The modules in (33) can be computed for all standard rings of operators in systems theory via Gröbner bases and constructive primary decompositions.
2. Let $\mathfrak{C}_{\mathrm{fin}}$ be the category of all B-modules whose f.g. submodules are of finite length (f.l.). Then $\mathfrak{P}_{\mathrm{fin}} := \mathfrak{P}_1(\mathfrak{C}_{\mathrm{fin}}) = \max(B)$ [4, Prop. IV.2.7] and $\mathrm{Ra}_{\mathrm{fin}}(M) := \mathrm{Ra}_{\mathfrak{C}_{\mathrm{fin}}}(M) \subseteq \mathrm{tor}(M)$ is the largest submodule of M in $\mathfrak{C}_{\mathrm{fin}}$. If M is f.g. then $\mathrm{Ra}_{\mathrm{fin}}(M)$ is of f.l.. If B is F-affine then the Nullstellensatz shows that the modules of f.l. are exactly the F-finite dimensional (f.d.) ones. The decomposition $D(M) = D(M/U_1) + D(M/\mathrm{Ra}_{\mathrm{fin}}(M))$ is such that M/U_1 is of f.l. (f.d.) whereas $M/\mathrm{Ra}_{\mathrm{fin}}(M)$ has no submodules of f.l. (f.d.). If B is F-affine the maximal ideals are exactly the prime ideals $\mathfrak{q}$ with $\dim_F(B/\mathfrak{q})$. This and therefore the modules U_1 and U_2 can be constructively determined for all F-affine domains via Gröbner bases.
3. Let $T \subseteq B \setminus \{0\}$ be a multiplicatively closed subset which induces the quotient ring $B_T := \left\{\frac{b}{t};\ b \in B, t \in T\right\} \subseteq \mathrm{quot}(B)$(:=quotient field of B) and the exact quotient module functor $\mathrm{Mod}_B \to \mathrm{Mod}_{B_T},\ M \mapsto M_T$. Let $\mathfrak{C}(T)$ be the Serre category of all B-modules C with $C_T = 0$. Then

$$\begin{gathered} \mathfrak{T}(T) := \mathfrak{T}(\mathfrak{C}(T)) = \left\{\mathfrak{b} \subseteq B;\ \mathfrak{b} \bigcap T \neq \emptyset\right\},\ \mathfrak{P}_1(T) := \mathfrak{T}(T) \bigcap \mathrm{spec}(B), \\ \mathrm{Ra}_{\mathfrak{C}(T)}(M) = \mathrm{tor}_T(M) := \{x \in M;\ \exists t \in T \text{ with } tx = 0\} \subseteq \mathrm{tor}(M). \end{gathered}$$

The $\mathfrak{C}(T)$-radical is also called the *T-torsion submodule* of M. If ${}_BM$ is f.g. then $M \in \mathfrak{C}(T)$ if and only if there is $t \in T$ with $tM = 0$. The decision $\mathfrak{b} \cap T \neq \emptyset$ is not always constructive, but is so in the case of Def. and Cor. 3.3, see Cor. 3.8.

3 Time-autonomy

Assumption 2.6 is in force and its subsequent data are used, in particular the injective cogenerator ${}_AW$, the univariate polynomial algebra $B := A[\sigma]$ and the injective cogenerator ${}_BW^{\mathbb{N}}$. We identify $\left(W^{\mathbb{N}}\right)^{\ell} = \left(W^{\ell}\right)^{\mathbb{N}}$.

Definition 3.1. Consider a $W^{\mathbb{N}}$-behavior

$$\begin{gathered} \mathscr{B} = D(M) := D_{W^{\mathbb{N}}}(M) = U^{\perp} = \left\{w \in \left(W^{\mathbb{N}}\right)^{\ell};\ R \circ w = 0\right\} \text{ where} \\ R \in B^{k \times \ell},\ U = B^{1 \times k} R \subseteq B^{1 \times \ell},\ M = B^{1 \times \ell}/U. \end{gathered}$$

The behavior $\mathscr{B}$ is called time-autonomous (ta) if there is a time instant $d \in \mathbb{N}$ such that the A-linear map

$$\mathscr{B} \to W^d,\ w \mapsto (w(0),\cdots,w(d-1)),\ \text{is injective or}$$
$$w \in \mathscr{B} \text{ and } w(0) = \cdots = w(d-1) = 0 \text{ imply } w = 0.$$

This signifies that the whole trajectory $w \in \mathscr{B}$ is determined by its values in finitely many instants.

Lemma 3.2. *For the behavior $\mathscr{B}$ from Definition 3.1 assume that there is a*

$$\text{monic } \textit{polynomial } f = \sigma^d + \sum_{i=0}^{d-1} f_i\sigma^i \in \mathrm{ann}_B(M) := \{g \in B;\ gM = 0\} =$$
$$\mathrm{ann}_B(\mathscr{B}) := \{g \in B;\ g \circ \mathscr{B} = 0\}.$$

Then $\mathscr{B}$ is time-autonomous. The equality $\mathrm{ann}_B(M) = \mathrm{ann}_B(\mathscr{B})$ follows from the injective cogenerator property of $W^{\mathbb{N}}$. The result remains true, of course, if the leading coefficient $\mathrm{lc}_\sigma(f) \in A$ of f is invertible in A.

Proof. The condition $f \circ w = 0$ for $w \in \mathscr{B}$ signifies that

$$w(t+d) = -\sum_{i=0}^{d-1} f_i \circ w(t+i),\ t \in \mathbb{N}.$$

By induction on t the values $w(t) \in W$, $t \geq d$, can be computed from $w(i)$, $i \leq d-1$. Hence $\mathscr{B} \to W^d$, $w \mapsto (w(0),\cdots,w(d-1))$, is injective and $\mathscr{B}$ is ta. □

The preceding lemma gives rise to the following definitions.

Definition and Corollary 3.3. Let $\mathrm{U}(A)$ denote the group of invertible elements of A and

$$T := \left\{ f = \sum_{i=0}^{d} f_i\sigma^i \in B;\ \mathrm{lc}_\sigma(f) := f_d \in \mathrm{U}(A). \right\}$$

The subset $T \subset B$ is multiplicatively closed and saturated, i.e., each divisor of $t \in T$ also belongs to T, especially $\mathrm{U}(A) \subset T$. The set T gives rise to the quotient ring $B_T := \left\{ \frac{f}{t};\ f \in B, t \in T \right\} =: A < \sigma >$ (compare [9, pp.139-143]), the Serre subcategory $\mathfrak{C} := \mathfrak{C}(T)$ from Ex. 2.10,(3), and its radicals $\mathrm{Ra}_{\mathfrak{C}(T)}(M) = \mathrm{tor}_T(M)$ for $M \in \mathrm{Mod}_B$. A f.g. module M belongs to $\mathfrak{C}$ if and only if there is a $t \in T$ with $tM = 0$ (see Ex. 2.10,(3)).

Remark 3.4. Recall that a behavior $\mathscr{B} := D(M)$ is autonomous if and only if M is a f.g. torsion module, i.e., if there is a nonzero $f \in \mathrm{ann}_B(M) = \mathrm{ann}_B(\mathscr{B})$. The behavioral interpretation of this module fact has been treated by many colleagues and is the following: By duality there is the isomorphism

$$M \underset{\text{ident.}}{=} \mathrm{Hom}_B(B,M) \cong \mathrm{Hom}(D(M),D(B)) = \mathrm{Hom}(\mathscr{B},W^{\mathbb{N}}),\ x \mapsto x\circ, \tag{34}$$

where $\mathrm{Hom}(\mathscr{B}_1,\mathscr{B}_2)$ is the B-module of behavior morphisms from $\mathscr{B}_1$ to $\mathscr{B}_2$. Pommaret introduced the term *observables* of $\mathscr{B}$ for the elements of the module M in (34). By duality an observable $x : \mathscr{B} \to W^{\mathbb{N}}$, $x \in M$, is surjective, i.e., $x \circ \mathscr{B} = W^{\mathbb{N}}$, if and only if x is free or linearly independent in M, i.e., $B \cong Bx$, or x is not a torsion element of ${}_BM$. So ${}_BM$ is a torsion module if and only if $\mathscr{B}$ has *no free observables* (standard terminology: *no free variables*). In [13, Thm 2.69, (7.76-77)] it was shown that this is the case if and only if no projection

$$\mathscr{B} \subseteq \left(W^{\mathbb{N}}\right)^{\ell} \to W^{\mathbb{N}},\ w = (w_1, \cdots, w_\ell)^\top \mapsto w_i,$$

is free or surjective. Thm. 3.7 below implies that for the signal space $W = A^*$ a time-autonomous behavior is autonomous, but not conversely.

Lemma 3.5. *If M is F-f.d., i.e., $M = \mathrm{Ra}_{\mathrm{fin}}(M)$, then $M \in \mathfrak{C}(T)$ and hence $D(M)$ is ta. For every module this implies $\mathrm{Ra}_{\mathrm{fin}}(M) \subseteq \mathrm{tor}_T(M)$.*

Proof. Let $x = (x_1, \cdots, x_n)$ be an F-basis of M and $\sigma x = x\Delta$ with $\Delta \in F^{n\times n}$. The characteristic polynomial $t := \det(\sigma\,\mathrm{id}_n - \Delta) \in F[\sigma] \subseteq A[\sigma]$ is monic and $t \circ M = 0$. □

Lemma 3.2 shows that $\mathscr{B}$ is time-autonomous if $M \in \mathfrak{C}$. The main theorem below shows that for the injective A-cogenerator $W := A^*$ as in all discrete standard cases the condition $M \in \mathfrak{C}$ is also *necessary* for time-autonomy of $\mathscr{B} = D(M)$ and thus characterizes this property.

The proof of this main theorem is first carried out for the polynomial algebra $A = F[s_1, \cdots, s_m]$ and then extended to arbitrary A by means of Example 2.4. We use the following special notations:

$$\begin{gathered}\mathbb{N}^{1+m} = \mathbb{N} \times \mathbb{N}^m \ni \mu = (\mu_0, \cdots, \mu_n) = (\mu_0, \mu'),\ \mu' = (\mu_1, \cdots, \mu_n) \in \mathbb{N}^m,\\ s = (s_0, \cdots, s_m) = (s_0, s'),\ A = F[s'] \subset B = A[s_0] = F[s],\\ W := F[s']^* = F^{\mathbb{N}^m},\ B^* = F[s]^* = F^{\mathbb{N}^{1+m}} = F^{\mathbb{N}\times\mathbb{N}^m} = \left(F^{\mathbb{N}^m}\right)^{\mathbb{N}} = W^{\mathbb{N}}.\end{gathered} \tag{35}$$

So the indeterminate σ from above is now denoted by s_0. Vectorial signals in $\left(W^{\mathbb{N}}\right)^{\ell} = \left(W^{\ell}\right)^{\mathbb{N}}$ are written as

$$\begin{gathered} w = \left(w_i(\mu_0, \mu')\right)_{(i,\mu_0,\mu')\in[\ell]\times\mathbb{N}^{1+m}} = w(t)_{t\in\mathbb{N}},\ [\ell] := \{1, \cdots, \ell\}\\ w(t) := \left(w_1(t, \mu'), \cdots, w_\ell(t, \mu')\right)^\top_{\mu'\in\mathbb{N}^m}\end{gathered} \tag{36}$$

We need the Gröbner basis theory. Let $<$ be any term (well-)order on $\mathbb{N}^m$. With this we define the lexicographic term order on $\mathbb{N}^{1+m} = \mathbb{N} \times \mathbb{N}^m$ by

$$(\mu_0, \mu') < (\nu_0, \nu') \text{ in } \mathbb{N} \times \mathbb{N}^m :\Longleftrightarrow \mu_0 < \nu_0 \text{ or } \left(\mu_0 = \nu_0 \text{ and } \mu' < \nu'\right). \tag{37}$$

For a nonzero

$$\begin{gathered} f = \sum_{\mu\in\mathbb{N}^{1+m}} f_\mu s^\mu = \sum_{k\in\mathbb{N}} f_k s_0^k \in B = F[s],\ f_k := \sum_{\mu'\in\mathbb{N}^m} f_{(k,\mu')} s'^{\mu'} \in A, \text{ define}\\ d = (d_0, d') = \deg(f) := \max\left\{\mu \in \mathbb{N}^{1+m};\ f_\mu \neq 0\right\} \in \mathbb{N} \times \mathbb{N}^m\\ \deg_{s_0}(f) := \max\{k;\ f_k \neq 0\} \in \mathbb{N}\end{gathered} \tag{38}$$

as usual. In the same fashion we define $\deg(g)$ for $0 \neq g \in A = F[s']$ with the given term order on $\mathbb{N}^m$. Then the degrees and leading coefficients satisfy

$$\begin{gathered} d_0 = \deg_{s_0}(f),\ d' = \deg(f_{d_0})\\ \mathrm{lc}(f) := f_d \in F,\ \mathrm{lc}_{s_0}(f) := f_{d_0} \in A,\ \mathrm{lc}(f) = \mathrm{lc}(f_{d_0}) \text{ or}\\ f = \mathrm{lc}(f) s^d + \cdots = f_{d_0} s_0^{d_0} + \cdots \in B = F[s] = A[s_0],\\ f_{d_0} = \mathrm{lc}(f) s^{d'} + \cdots \in A = F[s'].\end{gathered} \tag{39}$$

These relations follow directly from the chosen lexicographic term order on $\mathbb{N}^{1+m} = \mathbb{N}\times\mathbb{N}^m$. For a nonzero $f\in B$ one likewise obtains

$$\begin{gathered} f\in A \iff \deg_{s_0}(f)=0 \iff \deg(f)\in\{0\}\times\mathbb{N}^m \text{ and}\\ f\in T \iff \deg(f)\in\mathbb{N}\times\{0\}\subset\mathbb{N}\times\mathbb{N}^m. \end{gathered} \tag{40}$$

We also need the degree for polynomial vectors in $B^{1\times\ell}$. For this purpose we define a term-over-position (well-)order on the index set $[\ell]\times\mathbb{N}\times\mathbb{N}^m$ by

$$\begin{gathered} (i,\mu_0,\mu')<(j,\nu_0,\nu') :\iff \mu_0<\nu_0 \text{ or } \big(\mu_0=\nu_0 \text{ and } \mu'<\nu'\big) \text{ or}\\ \big(\mu_0=\nu_0,\ \mu'=\nu' \text{ and } i<j\big). \end{gathered} \tag{41}$$

With the standard basis $\delta_i := (0,\cdots,0,\overset{i}{1},0,\cdots,0)$ of $B^{1\times\ell}$ the degree of *monomials* is $\deg(s^\mu\delta_i) =: (i,\mu)$. The componentwise (cw) partial order on $[\ell]\times\mathbb{N}^{1+m}$ is defined by

$$(i,\mu_0,\mu')\le_{\text{cw}}(j,\nu_0,\nu') :\iff i=j \text{ and } (\nu_0,\nu')\in(\mu_0,\mu')+\mathbb{N}^{1+m}. \tag{42}$$

The following considerations on the Cauchy problem are taken from [13] and in improved form from [14, Thm. 5]. Consider a behavior $\mathscr{B}$ defined by

$$\begin{gathered} U\subseteq B^{1\times\ell},\ M:=B^{1\times\ell}/U \text{ and } \mathscr{B}:=D(M)=U^\perp\subseteq\big(W^{\mathbb{N}}\big)^\ell. \text{ Then}\\ \deg(U):=\{\deg(u);\ 0\ne u\in U\} \end{gathered} \tag{43}$$

is the degree set of the submodule U. It has the form

$$\deg(U)=\biguplus_{i=1}^{\ell}\{i\}\times N_i \text{ with cw order ideals } N_i=N_i+\mathbb{N}^{1+m}\subseteq\mathbb{N}^{1+m}. \tag{44}$$

Its complement is called the *initial region* and is of the form

$$\Gamma:=\big([\ell]\times\mathbb{N}^{1+m}\big)\setminus\deg(U)=\biguplus_{i=1}^{\ell}\{i\}\times\Gamma_i,\ \Gamma_i\biguplus N_i=\mathbb{N}^{1+m}. \tag{45}$$

The restriction map

$$\mathscr{B}\to F^\Gamma,\ w\mapsto w|\Gamma=(w_i(\mu))_{(i,\mu)\in\Gamma}, \tag{46}$$

is an isomorphism, i.e., for every $x\in F^\Gamma$ there is a unique trajectory $w\in\mathscr{B}$ satisfying the initial condition $w|\Gamma=x$. Thus w is the unique solution of the *initial value or Cauchy problem*.

In the proof of the main theorem we also need the ring $A[[s_0^{-1}]]$ of power series and that of Laurent series in s_0^{-1} with coefficients in A, i.e.,

$$\begin{gathered} A[[s_0^{-1}]]\subset A<<s_0^{-1}>>:=\bigcup_{k=0}^{\infty}A[[s_0^{-1}]]s_0^k=A[s_0]\bigoplus A[[s_0^{-1}]]_+,\\ A[[s_0^{-1}]]_+:=A[[s_0^{-1}]]s_0^{-1}. \end{gathered} \tag{47}$$

Theorem 3.4 of [18] is an analogue of the next theorem in the continuous case, but only for behaviors $(Bf)^\perp$ defined by *principal ideals* $Bf,\ f\in B$.

Theorem 3.6. *(Compare [18, Thm. 3.4]) Data from above, especially* $A = F[s']$, $B = F[s]$ *and* $W = A^* = F^{\mathbb{N}^m}$. *The behavior* $\mathscr{B} = D(M)$ *is time-autonomous if and only* $M \in \mathfrak{C}$, *i.e., if and only if there is a monic polynomial* $f = s^d + f_{d-1}s^{d-1} + \cdots f_0 \in T \subset B = A[s_0] = F[s]$ *with* $fM = 0$ *or* $f \circ \mathscr{B} = 0$.

Proof. Due to Lemma 3.2 it remains to show that time-autonomy of $\mathscr{B}$ implies $fM = 0$ for some $f \in T$. Assume that $d \in \mathbb{N}$ is a finite instant such that $w \in \mathscr{B}$ and $w(0) = \cdots = w(d-1) = 0$ imply $w = 0$.
1. We show first that each order ideal N_i from (44) has a non-empty intersection with $\mathbb{N} \times \{0\} \subset \mathbb{N} \times \mathbb{N}^m = \mathbb{N}^{1+m}$. We argue indirectly and assume that some N_i has an empty intersection with this set, say N_1 for simplicity, thus

$$\mathbb{N} \times \{0\} \subseteq \Gamma_1 \text{ and } \{1\} \times \mathbb{N} \times \{0\} \subset \Gamma.$$

For $(i,\mu) \in \Gamma$ we choose the initial condition vector $x \in F^\Gamma$ by

$$x(i,\mu) := \begin{cases} 0 & \text{for } \mu_0 \le d-1 \\ 1 & \text{for } i = 1,\ d \le \mu_0,\ \mu' = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Let $w \in \mathscr{B}$ the unique trajectory with $w \mid \Gamma = x$, in particular

$$\begin{aligned} w_i(\mu) = x(i,\mu) = 0 \text{ for } (i,\mu) \in \Gamma \text{ and } \mu_0 \le d-1 \text{ and} \\ w_1(t,0) = x(1,t,0) = 1, \text{ hence } w(t) \ne 0 \text{ for } t \ge d. \end{aligned} \tag{48}$$

We are now going to show $w(t) = 0$ for $t \le d-1$ and thus obtain a contradiction to the assumed time-autonomy: By transfinite induction on $(i,\mu) \in [\ell] \times \mathbb{N}^{1+m}$ with $\mu_0 \le d-1$ we show that $w_i(\mu) = 0$. For $(i,\mu) \in \Gamma$ this is true by definition of x and w. If in contrast $(i,\mu) \in \deg(U),\ \mu_0 \le d-1$, then there is a vector

$$\begin{aligned} u = s^\mu \delta_i + \sum_{(j,\nu)<(i,\mu)} u_{(j,\nu)} s^\nu \delta_j \in U \underset{w \in \mathscr{B} = U^\perp}{\Longrightarrow} u \circ w = 0 \Longrightarrow \\ 0 = (u \circ w)(0) = w_i(\mu) + \sum_{(j,\nu)<(i,\mu)} u_{(j,\nu)} w_j(\nu). \end{aligned}$$

But $(j,\nu) < (i,\mu)$ implies $\nu_0 \le \mu_0 \le d-1$ with respect to the term order from (41) and therefore by (transfinite) induction $w_j(\nu) = 0$ for these (j,ν), hence also $w_i(\mu)$=0.
2. We have thus shown that for each $i = 1, \cdots, \ell$ there is a polynomial vector $u_i \in U$ of degree $\deg(u_i) = (i, d(i), 0) \in [\ell] \times \mathbb{N} \times \mathbb{N}^m$. Again the term order from (41) implies the representations

$$u_i = s_0^{d(i)} \sum_{j \le i} f_{ij} \delta_j + v_i,\ f_{ij} \in F,\ f_{ii} \ne 0,\ \deg_{s_0}(v_i) < d(i). \tag{49}$$

With $f_{ij} := 0$ for $i < j$ the square matrices P resp. $P_{\mathrm{lc}} \in A[s_0]^{\ell \times \ell}$ with rows $P_{i-} := u_i$ resp. $(P_{\mathrm{lc}})_{i-} = v_i$ thus have the form

$$\begin{aligned} P = \Delta P_{hc} + P_{lc},\ \Delta := \operatorname{diag}(s_0^{d(1)}, \cdots, s_0^{d(\ell)}), \\ P_{hc} := (f_{ij})_{ij} \in \mathrm{Gl}_\ell(F),\ \deg_{s_0}((P_{\mathrm{lc}})_{i-}) < d(i). \end{aligned}$$

The matrix $P_{\mathrm{hc}} \in F^{\ell \times \ell}$ of *highest coefficients* is invertible as lower triangular matrix with nonzero elements in the main diagonal. Since the rows of P lie in U we have

$$\begin{aligned} P \circ w = 0, \text{ hence also } \det(P) \circ w = 0 \text{ for } w \in \mathscr{B} = U^\perp \subseteq \left(W^{\mathbb{N}}\right)^\ell \\ \text{and } \det(P)M = 0,\ M = B^{1\times\ell}/U. \end{aligned}$$

We finally show $\det(P) \in T$ and thus $M \in \mathfrak{C}$: The following matrix calculations are performed over the coefficient ring $A << s_0^{-1} >>$ of Laurent power series:

$$
\begin{gathered}
P = \Delta P_{\rm hc} + P_{\rm lc} = \Delta\left(\mathrm{id}_\ell + \Delta^{-1} P_{\rm lc} P_{\rm hc}^{-1}\right) P_{\rm hc}. \text{ But} \\
\deg_{s_0}\left((P_{\rm lc})_{i-}\right) < d(i) \Longrightarrow \Delta^{-1} P_{\rm lc} P_{\rm hc}^{-1} \in \left(s_0^{-1} A[[s_0^{-1}]]\right)^{\ell\times\ell} \Longrightarrow \\
\det\left(\mathrm{id}_\ell + \Delta^{-1} P_{\rm lc} P_{\rm hc}^{-1}\right) = 1 + s_0^{-1} g \in 1 + s_0^{-1} A[[s_0^{-1}]] \Longrightarrow \\
\det(P) = \det(P_{\rm hc}) \det(\Delta)(1 + s_0^{-1} g) = \det(P_{\rm hc}) s_0^{d(1)+\cdots+d(\ell)}(1 + s_0^{-1} g) \in A[s_0] \\
\Longrightarrow \det(P) = \det(P_{\rm hc}) s_0^{d(1)+\cdots+d(\ell)} + \cdots \in T.
\end{gathered}
$$

This completes the proof of the theorem. □

With the trick from Example 2.4 we extend the preceding theorem to arbitrary affine integral domains of the form

$$
\begin{gathered}
A = F[s']/I \subset B := A[s_0],\ s = (s_0, s') = (s_0, \cdots, s_m),\ I \in \mathrm{spec}(F[s']), \\
A' := F[s'] \subset B' := A'[s_0] = F[s], \text{ hence} \\
B = B'/B'I,\ B'I = I[s_0] \in \mathrm{spec}(F[s]),
\end{gathered} \tag{50}
$$

with injective cogenerators

$$
{}_A W := A^*,\ {}_B W^{\mathbb{N}} = B^*,\ {}_{F[s']} F^{\mathbb{N}^m} = A'^*,\ {}_{F[s]} F^{\mathbb{N}^{1+m}} = \left(F^{\mathbb{N}^m}\right)^{\mathbb{N}} = B'^* \tag{51}
$$

with the identifications discussed in Examples 2.2-2.5 We consider a ${}_{A[s_0]} W^{\mathbb{N}}$-behavior

$$
\mathscr{B} := D(M),\ M = B^{1\times\ell}/U,\ U \subseteq B^{1\times\ell}. \tag{52}
$$

Theorem 3.7. *For the data from* (50)-(52) *the behavior* $\mathscr{B}$ *is time-autonomous if and only if* $M \in \mathfrak{C}$ *(compare Def. and Cor. 3.3). This applies in particular to* $A = F[N]$ *and* $W = F^N$ *from Example 2.5.*

Proof. We apply Example 2.4 to the canonical epimorphism

$$
\begin{gathered}
\mathrm{can} : B' = F[s] \to B'/B'I = F[s]/F[s]I = A[s_0], \text{ especially} \\
W^{\mathbb{N}} = A[s_0]^* \cong (F[s]I)^\perp = \left(I^\perp\right)^{\mathbb{N}} \subseteq F^{\mathbb{N}^{1+m}} = \left(F^{\mathbb{N}^m}\right)^{\mathbb{N}} \\
w \mapsto w',\ w(t)(s'^{\mu'} + I) = w'(t)(\mu') = w'(t,\mu'),\ (t,\mu') \in \mathbb{N}\times\mathbb{N}^m.
\end{gathered}
$$

The module U induces the inverse image

$$
U' := \mathrm{can}^{-1}(U) \subseteq B'^{1\times\ell} \text{ with behavior } \mathscr{B}' := U'^\perp \subseteq \left(F^{\mathbb{N}^{1+m}}\right)^\ell
$$

and the B- and B'-isomorphisms

$$
\begin{gathered}
\mathrm{can}_{\rm ind} : M' := B'^{1\times\ell}/U' \cong M = B^{1\times\ell}/U \text{ and} \\
\mathscr{B} \cong \mathscr{B}',\ w \mapsto w',\ w(t)(s'^{\mu'} + I) = w'(t)(\mu') = w'(t,\mu').
\end{gathered} \tag{53}
$$

Since $\mathscr{B}$ is time-autonomous by assumption so is $\mathscr{B}'$. This follows directly from the special form of the isomorphism (53). From the preceding theorem we infer the existence of a monic polynomial

$$f' := s_0^d + \sum_{i=0}^{d-1} f_i' s_0^i \in B' = A'[s_0] \text{ with } f'M' = 0, \text{ hence also } f'M = 0.$$

Since M is a $B = B'/B'I$-module this implies the desired property

$$fM = 0 \text{ for } f := f' + B'I = s_0^d + \sum_{i=0}^{d-1} (f_i + I)s_0^i \in B = A[s_0],\ A = F[s']/I.$$

□

Corollary 3.8. *(*Computation of $\mathrm{tor}_T(M)$*). Consider the situation of Ex. 2.10,(3) with T from Def. and Cor. 3.3. For any ideal $\mathfrak{b}$ of $B = F[s'][\sigma]$, $\sigma = s_0$, and the term order* (37) *we have the equivalence*

$$\mathfrak{b} \in \mathfrak{T}(T),\ \textit{i.e.,}\ T \cap \mathfrak{b} \neq \emptyset \iff \mathbb{N} \times \{0\} \cap \deg(\mathfrak{b}) \neq \emptyset,\ \mathbb{N} \times \{0\} \subset \mathbb{N}^{m+1} \tag{54}$$

as in the proof of Thm. 3.6. This can be checked via Gröbner bases. For arbitrary F-affine B one proceeds as in Thm. 3.7. Hence the inclusion $\mathfrak{q} \in \mathfrak{P}_i(T)$ can be decided and therefore all modules in Ex. 2.10,(3), especially, $U_2 = \mathrm{tor}_T(M)$ can also be computed via Gröbner bases. Since

$$\mathrm{tor}_T(M) = \begin{cases} M & \textit{if and only } \mathrm{ass}(M) \subset \mathfrak{P}_1(T) \\ 0 & \textit{if and only } \mathrm{ass}(M) \subset \mathfrak{P}_2(T) \end{cases}$$

also $\mathrm{tor}_T(M) = M$ *and* $\mathrm{tor}_T(M) = 0$ *can be decided via Gröbner bases.*

4 Time-controllability

The program of this section was described in the Introduction. Assumption 2.6 with its derived data remains in force.

The following definition of time-controllability (tc) is again inspired by the models in [18] and [3, Def. 4.1].

Definition 4.1. A behavior $\mathscr{B}$ (as in Def. 3.1) is called time-controllable (tc) if there is a time-lag $L \in \mathbb{N}$ such that for all $p \geq 0$ and trajectories $w_1, w_2 \in \mathscr{B}$ there is another trajectory $w \in \mathscr{B}$ such that

$$w(t) = \begin{cases} w_1(t) & \text{for } t \leq p-1 \\ w_2(t-(p+L)) & \text{for } t \geq p+L \end{cases}$$

The second equation for $t \geq p+L$ can also be expressed as

$$(w(p+L), w(p+L+1, w(p+L+2), \cdots) = \sigma^{p+L} \circ w = w_2 = \\ (w_2(0), w_2(1), w_2(2), \cdots).$$

These conditions signify that in the time interval $\{p, \cdots, p+L-1\} \subset \mathbb{N}$ the two trajectories w_1 starting at $t = 0$ and w_2 starting at $t = p+L$ can be concatenated to another trajectory in $\mathscr{B}$.

The next lines are variants of standard arguments in behavioral duality theory. We need them for the behavioral treatment of subspaces

$$\left\{w \in W^{\mathbb{N}};\ w(0) = \cdots = w(p-1) = 0\right\}$$

which are A-submodules, but not B-submodules, i.e., not shift-invariant. We use the *Gelfand map*

$$\begin{aligned} &\rho_M : M \to \mathrm{Hom}_A\left(\mathrm{Hom}_A(M,W),W\right) \subseteq W^{\mathrm{Hom}_A(M,W)},\ x \mapsto \rho_M(x),\\ &\rho_M(x)(\varphi) = \varphi(x),\ x \in M,\ \varphi \in \mathrm{Hom}_A(M,W),\\ &\text{with kernel } \ker(\rho_M) = \{x \in M,\ \forall \varphi \in \mathrm{Hom}_A(M,W):\ \varphi(x) = 0\}. \end{aligned} \tag{55}$$

Recall that the cogenerator property of ${}_AW$ is equivalent with the injectivity of ρ_M for all ${}_AM$. For the extension of the module-behavior duality we define the A-bilinear form

$$A[\sigma] \times W^{\mathbb{N}} \to W,\ < \sum_{t\in\mathbb{N}} f_t \sigma^t, w > := \sum_{t\in\mathbb{N}} f_t \circ w(t), \tag{56}$$

which induces the isomorphism

$$\mathrm{Hom}_A(A[\sigma],W) \cong W^{\mathbb{N}},\ \varphi = (f \mapsto < f,w >) \leftrightarrow w = \left(\varphi(\sigma^t)\right)_{t\in\mathbb{N}}. \tag{57}$$

For an A-submodule $\mathfrak{b} \subseteq B = A[\sigma]$, which is not necessarily a B-submodule or ideal of B, and an A-submodule $\mathscr{B} \subseteq W^{\mathbb{N}}$ we define the orthogonal submodules

$$\mathfrak{b}^{\perp} := \left\{w \in W^{\mathbb{N}};\ < \mathfrak{b}, w >= 0\right\} \text{ and } \mathscr{B}^{\perp} := \{f \in B = A[\sigma];\ < f, \mathscr{B} >= 0\} \tag{58}$$

which form a Galois correspondence as usual. If $\mathfrak{b}$ is an ideal of B then (see (11))

$$\mathfrak{b}^{\perp} = \left\{w \in W^{\mathbb{N}};\ \mathfrak{b} \circ w = 0\right\} = \left\{w \in W^{\mathbb{N}};\ < \mathfrak{b}, w >= 0\right\}. \tag{59}$$

Via the homomorphism theorem the isomorphism (57) induces the isomorphism

$$\mathrm{Hom}_A(A[\sigma]/\mathfrak{b},W) \cong \mathfrak{b}^{\perp} (\subseteq W^{\mathbb{N}}),\ \varphi \mapsto w,\ \varphi(f+\mathfrak{b}) = < f,w > \tag{60}$$

by which we identify the two modules.

Lemma 4.2. *For an A-submodule $\mathfrak{b}$ of B:*

$$\ker\left(\rho_{B/\mathfrak{b}} : B/\mathfrak{b} \to W^{\mathrm{Hom}_A(B/\mathfrak{b},W)} \cong W^{\mathfrak{b}^{\perp}}\right) = \mathfrak{b}^{\perp\perp}/\mathfrak{b},\ \textit{hence } \mathfrak{b}^{\perp\perp} = \mathfrak{b}$$

since ${}_AW$ is a cogenerator and $\rho_{B/\mathfrak{b}}$ is injective.

Proof. $f + \mathfrak{b} \in \ker(\rho_{B/\mathfrak{b}}) \iff \forall \varphi \in \mathrm{Hom}_A(B/\mathfrak{b},W):\ \varphi(f+\mathfrak{b}) = 0 \iff$ $\forall w \in \mathfrak{b}^{\perp} : < f,w >= 0 \iff f \in \mathfrak{b}^{\perp\perp}$. ☐

The preceding arguments can be directly extended to A-submodules $U \subseteq B^{1\times\ell}$, $\ell \in \mathbb{N}$. The A-bilinear form $< -,- >$ is extended to

$$B^{1\times\ell} \times \left(W^{\mathbb{N}}\right)^{\ell} \to W,\ < (f_1,\cdots,f_\ell),(w_1,\cdots,w_\ell)^{\top} > := \sum_{i=1}^{\ell} < f_i, w_i >. \tag{61}$$

The A-submodule $U \subseteq B^{1\times\ell}$ gives rise to its orthogonal A-submodule

$$U^{\perp} = \left\{w \in \left(W^{\mathbb{N}}\right)^{\ell};\ < U, w >= 0\right\} \tag{62}$$

and again the cogenerator property of ${}_AW$ implies $U = U^{\perp\perp}$. The isomorphism (60) generalizes to $\mathrm{Hom}_A(B^{1\times\ell}/U, W)\} \cong U^{\perp}$. If U is a B-submodule then $U^{\perp}$ is the usual behavior. As in standard behavior theory the duality relations

$$(U_1+U_2)^{\perp} = U_1^{\perp}\bigcap U_2^{\perp},\ \left(U_1\bigcap U_2\right)^{\perp} = U_1^{\perp}+U_2^{\perp},\ U_i \subseteq B^{1\times\ell}, \qquad (63)$$

hold where the first is obvious and the second follows from the injectivity of ${}_AW$.

Remark 4.3. Notice that $B = A[\sigma]$ is a noetherian ring or B-module, but not a f.g. or noetherian A-module. Nor are the A-submodules $U \subseteq B^{1\times\ell}$ and their factor A-modules $B^{1\times\ell}/U$ f.g. or noetherian, but the duality holds nevertheless. We will call $U^{\perp}$ the A(!!)-behavior defined by the equation module U. These new behaviors are proper generalizations of the usual B-behaviors $V^{\perp} \subseteq \left(W^{\mathbb{N}}\right)^{\ell}$ for B-submodules $V \subseteq B^{1\times\ell}$.

Example 4.4. For $p \in \mathbb{N}$ consider the A-submodule

$$A[\sigma]_p := \oplus_{t=0}^{p-1} A\sigma^t \subset B = A[\sigma]$$

of polynomials of σ-degree less than p. Then

$$A[\sigma]_p^{\perp} = \left\{w \in W^{\mathbb{N}};\ w(t) = <\sigma^t, w> = 0 \text{ for } 0 \le t \le p-1\right\} = \left\{w = (0, \cdots, \overset{p-1}{0}, w(p), \cdots) \in W^{\mathbb{N}}\right\}.$$

In this fashion initial conditions $w(0) = \cdots = w(p-1) = 0$ for trajectories can be formulated on the module side.

Lemma 4.5. *Let U be a B-submodule of $B^{1\times\ell}$, $M := B^{1\times\ell}/U$ and $\mathscr{B} := D(M) = U^{\perp} \subseteq \left(W^{\mathbb{N}}\right)^{\ell}$. Then the following conditions are equivalent:*
1. $\mathscr{B}$ is time-controllable (tc).
2. There is a time lag $L \ge 0$ such that for any $p \ge 0$ the map

$$\sigma^{p+L}\circ : \{w \in \mathscr{B};\ w(0) = \cdots = w(p-1) = 0\} \to \mathscr{B} \qquad (64)$$

is surjective.
3. (i) $\sigma\circ : \mathscr{B} \to \mathscr{B}$ is surjective or, by duality, $\sigma : M \to M$ is injective. In other words this signifies that $\sigma \in B$ is a non-zero-divisor of M and this can be constructively checked by standard computer algebra.
(ii) There is a time lag L such that for every $p \ge 0$ and $w_1 \in \mathscr{B}$ there is a $w \in \mathscr{B}$ with $w(t) = w_1(t)$ for $t \le p-1$ and $\sigma^{p+L} \circ w = 0$. This property is called zero time controllability *(ztc) of $\mathscr{B}$ by Bisiacco and Valcher [3, Def. 4.1].*

Proof. 1. $\Longrightarrow$ 2. : The condition follows from Def. 3.1 by the choice $w_1 = 0$.
2. $\Longrightarrow$ 1. : Assume $p \ge 0$ and $w_1, w_2 \in \mathscr{B}$. From 2. we infer that there is $w' \in \mathscr{B}$ with $w'(0) = \cdots = w'(p-1) = 0$ and $\sigma^{p+L} \circ w' = w_2 - \sigma^{p+L} \circ w_1$. But then $w := w_1 + w'$ satisfies $w(t) = w_1(t)$ for $t \le p-1$ and $\sigma^{p+L} \circ w = w_2$ as required.
1. $\Longrightarrow$ 3. : (i) From 2. we infer the surjectivity of $\sigma^{p+L} : \mathscr{B} \to \mathscr{B}$ for all $p > 0$. But this especially implies the surjectivity of $\sigma\circ : \mathscr{B} \to \mathscr{B}$.
(b) This follows from the definition of tc by the choice $w_2 = 0$.
3. $\Longrightarrow$ 1. : Let $p \ge 0$ and $w_1, w_2 \in \mathscr{B}$. By condition (i) we can choose $w' \in \mathscr{B}$ with $\sigma^{p+L} \circ w' = w_2$. By condition (ii) we get $w'' \in \mathscr{B}$ with $w''(t) = w_1(t) - w'(t)$ for $t \le p-1$ and $\sigma^{p+L} \circ w'' = 0$. Then $w := w' + w''$ satisfies $w(t) = w_1(t)$ for $t \le p-1$ and $\sigma^{p+L} \circ w = w_2$. □

The next lemma is a variant of the fact that a both autonomous and controllable behavior is zero. In the continuous case it was shown in [18, Lemma 4.1].

Lemma 4.6. *A both time-autonomous and time-controllable behavior is zero.*

Proof. Assume that L is a time lag for the tc $\mathscr{B}$ and that $w \in \mathscr{B}$ is zero if $w(0) = \cdots = w(d-1) = 0$. From the preceding lemma we infer the surjection

$$\sigma^{d+L} : \{w \in \mathscr{B};\ w(0) = \cdots = w(d-1) = 0\} \to \mathscr{B}. \text{ But}$$
$$\{w \in \mathscr{B};\ w(0) = \cdots = w(d-1) = 0\} = 0, \text{ hence } \mathscr{B} = 0.$$

□

The following lemma is a variant of results in [24] for standard controllability.

Lemma 4.7. *The image of a time-controllable behavior is again such, i.e., if a behavior $\mathscr{B}' \subseteq \left(W^{\mathbb{N}}\right)^{\ell'}$ is tc and if $P \in B^{\ell \times \ell'}$ then also $\mathscr{B} := P \circ \mathscr{B}'$ is tc. In particular, a controllable behavior is also tc.*

Proof. Let L' be a time lag of $\mathscr{B}'$ and $d := \deg_\sigma(P)$. Then P has the form

$$P = \sum_{k \le d} P_k \sigma^k \in B^{\ell \times \ell'},\ P_k \in A^{\ell \times \ell'}, \text{ hence}$$
$$(P \circ w')(t) = \sum_{k \le d} P_k \circ w'(t+k) \text{ for } w' \in \mathscr{B}' \text{ and } t \in \mathbb{N}.$$

We show that $L := d + L'$ is a suitable time lag for $\mathscr{B}$: Let $p \geq 0$ and $w_1, w_2 \in \mathscr{B} = P \circ \mathscr{B}'$. Choose $w_1', w_2' \in \mathscr{B}'$ with $P \circ w_i' = w_i$. Since $\mathscr{B}'$ is tc there are $w' \in \mathscr{B}'$ and $w := P \circ w' \in \mathscr{B}$ with

$$w'(t) = w_1'(t) \text{ for } t \le p+d-1 \text{ and } \sigma^{p+d+L'} \circ w' = w_2' \Longrightarrow$$
$$\forall t \le p-1 :\ w(t) = (P \circ w')(t) = \sum_{k \le d} P_k \circ w'(t+k) =$$
$$\sum_{k \le d} P_k \circ w_1'(t+k) = (P \circ w_1')(t) = w_1(t) \text{ and } \sigma^{p+L} \circ w =$$
$$\sigma^{p+d+L'} \circ w = \sigma^{p+d+L'} \circ P \circ w' = P \circ \sigma^{p+d+L'} \circ w' = P \circ w_2' = w_2.$$

The signal space $W^{\mathbb{N}}$ is tc with lag $L = 0$. Since a controllable behavior $\mathscr{B}$ admits an image representation, i.e., an epimorphism $P \circ : \left(W^{\mathbb{N}}\right)^{\ell'} \to \mathscr{B}$, it is tc by the present lemma.

□

The next theorem is a discrete analogue of Thm. 4.3 of [18]. We use the data of Def. and Cor. 3.3.

Theorem 4.8. *If a behavior $D(M)$ is time-controllable (tc) then* $\mathrm{tor}_T(M) = 0$*, hence also* $\mathrm{Ra}_{\mathrm{fin}}(M) = 0$ *by Lemma 3.5. Due to Example 4.17 below the reverse implication* $(\mathrm{tor}_T(M) = 0 \Longrightarrow D(M)$ *tc) does* not *hold. Due to Cor. 3.8 the necessary condition* $\mathrm{tor}_T(M) = 0$ *for tc can be checked constructively.*

Proof. The injection $\mathrm{inj} : \mathrm{tor}_T(M) \to M$ induces the surjection $D(\mathrm{inj}) : D(M) \to D(\mathrm{tor}_T(M))$. Since $D(M)$ is tc so is $D(\mathrm{tor}_T(M))$ by Lemma 4.7. But the latter behavior is also time-autonomous since $\mathrm{tor}_T(M) \in \mathfrak{C}$. With Lemma 4.6 we conclude $D(\mathrm{tor}_T(M)) = 0$ and finally $\mathrm{tor}_T(M) = 0$. □

The next theorem exhibits a large class of time-controllable behaviors. Notice that $B = A[\sigma] = \oplus_{t\in\mathbb{N}} A\sigma^t$ is a free A-module with the A-basis $\sigma^t,\ t \in \mathbb{N}$. In particular, the extension $A \subset B$ is faithfully flat, i.e., the functor $B \otimes_A (-) : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is faithfully exact. For A-modules $U' \subseteq M'$ this implies

$$\begin{aligned} M'[\sigma] := B\otimes_A M' = BM' \supseteq M' \underset{\text{ident.}}{=} A\otimes_A M', \\ B\otimes_A U' = U'[\sigma] = BU' \subseteq BM', \\ B\otimes_A \left(M'/U'\right) \underset{\text{ident.}}{=} BM'/BU'. \end{aligned} \tag{65}$$

For the construction of tc behaviors we need the so-called extended modules: Let

$$\begin{aligned} R \in A^{k\times\ell},\ U' := A^{1\times k}R \subseteq A^{1\times\ell},\ M' := A^{1\times\ell}/U' \text{ and} \\ BU' = U'[\sigma] = \oplus_{t\in\mathbb{N}} U'\sigma^t = B^{1\times k}R \subseteq B^{1\times\ell} = \oplus_{t\in\mathbb{N}} A^{1\times\ell}\sigma^t, \\ B^{1\times\ell}/BU' \underset{\text{ident.}}{=} B\otimes_A M'. \end{aligned} \tag{66}$$

The f.g. $A[\sigma]$-modules of the form $A[\sigma]\otimes_A M'$ are called *extended*. They play an essential part in the solution of Serre's problem on projective modules [9, p.5]. The associators of M' and of $B\otimes_A M'$ are related [4, ch.IV, §2.6, Thm.2, Cor.1] via

$$\mathrm{ass}(B\otimes_A M') = \left\{B\mathfrak{p} = \mathfrak{p}[\sigma];\ \mathfrak{p}\in \mathrm{ass}(M')\right\},\ \mathfrak{p} = A\cap\mathfrak{p}[\sigma]. \tag{67}$$

Notice here that $B/B\mathfrak{p} = A[\sigma]/\mathfrak{p}[\sigma] = (A/\mathfrak{p})[\sigma]$ is an integral domain and hence $\mathfrak{p}[\sigma]$ is indeed a prime ideal of B, but never maximal since a polynomial ring is not a field.

Lemma 4.9. *For the modules from* (66) *the* ${}_BW^{\mathbb{N}}$*-behavior*

$$\mathscr{B} := D(B\otimes_A M') = (BU')^{\perp} = \left\{w\in W^{\mathbb{N}};\ R\circ w = 0\right\} = \left(U'^{\perp}\right)^{\mathbb{N}}$$

is time-controllable with time lag $L = 0$.

Proof. The decisive fact is that the matrix R has coefficients in A only, i.e., no shifts. For $w \in \left(W^{\mathbb{N}}\right)^{\ell}$ this implies

$$R\circ w = 0 \iff \forall t\in\mathbb{N}:\ R\circ w(t) = 0,\ \text{hence } (BU')^{\perp} = \left(U'^{\perp}\right)^{\mathbb{N}}$$
$$\text{where } U'^{\perp}\subseteq W^{\ell} \text{ and } \mathscr{B} = (BU')^{\perp} \subseteq \left(W^{\mathbb{N}}\right)^{\ell}.$$
$$\text{If} p\geq 0 \text{ and } w_1, w_2\in\mathscr{B} = \left(U'^{\perp}\right)^{\mathbb{N}} \text{ then } w(t) := \begin{cases} w_1(t) & \text{for } t\leq p-1 \\ w_2(t) & \text{for } t\geq p = p+0 \end{cases}$$

is a trajectory of $\mathscr{B}$ which concatenates the trajectories w_1 and w_2 of $\mathscr{B}$.

□

Theorem 4.10. *Data from* (66) *and Lemma 4.9. In addition let* $P\in B^{k\times\ell}$ *and*

$$\begin{aligned} U := (\circ P)^{-1}(BU') \subseteq B^{1\times k},\ \textit{hence} \\ (\circ P)_{\mathrm{ind}} : M := B^{1\times k}/U \to B^{1\times\ell}/BU' = B\otimes_A M' \textit{ is injective.} \end{aligned}$$

Indeed each submodule of $B\otimes_A M'$ *is the image of* $(\circ P)_{\mathrm{ind}}$ *for some* P*. Then:*

1. *The behavior* $\mathscr{B} := U^{\perp}$ *is time-controllable.*

2. *Each prime ideal* $\mathfrak{q} \in \operatorname{ass}(M)$ *is extended, i.e., of the form* $\mathfrak{q} = B(A \bigcap \mathfrak{q}) = (A \bigcap \mathfrak{q})[\sigma]$.

3. *There is a sum decomposition* $\mathscr{B} = \mathscr{B}_{\mathrm{cont}} + \mathscr{B}_{\mathrm{aut,tc}}$ *with a both autonomous and time-controllable behavior* $\mathscr{B}_{\mathrm{aut},tc}$. *The proof of the theorem below constructs the decomposition via primary decompositions which can be realized in standard Computer Algebra systems.*

Proof. 1. By duality the injection $(\circ P)_{\mathrm{ind}}$ induces the surjection

$$P\circ : \mathscr{B}' := (BU')^{\perp} \to \mathscr{B} := U^{\perp}, \text{ hence } P\circ\mathscr{B}' = \mathscr{B}.$$

By Lemma 4.9 $\mathscr{B}'$ is tc and by Lemma 4.7 $\mathscr{B}$ is too.
2. Since M is a submodule of $B \otimes_A M'$, $M' = A^{1\times\ell}/U'$, up to isomorphism, the inclusion

$$\operatorname{ass}(M) \subseteq \operatorname{ass}(B \otimes_A M') = \left\{B\mathfrak{p} = \mathfrak{p}[\sigma];\ \mathfrak{p} \in \operatorname{ass}(M')\right\}$$

holds where the last equality follows from (67).
3. The images of controllable, autonomous resp. tc behaviors are of the same type. Since $P\circ\mathscr{B}' = \mathscr{B}$ it thus suffices to show that $\mathscr{B}'$ admits a decomposition as in 3. We apply Ex. 2.10,(1), to the f.g. A-module M' and obtain a decomposition

$$0 = U_1' \bigcap U_2',\ U_i' \subset M',\ U_2' = \operatorname{tor}({}_AM'),\ M'/U_1' = \operatorname{tor}(M'/U_1').$$

With the identifications from (65) we conclude

$$BM' = B \otimes_A M',\ 0 = BU_1' \bigcap BU_2',\ B \otimes_A (M'/U_i') = BM'/BU_i', \text{ and}$$
$$D(B \otimes_A M') = D(BM'/BU_1') + D(BM'/BU_2')$$

Since the $B \otimes_A (M'/U_i')$ are extended the dual behaviors are tc. Since M'/U_1' is torsion so is $B \otimes_A (M'/U_1')$, hence $D(BM'/BU_1')$ is autonomous and tc. The injection $BU_2' \to BM'/BU_1'$ implies that also BU_2' is a torsion module. Since M'/U_2' is torsionfree there is an embedding $M'/U_2' \to A^{1\times m}$. The exactness of $B \otimes_A (-)$ implies the embedding

$$BM'/BU_2' = B \otimes_A \left(M'/U_2'\right) \to B \otimes_A A^{1\times m} = B^{1\times m}.$$

Hence BM'/BU_2' is torsionfree which implies $BU_2' = \operatorname{tor}(BM')$, $D(BM'/BU_2') = D(BM')_{\mathrm{cont}}$ and the asserted decomposition. □

Lemma 4.11. *For an affine integral domain A,* $W := A^*$, $B = A[\sigma]$ *and* $W^{\mathbb{N}} \cong B^*$ *a subbehavior of an autonomous and time-controllable* $W^{\mathbb{N}}$*-behavior is in general not of the same type. So the torsion modules with tc* $D(M)$ *do not generate a Serre subcategory and do not satisfy the intuitive properties of "small or negligible" behaviors, compare Remark 4.16.*

Proof. Let $\mathfrak{a}$ be a nonzero proper ideal of A and $\mathfrak{b} := B\mathfrak{a} = \mathfrak{a}[\sigma]$, hence $0 \subsetneq \mathfrak{b} \subsetneq B$. By the preceding lemma $\mathfrak{b}^{\perp} \subset W^{\mathbb{N}}$ is tc. Let $\mathfrak{n} \in \max(B)$ be a maximal ideal with $\mathfrak{n} \supset \mathfrak{b}$ and hence $0 \neq \mathfrak{n}^{\perp} \subset \mathfrak{b}^{\perp}$. By Hilbert's Nullstellensatz $B/\mathfrak{n}$ is F-finite-dimensional. From Lemma 3.5 we infer that $\mathfrak{n}^{\perp}$ is time-autonomous and then from Lemma 4.6 that $\mathfrak{b}^{\perp}$ is not time-controllable. □

The following, presently non-constructive, algebraic characterization of time-controllability is restricted to autonomous behaviors $\mathfrak{b}^\perp,\ 0\neq\mathfrak{b}\subseteq B$, for simplicity. For $L,p\in\mathbb{N}$ define the A-submodule

$$\begin{gathered}\mathfrak{b}(L,p):=\left\{g\in B=A[\sigma];\ g\sigma^{L+p}\in A[\sigma]_p+\mathfrak{b}\right\}\subseteq B.\\ \text{For } p=0 \text{ this gives } \mathfrak{b}(L,0)=(\mathfrak{b}:\sigma^L):=\left\{g\in B;\ g\sigma^L\in\mathfrak{b}\right\}.\end{gathered}\tag{68}$$

Since $\sigma A[\sigma]_p\subset A[\sigma]_{p+1}$ and $\sigma\mathfrak{b}\subseteq\mathfrak{b}$ one gets the increasing sequence

$$\mathfrak{b}\subseteq\mathfrak{b}(L,0)=(\mathfrak{b}:\sigma^L)\subseteq\mathfrak{b}(L,1)\subseteq\mathfrak{b}(L,2)\subseteq\cdots.\tag{69}$$

Below we are going to use the duality theory from (56)- (63) and Results 4.2-4.4.

Theorem 4.12. *The following properties of an autonomous behavior $\mathscr{B}:=\mathfrak{b}^\perp,\ 0\neq\mathfrak{b}\subseteq B$, are equivalent:*
1. The behavior is time-controllable.
2. There is $L\geq 0$ such that for all $p\geq 0$ the equality $\mathfrak{b}(L,p)=\mathfrak{b}$ and moreover $(\mathfrak{b}:\sigma):=\{f\in B;\ f\sigma\in\mathfrak{b}\}=\mathfrak{b}$ hold. The last condition signifies that $\sigma: B/\mathfrak{b}\to B/\mathfrak{b}$ is injective or, equivalently by duality, that $\sigma\circ:\mathfrak{b}^\perp\to\mathfrak{b}^\perp$ is surjective (compare Lemma 4.5).
3. There is $L\geq 0$ such that for all $p\geq 0$ the inclusion $A[\sigma]_p\bigcap\left(B\sigma^{p+L}+\mathfrak{b}\right)\subset\mathfrak{b}$ and moreover $(\mathfrak{b}:\sigma)=\mathfrak{b}$ hold.
If the conditions in 2. or 3. hold for L then also for all $L'\geq L$.

Proof. 1. $\Longleftrightarrow$ 2 : Recall from Lemma 4.5 that $\mathfrak{b}^\perp$ is tc if and only if there exists $L\geq 0$ such that for all $p\geq 0$ the map

$$\sigma^{p+L}:\left\{w\in\mathfrak{b}^\perp;\ \forall t\leq p-1:\ w(t)=0\right\}\to\mathfrak{b}^\perp\underset{\text{ident.}}{=}\operatorname{Hom}_A(A[\sigma]/\mathfrak{b},W)\tag{70}$$

is surjective. But

$$\begin{gathered}\mathscr{B}=\mathfrak{b}^\perp \text{ and } \left\{w\in W^{\mathbb{N}};\ w(0)=\cdots=w(p-1)=0\right\}=A[\sigma]_p^\perp\Longrightarrow\\ \{w\in\mathscr{B};\ \forall t\leq p-1:\ w(t)=0\}=(A[\sigma]_p+\mathfrak{b})^\perp\\ \underset{\text{ident.}}{=}\operatorname{Hom}_A\left(A[\sigma]/\left(A[\sigma]_p+\mathfrak{b}\right),W\right)\end{gathered}$$

By duality (70) is surjective if and only if $\sigma^{p+L}: A[\sigma]/\mathfrak{b}\to A[\sigma]/\left(A[\sigma]_p+\mathfrak{b}\right)$ is injective. But the kernel of this map is

$$\left\{g+\mathfrak{b}\in A[\sigma]/\mathfrak{b};\ g\sigma^{p+L}\in A[\sigma]_p+\mathfrak{b}\right\}=\mathfrak{b}(L,p)/\mathfrak{b},$$

hence the equivalence of conditions 1. and 2.
2. $\Longrightarrow$ 3.: Let

$$\begin{gathered}g=f\sigma^{p+l}+h\in A[\sigma]_p\bigcap\left(B\sigma^{p+L}+\mathfrak{b}\right)\Longrightarrow\\ f\sigma^{p+L}=g-h\in A[\sigma]_p+\mathfrak{b}\Longrightarrow f\in\mathfrak{b}(L,p)\underset{2.}{=}\mathfrak{b}\Longrightarrow g\in\mathfrak{b}.\end{gathered}$$

3. $\Longrightarrow$ 2.: The equality $(\mathfrak{b}:\sigma)=\mathfrak{b}$ or injectivity of $\sigma: B/\mathfrak{b}\to B/\mathfrak{b}$ implies the injectivity of $\sigma^{p+L}: B/\mathfrak{b}\to B/\mathfrak{b}$ and hence $\left(\mathfrak{b}:\sigma^{p+L}\right)=\mathfrak{b}$. Let

$$\begin{gathered}f\in\mathfrak{b}(L,p)\Longrightarrow f\sigma^{p+L}=g+h\in A[\sigma]_p+\mathfrak{b}\Longrightarrow\\ g=f\sigma^{p+L}-h\in A[\sigma]_p\bigcap\left(B\sigma^{p+L}+\mathfrak{b}\right)\underset{3.}{\subseteq}\mathfrak{b}\Longrightarrow\\ f\sigma^{p+L}=g+h\in\mathfrak{b}\Longrightarrow f\in\left(\mathfrak{b}:\sigma^{p+L}\right)=\mathfrak{b}\Longrightarrow\mathfrak{b}(L,p)=\mathfrak{b}.\end{gathered}$$

□

Remark 4.13. For fixed L, p the condition 3. of the preceding theorem can be constructively checked via Gröbner bases in all standard cases as we are going to show promptly. Nevertheless the preceding algebraic characterization of time-controllability is not constructive because condition 3. has to be checked for possibly infinitely many L and infinitely many p.
Consider the special case that

$$A = F[s'] = F[s_1, \cdots, s_m] \text{ and } B = A[s_0] = F[s] = F[s_0, s'],\ s_0 := \sigma.$$

We use the term order and ensuing degree from from (37) and (39). Consider any nonzero ideal $\mathfrak{b} \subseteq B$ and any pair (L, p). Let $G := G(L, p)$ be the unique reduced Gröbner basis of $B\sigma^{p+L} + \mathfrak{b}$ with respect to this term order. For any $f \in B\sigma^{p+L} + \mathfrak{b}$ constructive division without remainder furnishes a representation

$$\begin{gathered} f = \sum_{g\in G} h_g g,\ h_g \in B,\ \deg(h_g g) = \deg(h_g) + \deg(g) \leq \deg(f) \Longrightarrow \\ \deg_\sigma(h_g g) = \deg_\sigma(h_g) + \deg_\sigma(g) \leq \deg_\sigma(f) \Longrightarrow \\ \forall f \in A[\sigma]_p \bigcap \left(B\sigma^{p+L} + \mathfrak{b}\right) : f = \sum \left\{h_g g;\ g \in G,\ \deg_\sigma(g) < p\right\}. \text{ Hence} \\ A[\sigma]_p \bigcap \left(B\sigma^{p+L} + \mathfrak{b}\right) \subset \mathfrak{b} \Longleftrightarrow \forall g \in G(L,p) \text{ with } \deg_\sigma(g) < p:\ g \in \mathfrak{b}. \end{gathered} \tag{71}$$

Hence condition 3. of Thm. 4.12 can be constructively checked for each pair (L, p).

The next results furnish further interesting necessary conditions for tc.

Lemma 4.14. *Consider an ideal*

$$\begin{gathered} \mathfrak{b} \subseteq B \text{ with } \mathfrak{b}^\perp = D(B/\mathfrak{b}) \subseteq W^{\mathbb{N}} \text{ and } \mathfrak{a} := A \bigcap \mathfrak{b},\ \text{hence } B\mathfrak{a} = \mathfrak{a}[\sigma] \subseteq \mathfrak{b}. \\ \text{Then } \mathfrak{a}^\perp = \mathrm{proj}_0(\mathfrak{b}^\perp) := \left\{w(0);\ w \in \mathfrak{b}^\perp\right\}. \end{gathered} \tag{72}$$

Proof. The injection $A \subset B = A[\sigma]$ induces the injection $\mathrm{inj}_{\mathrm{ind}} : A/\mathfrak{a} \to B/\mathfrak{b}$ and the commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{inj}} & B \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ A/\mathfrak{a} & \xrightarrow[\mathrm{inj}_{\mathrm{ind}}]{} & B/\mathfrak{b} \end{array} \qquad \text{and}$$

$$\begin{array}{ccc} W = \mathrm{Hom}_A(A,W) & \xleftarrow{\mathrm{proj}_0 = \mathrm{Hom}(\mathrm{inj},W)} & W^{\mathbb{N}} = \mathrm{Hom}_A(B,W) \\ {\scriptstyle \subseteq = \mathrm{Hom}(\mathrm{can},W)}\uparrow & & \uparrow{\scriptstyle \subseteq = \mathrm{Hom}(\mathrm{can},W)} \\ \mathfrak{a}^\perp = \mathrm{Hom}_A(A/\mathfrak{a},W) & \xleftarrow[\mathrm{proj}_0 = \mathrm{Hom}(\mathrm{inj}_{\mathrm{ind}},W)]{} & \mathfrak{b}^\perp = \mathrm{Hom}_A(B/\mathfrak{b},W) \end{array}$$

Since $\mathrm{inj}_{\mathrm{ind}}$ is injective $\mathrm{proj}_0 = \mathrm{Hom}(\mathrm{inj}_{\mathrm{ind}}, W) : \mathfrak{b}^\perp \to \mathfrak{a}^\perp$ is surjective by duality, hence $\mathfrak{a}^\perp = \mathrm{proj}_0(\mathfrak{b}^\perp)$. ☐

Recall [4, Ch.IV, §1, Prop.9, Cor.2 of Prop.2] that for any f.g. B-module M with annihilator $\mathfrak{b} := \mathrm{ann}_B(M) \subseteq B$, for instance $M = B/\mathfrak{b}$, the identities

$$\begin{aligned}\sqrt{\mathfrak{b}} &:= \{f \in B;\ \exists n \in \mathbb{N} \text{ with } f^n \in \mathfrak{b}\} = \bigcap_{\mathfrak{q} \in \mathrm{ass}(M)} \mathfrak{q} \\ \{f \in B;\ f : M \to M \text{ injective}\} &= \bigcap_{\mathfrak{q} \in \mathrm{ass}(M)} (B \setminus \mathfrak{q})\end{aligned} \tag{73}$$

hold. Also let $\mathrm{lc}(0) := 0$ and

$$\mathrm{lc}(\mathfrak{b}) := \left\{\mathrm{lc}(f) \in A;\ f = \mathrm{lc}(f)\sigma^{\deg_\sigma(f)} + \cdots \in \mathfrak{b}\right\} \supset \mathfrak{a} = A\bigcap \mathfrak{b} \tag{74}$$

be the ideal in A of all leading coefficients of polynomials in $\mathfrak{b}$. It is easy to see that

$$\mathrm{lc}(\mathfrak{b}) = A\bigcap \mathfrak{b} \iff \mathfrak{b} = (A\bigcap \mathfrak{b})[\sigma], \tag{75}$$

i.e., $\mathfrak{b}$ is extended from A.

Theorem 4.15. *1. Assume $\mathfrak{b} \subseteq B$ and that $\mathscr{B} := \mathfrak{b}^\perp$ is time-controllable with time lag L. Let $\mathfrak{a} := A\bigcap \mathfrak{b}$. Then $a^{L+1} \in \mathfrak{a}$ for all $a \in \mathrm{lc}(\mathfrak{b})$, hence $\mathrm{lc}(\mathfrak{b}) \subseteq \sqrt{\mathfrak{a}}$. If $\mathfrak{b}^\perp$ is autonomous, i.e., $\mathfrak{b} \neq 0$, then also $\mathfrak{a} \neq 0$.*

2. If $\mathfrak{b} = \sqrt{\mathfrak{b}}$, especially if $\mathfrak{b} \in \mathrm{spec}(B)$, then $\mathfrak{b} = B\mathfrak{a} = \mathfrak{a}[\sigma]$, i.e., $\mathfrak{b}$ is extended from A.

3. Assume that M is any f.g. B-module with time-controllable behavior $\mathscr{B} := D(M)$ and let $x \in M$ be any observable with annihilator $\mathfrak{b} := \mathrm{ann}_B(x) = \mathrm{ann}_B(x \circ \mathscr{B})$. If $\sqrt{\mathfrak{b}} = \mathfrak{b}$ then $\mathfrak{b}$ is extended from A. In particular, any prime $\mathfrak{q} \in \mathrm{ass}(M)$ is extended from A. This property can be constructively checked via Gröbner bases and was proven for submodules of extended modules in Thm. 4.10,(2).

Proof. 1. The assertion is obvious for $\mathfrak{b} = 0$, so assume $\mathfrak{b} \neq 0$. Let $f = f_d\sigma^d + \cdots \in \mathfrak{b}$, $0 \neq f_d \in A$, and $v \in \mathfrak{a}^\perp = \mathrm{proj}_0(\mathfrak{b}^\perp)$. (see Lemma 4.14). The tc implies the surjection

$$\sigma^{d+L} : \left\{w \in \mathfrak{b}^\perp;\ \forall t \le d-1 :\ w(t) = 0\right\} \to \mathfrak{b}^\perp,$$

hence there is a $w \in \mathfrak{b}^\perp$ with $w(t) = 0$ for $t \le d-1$ and $w(d+L) = v$. But

$$f \circ w = 0 \Longrightarrow \forall t : f_d \circ w(t+d) = -\sum_{i=0}^{d-1} f_i \circ w(t+i) \in \sum_{i=0}^{d-1} A \circ w(t+i) =$$

$$\sum_{i=0}^{d-1} A \circ w(t+d-1-i) \Longrightarrow f_d^2 \circ w(t+d) \in \sum_{i=0}^{d-1} A \circ w(t+d-2-i) \underset{\text{induction}}{\Longrightarrow}$$

$$f_d^{L+1} \circ w(t+d) \in \sum_{i=0}^{d-1} A \circ w(t+d-(L+1)-i) \text{ if } t \ge L \Longrightarrow$$

$$\text{for } t := L \text{ and } w(L+d) = v :\ f_d^{L+1} \circ v \in \sum_{i=0}^{d-1} A \circ w(d-1-i) = 0,$$

the last equality following from $w(t) = 0$ for $t \le d-1$. We conclude

$$\forall a = f_d = \mathrm{lc}(f) \in \mathrm{lc}(\mathfrak{b}) :\ a^{L+1} \circ \mathfrak{a}^\perp = 0 \Longrightarrow a^{L+1} \in \mathfrak{a}^{\perp\perp} = \mathfrak{a}.$$

2. The equality $\sqrt{\mathfrak{a}} = A\bigcap\sqrt{\mathfrak{b}}$ is simple, hence $\sqrt{\mathfrak{b}} = \mathfrak{b}$ implies $\sqrt{\mathfrak{a}} = \mathfrak{a}$ and then, by 1., $\mathrm{lc}(\mathfrak{b}) = \mathfrak{a}$. With (75) we conclude $\mathfrak{b} = \mathfrak{a}[\sigma] = B\mathfrak{a}$.
3. Recall that $M \cong \mathrm{Hom}_B(B,M) \cong \mathrm{Hom}(D(M),W^{\mathbb{N}})$ by which each $x \in M$ can be interpreted as a difference operator $x\circ : D(M) \to W^{\mathbb{N}}$. The injection $B/\mathfrak{b} \cong Bx \subseteq M$ gives rise to the surjection $x\circ : D(M) \to D(Bx) \cong D(B/\mathfrak{b}) = \mathfrak{b}^{\perp}$ by duality, hence $x\circ \mathscr{B} = \mathfrak{b}^{\perp}$. As image of the tc $\mathscr{B}$ also $\mathfrak{b}^{\perp}$ is tc. The assertion now follows from 2.. □

Lemma 4.16. *The behavior* $\mathfrak{b}^{\perp}$, $\mathfrak{b} \subseteq B$, *is time-controllable with time lag* $L = 0$ *if and only if* $\mathfrak{b}$ *is extended, i.e.,* $\mathfrak{b} = \mathfrak{a}[\sigma]$, $\mathfrak{a} := A\bigcap\mathfrak{b}$, *or* $\mathrm{lc}(\mathfrak{b}) = \mathfrak{a}$.

Proof. By Lemma 4.9 the condition is sufficient for tc with lag $L = 0$. For the necessity we use the data from Thm. 4.12. Let $\mathfrak{b}^{\perp}$ be tc with $L = 0$. For each $p \geq 0$ this implies $\mathfrak{b}(0,p) = \left\{f \in B;\ f\sigma^p \in A[\sigma]_p + \mathfrak{b}\right\} = \mathfrak{b}$. Let

$$f = f_d\sigma^d + r \in \mathfrak{b},\ r \in A[\sigma]_d, \text{ i.e., } \deg_\sigma(r) < d,\ f_d = \mathrm{lc}(f) \in \mathrm{lc}(\mathfrak{b}) \Longrightarrow$$
$$f_d\sigma^d = -r + f \in A[\sigma]_d + \mathfrak{b} \Longrightarrow f_d \in \mathfrak{b}(0,d) = \mathfrak{b} \Longrightarrow \mathrm{lc}(\mathfrak{b}) \subseteq A\bigcap\mathfrak{b} = \mathfrak{a}.$$

□

Example 4.17. Consider the bivariate polynomial ring

$$B = F[s_0,s_1] = F[s_1][s_0] \text{ with } \sigma := s_0,\ W = F^{\mathbb{N}},\ W^{\mathbb{N}} = F^{\mathbb{N}^2},$$

the irreducible polynomial $q := s_0s_1 + 1$, the prime ideal $\mathfrak{q} = F[s_0,s_1](s_0s_1 + 1)$ and autonomous behavior $\mathfrak{q}^{\perp} = \left\{w \in F^{\mathbb{N}^2};\ w(\mu_0+1,\mu_1+1) + w(\mu_0,\mu_1) = 0\right\}$. Consider the saturated multiplicatively closed set T from Def. and Cor. 3.3. The polynomial q does not belong to T and hence $\mathfrak{q} \cap T = \emptyset$ due to the saturation of T. Thus for all $t \in T$ the multiplication $t\circ : B/\mathfrak{q} \to B/\mathfrak{q}$ is injective which implies $\mathrm{tor}_T(B/\mathfrak{q}) = 0$. The prime ideal $\mathfrak{q}$ is not extended from $F[s_1]$ since $q \notin F[s_1]$. From $\mathrm{ass}(B/\mathfrak{q}) = \{\mathfrak{q}\}$ and Thm. 4.15,(3), we infer that $\mathfrak{q}^{\perp}$ is not time-controllable. This example shows that the converse of Thm. 4.8 does not hold.

5 Krull dimension two

The goal of this Section is a slight generalization and sharpening of Bisiacco's and Valcher's characterization of two-dimensional time-controllability [3, Thm. 5.1] and to give a new, constructive *and shorter module theoretic proof of this result.*
We assume that A is an F-affine principal ideal domain, $B = A[\sigma]$ and ${}_AW$ an injective cogenerator. The case of Bisiacco and Valcher [loc.cit.] is $A = F[s_1,s_1^{-1}]$. The Krull dimension (dim) of B is $\dim(B) = \dim(A[\sigma]) = \dim(A) + 1 = 2$ [10, Thm. 15.4]. For each maximal ideal $\mathfrak{n}$ of B one has $\dim(B_{\mathfrak{n}}) = 2$ [10, Thm. 15.1]. Since A is factorial so is B (Gauß), hence the minimal nonzero prime ideals are of the form Bq with a prime $q \in B$. The only maximal chains of prime ideals are of the form $0 \subsetneq Bq \subsetneq \mathfrak{n}$ where q is a prime element of B and $\mathfrak{n}$ is maximal. This implies

$$\mathrm{spec}(B) \setminus \max(B) = \{0\} \uplus \{Bq;\ q \in B \text{ prime}\}. \tag{76}$$

According to Result 2.8 and Ex. 2.10,(2), we conclude for a f.g. B-module M:

$$\begin{aligned} M = \mathrm{Ra}_{\mathrm{fin}}(M) &\Longleftrightarrow \mathrm{ass}(M) \subset \max(B), \\ \mathrm{Ra}_{\mathrm{fin}}(M) = 0 &\Longleftrightarrow \mathrm{ass}(M) \subseteq \{0\} \uplus \{Bq;\ q \in B \text{ prime}\}. \end{aligned} \tag{77}$$

Notice that the zero ideal is associated with M if and only if M is not a torsion module. The ring B is regular [10, Thm. 19.5]. This implies that its global dimension is two [10, Thm. 19.2]. Each f.g. projective B- module is free [9, Quillen-Suslin Thm. V.2.9]. We need the following

Result 5.1. *A f.g. torsion module M has $\mathrm{Ra}_{\mathrm{fin}}(M) = 0$ if and only if M is of the form $M \cong B^{1\times r}/B^{1\times r}\Delta$ with a* square matrix Δ *of* $\mathrm{rank}(\Delta) = r$. *The associated behavior $D(M)$ is then called* square autonomous *in [21], [12] and [3]. At the end of this section we will give a new constructive and short proof of this result in greater generality.*

For a prime $q \in B$ and $m > 0$

$$Bq^{m-1}/Bq^m \cong B/Bq \Longrightarrow \mathrm{ass}\left(Bq^{m-1}/Bq^m\right) = \mathrm{ass}(B/Bq) = \{Bq\} \Longrightarrow$$
$$\text{the filtration } B \supset Bq \supset \cdots \supset Bq^m \text{ implies } \mathrm{ass}(B/Bq^m) = \{Bq\}\,.$$

If $0 \neq b \in B$ has the prime factor decomposition $b = \prod_{i=1}^r q_i^{m_i}$, $m_i > 0$, the isomorphism $Bbq_i^{-1}/Bb \cong B/Bq_i$ implies $Bq_i \in \mathrm{ass}(B/Bb)$. On the other hand

$$Bb = \bigcap_{i=1}^r Bq_i^{m_i} \Longrightarrow B/Bb \overset{\text{ident.}}{\subseteq} \prod_{i=1}^r B/Bq_i^{m_i} \Longrightarrow \mathrm{ass}(B/Bb) \subseteq \mathrm{ass}\left(\prod_{i=1}^r B/Bq_i^{m_i}\right) =$$
$$\bigcup_{i=1}^r \mathrm{ass}\left(B/Bq_i^{m_i}\right) = \{Bq_i;\ i = 1,\cdots,r\} \Longrightarrow \mathrm{ass}(B/Bb) = \{Bq_i;\ i = 1,\cdots,r\}\,. \tag{78}$$

Corollary 5.2. *Let M be a torsion module with (nonzero) annihilator $\mathfrak{b} = \mathrm{ann}_B(M)$. If $\mathrm{Ra}_{\mathrm{fin}}(M) = 0$, i.e., if Result 5.1 is applicable to M, then $\mathfrak{b}$ is a principal ideal.*

Proof. We use (77). The zero ideal is not associated to M. Let $M = \sum_{i=1}^k Bx_i$. The map $B/\mathfrak{b} \to M^k$, $b + \mathfrak{b} \mapsto (bx_i)_{i=1,\cdots,k}$ is injective and hence $\mathrm{Ra}_{\mathrm{fin}}(B/\mathfrak{b}) = 0$ since $\mathrm{Ra}_{\mathrm{fin}}(M) = 0$. Let d be the greatest common divisor of the elements of $\mathfrak{b}$ or, in other words, let Bd be the least *principal ideal* which contains $\mathfrak{b}$. Thus $\mathfrak{b} = d\mathfrak{c}$ with an ideal $\mathfrak{c}$ which is contained in no principal ideal. In particular, no associated prime of $B/\mathfrak{c}$ is principal and therefore $\mathrm{ass}(B/\mathfrak{c}) \subseteq \max(B)$ and $B/\mathfrak{c} = \mathrm{Ra}_{\mathrm{fin}}(B/\mathfrak{c})$ by (77). On the other hand, the map $B/\mathfrak{c} \to B/\mathfrak{b}, b + \mathfrak{c} \mapsto bd + \mathfrak{b}$, is injective, thus $\mathrm{Ra}_{\mathrm{fin}}(B/\mathfrak{b}) = 0$ implies $\mathrm{Ra}_{\mathrm{fin}}(B/\mathfrak{c}) = 0$ and hence $B/\mathfrak{c} = 0$, $\mathfrak{c} = B$ and $\mathfrak{b} = Bd$. □

The next theorem is the announced slight generalization and sharpening of [3, Thm. 5.1] with a different, constructive and shorter proof.

Theorem 5.3. (Characterization of time-controllable behaviors in dimension two) *Let A be an F-affine principal ideal domain, $B = A[\sigma]$ and M a f.g. B-module with behavior $\mathscr{B} := D(M)$. The following properties are equivalent:*

1. *M is a submodule of an extended module. The proof shows how an embedding of M into an extended module can be constructed.*
2. *$\mathscr{B}$ admits a decomposition $\mathscr{B} = \mathscr{B}_{\mathrm{cont}} + \mathscr{B}_{\mathrm{aut,tc}}$ with an autonomous and time-controllable behavior $\mathscr{B}_{\mathrm{aut,tc}}$. This decomposition was constructed in Thm. 4.10,(3).*
3. *$\mathscr{B}$ is time-controllable.*
4. *Each associated prime ideal of M is extended, i.e., zero or of the form Bp with a prime of A, and this can again be constructively checked via Gröbner bases.*

Proof. 1. $\Longrightarrow$ 2.: Theorem 4.10.
2. $\Longrightarrow$ 3.: It follows directly from the definition of tc that the sum of two tc behaviors is again such.
3. $\Longrightarrow$ 4.: By Thm. 4.15 each $\mathfrak{q}\in\operatorname{ass}(M)$ is extended and therefore zero or of the form $\mathfrak{q}=Bp$ with a prime p of A since no maximal ideal is extended.
4. $\Longrightarrow$ 1.: (i) Assume first that M is a torsion module, hence the zero ideal is not associated to M. From (77) and Cor. 5.2 we infer that $\mathrm{Ra}_{\mathrm{fin}}(M)=0$ and that the annihilator $\mathfrak{b}=\operatorname{ann}_B(M)=Bb$ is principal with $\operatorname{ass}(B/Bb)\subseteq\operatorname{ass}(M)$, hence $\operatorname{ass}(B/Bb)=\{Bp_i;\ i=1,\cdots,k\}$ where the p_i are primes in A and in B. From (78) we infer $\mathfrak{b}=Ba,\ a:=\prod_{i=1}^k p_i^{m_i}$. With $\mathrm{Ra}_{\mathrm{fin}}(M)=0$ Result 5.1 furnishes a representation $M\underset{\text{ident.}}{=}B^{1\times r}/B^{1\times r}\Delta$ with a *square matrix* Δ. The equation $aM=0$ implies

$$aB^{1\times r}\subseteq B^{1\times r}\Delta\Longrightarrow a\,\mathrm{id}_r=X\Delta\Longrightarrow a^r=\det(a\,\mathrm{id}_r)=\det(X)\det(\Delta)\in A\underset{B=A[\sigma]}{\Longrightarrow}$$
$$\det(\Delta)\in A\Longrightarrow B/B\det(\Delta)\text{ and }(B/B\det(\Delta))^{1\times r}\text{ are extended.}$$

The adjoint matrix Δ_{adj} induces the injection

$$M=B^{1\times r}/B^{1\times r}\Delta\overset{(\circ\Delta_{\mathrm{adj}})_{\mathrm{ind}}}{\longrightarrow}(B/B\det(\Delta))^{1\times r}$$

and therefore M is a submodule of an extended one.
(ii) Assume that M is not a torsion module so that the zero ideal belongs to $\operatorname{ass}(M)$. Result 2.8 and Ex. 2.10,(1), furnish a representation

$$0=U_1\bigcap U_2,\ U_2=\operatorname{tor}(M),\ M/U_1=\operatorname{tor}(M/U_1),$$
$$\operatorname{ass}(M/U_1)=\operatorname{ass}(M)\setminus\{0\}\subset\{Bp;\ p\in A\text{ prime}\}.$$

By (i) the module M/U_1 is a submodule of an extended module N_1. As torsionfree module M/U_2 is a submodule of a free and thus extended module N_2. But then $M\underset{\text{ident.}}{\subseteq}M/U_1\times M/U_2\subseteq N_1\times N_2$ and therefore M is a submodule of the extended module $N_1\times N_2$.

□

Corollary 5.4. *If M satisfies the equivalent conditions of the preceding theorem then each annihilator $\operatorname{ann}_B(x)$ of $x\in M$ is extended. In particular, if $\mathfrak{b}$ is an ideal of B and $\mathfrak{b}^\perp$ is tc then $\mathfrak{b}$ is extended, i.e., of the form $\mathfrak{b}=Ba$ with $a\in A$.*

Proof. If $x\notin\operatorname{tor}(M)$ then the annihilator is zero. If $x\in\operatorname{tor}(M)$ then $B/\operatorname{ann}_B(x)\cong Bx\subseteq M$ and hence $D(B/\operatorname{ann}_B(x))$ is tc as image of $D(M)$. Its annihilator $\operatorname{ann}_B(x)$ is extended according to 4. $\Longrightarrow$ 1.,(i), of the proof of Thm. 5.3. □

In dimension higher than two the preceding corollary does not hold.

Example 5.5. The ring $A/\mathfrak{c}$ of this example is discussed in [8, p. 419,(5)] as a Frobenius algebra. Let

$$A=F[s_1,s_2]\subset B=A[s_0]=F[s_0,s_1,s_2],$$
$$\mathfrak{c}:=As_1s_2+A(s_1^2-s_2^2)\subset\mathfrak{c}[s_0]=B\mathfrak{c}=Bs_1s_2+B(s_1^2-s_2^2).\text{ Define}$$
$$P:=s_0s_1+s_2,\ \mathfrak{b}:=(\mathfrak{c}[s_0]:P):=\{b\in B;\ bP\in\mathfrak{c}[s_0]\},\ \mathfrak{a}:=A\cap\mathfrak{b}.$$

The behavior $\mathfrak{b}^\perp \subseteq F^{\mathbb{N}^3}$ is tc by Thm. 4.10,(1). With respect to the pure lexicographic term order $s_0 > s_1 > s_2$ the reduced Gröbner basis of $\mathfrak{b}$ is s_2^2, s_1s_2, s_1^2, $s_0s_2 - s_1$, hence

$$\mathfrak{b}^\perp = \left\{ w \in F^{\mathbb{N}^3};\ \begin{cases} w(\mu_0,\mu_1,\mu_2) = 0 \text{ for } \mu_1 \geq 2 \\ w(\mu_0,\mu_1,\mu_2) = 0 \text{ for } \mu_2 \geq 2, \\ w(\mu_0,\mu_1,\mu_2) = 0 \text{ for } \mu_1 \geq 1,\ \mu_2 \geq 1 \\ w(\mu_0+1,\mu_1,\mu_2+1) - w(\mu_0,\mu_1+1,\mu_2) = 0 \end{cases} \right\}.$$

Due to the choice of the term order as in (37) a Gröbner basis of $\mathfrak{a} := A \cap \mathfrak{b}$ is

$$s_2^2, s_1s_2, s_1^2, \text{ thus } B\mathfrak{a} = \mathfrak{a}[s_0] = \oplus_{k=0}^{\infty} \mathfrak{a} s_0^k \not\ni s_0s_2 - s_1 \text{ and } B(A \cap \mathfrak{b}) \subsetneq \mathfrak{b}.$$

This shows that $\mathfrak{b}$ is not extended although $\mathfrak{b}^\perp$ is tc.

Remark 5.6. (*Remarks on constructiveness*) Theorem 5.3 in dimension two is a fully constructive characterization of tc. In arbitrary dimensions the necessary conditons for tc in Thms. 4.8 and 4.15 are also fully constructive. But there are presently no constructive necessary and sufficient conditons for tc. According to the (computer) algebraists mentioned at the end of the Introduction there seems to be no algorithm to decide whether a module is a submodule of an extended one. We have presently no counter-example to Thm. 5.3 in dimensions higher than two.

We finally prove Result 5.1 and, more generally, the matrix decompositions which are an essential mathematical tool for the two-dimensional results in [21], [21], [3], [12]. See the references in [21] for the origin of these decompositions. The theorems below are constructive via the algorithms in [6] and permit to find these decompositions via Computer Algebra.
We assume a (not necessarily F-affine!) regular noetherian domain of pure dimension two, i.e., for all maximal ideals $\mathfrak{n}$ of B the local ring B is regular of Krull dimension and hence global dimension two. We also assume that every f.g. projective module is free. The ring B is factorial [10, Thm. 20.4] and has all the properties which we described before Result 5.1 with the difference that F-finite dimensional B-modules are replaced by modules of finite length (f.l.). The assumptions are satisfied for $\mathbb{Z}[\sigma]$. In particular, (76) and (77) are valid.
Consider any matrix $R \in B^{k\times\ell}$ of $\mathrm{rank}(R) = r$ and the f.g. module $M = B^{1\times\ell}/B^{1\times k}R$ of $\mathrm{rank}(M) = \ell - r$. That B has global dimension two and that each f.g. projective module is free signifies that for each R the kernels

$$\begin{gathered} \ker\left(\circ R : B^{1\times k} \to B^{1\times\ell},\ x \mapsto xR\right) \text{ and } \ker\left(R\circ : B^{\ell} \to B^{k},\ y \mapsto Ry\right) \text{ are free,} \\ \text{hence } \ker(\circ R) = B^{1\times(k-r)}X,\ X \in B^{(k-r)\times k},\ \ker(R\circ) = YB^{\ell-r},\ Y \in B^{\ell\times(\ell-r)}, \\ \mathrm{rank}(X) = k - r,\ \mathrm{rank}(Y) = \ell - r. \end{gathered} \tag{79}$$

This argument was already used in [13, Thm. 7.42]. By definition X resp. Y are a *universal left resp. right annihilator* of R and unique up to row resp. column equivalence. In the standard cases the matrices X and Y can be computed by the *constructive* Quillen-Suslin-theorem [6]. Let $R_2 \in B^{r\times\ell}$ be a universal left annihilator of Y of

$$\begin{gathered} \mathrm{rank}(R_2) = \ell - \mathrm{rank}(Y) = \ell - (\ell - r) = r = \mathrm{rank}(R). \text{ Then} \\ \mathrm{tor}(M) \underset{[13, Thm. 7.24]}{=} B^{1\times r}R_2/B^{1\times k}R \subseteq M = B^{1\times\ell}/B^{1\times k}R, \text{ hence} \\ B^{1\times\ell}/B^{1\times r}R_2 = M/\mathrm{tor}(M) \text{ and } R = R'R_2,\ R' \in B^{k\times r},\ \mathrm{rank}(R') = r. \end{gathered} \tag{80}$$

Moreover the matrix R_2 is *left factor prime* (lFP) [13, Thm. 7.21].

Lemma 5.7. *The module $N := B^{1\times r}/B^{1\times \ell}R_2^\top$ is of finite length.*

Proof. For a minimal nonzero prime ideal $\mathfrak{q} = Bq$, q prime, the local ring $B_\mathfrak{q}$ is a principal ideal domain with the unique prime q over which all f.g torsionfree modules are free. The canonical surjection

$$B^{1\times \ell} \xrightarrow{\text{can}} M/\operatorname{tor}(M) = B^{1\times \ell}/B^{1\times r}R_2$$

induces the surjection

$$B_\mathfrak{q}^{1\times \ell} \xrightarrow{\text{can}} (M/\operatorname{tor}(M))_\mathfrak{q} = B_\mathfrak{q}^{1\times \ell}/B_\mathfrak{q}^{1\times r}R_2.$$

Since the module on the right is free the matrix R_2 has a right inverse in $B_\mathfrak{q}^{\ell\times r}$. Therefore $R_2^\top$ has a left inverse in $B_\mathfrak{q}^{r\times \ell}$ which implies $N_\mathfrak{q} = B_\mathfrak{q}^{1\times r}/B_\mathfrak{q}^{1\times \ell}R_2^\top = 0$. In the sense of [4, §VII.4.4] the module N is pseudo-zero. According to [13, Thm. 7.74] this is equivalent with $\dim(N) \le \dim(B) - 2 = 0$ and therefore with N being of f.l.. Since $\operatorname{supp}(N) \subseteq \max(B)$ this follows also from (77). □

Remark 5.8. The module N and its groups $\operatorname{Ext}_B^i(N,B)$ were already considered in arbitrary dimensions by Palamodov and Pommaret (see references in [6]), Quadrat [6, §2] and Zerz [25]. The papers [25] and [6] discuss constructive methods.

Application of the preceding considerations to $R'^\top \in B^{r\times k}$ with $\operatorname{rank}(R'^\top) = r$ (see (80)) furnishes

$$\begin{gathered} R'^\top = \Delta^\top R_1^\top,\ \Delta^\top \in B^{r\times r},\ R_1^\top \in B^{r\times k},\ \text{ hence } R = R'R_2 = R_1\Delta R_2 \text{ with} \\ \operatorname{rank}(R) = r = \operatorname{rank}(R') = \operatorname{rank}(\Delta) = \operatorname{rank}(R_1),\ R_1 \text{ right factor prime (rFP)}, \\ B^{1\times r}/B^{1\times k}R_1 \text{ of finite length},\ R_1 = (R_1^\top)^\top. \end{gathered} \tag{81}$$

Summing up we obtain

Theorem 5.9. *Assume that B is a regular noetherian domain of pure Krull dimension two and assume that each f.g. projective B-module is free, for instance*

$$B = F[s_1,s_2],\ F[s_1,s_2,s_2^{-1}],\ F[s_1,s_1^{-1},s_2,s_2^{-1}],\ \mathbb{Z}[\sigma].$$

Let $R \in B^{k\times \ell}$ and $M = B^{1\times \ell}/B^{1\times k}R$, $r := \operatorname{rank}(R) > 0$, hence $\operatorname{rank}(M) = \ell - r$. The matrix R admits a decomposition or factorization

$$\begin{gathered} R = R_1\Delta R_2,\ R_1 \in B^{k\times r},\ \Delta \in B^{r\times r},\ R_2 \in B^{r\times \ell},\ r = \operatorname{rank}(R_1) = \operatorname{rank}(\Delta) = \operatorname{rank}(R_2), \\ R_1 \textit{ rFP},\ R_2 \textit{ lFP},\ B^{1\times r}/B^{1\times k}R_1 \textit{ and } B^{1\times r}/B^{1\times \ell}R_2^\top \textit{ of finite length}. \end{gathered} \tag{82}$$

If B is F-affine the modules of finite length are exactly the F-finite dimensional ones. All matrices can be computed via Gröbner bases with the help of the constructive Quillen-Suslin theorem [6].
The theorem is an essential tool in the work of Bisiacco and Valcher [3] and others.

The decomposition of R has a module theoretic counter-part which shows that the decomposition is essentially unique. It induces the two exact sequences

$$\begin{gathered} 0 \to B^{1\times r}/B^{1\times k}R_1\Delta \stackrel{(\circ R_2)_{\mathrm{ind}}}{\longrightarrow} M = B^{1\times \ell}/B^{1\times k}R \stackrel{\mathrm{can}}{\longrightarrow} M/\operatorname{tor}(M) = B^{1\times \ell}/B^{1\times r}R_2 \to 0 \\ \text{hence } B^{1\times r}/B^{1\times k}R_1\Delta \stackrel{(\circ R_2)_{\mathrm{ind}}}{\cong} \operatorname{tor}(M) \subseteq M = B^{1\times \ell}/B^{1\times k}R, \text{ and} \\ 0 \to B^{1\times r}/B^{1\times k}R_1 \stackrel{(\circ\Delta)_{\mathrm{ind}}}{\longrightarrow} B^{1\times r}/B^{1\times k}R_1\Delta \stackrel{\mathrm{can}}{\longrightarrow} B^{1\times \ell}/B^{1\times r}\Delta \to 0. \end{gathered} \tag{83}$$

Let Δ_{adj} be the adjoint matrix and

$$d := \det(\Delta) \in B. \text{ Then } \left(\circ \left(\Delta_{\mathrm{adj}}\right)_{\mathrm{ind}} : B^{1\times \ell}/B^{1\times r}\Delta \to (B/Bd)^{1\times \ell}\right.$$

is injective. As in (78) we conclude that

$$\operatorname{ass}(B^{1\times \ell}/B^{1\times r}\Delta) \subseteq \operatorname{ass}(B/Bd) \subset \operatorname{spec}(B) \setminus \max(B)$$

and hence that $\mathrm{Ra}_{\mathrm{fin}}\left(B^{1\times \ell}/B^{1\times r}\Delta\right) = 0$ by (77).

Theorem 5.10. *Assumptions and data from Thm.* (5.9). *Then*

$$\begin{gathered} B^{1\times \ell}/B^{1\times r}R_2 \cong M/\operatorname{tor}(M),\ B^{1\times r}/B^{1\times k}R_1\Delta \stackrel{(\circ R_2)_{\mathrm{ind}}}{\cong} \operatorname{tor}(M) \\ B^{1\times r}/B^{1\times k}R_1 \stackrel{(\circ \Delta R_2)_{\mathrm{ind}}}{\cong} \mathrm{Ra}_{\mathrm{fin}}(\operatorname{tor}(M)) = \mathrm{Ra}_{\mathrm{fin}}(M), \\ B^{1\times r}/B^{1\times r}\Delta \stackrel{(\circ R_2)_{\mathrm{ind}}}{\cong} \operatorname{tor}(M)/\,\mathrm{Ra}_{\mathrm{fin}}(M). \end{gathered}$$

This shows in particular that the matrices R_1, Δ, R_2 are essentially unique and depend on the module M only. If M is a torsion module with $\mathrm{Ra}_{\mathrm{fin}}(M) = 0$ we get $M = \operatorname{tor}(M)/\,\mathrm{Ra}_{\mathrm{fin}}(M) \cong B^{1\times r}/B^{1\times r}\Delta$ and this is Result 5.1.

Proof. Consider the second exact sequence in (83). Since $B^{1\times r}/B^{1\times k}R_1$ is of f.l. and $\mathrm{Ra}_{\mathrm{fin}}\left(B^{1\times \ell}/B^{1\times r}\Delta\right) = 0$ Result 2.8 furnishes

$$\begin{gathered} B^{1\times r}/B^{1\times k}R_1 \stackrel{(\circ\Delta)_{\mathrm{ind}}}{\cong} \mathrm{Ra}_{\mathrm{fin}}\left(B^{1\times r}/B^{1\times k}R_1\Delta\right), \text{ hence} \\ B^{1\times r}/B^{1\times k}R_1 \stackrel{(\circ \Delta R_2)_{\mathrm{ind}}}{\cong} \mathrm{Ra}_{\mathrm{fin}}(\operatorname{tor}(M)) = \mathrm{Ra}_{\mathrm{fin}}(M). \end{gathered}$$

□

Remark 5.11. In forthcoming work we consider Serre subcategories $\mathfrak{C} \subset \mathrm{Mod}_B$ of torsion modules and the corresponding $\mathfrak{C}$-*small or negligible* autonomous behaviors $D(C)$, $C \in \mathfrak{C}$, and call a behavior $\mathscr{B} = D(M)$ $\mathfrak{C}$-controllable if the torsion module $\operatorname{tor}(M)$ belongs to $\mathfrak{C}$ or, equivalently, if $\mathscr{B}$ admits a sum representation $\mathscr{B} = \mathscr{B}_{\mathrm{cont}} + \mathscr{B}_{\mathrm{aut},\mathfrak{C}}$ with its controllable part $\mathscr{B}_{\mathrm{cont}} = D(M/\operatorname{tor}(M))$ and a small behavior $\mathscr{B}_{\mathrm{aut},\mathfrak{C}}$. We also generalize the characterization of discrete controllable behaviors by concatenability of trajectories [15],[23], [24] to $\mathfrak{C}$-controllability. This approach has proved suitable for multidimensional stabilizability and stabilization[19], establishes a strong relation between the notions of generalized controllability and negligibility as in the one-dimensional case and in the two-dimensional one (Thm. 5.3, [3, Thm. 5.1]) and will be applied to multidimensional controller design which in the 2D case is treated in [3]. Due to Lemma 4.11 and other results of Section 4 time-controllability seems (to us) less suitable for controller design in dimension higher than two. Multidimensional observers are treated in [1] in the discrete 2D case and in the submitted paper [20] for continuous and discrete behaviors in arbitrary dimensions.

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