Radon Series Comp. Appl. Math **1**, 1–19

Solving systems of linear partial difference and differential equations with constant coefficients using Gröbner bases

Ulrich Oberst and Franz Pauer

Key words. partial difference equation, partial differential equation, Gröbner basis

AMS classification. 13P10, 39A10, 93C20

0.1 Introduction

The main objective of this tutorial paper is to give an elementary presentation of a method developed in [19] to solve systems of linear partial difference equations with constant coefficients using Gröbner bases. This is done in Sections 1, 2 and 4, these sections are self-contained. In Sections 3 and 5 related results of [19] and [22] are described without proofs. The last section contains some historical remarks.

0.2 The Problem

Let F be a field and let k, ℓ, m, n be positive integers. By $M^{\ell\times m}$ we denote the set of all $\ell \times m$-matrices with entries in a set M.

Definition 0.1 A *system of* k *linear partial difference equations with constant coefficients for* 1 *unknown function in* n *variables* is given by

(1) a family

$$(R(\alpha))_{\alpha\in\mathbb{N}^n}$$

of columns $R(\alpha) \in F^{k\times 1}$,
where only finitely many $R(\alpha)$ are different from 0,

and

(2) a family

$$(v(\alpha))_{\alpha\in\mathbb{N}^n}$$

of columns $v(\alpha)\in F^{k\times 1}$.

A *solution* of this system is a family

$$(w(\gamma))_{\gamma\in\mathbb{N}^n}$$

of elements $w(\gamma)\in F$ (i.e. a function $w:\mathbb{N}^n\longrightarrow F\,,\ \gamma\mapsto w(\gamma)$) such that for all $\beta\in\mathbb{N}^n$

$$\sum_{\alpha\in\mathbb{N}^n} w(\alpha+\beta)R(\alpha)=v(\beta).$$

The system is *homogeneous* if and only if $v(\beta)=0$, for all $\beta\in\mathbb{N}^n$.

Example 0.2 The problem "Find a function $w:\mathbb{N}^2\longrightarrow\mathbb{Q}$ such that for all $\beta\in\mathbb{N}^2$

$$\begin{array}{rcl} 2w((2,1)+\beta)+w(\beta) &=& v_1(\beta)\\ 3w((1,2)+\beta)+2w(\beta) &=& v_2(\beta)\end{array},$$

where

$$v(\beta):=\begin{pmatrix}v_1(\beta)\\ v_2(\beta)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\ \text{for } \beta\in\mathbb{N}^2\setminus\{(1,1),(0,2),(2,3)\}$$

and

$$v(1,1)=\begin{pmatrix}0\\15\end{pmatrix},\ v(0,2)=\begin{pmatrix}13\\6\end{pmatrix},\ v(2,3)=\begin{pmatrix}5\\10\end{pmatrix}\text{"}$$

is a system of 2 linear partial difference equations with constant coefficients for 1 unknown function in 2 variables. It is defined by the family

$$R(\alpha):=\begin{pmatrix}0\\0\end{pmatrix}\ \text{for } \alpha\in\mathbb{N}^2\setminus\{(0,0),(1,2),(2,1)\}$$

and

$$R(0,0)=\begin{pmatrix}1\\2\end{pmatrix},\ R(2,1)=\begin{pmatrix}2\\0\end{pmatrix},\ R(1,2)=\begin{pmatrix}0\\3\end{pmatrix}$$

and the family $(v(\alpha))_{\alpha\in\mathbb{N}^2}$.

A solution of the system in Definition 0.1 is a function $w:\mathbb{N}^n\longrightarrow F\,,\ \alpha\mapsto w(\alpha)$. We write $F^{\mathbb{N}^n}$ for the vector space of all functions from $\mathbb{N}^n$ to F. There are other useful interpretations of the function w:

- If we choose n letters $z_1, \ldots, z_n$, the family w can be written as the power series
$$\sum_{\alpha \in \mathbb{N}^n} w(\alpha) z^\alpha ,$$
where $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$. If only finitely many $w(\alpha)$ are different from 0, then $\sum_{\alpha \in \mathbb{N}^n} w(\alpha) z^\alpha$ is a polynomial.
Let $F[[z]] := F[[z_1, \ldots, z_n]]$ be the F-algebra of power series in $z_1, \ldots, z_n$.
- Consider the n-variate polynomial ring $F[s] := F[s_1, \ldots, s_n]$. Then w defines a linear function
$$\varphi : F[s] \longrightarrow F$$
by
$$\varphi(s^\alpha) := w(\alpha) \,, \text{ for all } \alpha \in \mathbb{N}^n \,.$$
where $s^\alpha := s_1^{\alpha_1} s z_2^{\alpha_2} \cdots s_n^{\alpha_n}$.
We denote by $\mathrm{Hom}_F(F[s], F)$ the vector-space of all linear maps from $F[s]$ to F.

The vector spaces $F^{\mathbb{N}^n}$, $F[[z]]$, and $\mathrm{Hom}_F(F[s], F)$ are $F[s]$-modules in a natural way: Let $s^\alpha \in F[s]$, $w \in F^{\mathbb{N}^n}$, $\sum_{\beta \in \mathbb{N}^n} c_\beta z^\beta \in F[[z]]$, and $\varphi \in \mathrm{Hom}_F(F[s], F)$. Then
$$(s^\alpha \circ w)(\beta) := w(\alpha + \beta), \text{ for all } \beta \in \mathbb{N}^n,$$
$$s^\alpha \circ \sum_{\beta \in \mathbb{N}^n} c_\beta z^\beta := \sum_{\beta \in \mathbb{N}^n} c_{\alpha+\beta} z^\beta = \sum_{\beta \in \alpha + \mathbb{N}^n} c_\beta z^{\beta - \alpha} ,$$
and
$$(s^\alpha \circ \varphi)(\beta) := \varphi(s^{\alpha+\beta}), \text{ for all } \beta \in \mathbb{N}^n.$$
It is easy to verify that the maps
$$F[[z]] \longrightarrow F^{\mathbb{N}^n} \longrightarrow \mathrm{Hom}_F(F[s], F)$$
$$\sum_{\beta \in \mathbb{N}^n} c_\beta z^\beta \mapsto (c_\beta)_{\beta \in \mathbb{N}^n} \mapsto \varphi \text{ (where } \varphi(s^\alpha) = c_\alpha)$$
are isomorphisms of $F[s]$-modules.

Thus, Definition 0.1 could equivalently be formulated as:

Definition 0.3 A *system of k linear partial difference equations with constant coefficients for 1 unknown function in n variables* is given by

(1') a column
$$\begin{pmatrix} R_1 \\ \vdots \\ R_k \end{pmatrix} \in F[s]^{k \times 1}$$
of polynomials $R_i := \sum_{\alpha \in \mathbb{N}^n} R_i(\alpha) s^\alpha \in F[s]$,

and

(2') a column

$$\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} \in F[[z]]^{k\times 1}$$

of power series $v_i := \sum_{\alpha\in\mathbb{N}^n} v_i(\alpha)z^\alpha \in F[[z]]$.

A *solution* of this system is a power series

$$w := \sum_{\alpha\in\mathbb{N}^n} w(\alpha)z^\alpha$$

such that

$$R_i \circ w = v_i \,,\; 1 \le i \le k \,.$$

Equivalently, we could replace (2') by

(2") a column

$$\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} \in \mathrm{Hom}_F(F[s], F)^{k\times 1}$$

of linear functions $v_i : F[s] \longrightarrow F \,,\; 1 \le i \le k$.

Then a *solution* of this system is a linear function $\varphi : F[s] \longrightarrow F$ such that

$$R_i \circ \varphi = v_i \,,\; 1 \le i \le k.$$

Example 0.4 The data for the system of partial difference equations in Example 0.2 can be written as

$$R := \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 2s_1^2 s_2 + 1 \\ 3s_1 s_2^2 + 2 \end{pmatrix}$$

and

$$v := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 13z_2^2 + 5z_1^2 z_2^3 \\ 6z_2^2 + 10z_1^2 z_2^3 + 15z_1 z_2 \end{pmatrix} .$$

A solution is a power series $w := \sum_{\alpha\in\mathbb{N}^2} w(\alpha_1,\alpha_2) z_1^{\alpha_1} z_2^{\alpha_2}$ such that $R_i \circ w = v_i$, $1 \le i \le 2$. Or, equivalently, a linear function $\varphi : \mathbb{Q}[s] \longrightarrow \mathbb{Q}$ such that $\varphi(s_1^{\alpha_1} s_2^{\alpha_2} R_i) = v_i(\alpha_1,\alpha_2)$, for all $\alpha \in \mathbb{N}^2$, $1 \le i \le 2$.

Remark 0.5 The *inverse system* of the ideal $\langle R_1, \ldots, R_k \rangle$ generated by $R_1, \ldots, R_k$ in $F[s]$ is the set of all linear functions $\varphi : F[s] \longrightarrow F$ such that

$$\varphi|_{\langle R_1, \ldots, R_k \rangle} = 0\,.$$

This notion was introduced by F.S. Macaulay in [15]. Hence the inverse system is the set of solutions of the homogeneous system of partial difference equations given by $R_1, \ldots, R_k \in F[s]$. Inverse Systems have been studied e.g. in [15], [19], [14], [18], and [10].

It is clear that the set of solutions of a homogeneous system is an $F[s]$-submodule of $F^{\mathbb{N}^n}$ resp. $F[[z]]$ resp. $\mathrm{Hom}_F(F[s], F)$. Hence, if w is a solution of a system, then all solutions can be obtained by adding solutions of the homogeneous system to w.

Nevertheless, since a solution is an infinite family, and - even worse - the space of all solutions is in general infinite-dimensional over F, at first view it is not clear how to describe the solutions by finitely many data. We shall describe a family

$$(w(\alpha))_{\alpha \in \mathbb{N}^n} \in F^{\mathbb{N}^n}$$

by an algorithm which permits us to compute $w(\alpha)$ for any $\alpha \in \mathbb{N}^n$.

To *solve* a system of difference equations means

- Decide if there is a solution or not.
- If there are solutions: Determine a *canonical subset* $\Gamma \subseteq \mathbb{N}^n$ such that for every *initial condition* $x := (x(\gamma))_{\gamma \in \Gamma}$ there is exactly one solution w_x such that $w_x(\gamma) = x(\gamma)$, for all $\gamma \in \Gamma$.
- Give an algorithm to compute $w_x(\alpha)$ for any $\alpha \in \mathbb{N}^n$ and any initial condition x.

Remark 0.6

For $\alpha \in \mathbb{N}^n,\ u \in \mathbb{C}[[z]]$ consider

$$s^\alpha \bullet u := \partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}\ .$$

If we replace $R_i \circ w$ in Definition 0.3 by $R_i \bullet w$, we get the definition of a *system of k linear partial differential equations with constant coefficients for 1 unknown function in n variables*. The map

$$\mathbb{C}[[z]] \to \mathbb{C}[[z]]\,,\ \sum_\alpha u(\alpha) z^\alpha \mapsto \sum_\alpha \frac{u(\alpha)}{\alpha!} z^\alpha, \qquad (0.1)$$

is an isomorphism of the $\mathbb{C}[s]-$modules $(\mathbb{C}[[z]], \circ)$ and $(\mathbb{C}[[z]], \bullet)$.

Results for systems of partial difference equations (“discrete case”) can thus be transferred to systems of partial differential equations (“continuous case”) .

Example 0.7 The system of partial differential equations corresponding to that of partial difference equations in Examples 0.2 and 0.4 is:
"Find a power series $w \in \mathbb{C}[[z]]$ such that

$$2\frac{\partial^3 w}{\partial z_1^2 \partial z_2} + w = \frac{13}{2} z_2^2 + \frac{5}{12} z_1^2 z_2^3$$

and

$$3\frac{\partial^3 w}{\partial z_1 \partial z_2^2} + 2w = 3z_2^2 + \frac{5}{6} z_1^2 z_2^3 + 15 z_1 z_2 \text{."}$$

A power series $\sum_{\alpha\in\mathbb{N}^2} w(\alpha) z^\alpha$ is a solution of this system of differential equations if and only if $\sum_{\alpha\in\mathbb{N}^2} \alpha_1! \alpha_2! w(\alpha) z^\alpha$ is so for the system of difference equations in Example 0.2.

0.3 The Algorithm

Let $R_1, \dots, R_k \in F[s]$ be polynomials and let I be the ideal generated by them in $F[s]$. Let $\le$ be a term order on $\mathbb{N}^n$, $\deg(I) \subseteq \mathbb{N}^n$ the set of degrees of all polynomials in I, and

$$\Gamma := \mathbb{N}^n \setminus \deg(I) \,. \tag{0.2}$$

Then

$$F[s] = I \oplus \bigoplus_{\gamma\in\Gamma} F s^\gamma \tag{0.3}$$

i.e. any polynomial $h \in F[s]$ can uniquely be written as

$$h = h_I \, + \, \mathrm{nf}(h) \,,$$

where $h_I \in I$ and $\mathrm{nf}(h) \in \bigoplus_{\gamma\in\Gamma} F s^\gamma$. The polynomial $\mathrm{nf}(h)$ is the *normal form* of h with respect to I and $\le$.
The set $\deg(I)$ and the polynomial $\mathrm{nf}(h)$ can be computed via a Gröbner basis ([3], [4]) of I with respect to $\le$. Moreover, using Gröbner bases we can compute a system of generators

$$L_i := (L_{i1}, \dots, L_{ik}) \in F[s]^{1\times k} \,,\; 1 \le i \le p \,,$$

of the module of syzygies of $(R_1, \dots, R_k)$, i.e. of the $F[s]$-submodule

$$\{u \in F[s]^{1\times k} \mid \sum_{j=1}^k u_j R_j = 0\} \le F[s]^{1\times k} \,.$$

Theorem 0.8 *[22, Th. 3 and Th. 5]*

(1) There exists a function $w \in F^{\mathbb{N}^n}$ *such that*

$$R_i \circ w = v_i \, , \, 1 \le i \le k \, ,$$

if and only if

$$\text{(S)} \qquad \sum_{j=1}^{k} L_{ij} \circ v_j = 0 \; , \; 1 \le i \le p \; .$$

(2) If condition (S) holds then for any initial condition $x : \Gamma \longrightarrow F$ *there is a unique function* $w \in F^{\mathbb{N}^n}$ *such that*

$$w|_\Gamma = x \quad \text{ and } \quad R_i \circ w = v_i \, , \, 1 \le i \le k \, .$$

For any $\alpha \in \mathbb{N}^n$ *the element* $w(\alpha) \in F$ *can be computed as follows:*
Compute $c_\gamma \in F$*, where* $\gamma \in \Gamma$*, and* $d_{\beta,i} \in F$*, where* $\beta \in \mathbb{N}^n$*,* $1 \le i \le k$*, such that*

$$\mathrm{nf}(s^\alpha) = \sum_{\gamma \in \Gamma} c_\gamma s^\gamma$$

and

$$s^\alpha - \mathrm{nf}(s^\alpha) = \sum_{\beta,i} d_{\beta,i} s^\beta R_i \, .$$

Then

$$w(\alpha) = \sum_{\gamma \in \Gamma} c_\gamma x(\gamma) + \sum_{\beta,i} d_{\beta,i} v_i(\beta) \, .$$

Proof. First we show that the existence of a solution w implies condition (S):

$$\sum_{j=1}^{k} L_{ij} \circ v_j = \sum_{j=1}^{k} L_{ij} \circ (R_j \circ w) = (\sum_{j=1}^{k} L_{ij} R_j) \circ w = 0 \, .$$

Now we assume that (S) holds.
If a solution w exists, we consider the associated linear function $\varphi : F[s] \longrightarrow F$, where $\varphi(s^\alpha) = w(\alpha)$. Then

$$w(\alpha) = \varphi(s^\alpha) = \varphi(\sum_{\gamma \in \Gamma} c_\gamma s^\gamma + \sum_{\beta,i} d_{\beta,i} s^\beta R_i) = \sum_{\gamma \in \Gamma} c_\gamma x(\gamma) + \sum_{\beta,i} d_{\beta,i} v_i(\beta) \, .$$

To show the existence of a solution w resp. φ, we define the linear function φ as follows: Since $\{s^\beta R_i \mid \beta \in \mathbb{N}^n, \, 1 \le i \le k\}$ is a system of generators of the F-vector-space I, we may choose a subset B of it which is an F-basis of I. Then define

$$\varphi(s^\gamma) = x(\gamma) \, , \, \gamma \in \Gamma \, ,$$

and

$$\varphi(s^\beta R_i) = v_i(\beta) \, , \; \text{ for all } s^\beta R_i \in B.$$

Now we only have to show that

$$\varphi(s^\alpha R_j) = v_j(\alpha)\,,\ \ \text{for all } s^\alpha R_j \notin B.$$

Let $s^\alpha R_j \notin B$ and let $a_{\beta,i} \in F$ such that

$$s^\alpha R_j = \sum_{s^\beta R_i \in B} a_{\beta,i} s^\beta R_i\,.$$

Then

$$0 = \sum_{s^\beta R_i \in B} a_{\beta,i} s^\beta R_i - s^\alpha R_j =$$

$$= \sum_{\substack{i \\ i \neq j}} \left(\sum_{\substack{\beta \\ s^\beta R_i \in B}} a_{\beta,i} s^\beta \right) R_i + \left(\left(\sum_{\substack{\beta \\ s^\beta R_j \in B}} a_{\beta,j} s^\beta \right) - s^\alpha \right) R_j\,.$$

Since (S) holds, it is easy to verify that $\sum_{j=1}^k u_j \circ v_j = 0$ for any syzygy u of $(R_1, \ldots, R_k)$. Hence

$$0 = \sum_{\substack{i \\ i \neq j}} \left(\sum_{\substack{\beta \\ s^\beta R_i \in B}} a_{\beta,i} s^\beta \right) \circ v_i + \left(\left(\sum_{\substack{\beta \\ s^\beta R_j \in B}} a_{\beta,j} s^\beta \right) - s^\alpha \right) \circ v_j$$

and

$$0 = \left(\sum_{\substack{i \\ i \neq j}} \left(\sum_{\substack{\beta \\ s^\beta R_i \in B}} a_{\beta,i} s^\beta \right) \circ v_i + \left(\left(\sum_{\substack{\beta \\ s^\beta R_j \in B}} a_{\beta,j} s^\beta \right) - s^\alpha \right) \circ v_j \right)(0) =$$

$$= \sum_{\substack{i \\ i \neq j}} \left(\sum_{\substack{\beta \\ s^\beta R_i \in B}} a_{\beta,i} \right) v_i(\beta) + \sum_{\substack{\beta \\ s^\beta R_j \in B}} a_{\beta,j} v_j(\beta) - v_j(\alpha)\,.$$

Thus

$$v_j(\alpha) = \sum_{\substack{i \\ i \neq j}} \left(\sum_{\substack{\beta \\ s^\beta R_i \in B}} a_{\beta,i} \right) \varphi(s^\beta R_i) + \sum_{\substack{\beta \\ s^\beta R_j \in B}} a_{\beta,j} \varphi(s^\beta R_j) = \varphi(s^\alpha R_j) \,.$$

□

Example 0.9 We present the solution of the system of linear difference equations in Example 0.2 resp. Example 0.4. Let

$$R := \begin{pmatrix} 2s_1^2 s_2 + 1 \\ 3s_1 s_2^2 + 2 \end{pmatrix}$$

and

$$v := \begin{pmatrix} 13z_2^2 + 5z_1^2 z_2^3 \\ 6z_2^2 + 10z_1^2 z_2^3 + 15z_1 z_2 \end{pmatrix} \,.$$

The $\mathbb{Q}[s]$-module of syzygies is generated by the element

$$L = (-R_2, R_1) = (-3s_1 s_2^2 - 2, 2s_1^2 s_2 + 1) \,.$$

We compute $L \circ v = 0$, hence this system is solvable.
A Gröbner basis of the ideal

$$I := \langle R_1, R_2 \rangle = \langle 2s_1^2 s_2 + 1,\, 3s_1 s_2^2 + 2 \rangle$$

in $\mathbb{Q}[s_1, s_2]$ with respect to the graded lexicographic order ($s_1 > s_2$) is

$$\{4s_1 - 3s_2, 9s_2^3 + 8\}$$

and hence

$$\Gamma = \{(0,0), (0,1), (0,2)\} \,.$$

Let $\alpha := (2,3)$. Then

$$s_1^2 s_2^3 = -\frac{1}{2} s_2^2 + (-\frac{3}{4} s_1 s_2^4) R_1 + (\frac{1}{2} s_1^2 s_2^3 + \frac{1}{4} s_2^2) R_2 \,.$$

Hence

$$w(2,3) =$$
$$= -\frac{1}{2} x(0,2) - \frac{3}{4} v_1(1,4) + \frac{1}{2} v_2(2,3) + \frac{1}{4} v_2(0,2) =$$
$$= -\frac{1}{2} x(0,2) + \frac{13}{2} \,.$$

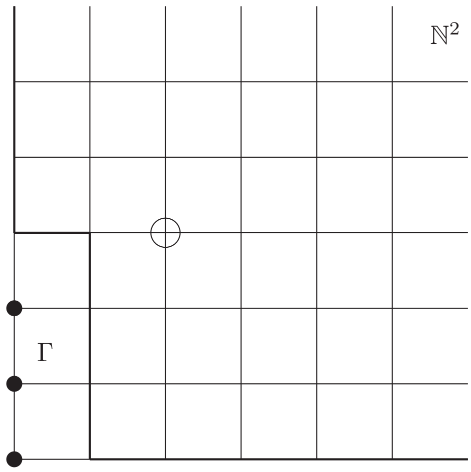

Figure 0.1 The elements • of Γ and $\bigcirc = (2,3) \in \deg(I)$.

0.4 Riquier's decomposition and the Hilbert function

Let I be an ideal in $F[s]$ and $\deg(I)$ and $\Gamma := \mathbb{N}^n \setminus \deg(I)$ as in the preceding section. We construct disjoint decompositions of Γ, due to Riquier [27], and of $\deg(I)$, due to Janet [12], and apply the first to the computation of the Hilbert function of the ideal I. The details of these considerations are contained in [22] and [20]. In [20] Riquier's decomposition is used for the construction of the canonical state representation of a multidimensional system. In Section 0.7 we make some further remarks on the history of this decomposition.

Let, more generally, N be an *order ideal* of $\mathbb{N}^n$, i.e. a subset satisfying

$$N = N + \mathbb{N}^n,$$

$\Gamma := \mathbb{N}^n \setminus N$ its complement and

$$\min(N) := \{\alpha \in N \mid \text{there are no } \beta \in N, \gamma \in \mathbb{N}^n \setminus \{0\} \text{ such that } \alpha = \beta + \gamma\}$$

the set of minimal elements of N. The set $\min(N)$ is finite (Dickson's lemma) and satisfies $N = \min(N) + \mathbb{N}^n$. The degree set (with respect to a term order) $N := \deg(I)$ of an ideal I is such an order ideal, and Buchberger's algorithm especially furnishes the finite discrete set $\min(N)$. In the sequel we assume that the order ideal N is given in the form $N = D + \mathbb{N}^n$ where D is a finite subset of $\mathbb{N}^n$. Then $\min(N) \subseteq D$.

For a subset S of $\{1, \cdots, n\}$ with its complement $S' := \{1, \cdots, n\} \setminus S$ we *identify* $\mathbb{N}^S$ as subset of $\mathbb{N}^n$ via

$$\begin{aligned} \mathbb{N}^S &= \{x = (x_1, \cdots, x_n) \in \mathbb{N}^n \mid x_i = 0 \text{ for all } i \notin S\} \text{ and} \\ \mathbb{N}^n &= \mathbb{N}^S \times \mathbb{N}^{S'} \ni x = ((x_i)_{i \in S}, (x_i)_{i \in S'}). \end{aligned} \tag{0.4}$$

We are going to construct disjoint decompositions of N and of Γ by induction on n. These decompositions are given by *finite* subsets

$$\begin{gathered} A_N \subset N \text{ and } A_\Gamma \subset \Gamma \text{ and subsets} \\ S(\alpha) \subseteq \{1, \cdots, n\},\ \alpha \in A_N \biguplus A_\Gamma, \text{ such that} \\ N = \biguplus_{\alpha \in A_N} \left(\alpha + \mathbb{N}^{S(\alpha)}\right) \text{ and } \Gamma = \biguplus_{\alpha \in A_\Gamma} \left(\alpha + \mathbb{N}^{S(\alpha)}\right). \end{gathered} \tag{0.5}$$

The recursive algorithm
Induction beginning: $n = 1$:
(i) If $N = D = \emptyset$, we choose $A_N := \emptyset$, $A_\Gamma = \{0\}$ and $S(0) := \{1\}$ and obtain $\mathbb{N} = \emptyset \uplus (0 + \mathbb{N})$.
(ii) If $N \neq \emptyset$ and $\{d\} := \min(N)$ we define

$$\begin{gathered} A_N := \{d\},\ S(d) := \{1\},\ A_\Gamma := \Gamma = \{0, \cdots, d-1\},\ S(\alpha) := \emptyset \text{ for } \alpha \in A_\Gamma, \\ \text{hence } \mathbb{N} = (d + \mathbb{N}) \uplus \{0\} \uplus \cdots \uplus \{d-1\}. \end{gathered}$$

The induction step: For $n > 1$ we assume that the decomposition has already been constructed for order ideals in $\mathbb{N}^{n-1}$. We identify

$$\mathbb{N}^n = \mathbb{N} \times \mathbb{N}^{n-1} \ni \alpha = (\alpha_1, \alpha_{II}),\ \alpha_{II} := (\alpha_2, \cdots, \alpha_n),$$

consider the projection $\mathrm{proj} : \mathbb{N}^n \to \mathbb{N}^{n-1},\ \alpha = (\alpha_1, \alpha_{II}) \mapsto \alpha_{II}$, and define

$$\begin{gathered} N_{II} := \mathrm{proj}_{II}(N) := \{\alpha_{II} \mid (\alpha_1, \alpha_{II}) \in N \text{ for some } \alpha_1 \in \mathbb{N}\} \text{ resp.} \\ D_{II} := \mathrm{proj}_{II}(D) := \{\alpha_{II} \mid (\alpha_1, \alpha_{II}) \in D \text{ for some } \alpha_1 \in \mathbb{N}\}. \end{gathered}$$

Since proj is an epimorphism the representation $N = D + \mathbb{N}^n$ implies $N_{II} = D_{II} + \mathbb{N}^{n-1}$ and in particular that N_{II} is an order ideal in $\mathbb{N}^{n-1}$. Moreover

$$\begin{gathered} N = \biguplus_{i=0}^{\infty} (\{i\} \times N_{II}(i)) \text{ with} \\ N_{II}(i) := \{\alpha_{II} \mid (i, \alpha_{II}) \in N\} = D_{II}(i) + \mathbb{N}^{n-1} \text{ where} \\ D_{II}(i) := \{\alpha_{II} \mid (\alpha_1, \alpha_{II}) \in D \text{ for some } \alpha_1 \leq i\}. \end{gathered}$$

The $N_{II}(i)$ and $D_{II}(i)$ are increasing, i.e.

$$\begin{gathered} N_{II}(0) \subseteq N_{II}(1) \subseteq \cdots \subseteq N_{II} = \bigcup_{i=0}^{\infty} N_{II}(i) \\ D_{II}(0) \subseteq D_{II}(1) \subseteq \cdots \subseteq D_{II} = \bigcup_{i=0}^{\infty} D_{II}(i). \end{gathered}$$

Since D and hence D_{II} are finite the sequences of the $D_{II}(i)$ and hence of the $N_{II}(i) = D_{II}(i) + \mathbb{N}^n$ become stationary and we define

$$k := \min\{i \in \mathbb{N} \mid N_{II}(i) = N_{II}\}.$$

By induction applied to

$$N_{II}(i) = D_{II}(i) + \mathbb{N}^{n-1} \subset \mathbb{N}^{n-1} = \mathbb{N}^{\{2,\cdots,n\}},\ 0 \le i \le k,$$

we obtain

$$\mathbb{N}^{n-1} = N_{II}(i) \uplus \Gamma_{II}(i) =$$
$$\left[\biguplus_{\alpha_{II}\in B_{II}(i)} (\alpha_{II} + \mathbb{N}^{S_{II}(i,\alpha_{II})})\right] \biguplus \left[\biguplus_{\alpha_{II}\in C_{II}(i)} (\alpha_{II} + \mathbb{N}^{S_{II}(i,\alpha_{II})})\right]$$

where

$$B_{II}(i) := A_{N_{II}(i)},\ C_{II}(i) := A_{\Gamma_{II}(i)},$$
$$S_{II}(i,\alpha_{II}) \subset \{2,\cdots,n\} \text{ for } \alpha_{II} \in B_{II}(i) \uplus C_{II}(i).$$

With these data in dimension $n-1$ we define

$$A_N := \{(i,\alpha_{II});\ 0 \le i \le k,\ \alpha_{II} \in B_{II}(i)\}$$
$$A_\Gamma := \{(i,\alpha_{II});\ 0 \le i \le k,\ \alpha_{II} \in C_{II}(i)\}$$
$$A := A_N \uplus A_\Gamma$$
$$S(i,\alpha_{II}) := S_{II}(i,\alpha_{II}) \text{ for } (i,\alpha_{II}) \in A,\ 0 \le i \le k-1$$
$$S(k,\alpha_{II}) := \{1\} \uplus S_{II}(k,\alpha_{II}) \text{ for } (k,\alpha_{II}) \in A.$$

The desired decomposition in dimension n is

$$\mathbb{N}^n = N \uplus \Gamma = \left[\biguplus_{\alpha\in A_N} \left(\alpha + \mathbb{N}^{S(\alpha)}\right)\right] \biguplus \left[\biguplus_{\alpha\in A_\Gamma} \left(\alpha + \mathbb{N}^{S(\alpha)}\right)\right]. \tag{0.6}$$

According to Janet [12] the indices i in the set $S(\alpha),\ \alpha \in A$, resp. the corresponding indeterminates s_i are called *multiplicative* with respect to α and the others are called *non-multiplicative*. Notice that the decomposition (0.6) depends on N only and not on the special choice of the finite set D. The only required computations are comparisons of finitely many vectors in $\mathbb{N}^n$ and $\mathbb{N}^{n-1}$ and reading off first and second components α_1 and α_{II} of vectors α in $\mathbb{N}^n$. These manipulations are simple and easy to implement and do not use more complex methods like Buchberger's algorithm or involutive division.

Figure 2 illustrates the preceding theorem for the order ideal $N = D + \mathbb{N}^2$ with $D = \min(N) := \{(2,4),(4,2),(5,1)\}$.

We denote by $F[s|S(\alpha)]$ resp. $F[[z|S(\alpha)]]$ the polynomial ring resp. power series ring $F[s_i\,;\ i \in S(\alpha)]$ resp. $F[[z_i\,;\ i \in S(\alpha)]]$ in the variables s_i resp. $z_i,\ i \in S(\alpha) \subseteq \{1,\ldots,n\}$. Application of the preceding decomposition to $N := \deg(I)$ from Section 0.3 and the direct sum decomposition $F[s] = I \oplus \oplus_{\gamma\in\Gamma} F s^\gamma$ from (0.3) imply

$$\oplus_{\gamma\in\Gamma} F s^\gamma = \oplus_{\alpha\in A_\Gamma} F[s|S(\alpha)] s^\alpha,$$
$$F[s]/I = \oplus_{\alpha\in A_\Gamma} F[s|S(\alpha)]\overline{s^\alpha} \cong \prod_{\alpha\in A_\Gamma} F[s|S(\alpha)] \tag{0.7}$$
$$\text{and } F^\Gamma = \oplus_{\alpha\in A_\Gamma} F[[z|S(\alpha)]] z^\alpha \subset F[[z]]$$

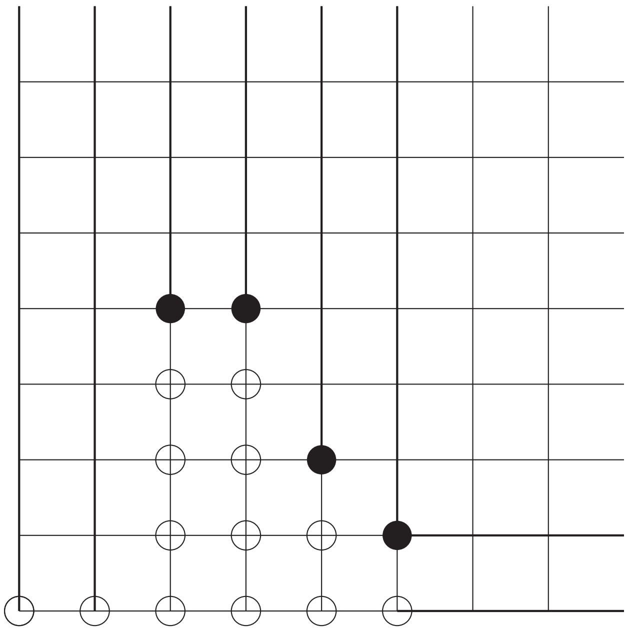

Figure 0.2 The decomposition $\mathbb{N}^2 = N \uplus \Gamma$, the points $\bullet$ of A_N and $\circ$ of A_Γ.

where $\overline{s^\alpha}$ denotes the residue class of s^α in $F[s]/I$. A direct sum decomposition of the factor ring $F[s]/I$ as in (0.7) is called a *Stanley decomposition* in [30, Def. 1.4.1], probably after [29, Th. 5.13]. The decomposition of Γ, due to Riquier, improves Macaulay's result that the residue classes $\overline{s^\gamma},\ \gamma \in \Gamma$, are an F-basis of the factor ring. The decomposition $F^\Gamma = \oplus_{\alpha\in A_\Gamma} F[[z|S(\alpha)]]z^\alpha$ signifies that the solution w of the Cauchy problem $R \circ w = 0,\ w|\Gamma = x$, depends on $|A_\Gamma|$ power series (functions) in the variables $z_i,\ i \in S(\alpha), \alpha \in A_\Gamma$.

Riquier's decomposition also leads to a computation of the Hilbert function of the ideal I. For this result we need a little preparation. For $\alpha \in \mathbb{N}^n$ define $\mid \alpha \mid := \alpha_1 + \cdots + \alpha_n$. This is the total degree of the monomial s^α. For $m \in \mathbb{N}$ the space

$$F[s]_{\le m} := \oplus \{Fs^\alpha;\ \alpha \in \mathbb{N}^n,\ \mid \alpha \mid \le m\}$$

is the finite-dimensional subspace of $F[s]$ of polynomials of total degree at most m. The function

$$HF_I : \mathbb{N} \to \mathbb{N},\ m \mapsto HF_I(m) := \dim_F \left(F[s]_{\le m}/(F[s]_{\le m} \cap I)\right),$$

is called the *Hilbert function* of the ideal I. It is a polynomial function for large m, i.e. there is a unique polynomial $HP_I \in \mathbb{Q}[s]$, the *Hilbert polynomial* of I, such that $HF_I(m) = HP_I(m)$ for almost all m. The power series $HS_I := \sum_{m=0}^{\infty} HF_I(m)t^m$ in one variable t is called the *Hilbert series* of I and is a rational function of t.

Theorem 0.10 *[20, Th. 3.6] Consider a* graded *term order on* $\mathbb{N}^n$ *(i.e.:* $\sum_{i=1}^{n} \alpha_i < \sum_{i=1}^{n} \beta_i$ *implies* $\alpha \leq \beta$ *for all* $\alpha, \beta \in \mathbb{N}^n$*). Denote by* $\#(S(\alpha))$ *the number of elements of* $S(\alpha)$*. Then the dimension of* I*, i.e. the Krull dimension of the factor ring* $F[s]/I$*, is*

$$\dim(F[s]/I) = \max_{\alpha \in A_\Gamma} \#(S(\alpha)).$$

The Hilbert function, polynomial and series of the ideal I *are given as*

$$HF_I(m) = \sum \left\{ \binom{m- \mid \alpha \mid + \#(S(\alpha))}{\#(S(\alpha))} \mid \alpha \in A_\Gamma, \mid \alpha \mid \leq m \right\}$$

$$HP_I(m) = \sum_{\alpha \in A_\Gamma} \binom{m- \mid \alpha \mid + \#(S(\alpha))}{\#(S(\alpha))}$$

$$HS_I = \sum_{\alpha \in A_\Gamma} t^{|\alpha|}(1-t)^{-\#(S(\alpha))-1}.$$

0.5 Difference equations for ℓ unknowns

Definition 0.11 A *system of* k *linear partial difference equations with constant coefficients for* ℓ *unknown function in* n *variables* is given by

(1) a family

$$(R(\alpha))_{\alpha \in \mathbb{N}^n}$$

of matrices $R(\alpha) \in F^{k \times \ell}$,
where only finitely many matrices $R(\alpha)$ are different from 0,
and

(2) a family

$$(v(\alpha))_{\alpha \in \mathbb{N}^n}$$

of columns $v(\alpha) \in F^{k \times 1}$.

A *solution* of this system is a family

$$(w(\gamma))_{\gamma \in \mathbb{N}^n}$$

of columns $w(\gamma) \in F^{\ell \times 1}$ (i.e. a function $w : \mathbb{N}^n \longrightarrow F^\ell$, $\gamma \mapsto \begin{pmatrix} w_1(\gamma) \\ \vdots \\ w_\ell(\gamma) \end{pmatrix}$) such that for all $\beta \in \mathbb{N}^n$

$$\sum_{\alpha \in \mathbb{N}^n} R(\alpha) w(\alpha + \beta) = v(\beta)$$

(i.e.

$$\sum_{j=1}^{\ell} \sum_{\alpha \in \mathbb{N}^n} R_{ij}(\alpha) w_j(\alpha + \beta) = v_i(\beta) \,,$$

for $i = 1, \cdots, k$ and all $\beta \in \mathbb{N}^n$).
The system is *homogeneous* if and only if $v(\beta) = 0$, for all $\beta \in \mathbb{N}^n$.

As in the case of one unknown function we can give another interpretation of the data and the solutions:
A function $w : \mathbb{N}^n \longrightarrow F^\ell$, $\alpha \mapsto w(\alpha)$, can be written as a column (with ℓ rows) of power series

$$\begin{pmatrix} \sum_{\alpha\in\mathbb{N}^n} w_1(\alpha) z^\alpha \\ \vdots \\ \sum_{\alpha\in\mathbb{N}^n} w_\ell(\alpha) z^\alpha \end{pmatrix} ,$$

and the family of matrices $(R(\alpha))_{\alpha\in\mathbb{N}^n}$ can be represented by a matrix $R \in F[s]^{k\times\ell}$, where $R_{ij} := \sum_{\alpha\in\mathbb{N}^n} R_{ij}(\alpha) s^\alpha$, $1 \le i \le k$, $1 \le j \le \ell$. Then a solution is a column

$$\begin{pmatrix} w_1 \\ \vdots \\ w_\ell \end{pmatrix}$$

of power series such that

$$R \circ \begin{pmatrix} w_1 \\ \vdots \\ w_\ell \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} .$$

Let U be the $F[s]$-submodule generated by the rows $R_{1-}, \ldots, R_{k-}$ of R in $F[s]^{1\times\ell}$. Let $\le$ be a term order on $(\mathbb{N}^n)^\ell$, $\deg(U) \subseteq (\mathbb{N}^n)^\ell$ the set of degrees of all elements in U, and

$$\Gamma := (\mathbb{N}^n)^\ell \setminus \deg(U) .$$

Using the theory of Gröbner bases for $F[s]$-modules we can compute Γ, the normal form of any element in $F[s]^\ell$ and the syzygies for $(R_{1-}, \ldots, R_{k-})$. The results of Theorem 0.8 can be transferred to this case in an obvious way.

0.6 Convergent solutions and solutions of exponential type

Let

$$\mathbb{C}\langle z\rangle := \mathbb{C}\langle z_1, \ldots, z_n\rangle$$

be the algebra of (locally) convergent power series (i.e. power series $\sum_\alpha u(\alpha) z^\alpha$ such that there are $C > 0$ and $d_1 > 0, \ldots, d_n > 0$ with $|u(\alpha)| \le C d^\alpha$ for all $\alpha \in \mathbb{N}^n$). Consider functions from $\mathbb{N}^n$ to $\mathbb{C}$ as power series. Then the solution of

$$R_i \circ w = v_i \, , 1 \le i \le k, \; w|\Gamma = x$$

is convergent if the data x and $v_i, 1 \le i \le k$, are so [22, Theorem 24].

Let

$$O(\mathbb{C}^n; \exp)$$

be the algebra of entire holomorphic functions of *exponential type* (i.e. entire holomorphic functions $b = \sum_{\alpha\in\mathbb{N}^n} b(\alpha)z^\alpha$ on $\mathbb{C}^n$ such that there are $C > 0$ and $d_1 > 0, \ldots, d_n > 0$ with $|b(z)| \leq C\exp(\sum_{i=1}^n d_i|z_i|)$, for all $z \in \mathbb{C}^n$).
The isomorphism (0.1) in Section 1 induces the isomorphism

$$(\mathbb{C}\langle z\rangle, \circ) \cong (O(\mathbb{C}^n; \exp), \bullet) .$$

Hence the Cauchy (or initial value) problem for partial differential equations and entire functions of exponential type can be solved constructively. The solution of the corresponding problem for *locally convergent power series* is due to Riquier [27, Th. d'Existence, p. 254]. This theorem requires that the term order is graded (i.e.: $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$ implies $\alpha \leq \beta$ for all $\alpha, \beta \in \mathbb{N}^n$), as the following example [5, ch.1, §2.2] shows:
Consider the two-dimensional heat equation

$$(s_1 - s_2^2) \bullet y = \frac{\partial y}{\partial z_1} - \frac{\partial^2 y}{\partial z_2^2} = 0,\ y \in \mathbb{C}[[z_1, z_2]],$$

where z_1 resp. z_2 are the time resp. the position. As term order on $\mathbb{C}[s_1, s_2]$ take the lexicographic one with $s_1 > s_2$, hence

$$\deg(\langle s_1 - s_2^2\rangle) = (1, 0) + \mathbb{N}^2 \text{ and } \Gamma = \mathbb{N}^2 \setminus \deg(\langle s_1 - s_2^2\rangle) = \{0\} \times \mathbb{N}.$$

For the locally convergent initial condition $y(0, z_2) := (1 - z_2)^{-1} \in \mathbb{C}\langle z_2\rangle$ the unique power series solution $y(z_1, z_2) \in \mathbb{C}[[z_1, z_2]]$ according to Theorem 0.8 satisfies $y(z_1, 0) = \sum_{k=0}^\infty (2k)!z_1^k$ [5, ch.1, §2.2] and is not convergent.
According to [11, Th. 8.6.7] the heat equation also has a C^∞-solution y whose support is exactly the half-space $\{z \in \mathbb{R}^2 \,|\, z_1 \leq 0\}$ and thus satisfies the initial condition $y(0, z_2) = 0$. In contrast to Theorem 0.8 there is no uniqueness in the C^∞-Cauchy problem. The reason is that $\{0\} \times \mathbb{R}$ is a *characteristic* hyper-plane (line).

0.7 History

The problem introduced in Section 1 is known as the Cauchy problem for linear systems of partial *difference* equations with constant coefficients over the lattice $\mathbb{N}^n$. Its constructive solution was first developed in [19, §5, Th. 41, Cor. 44] after some inspirations through [9]. The inhomogeneous Cauchy problem $R \circ w = u,\ w|\Gamma = x$, with $u \neq 0$ was solved recursively. The direct algorithmic solution of the inhomogeneous problem according to Theorem 0.8,(2), is less complex and comes from [22, Th.5]. Extensions of these results to other lattices like $\mathbb{Z}^r$ are contained in [32], [31], [22], [24]. The proof for *locally convergent* power series in $\mathbb{C}\langle z\rangle$ (see Section 5) was given in [22, Th. 24] with some support from [28]. By means of the isomorphism (0.1) (Remark 2 in Section 1) this result can be transferred to partial *differential* equations for entire functions of exponential type. Systems of partial difference equations over *quasi-Frobenius* *base rings* instead of fields, for instance over the finite rings $\mathbb{Z}/\mathbb{Z}n,\ n > 0$, and their Galois extensions, are treated in [13].

The history of the solution of systems of partial *differential* equations is highly interesting. An important early result was the *Cauchy-Kovalevskaya* theorem. The by far most important contribution to the field in the beginning of the last century was Riquier's book [27], published in 1910, which treated the solution of non-linear analytic systems of partial differential equations in spaces of analytic functions. This book was far ahead of its time, did not receive the recognition and distribution it deserved, and many prominent later researchers in the field did not mention it. Essentially Riquier [27, Th. d'Existence, p.254] proved the analogue of Theorem 0.8 for so-called passive orthonomic, non-linear, analytic systems of partial differential equations, but not fully constructively. The analogues of the conditions $L_i \circ v = 0$ in part (1) of Theorem 0.8 were called the necessary *integrability or compatibility conditions*. However, Riquier's proofs and algorithms for non-linear systems and for linear systems with variable coefficients are valid generically only, i.e. they can be applied in many cases, but not in all. For instance, the simple ordinary differential equation $zy'(z) = 1$ has no power series solution although the necessary integrability conditions are trivially satisfied (there are none). With a different terminology Riquier was the first to use (the most general) term orders (total orderings with *cotes*), the sets Γ and $\deg(I)$ from Section 2 (*parametric and principal derivatives*), and S-polynomials (*cardinal derivatives*). In 1920 Janet [12] gave a simplified, more algebraic and more algorithmic presentation of Riquier's work and especially developed the theory of involutive division and Janet bases (in today's language) as a very early alternative to Gröbner bases. Riquier's Théorème d'Existence was also exposed by Ritt [28, Ch.IX]. For the linear, constant coefficient case we gave a shorter proof in modern language of Riquier's theorem in [22, Th.29]. A different and independent version of the theory of analytic systems of partial differential equations is the Cartan-Kähler theory and is exposed, for instance, by Malgrange in his recent survey article [17] with many historical comments.

The prominent algebraists Macaulay and later Gröbner did not refer to the important algebraic (in Riquier's case in disguise) work of their contemporaries Riquier and Janet. The revival of the fundamental work of Riquier and Janet, its exposition in modern language and its application in theoretical physics and later in control theory is due to Pommaret [25] (dedicated to Janet) and [26]. Pommaret also pointed out this important work to us in the middle of the 1990s. At the time of writing [19] (1988/89) it was unknown to us. In the 1990s Gerdt et al. [8] took up the work of Janet and Pommaret to develop the modern theory of involutive division and involutive bases.

The analogue of part (1) of Theorem 0.8 for linear systems of partial differential equations with constant coefficients in spaces of C^∞*-functions or distributions* (instead of analytic functions in [27]) was proven by Ehrenpreis [6], Malgrange [16] and Palamodov [23] in the beginning 1960s. That the necessary integrability conditions are also sufficient for the solvability of such a system is called the *Fundamental Principle*. Again, these authors did not mention the work of Riquier or Janet. Compare, however, Malgrange's recent historical remarks [17].

The disjoint decomposition (0.6) in Section 3 of Γ is due to Riquier [27, pp.143-168]. Janet [12, pp.74-91] constructed it both for $\deg(I)$ and for Γ and used it for the derivation of his early version of the Gröbner basis theory. We gave a modern and shorter

inductive proof in [22, pp.269-274], but used Buchberger's algorithm for the computation of $\deg(I)$. The decomposition was also derived by Gerdt [8, Decomposition Lemma] by means of involutive division. Various forms of the decomposition are used for the computation of Hilbert functions and polynomials in [25], [26], [8], [20]. The Hilbert function and the Hilbert polynomial are treated in several standard textbooks on commutative algebra. Computations by means of Stanley decompositions are described, for instance, in [1], [2] and [30, Prop 1.4.3]. All quoted references treat the more general case of submodules of free polynomial modules instead of ideals.
Further historical comments are contained in [21].

Bibliography

[1] J. Apel, *The Theory of Involutive Divisions and an Application to Hilbert Function Computations*, J. Symb. Computation 25 (1998), 683-704

[2] J. Apel, *On a conjecture of R.P. Stanley II: Quotients modulo monomoial ideals*, J. Algebr. Comb. 17 (2003), 57-74

[3] B. Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*, Dissertation, Innsbruck, 1965

[4] B. Buchberger, *Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems*, Aequationes Math. 4 (1970), 374-383

[5] Y.V. Egorov, M.A. Shubin, *Linear Partial Differential Equations. Foundations of the Classical Theory*, in Y.V. Egorov, M.A. Shubin (Editors), *Partial Differential Equations I*, Springer, Berlin, 1992

[6] L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Wiley-Interscience Publ., New York, 1970

[7] V.G. Ganzha, E.W. Mayr, E.V. Vorozhtsov (Editors), *Computer Algebra in Scientific Computing*, Springer, Berlin, 1999

[8] V.P. Gerdt, *Completion of Linear Differential Systems to Involution*, in [7], 115-137

[9] J. Gregor, *Convolutional solutions of partial difference equations*, Math. of Control, Signals and Systems 2 (1991), 205-215

[10] W. Heiß, U. Oberst, F. Pauer, *On inverse systems and squarefree decomposition of zero-dimensional polynomial ideals*, J. Symbolic Computation 41, 261-284 (2006)

[11] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, Berlin, 1983

[12] M. Janet, Sur les Systèmes d'Équations aux Dérivées Partielles, J. de Mathématiques Pures et Appliquées 8 (1920), 65-151

[13] P. Lu, M. Liu, U. Oberst, *Linear Recurring Arrays, Linear Systems and Multidimensional Cyclic Codes over Quasi-Frobenius Rings*, Acta Appl. Math. 80 (2004), 175-198

[14] Marinari, M. G., Möller, H. M., Mora, T., *On Multiplicities in polynomial system solving*, Trans. Amer. Math. Soc. 348 (1996), 3283-3321.

[15] F.S. Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge University Press, Cambridge, 1916

[16] B. Malgrange, *Systèmes différentiels à coefficients constants*, Séminaire Bourbaki 1962/1963, 246.01-246.11

[17] B. Malgrange, *Systèmes Différentiels Involutifs*, Prépublication 636, Institut Fourier, Grenoble, 2004

[18] Mourrain, B., *Isolated points, duality and residues*. Journal of pure and applied algebra 117&118 (1997), 469-493

[19] U. Oberst, *Multidimensional Constant Linear Systems*, Acta Appl. Math. 20 (1990), 1 - 175

[20] U. Oberst, *Canonical State Representations and Hilbert Functions of Multidimensional Systems*, Acta Appl. Math., online publication September 2006, pp.1-53

[21] U. Oberst, *Canonical State Representations and Hilbert Functions of Multidimensional Systems (with historical comments and a discussion of Professor Pommaret's remarks)*, Talk at the conference D3, Linz, May 2006, homepage of the Gröbner-Semester.

[22] U. Oberst, F. Pauer, *The Constructive Solution of Linear Systems of Partial Difference and Differential Equations with Constant Coefficients*, Multidimensional Systems and Signal Processing 12 (2001), 253-308

[23] V.P. Palamodov, *Linear Differential Operators*, Springer, Berlin, 1970

[24] F. Pauer, A. Unterkircher, *Gröbner Bases for Ideals in Laurent Polynomial Rings and their Application to Systems of Difference Equations*, AAECC 9/4 (l999), 271-291

[25] J.-F. Pommaret, *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon and Breach, New York, 1978

[26] J.-F. Pommaret, *Partial Differential Control Theory, VolumeI: Mathematical Tools, Volume II: Control Systems*, Kluwer Academic Publishers, Dordrecht, 2001

[27] C. Riquier, *Les Systèmes d'Équations aux Dérivées Partielles*, Gauthiers - Villars, Paris, 1910

[28] J.F. Ritt: *Differential Equations From The Algebraic Standpoint*, American Mathematical Society, New York, 1932

[29] R.P. Stanley, *Combinatorics and Commutative Algebra*, 2.Edition, Birkhäuser, Boston, 1996

[30] W.V. Vasconcelos, *Computational Methods in Commutative Algebra and Algebraic Geometry*, Springer, Berlin, 1998

[31] S. Zampieri: *A Solution of the Cauchy Problem for Multidimensional Discrete Linear Shift - Invariant Systems*, Linear Algebra Appl. 202(1994), 143 - 162

[32] E. Zerz, U. Oberst: *The Canonical Cauchy Problem for Linear Systems of Partial Difference Equations with Constant Coefficients over the Complete r-Dimensional Integral Lattice* $\mathbb{Z}^r$, Acta Applicandae Mathematicae 31(1993), 249 - 273

Author information

Ulrich Oberst, Institut für Mathematik, Universität Innsbruck, Austria.
Email: Ulrich.Oberst@uibk.ac.at

Franz Pauer, Institut für Mathematik, Universität Innsbruck, Austria.
Email: Franz.Pauer@uibk.ac.at