The Constructive Solution of Linear Systems of Partial Difference and Differential Equations with Constant Coefficients, *Multidimensional Systems and Signal Processing 12(2001), 253-308*

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Abstract

This paper gives a survey of past work in the treated subject and also contains several new results. We solve the Cauchy problem for linear systems of partial difference equations on general integral lattices by means of suitable transfer operators and show that these can be easily computed with the help of standard implementations of Gröbner basis algorithms. The Borel isomorphism permits to transfer these results to systems of partial differential equations. We also solve the Cauchy problem for the function spaces of convergent power series and for entire functions of exponential type. The unique solvability of the Cauchy problem implies that the considered function spaces are large injective cogenerators for which the duality between finitely generated modules and behaviours holds. Already in the beginning of the last century C. Riquier considered and solved problems of the type discussed here.

Key words: partial difference equation, partial differential equation, Cauchy problem, Gröbner basis, fundamental principle, transfer operator, injective cogenerator, multidimensional behaviour.

AMS-classification: 35N05; 13P10, 35C10, 39A10, 93C35.

1 Introduction

In the second section of this paper we present a survey of the results in the papers [7], [8], [9], [11], [14], [15] with partially new and simpler proofs. The formulation of the *Cauchy problem* in [8] was influenced by an early version of J. Gregor's paper [5]. We solve this problem for linear systems of partial difference equations on general integral lattices by means of suitable *transfer*

operators and show that these can be easily computed with the help of standard implementations of the Gröbner basis algorithm. The new algorithm uses less storage space and is faster than those of the quoted papers which used a recursive solution method and recursive programming [7]. The Borel isomorphism permits to transfer the results to systems of partial *differential* equations. In the third section we extend these results to the function spaces of convergent power series and of entire functions of exponential type.
The unique solvability of the Cauchy problem implies that all considered function spaces are *injective cogenerators.* They are even *large* cogenerators and give rise to the categorical duality between finitely generated polynomial modules and behaviours [8].

In more detail, let F be a field, r a positive integer, and let $\mathcal{M}$ be a submonoid of $\mathbb{Z}^r$, for example $\mathcal{M} = \mathbb{Z}^r$ or $\mathbb{N}^r$. A linear system of partial difference equations defined over $\mathcal{M}$ is given by a family

$$(R_{ij}(\mu))_{1\leq i\leq k,\, 1\leq j\leq \ell,\, \mu\in\mathcal{M}}$$

in F with the property that almost all of them are zero, and by a signal vector $v : \mathcal{M} \to F^k$. We want to find all solutions of this system, i.e. all signal vectors $w : \mathcal{M} \to F^\ell$ such that

$$\sum_{j=1}^{\ell}\sum_{\mu\in\mathcal{M}} R_{ij}(\mu)w_j(\mu+\nu) = v_i(\nu) \text{ for } i = 1,\cdots,k \text{ and all } \nu\in\mathcal{M} \ . \qquad (1)$$

In the second section a complete solution of this problem is presented, i.e. algorithmic answers to the following questions are given:

- How can one decide the solvability of system (1)?
- How can one find a canonical subset $\Gamma \subseteq \mathcal{M}$ such that for every "initial condition" $x : \Gamma \to F^\ell$ there is exactly one solution w with $w|\Gamma = x$?
- If w is such a solution, how can one compute $w(\mu)$ for any $\mu \in \mathcal{M}$?

Let $\mathcal{A} = F^{\mathcal{M}}$ be the F-space of all signals on $\mathcal{M}$ and let $\mathcal{D} := \bigoplus_{\nu\in\mathcal{M}} Fs^\nu$ be the F-algebra of all Laurent polynomials in

$$F[s, s^{-1}] = F[s_1,\ldots,s_r,s_1^{-1},\ldots,s_r^{-1}]$$

with support in $\mathcal{M}$. We consider $\mathcal{A}$ as $\mathcal{D}$-module $(\mathcal{A}, \circ)$ by the left shift action

$$(s^\mu \circ a)(\nu) := a(\mu+\nu),\ \mu,\nu\in\mathcal{M},\ a\in\mathcal{A},$$

and define

$$R_{ij} := \sum_{\mu\in\mathcal{M}} R_{ij}(\mu)s^\mu \in \mathcal{D} \ ,$$

$$R := (R_{ij})_{1\leq i\leq k,1\leq j\leq\ell} \in \mathcal{D}^{k\times\ell} \ .$$

Thus equation (1) gets the simple form

$$R\circ w = v \ .$$

Let $L \in \mathcal{D}^{g\times k}$ be a universal left annihilator of R, i.e. a matrix whose rows generate the syzygy module of the rows of R. Theorem 3 asserts that there is a signal $w \in \mathcal{A}^\ell$ such that $R \circ w = v$ if and only if $L \circ v = 0$. This result is called the *fundamental principle* by L. Ehrenpreis [3].

Let U be the $\mathcal{D}$-submodule of $\mathcal{D}^{1\times\ell} = \bigoplus_{i=1}^{\ell} \mathcal{D}\delta_i = \bigoplus_{1\le i\le\ell,\mu\in\mathcal{M}} Fs^\mu\delta_i$ generated by the rows of R. We choose a well-order $\le$ on $[\ell] \times \mathcal{M}$ where $[\ell] := \{1, \cdots, \ell\}$. This enables to define the degree $\deg(\xi)$ for $0 \ne \xi \in \mathcal{D}^{1\times\ell}$ as usual,

$$\deg(U) := \{\deg(\xi) \mid 0 \ne \xi \in U\} \text{ and } \Gamma := ([\ell] \times \mathcal{M}) \setminus \deg(U) \ .$$

Then the *canonical Cauchy problem*

$$R \circ w = v, \quad w \mid \Gamma := (w_i(\mu))_{(i,\mu)\in\Gamma} = x, \quad L \circ v = 0,$$

has the unique solution $w = \mathcal{H}_s x + \mathcal{H}v$ where $\mathcal{H}_s$ and $\mathcal{H}$ are suitable row-finite matrices which are called the 0-input resp. the 0-state *transfer operators* of the system $R \circ w = v$ (theorem 5). We present an algorithm using Gröbner basis algorithms only to compute the matrix L, the set $\deg(U)$ for a conveniently chosen well-order (a term-order if $\mathcal{M} = \mathbb{N}^r$) and the operators $\mathcal{H}_s$ and $\mathcal{H}$.

In the case $\mathcal{M} = \mathbb{N}^r$ the algebra $\mathcal{D}$ is just the polynomial algebra $F[s]$ and $\mathcal{A}$ can be identified with the ring $F[[z]]$ of formal power series ($a = \sum_{\mu\in\mathbb{N}^r} a(\mu)z^\mu$). We reformulate a result of Riquier [11] which expresses the initial data as a finite family of formal power series (theorem 15).

The results for partial *difference* equations can, in the case $\mathcal{M} = \mathbb{N}^r$, $\mathcal{D} = F[s]$ and $char(F) = 0$, be translated into those for linear systems of partial *differential* equations with constant coefficients. The polynomial algebra $F[s]$ acts on $\mathcal{A} = F[[z]]$ by partial differential operators via

$$s^\mu \bullet b := \partial^\mu(b), \ s^\mu \bullet z^\nu = \prod_{i=1}^{r} \nu_i * (\nu_i - 1) * \cdots * (\nu_i - \mu_i + 1) z^{\nu-\mu} \ .$$

By means of the $F[s]$-linear *Borel isomorphism*

$$\widehat{(-)} : (F[[z]], \circ) \cong (F[[z]], \bullet)$$

$$a := \sum_{\nu\in\mathbb{N}^r} a(\nu)z^\nu \mapsto \hat{a} := \sum_{\nu\in\mathbb{N}^r} \frac{a(\nu)}{\nu!} z^\nu,$$

we get the analoga of theorems 3 and 5 for the Cauchy problem

$$R(\partial)w = v, \ w|\Gamma = x, \ L(\partial)v = 0$$

of partial differential equations (theorems 16 and 17).

A signal $b : \mathcal{M} \to \mathbb{C}$ is *convergent* if there are $C \in \mathbb{R}_{>0}$ and $\rho \in \mathbb{R}^r_{>0}$ such that

$$|b(\mu)| \le C\rho^{|\mu|}$$

for all $\mu \in \mathcal{M}$ (where $|\mu| := (|\mu_1|, \ldots, |\mu_r|)$). Section 3 treats the following question: If v and x are convergent, are the solutions of

$$R \circ w = v, \ w|\Gamma = x, \ L \circ v = 0$$

resp. (for $\mathcal{M} = \mathbb{N}^r$)

$$R(\partial)w = v, \ w|\Gamma = x, \ L(\partial)v = 0$$

convergent too? Theorems 24 and 29 give a positive answer for $\mathcal{M} = \mathbb{N}^r$, theorem 44 for $\mathcal{M} = \mathbb{N}^{r_1} \times \mathbb{Z}^{r_2}$. Theorem 29 was inspired by, but is different from C. Riquier's *existence theorem*, the main result of the book [11].

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2 The Cauchy problem for formal power series

2.1 Constructive solution of the Cauchy problem for partial difference equations

Let F be a field, r a positive integer, and let $\mathcal{M}$ be a submonoid of $\mathbb{Z}^r$, i.e. a subset closed under addition and containing $0 \in \mathbb{Z}^r$. The most important examples for $\mathcal{M}$ are $\mathbb{Z}^r$, $\mathbb{N}^r$ and $\mathbb{N}^{r-s} \times \mathbb{Z}^s$.
A *signal* on $\mathcal{M}$ with entries in F is a map from $\mathcal{M}$ to F or, in other words, a multi-sequence with entries in F and indices in $\mathcal{M}$. Let ℓ be a positive integer and $[\ell] := \{1, \dots, \ell\}$. A *signal vector* or *vector signal* is a map from $\mathcal{M}$ to F^ℓ. We denote by

$$\mathcal{A} := F^{\mathcal{M}} = \{a = (a(\nu))_{\nu \in \mathcal{M}}; a(\nu) \in F\} \tag{2}$$

the *signal space* of signals on $\mathcal{M}$ with entries in F. For

$$w : \mathcal{M} \longrightarrow F^\ell$$

we write

$$w = \begin{pmatrix} w_1 \\ \cdots \\ w_\ell \end{pmatrix}, \text{ where } w_1, \dots w_\ell \in \mathcal{A}$$

and

$$w(\nu) = \begin{pmatrix} w_1(\nu) \\ \cdots \\ w_\ell(\nu) \end{pmatrix} \in F^\ell .$$

Hence the *space of signal vectors* $\left(F^\ell\right)^{\mathcal{M}}$ can be identified with $\mathcal{A}^\ell = \left(F^{\mathcal{M}}\right)^\ell$ and with $F^{[\ell] \times \mathcal{M}}$.

A *linear system of partial difference equations defined over* $\mathcal{M}$ is given by a family

$$(R_{ij}(\mu))_{1 \le i \le k,\, 1 \le j \le \ell,\, \mu \in \mathcal{M}}$$

in F with the property that almost all of them are zero, and by a signal vector $v \in \mathcal{A}^k$. We want to find all signal vectors $w \in \mathcal{A}^\ell$ such that

$$\sum_{j=1}^{\ell} \sum_{\mu \in \mathcal{M}} R_{ij}(\mu) w_j(\mu + \nu) = v_i(\nu) \text{ for } i = 1, \cdots, k \text{ and all } \nu \in \mathcal{M} . \tag{3}$$

These signal vectors are *solutions* of (3).
The solution space of the homogeneous equation ($v = 0$)

$$\mathcal{B} := \{w \in \mathcal{A}^\ell; \text{Equation (3) is satisfied for } v = 0\} \tag{4}$$

is called an *ℓ-dimensional behaviour* (according to J.C. Willems [13]) or an *ℓ-dimensional system.*

The preceding data can be expressed in (Laurent-)polynomial form. Let

$$F[s,s^{-1}] := F[s_1,s_1^{-1}\cdots,s_r,s_r^{-1}] = \bigoplus_{\nu\in\mathbb{Z}^r} Fs^\nu,\quad s^\nu = s_1^{\nu_1}\cdot s_2^{\nu_2}\cdots s_r^{\nu_r}, \tag{5}$$

be the F-algebra of Laurent-polynomials in r indeterminates and let

$$\mathcal{D} := F[\mathcal{M}] := \bigoplus_{\nu\in\mathcal{M}} Fs^\nu$$

be its F-subspace of all Laurent-polynomials with support in $\mathcal{M}$. Since $\mathcal{M}$ is a submonoid of $\mathbb{Z}^r$, the subspace $\mathcal{D}$ is a subalgebra of $F[s,s^{-1}]$. The letters $\mathcal{D}$ resp. s_i are suggestive for *difference operators* resp. *shifts.* The ring $\mathcal{D}$ acts on the signal space $\mathcal{A}$ via left shifts:

$$(s^\mu\circ a)(\nu) := a(\mu+\nu),\ \mu,\nu\in\mathcal{M}, \tag{6}$$

and this action makes $\mathcal{A}$ a $\mathcal{D}-$ module. The left shift terminology comes from the one-dimensional situation:

$$s\circ(a(0),a(1),\cdots) = (a(1),a(2),\cdots).$$

The equations (3) thus obtain the form

$$\begin{gathered}\sum_{j,\mu} R_{ij}(\mu)w_j(\mu+\nu) = \sum_{j,\mu} R_{ij}(\mu)(s^\mu\circ w_j)(\nu) = \sum_j (R_{ij}\circ w_j)(\nu) = v_i(\nu),\\ \text{for } i=1,\dots,k \text{ and all } \nu\in\mathcal{M},\ \text{where } R_{ij} := \sum_\mu R_{ij}(\mu)s^\mu\in\mathcal{D},\\ \text{or}\\ R\circ w = v\ ,\quad\text{where}\\ R := (R_{ij})_{i,j}\in\mathcal{D}^{k\times\ell} \text{ and } R\circ w := (\sum_{j=1}^{\ell} R_{ij}\circ w_j)_{i=1,\cdots,k}\in\mathcal{A}^k.\end{gathered} \tag{7}$$

The associated behaviour is

$$\mathcal{B} = \ker(R\circ:\mathcal{A}^\ell\to\mathcal{A}^k) = \{w\in\mathcal{A}^\ell;\ R\circ w = 0\}. \tag{8}$$

It is a $\mathcal{D}$-submodule of $\mathcal{A}^\ell$ since the map $R\circ:\mathcal{A}^\ell\to\mathcal{A}^k,\ w\mapsto R\circ w$, is $\mathcal{D}-$ linear. This representation of $\mathcal{B}$ is called its *polynomial form* or *kernel representation.*

For every $\mathcal{D}-$ module M the dual vector space $M^* := \mathrm{Hom}_F(M,F)$ is a $\mathcal{D}-$ module, too, with the scalar multiplication

$$(d\circ\varphi)(x) := \varphi(dx),\ \varphi\in M^*,\ x\in M,\ d\in\mathcal{D}. \tag{9}$$

The dual space of $\mathcal{D}$ can be identified with $\mathcal{A}$ by means of the obvious $\mathcal{D}-$ isomorphism

$$\mathcal{D}^* = \mathrm{Hom}_F(\mathcal{D},F)\cong\mathcal{A} = F^{\mathcal{M}},\ \varphi\leftrightarrow a,\ \varphi(s^\nu) = a(\nu). \tag{10}$$

Likewise there is the canonical $\mathcal{D}-$ isomorphism

$$(\mathcal{D}^{1\times\ell})^* = \mathrm{Hom}_F(\mathcal{D}^{1\times\ell}, F) \cong \mathcal{A}^\ell,\ \varphi \leftrightarrow w,\ \varphi(s^\nu\delta_j) = w_j(\nu), \tag{11}$$

where $\delta_j := (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0) \in \mathcal{D}^{1\times\ell}$, $j = 1, \ldots, \ell$, is the standard $\mathcal{D}$-basis of $\mathcal{D}^{1\times\ell}$.
Any $F-$ linear map $f : M \to N$ induces its dual map

$$f^* := \mathrm{Hom}(f, F) : N^* \to M^*,\ \psi \mapsto \psi f$$

such that f^* is $\mathcal{D}-$ linear if f is. We will identify the two objects in (11). With this identification an easy computation yields

$$(\circ R)^* = R\circ : \mathcal{A}^\ell = (\mathcal{D}^{1\times l})^* \to \mathcal{A}^k = (\mathcal{D}^{1\times k})^*\,, \tag{12}$$

where $\circ R$ is the $\mathcal{D}$-linear map

$$\circ R : \mathcal{D}^{1\times k} \longrightarrow \mathcal{D}^{1\times\ell}\ ,\ \xi \longmapsto \xi R\ .$$

The following lemma is standard and easy to verify (use that any linear function on a subspace can be extended to the whole space).

Lemma 1. *A sequence*

$$U \xrightarrow{f} V \xrightarrow{g} W$$

of vector spaces and linear maps is exact (i.e. $\ker(g) = \mathrm{im}(f)$*) if and only if the dual sequence*

$$U^* \xleftarrow{f^*} V^* \xleftarrow{g^*} W^*$$

is exact.

Definition 2. Let $R \in \mathcal{D}^{k\times\ell}$. A matrix $L \in \mathcal{D}^{g\times k}$ is a *universal left annihilator* of R if and only if the sequence

$$\mathcal{D}^{1\times g} \xrightarrow{\circ L} \mathcal{D}^{1\times k} \xrightarrow{\circ R} \mathcal{D}^{1\times\ell} \tag{13}$$

is exact, i.e. that

$$\ker(\circ R) = \mathrm{im}(\circ L)\ .$$

This signifies that the rows of L generate the module of relations or syzygies of the rows of R, i.e. that

$$\{\xi \in \mathcal{D}^{1\times k};\ \sum_{i=1}^{k} \xi_i R_{i-} = 0\} = \{\sum_{i=1}^{g} \eta_i L_{i-};\ \eta \in \mathcal{D}^{1\times g}\} =: \mathcal{D}^{1\times g}L\ .$$

Theorem 3. (Fundamental principle). *Let $v \in \mathcal{A}^k$ be any signal vector, $R \in \mathcal{D}^{k\times\ell}$, and let $L \in \mathcal{D}^{g\times k}$ be a universal left annihilator of R. Then the difference system $R \circ w = v$ is solvable, i.e. has a solution $w \in \mathcal{A}^\ell$, if and only if the right side v satisfies the identity $L \circ v = 0$.*
The equation $L \circ v = 0$ or rather the equations $L_{i-} \circ v = 0,\ i = 1, \cdots, g$, are called the integrability conditions *of the system $R \circ w = v$.*

Proof. By means of lemma 1 and equations (11) and (12) the exact sequence (13) implies the exact sequence

$$\mathcal{A}^{1\times g} \xleftarrow{L\circ} \mathcal{A}^{1\times k} \xleftarrow{R\circ} \mathcal{A}^{1\times l}, \text{ hence}$$
$$\operatorname{im}(R\circ) = R\circ\mathcal{A}^k = \ker(L\circ) = \{\tilde{v} \in \mathcal{A}^k;\ L\circ\tilde{v} = 0\}.$$

But this means exactly that $R\circ w = v$ is solvable or $v \in \operatorname{im}(R\circ)$ if and only if $L\circ v = 0$. □

Definition 4. Let $R \in \mathcal{D}^{k\times\ell}$ and let $U := \mathcal{D}^{1\times k}R \subseteq \mathcal{D}^{1\times\ell}$ be the $\mathcal{D}$-submodule of $\mathcal{D}^{1\times\ell}$ generated by the rows of R. Let $\le$ be a well-order on $[\ell]\times\mathcal{M}$. For $0 \neq \xi = \sum_{j,\nu}\xi_j(\nu)s^\nu\delta_j \in \mathcal{D}^\ell$ we define the *degree of* ξ

$$\deg(\xi) := (j(\xi), d(\xi)) := \max\{(j,\nu) \in [\ell]\times\mathcal{M};\ \xi_j(\nu) \neq 0\} \in [\ell]\times\mathcal{M}$$

(with respect to $\le$) and the *degree set* of U

$$\deg(U) := \{\deg(\xi);\ 0 \neq \xi \in U\} \subseteq [\ell]\times\mathcal{M}\ .$$

The complement

$$\Gamma := ([\ell]\times\mathcal{M}) \setminus \deg(U) \tag{14}$$

is called the *canonical initial region* of the system $R\circ w = v$ and F^Γ its *state space* of *initial data*, this terminology being motivated by theorem 5 below. An easy result of linear algebra is the direct decomposition

$$\mathcal{D}^{1\times\ell} = \Big(\bigoplus_{(i,\mu)\in\Gamma} Fs^\mu\delta_i\Big) \oplus U \ni \xi = \xi_{norm} + (\xi - \xi_{norm})\ . \tag{15}$$

The component $\xi_{norm} \in \bigoplus_{(i,\mu)\in\Gamma} Fs^\mu\delta_i$ of $\xi \in \mathcal{D}^{1\times\ell}$ is called its *normal form.* The following terminology from [7] is motivated by the next theorem. We have to use $I\times J$-matrices $(M(i,j))_{(i,j)\in I\times J} \in F^{I\times J}$ with *infinite* index sets I and J. Such a matrix is called *row-finite* if for all $i \in I$ the row $M(i,-)$ has a finite support $\{j \in J\,;\ M(i,j) \neq 0\}$. The row-finite $([\ell]\times\mathcal{M})\times\Gamma$-matrix

$$\mathcal{H}_s \in F^{([\ell]\times\mathcal{M})\times\Gamma} \text{ such that}$$
$$(s^\nu\delta_j)_{norm} = \sum_{(i,\mu)\in\Gamma} \mathcal{H}_s((j,\nu),(i,\mu))s^\mu\delta_i \text{ for all } (j,\nu) \in [\ell]\times\mathcal{M} \tag{16}$$

is called the *zero-input transfer operator* of the system $R\circ w = 0$.
For every $(j,\nu) \in [\ell]\times\mathcal{M}$ we choose an element $\mathcal{H}_{(j,\nu)} \in \mathcal{D}^{1\times k}$ such that

$$s^\nu\delta_j - (s^\nu\delta_j)_{norm} = \mathcal{H}_{(j,\nu)}\circ R\ . \tag{17}$$

The choice of $\mathcal{H}_{(j,\nu)}$ is in general not unique. Let $\mathcal{H}((j,\nu),(i,\mu)) \in F$ be such that

$$\mathcal{H}_{(j,\nu)} = \sum_{(i,\mu)\in[k]\times\mathcal{M}} \mathcal{H}((j,\nu),(i,\mu))s^\mu\delta_i, \text{ for all } (j,\nu) \in [\ell]\times\mathcal{M}.$$

The row-finite $([\ell]\times\mathcal{M})\times([k]\times\mathcal{M})-$matrix

$$\mathcal{H} := \big(\mathcal{H}((j,\nu),(i,\mu))\big)_{(j,\nu)\in[\ell]\times\mathcal{M},(i,\mu)\in[k]\times\mathcal{M}} \in F^{([\ell]\times\mathcal{M})\times([k]\times\mathcal{M})} \tag{18}$$

is called the *(0-state) transfer operator* of the system $R\circ w = v,\ L\circ v = 0.$

Theorem 5. (Cauchy problem) *Let $R \in \mathcal{D}^{k\times\ell}$ and let $L \in \mathcal{D}^{g\times k}$ be a universal left annihilator of R. Let $v \in \mathcal{A}^k$ be a signal vector which satisfies the integrability conditions $L \circ v = 0$ and let $x = (x_i(\mu))_{(i,\mu)\in\Gamma}$ be arbitrary* initial data *in F^Γ. Then the* canonical Cauchy problem

$$R \circ w = v, \quad w \mid \Gamma := (w_i(\mu))_{(i,\mu)\in\Gamma} = x, \quad L \circ v = 0,$$

has the unique solution $w = \mathcal{H}_s x + \mathcal{H}v$, where

$$\begin{gathered} w_j(\nu) = \big(\mathcal{H}_s x\big)_j(\nu) + \big(\mathcal{H}v\big)_j(\nu)\ , \\ (\mathcal{H}_s x)_j(\nu) = \sum_{(i,\mu)\in\Gamma} \mathcal{H}_s((j,\nu),(i,\mu))x_i(\mu), \\ \text{and} \\ (\mathcal{H}v)_j(\nu) = \sum_{(i,\mu)\in[k]\times\mathcal{M}} \mathcal{H}((j,\nu),(i,\mu))v_i(\mu),\ (j,\nu)\in[\ell]\times\mathcal{M}. \end{gathered} \tag{19}$$

The signal $\mathcal{H}_s x \in \mathcal{A}^\ell$ is the unique solution of the homogeneous Cauchy problem

$$R \circ w = 0, \quad w \mid \Gamma = x,$$

the signal $\mathcal{H}v \in \mathcal{A}^\ell$ is the unique solution of the Cauchy problem with zero initial state

$$R \circ w = v, \quad w \mid \Gamma = 0, \quad L \circ v = 0.$$

In other terms: The map

$$\{\begin{pmatrix} w \\ v \end{pmatrix} \in \mathcal{A}^\ell \times \mathcal{A}^k \mid R \circ w = v\} \to F^\Gamma \times \ker(L\circ)\ ,\ \begin{pmatrix} w \\ v \end{pmatrix} \mapsto \begin{pmatrix} w|\Gamma \\ v \end{pmatrix}$$

is an isomorphism, its inverse maps

$$\begin{pmatrix} x \\ v \end{pmatrix} \ \text{to}\ \begin{pmatrix} \mathcal{H}_s x + \mathcal{H}v \\ v \end{pmatrix}.$$

As mentioned above the matrix $\mathcal{H}$ is not unique in general, however the operator $\{v \in \mathcal{A}^k \mid L \circ v = 0\} \longrightarrow \mathcal{A}^\ell\ ,\ v \longmapsto \mathcal{H}v\ ,$ is.

Proof. (i) We consider the homogeneous problem first. By Equation (15) the sequence

$$\mathcal{D}^{1\times k} \xrightarrow{\circ R} \mathcal{D}^{1\times l} \xrightarrow{(-)_{norm}} \bigoplus_{(i,\mu)\in\Gamma} F s^\mu \delta_i \to 0$$

is exact. By Lemma 1 and (12) it induces the exact sequence

$$\mathcal{A}^k = (\mathcal{D}^{1\times k})^* \xleftarrow{R\circ} \mathcal{A}^\ell = (\mathcal{D}^{1\times l})^* \xleftarrow{h} (\bigoplus_{(i,\mu)\in\Gamma} F s^\mu \delta_i)^* \leftarrow 0\ \ ,$$

where

$$\big(h(\varphi)\big)_j(\nu) = \varphi((s^\nu \delta_j)_{norm}) = \sum_{(i,\mu)\in\Gamma} \mathcal{H}_s((j,\nu),(i,\mu))\varphi(s^\mu \delta_i)\ .$$

Composing h with the isomorphism

$$F^\Gamma \cong \mathrm{Hom}_F(\bigoplus_{(i,\mu)\in\Gamma} Fs^\mu\delta_i, F)$$
$$(\varphi(s^\mu\delta_i))_{(i,\mu)\in\Gamma} \leftrightarrow \varphi$$

we get the F-linear isomorphism

$$F^\Gamma \cong \mathcal{B} = \ker(R\circ) \subseteq \mathcal{A}^\ell$$
$$x \mapsto \mathcal{H}_s x$$

and $\mathcal{H}_s x \mid \Gamma = x$. Hence the homogeneous problem has the unique solution $\mathcal{H}_s x$.

(ii) Consider now the inhomogeneous equation.
Since v satisfies the integrability conditions there is some solution w^1 of

$$R\circ w^1 = v\ .$$

Let w^2 be the unique solution of the homogeneous problem

$$R\circ w^2 = 0,\ w^2 \mid \Gamma = x - (w^1 \mid \Gamma)\ .$$

Then

$$R\circ(w^1+w^2) = v+0 = v \text{ and}$$
$$(w^1+w^2)\mid\Gamma = (w^1\mid\Gamma) + x - (w^1\mid\Gamma) = x,$$

i.e. $w := w^1 + w^2$ is a solution.
Since

$$s^\nu\delta_j = \sum_{(i,\mu)\in\Gamma} \mathcal{H}_s((j,\nu),(i,\mu))s^\mu\delta_i + \mathcal{H}_{(j,\nu)}\circ R =$$
$$= \sum_{(i,\mu)\in\Gamma} \mathcal{H}_s((j,\nu),(i,\mu))s^\mu\delta_i + \sum_{(i,\mu)\in[k]\times\mathcal{M}} \mathcal{H}((j,\nu),(i,\mu))s^\mu\delta_i\circ R\ ,$$

we have

$$w_j(\nu) = (s^\nu\delta_j\circ w)(0) = \sum_{(i,\mu)\in\Gamma} \mathcal{H}_s((j,\nu),(i,\mu))(s^\mu\delta_i\circ w)(0) + (\mathcal{H}_{(j,\nu)}\circ R\circ w)(0) =$$
$$= \sum_{(i,\mu)\in\Gamma} \mathcal{H}_s((j,\nu),(i,\mu))x_i(\mu) + (\mathcal{H}_{(j,\nu)}\circ v)(0) =$$
$$= \sum_{(i,\mu)\in\Gamma} \mathcal{H}_s((j,\nu),(i,\mu))x_i(\mu) + \sum_{(i,\mu)\in[k]\times\mathcal{M}} \mathcal{H}((j,\nu),(i,\mu))v_i(\mu)\ .$$

In particular, the solution is unique. □

In order to obtain an algorithm for the construction of the solution w whose existence was shown in the preceding theorem, we have to

(i) compute a universal left annihilator L of R,

(ii) determine the set $\deg(U)$,

(iii) compute a representation

$$s^\nu \delta_j = \sum_{(\mu,i)\in\Gamma} \mathcal{H}_s((j,\nu),(i,\mu)) s^\mu \delta_i + \sum_{1\le i\le k,\mu\in\mathcal{M}} \mathcal{H}((j,\nu),(i,\mu)) s^\mu R_{i-}$$

for every $(j,\nu) \in [\ell] \times \mathcal{M}$.

First we consider the case $\mathcal{M} = \mathbb{N}^r$. Then the F-algebra $\mathcal{D}$ is the polynomial ring $F[s] := F[s_1,\dots,s_r]$ and we can choose a *term order* $\le$ on $[\ell] \times \mathbb{N}^r$, i.e. a well-order on $[\ell] \times \mathbb{N}^r$ satisfying the following two conditions:

$$(i,0) \le (i,\mu), \text{ for all } \mu \in \mathbb{N}^r, i \in [\ell],$$
$$\text{and}$$
$$(i,\mu) \le (i',\nu) \text{ implies } (i,\mu+\lambda) \le (i',\nu+\lambda), \text{ for all } \mu,\nu,\lambda \in \mathbb{N}^r, i,i' \in [\ell]. \tag{20}$$

Well-known examples for term-orders are the pure lexicographic and the graded lexicographic orders.

For $(i,\mu) \in [\ell] \times \mathbb{N}^r$ we define

$$(i,\mu) + \mathbb{N}^r := \{(i,\mu+\nu) \mid \nu \in \mathbb{N}^r\} .$$

A *Gröbner basis* G of U (with respect to the term order $\le$) is a finite subset of U such that

$$\deg(U) = \bigcup_{g\in G} \big(\deg(g) + \mathbb{N}^r\big). \tag{21}$$

Let

$$\deg(U)_j := \{\mu \in \mathbb{N}^r \mid (j,\mu) \in \deg(U)\} \subseteq \mathbb{N}^r , \ j = 1,\dots,\ell. \tag{22}$$

For $r = 2$ these sets can be represented by echelon pictures for which an example is given by figure 1, the bold-face vertices of the staircase representing the elements $d(g) \in \deg(U)_j$, where $g \in G$, $\deg(g) = (j(g), d(g))$ and $j(g) = j$.

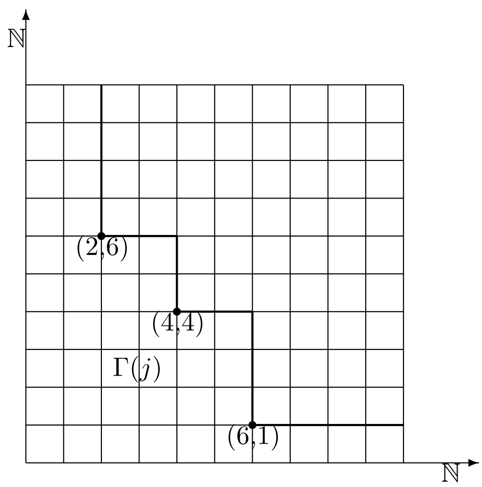

Figure 1: The j-th component $\deg(U)_j$ and its complementary initial region $\Gamma(j)$ below the staircase.

The Gröbner basis G can be computed by Buchberger's algorithm, which is implemented in several computer-algebra systems. Division of $s^\nu \delta_j$ by the Gröbner basis G yields the representation

$$s^\nu \delta_j = \sum_{(i,\mu)\in\Gamma} \mathcal{H}_s((j,\nu),(i,\mu)) s^\mu \delta_i + \sum_{g\in G} u_g g \ ,$$

where $u_g \in \mathcal{D},\ g \in G$. Since the elements of G and the rows of R generate the same $\mathcal{D}$-module U, there are polynomials $X_{g1}, \ldots, X_{g\ell} \in \mathcal{D}$ such that

$$g = \sum_{i=1}^{k} X_{gi} R_{i-} \quad \text{for all } g \in G \ .$$

Hence

$$s^\nu \delta_j = \sum_{(\mu,i)\in\Gamma} \mathcal{H}_s((j,\nu),(i,\mu)) s^\mu \delta_i + \sum_{i=1}^{k} \Big(\sum_{g\in G} u_g X_{gi}\Big) R_{i-} \ ,$$

and $\mathcal{H}((j,\nu),(i,\mu))$ is the coefficient of $\sum_{g\in G} u_g X_{gi}$ at $s^\mu \delta_i$.
The polynomial matrix $X := (X_{gi})_{g\in G, i\in[k]}$ and the Gröbner basis G of U can be obtained by *one* computation only of a Gröbner basis. Namely, we consider the matrix

$$(R \mid I_k) \in \mathcal{D}^{k\times(\ell\times k)}$$

and the $\mathcal{D}$-module V generated by its rows. We choose a term order $\leq$ on $\{\ell+1, \ldots, \ell+k\} \times \mathbb{N}^r$ and extend it and the term order on $[\ell] \times \mathbb{N}^r$ to a term order $\leq$ on $[\ell+k] \times \mathbb{N}^r$ by defining

$$(i,\mu) < (j,\nu) \text{ for all } (i,\mu) \in \{\ell+1, \ldots, \ell+k\} \times \mathbb{N}^r,\ (j,\nu) \in [k] \times \mathbb{N}^r \ .$$

Then for any vector $(\xi,\eta) \in \mathcal{D}^{1\times\ell} \times \mathcal{D}^{1\times k} = \mathcal{D}^{1\times(\ell+k)}$ with $\xi \neq 0$ we have

$$\deg(\xi,\eta) = \deg(\xi) \in [\ell] \times \mathbb{N}^r \subseteq [\ell+k] \times \mathbb{N}^r \ .$$

Let $X \in \mathcal{D}^{m\times k}$ be such that the rows of

$$X(R \mid I_k) = (XR \mid X) \in \mathcal{D}^{m\times(\ell+k)}$$

are a Gröbner basis of V. Due to the choice of the term order on $[\ell+k] \times \mathbb{N}^r$ the rows of $G := XR$ are a Gröbner basis of U. Hence the ℓ first components of a Gröbner basis of V furnish a Gröbner basis of U and the last k components the representation of the elements of G as $\mathcal{D}$-linear combinations of the rows of R.

Finally, the computation of L is a standard application of Gröbner bases, since the rows of L are a system of generators of the solution space of the system

$$(\xi_1, \ldots, \xi_k) R = 0$$

of linear equations with coefficients in $F[s]$ (see for example [1], chapter 3.7).

Example 6. Consider the system of partial difference equations over $\mathbb{N}^2$ given by

$$R := \begin{pmatrix} 2s_1^2 s_2 + 1 \\ 3s_1 s_2^2 + 2 \end{pmatrix} \quad \text{and the signal vector} \quad v := \begin{pmatrix} 13z_2^2 + 5z_1 z_2^3 \\ 6z_2^2 + 10z_1^2 z_2^3 + 15 z_1 z_2 \end{pmatrix} \ .$$

A Gröbner basis of the ideal $U := \langle R_1, R_2\rangle$ in $F[s_1, s_2]$ with respect to the graded lexicographic order ($s_1 > s_2$) is

$$\{G_1 := 4s_1 - 3s_2 = -3s_2R_1 + 2s_1R_2, G_2 := 9s_2^3 + 8 = 9s_2^3R_1 + (4 - 6s_1s_2^2)R_2\}$$

and hence $\Gamma = \{(0,0),(0,1),(0,2)\}$. The row $L := (-R_2, R_1)$ is a universal left annihilator of R. Since $L \circ v = 0$ the system $R \circ w = v$ is solvable for arbitrary initial data on Γ. For $\nu := (2,3)$ we get

$$s_1^2s_2^3 = -\frac{1}{2}s_2^2 + (-\frac{3}{4}s_1s_2^4 - \frac{9}{16}s_2^5)R_1 + (\frac{1}{2}s_1^2s_2^3 + \frac{3}{8}s_1s_2^4)R_2 \ .$$

Hence

$$(\mathcal{H}v)(2,3)) = -\frac{3}{4}v_1(1,4) - \frac{9}{16}v_1(0,5) + \frac{1}{2}v_2(2,3) + \frac{3}{8}v_2(1,4) = 5 \ .$$

For initial data $x(0,0) = 0, x(0,1) = 0, x(0,2) = 3$ we get

$$(\mathcal{H}_s x)(2,3) = -\frac{1}{2}x(0,2) = -\frac{3}{2} \ .$$

The computations in this example have been done with the help of the computer algebra system Maple 6.

Let now $\mathcal{M}$ be an arbitrary finitely generated submonoid of $\mathbb{Z}^r$. Then $F^{\mathcal{M}}$ is a finitely generated F-algebra. There are two known approaches to solve problems (i)-(iii).

The first approach to solve these problems chooses a surjective algebra homomorphism ψ from a polynomial ring P to $\mathcal{D} = F[\mathcal{M}]$. For example, if $\mathcal{M} = \mathbb{Z}^r$, we can choose the algebra homomorphism

$$\psi : P = F[t_0, t_1, \ldots, t_r] \longrightarrow F[s_1, s_1^{-1}, \ldots, s_r, s_r^{-1}] = \mathcal{D} \qquad (23)$$

given by $\psi(t_i) = s_i,\ 1 \le i \le r$, and $\psi(t_0) = s_1^{-1}s_2^{-1}\cdots s_r^{-1}$. The computations in $\mathcal{D}^\ell$ are "pulled back" to computations in P^ℓ, where Gröbner basis methods can be applied. In [7], [14], [15] this method was developed for the case $\mathcal{M} = \mathbb{N}^{r_1} \times \mathbb{Z}^{r_2}$. Its elaboration for an implementation in the computer algebra system Axiom 2.1 and examples can be found in [7]. Theorem 43 below reduces the computation of Gröbner bases in the Laurent polynomial algebra $F[\mathbb{N}^{r_1} \times \mathbb{Z}^{r_2}]$ to computations in the polynomial algebra

$$F[\mathbb{N}^{r_1+2r_2}] = F[s_1, \cdots, s_{r_1+r_2}, t_{r_1+1}, \cdots, t_{r_1+r_2}]$$

and proves the conjecture 3.10 of [7].

The second approach (see [9]) avoids the passage to a polynomial ring and develops a Gröbner basis theory for rings of Laurent polynomials. There is no total order on $[\ell] \times \mathbb{Z}^r$ which satisfies the conditions (20) (with $\mathbb{Z}^r$ instead of $\mathbb{N}^r$). Hence degrees of elements in $\mathcal{D}^{1\times\ell}$ must be defined with respect to a "generalized term order". To define it, we first choose a conic decomposition of $\mathcal{M}$ i.e. a finite family $\left(\mathcal{M}_j\right)_{j\in J}$ of finitely generated submonoids of $\mathcal{M}$ such that for all $j \in J$

$\mathcal{M}_j$ is a cone, i.e. 0 is the only invertible element,

the group generated by $\mathcal{M}_j$ contains $\mathcal{M}$,

and

$$\bigcup_{j\in J} \mathcal{M}_j = \mathcal{M} \ .$$

A *generalized term order* on $[\ell] \times \mathcal{M}$ is a total order $\leq$ satisfying the following conditions:

$$\begin{gathered}(i,0) \leq (i,\mu), \text{ for all } \mu \in \mathcal{M}, i \in [\ell], \\ \text{and} \\ (i,\mu) \leq (i',\nu) \text{ implies } (i,\mu+\lambda) \leq (i',\nu+\lambda), \\ \text{for all } \mu \in \mathcal{M}, i,i' \in [\ell], j \in J, \nu,\lambda \in \mathcal{M}_j.\end{gathered} \tag{24}$$

Every generalized term order is a well-order.
As an example, consider $\mathcal{M} = \mathbb{Z}^r$ and

$$m : \mathbb{Z}^r \longrightarrow \mathbb{Z} \ , \ \nu \longmapsto \min\{0, \nu_1, \ldots, \nu_r\} \ .$$

Define $\mathcal{M}_0 := \mathbb{N}^r$, $\mathcal{M}_j := \{\nu \in \mathbb{Z}^r \mid m(\nu) = \nu_j\}, 1 \leq j \leq r$, and

$$\begin{gathered}(i,\nu) \leq (j,\mu) :\Leftrightarrow \\ (i<j) \text{ or } (i=j \text{ and } m(\nu) > m(\mu)) \text{ or } (i=j,\, m(\nu)=m(\mu) \text{ and } \nu <_{lex} \mu),\end{gathered}$$

where $<_{lex}$ is the lexicographic order on $\mathbb{Z}^r$. Then $\leq$ is a generalized term order on $[\ell] \times \mathcal{M}$.
Note that in general $\deg(s^\mu g) \neq \mu + \deg(g)$.
A *Gröbner basis* G of U (with respect to a generalized term order $\leq$) is a finite subset of U such that

$$\deg(U) = \bigcup_{g \in G} \{\deg(s^\mu g) \mid \mu \in \mathcal{M}\} \ . \tag{25}$$

These Gröbner bases can be computed by an algorithm which is very similar to Buchberger's, but it is computationally more expensive. Likewise there is an analogon to the division algorithm [9, Prop. 3.1 and Prop.3.5]. Thus the problems (i)-(iii) could be solved without leaving the ring $\mathcal{D}$. Optimal implementations of this algorithm have still to be constructed. Examples were computed in [9, ch. 4].

Convention 7. We denote by $I_1 \uplus I_2$ the disjoint union of the sets I_1 and I_2. For an additive monoid G and index sets $I = J \uplus J'$ we identify

$$\begin{gathered}G^J = G^J \times 0 \subset G^I = G^J \times G^{J'} = G^J \oplus G^{J'} \\ g = (g \mid J, g \mid J') = g \mid J + g \mid J',\ g \in G^I \text{ and } g \mid J := (g(j))_{j \in J}.\end{gathered}$$

We apply this to

$$\begin{gathered}F^\Gamma \subset \mathcal{A}^\ell = F^{[\ell] \times \mathbb{N}^r} = F^\Gamma \times F^{\deg(U)} = F^\Gamma \oplus F^{\deg(U)} \text{ and later to} \\ \mathbb{N}^r = \mathbb{N}^S \times \mathbb{N}^{S'} \text{ where } S \uplus S' = [r] = \{1, \cdots, r\}.\end{gathered}$$

In general, several components of w are *free*, i.e. can be chosen arbitrarily [8, th. 2.69]. To see this let

$$p := \operatorname{rank}(R) \text{ resp. } m := \ell - \operatorname{rank}(R) \tag{26}$$

be the *output*- resp. the *input*- rank of $\mathcal{B} = \{w \in \mathcal{A}^\ell;\ R \circ w = 0\}$. Recall that $\operatorname{rank}(R)$ is the rank of the matrix R considered as a matrix with entries in the field $Quot(\mathcal{D}) = F(s)$. After choosing an *input/output (IO-)structure* of

R, i.e. p $F(s)-$linearly independent columns of R, and after a possible column permutation we write R in the IO-form

$$\begin{gathered} R = (P, -Q) \in \mathcal{D}^{k\times(p+m)},\ p = \operatorname{rank}(R) = \operatorname{rank}(P) \\ y \in \mathcal{A}^p,\ u \in \mathcal{A}^m,\ w := \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{A}^{p+m},\ \text{hence } R\circ w = P\circ y - Q\circ u. \end{gathered} \tag{27}$$

The rank condition implies the unique existence of the *transfer matrix*

$$H \in F(s)^{p\times m} \text{ with } PH = Q,\ P(I_p, -H) = (P, -Q) = R. \tag{28}$$

Theorem 8. (The Cauchy problem in IO-form, [7, th. 5.1]) *Let $P \in \mathcal{D}^{k\times p}$ and $Q \in \mathcal{D}^{k\times m}$ be matrices such that*

$$\operatorname{rank}(P) = p = \operatorname{rank}((P, -Q))$$

and let $L \in \mathcal{D}^{g\times k}$ be a universal left annihilator of P. Let $\leq$ be any well-order on $[p]\times\mathcal{M}$ and $\Gamma_P := ([p]\times\mathcal{M}) \setminus \deg(\mathcal{D}^{1\times k}P)$.
For every $u \in \mathcal{A}^m$ and $x_P \in F^{\Gamma_P}$ the Cauchy problem

$$P\circ y = Q\circ u,\ y \mid \Gamma_P = x_P$$

has a unique solution $y \in \mathcal{A}^p$.
In particular, there are unique row-finite matrices

$$\mathcal{H}_{P,s} \in F^{([p]\times\mathcal{M})\times\Gamma_P} \text{ and } \mathcal{H}_P \in F^{([p]\times\mathcal{M})\times([m]\times\mathcal{M})}$$

such that the maps

$$\begin{gathered} \{\begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{A}^{p+m};\ P\circ y = Q\circ u\} \cong F^{\Gamma_P}\times\mathcal{A}^m,\ \begin{pmatrix} y \\ u \end{pmatrix} \leftrightarrow \begin{pmatrix} x_P \\ u \end{pmatrix}, \\ y \mid \Gamma_P = x_P,\ y = \mathcal{H}_{P,s}x_P + \mathcal{H}_P u \end{gathered}$$

are inverse isomorphisms. Here $\mathcal{H}_{P,s}x_P$ resp. $\mathcal{H}_P u$ are the unique solutions of the Cauchy problems

$$P\circ y = 0,\ y \mid \Gamma_P = x_P$$

resp.

$$P\circ y = Q\circ u,\ y \mid \Gamma_P = 0\ .$$

The matrices $\mathcal{H}_{P,s}$ resp. $\mathcal{H}_P$ are called the zero-input *resp. the* (zero-state) transfer operator *of $\mathcal{B} := \{\binom{y}{u} \in \mathcal{A}^{p+m};\ P\circ y = Q\circ u\}$.*

Proof. We apply theorem 5 to the Cauchy problem $P\circ y = v,\ y \mid \Gamma_P = 0$ with $v := Q\circ u$ and need only show that $L\circ v = 0$. But with the transfer matrix H from (28) we have $LQ = L(PH) = (LP)H = 0H = 0$, hence

$$L\circ v = L\circ(Q\circ u) = (LQ)\circ u = 0\circ u = 0\ .$$

The existence of $\mathcal{H}_{P,s}$ and $\mathcal{H}_P$ are shown as in theorem 5. Since u is arbitrary the map $\mathcal{A}^m \longrightarrow \mathcal{A}^p,\ u \longmapsto \mathcal{H}_P u$, determines $\mathcal{H}_P$ uniquely.

□

2.2 The initial region and data according to Riquier

In the remainder of this paragraph we restrict the considerations to the case $\mathcal{M} = \mathbb{N}^r$ and the polynomial algebra $\mathcal{D} = F[s_1, \ldots, s_r]$. In analogy to C.Riquier [11, ch.V, pp.143-168] whose work was also exposed by J.F. Ritt in [12, ch.IX] the initial data can be expressed as a finite family of formal power series. For this purpose we use the formal power series notation for multi-sequences with indices in $\mathbb{N}^r$:

$$a = (a(\nu))_{\nu \in \mathbb{N}^r} = \sum_{\nu \in \mathbb{N}^r} a(\nu) z^\nu \tag{29}$$

and

$$F^{\mathbb{N}^r} = F[[z_1, \cdots, z_r]] =: F[[z]] \ .$$

For any subset $S \subseteq [r]$ we write

$$F[[z \mid S]] := F[[z_i; i \in S]] \subseteq F[[z]] \ .$$

By convention 7 we have

$$\mathbb{N}^S = \{(\nu_1, \ldots, \nu_r) \in \mathbb{N}^r \mid \text{ if } i \in [r] \setminus S \text{ then } \nu_i = 0\} \subseteq \mathbb{N}^r$$

and

$$F[[z \mid S]] = F^{\mathbb{N}^S} \subseteq F[[z]] = F^{\mathbb{N}^r} \ .$$

Let D be any finite subset of $\mathbb{N}^r$,

$$N := D + \mathbb{N}^r \text{ and } \Gamma := \mathbb{N}^r \setminus N \ . \tag{30}$$

By $\leq_{cw}$ we denote the componentwise order on $\mathbb{N}^r$, i.e. $\mu \leq_{cw} \nu$ if and only if $\mu_i \leq \nu_i$ for all $i \in [r]$. Then $\min_{cw}(N)$, the set of minimal elements in N with respect to the componentwise order, is a subset of D and indeed

$$\min_{cw}(D) = \min_{cw}(N) \text{ and } N = \min_{cw}(N) + \mathbb{N}^r \ .$$

For any translation-invariant subset N of $\mathbb{N}^r$ $\min_{cw}(N)$ is finite (Dickson's Lemma [2, ch.2, Theorem 5]) and $N = \min_{cw}(N) + \mathbb{N}^r$, i.e. N has the form (30).

The following inductive construction furnishes a disjoint decomposition

$$\begin{gathered}\mathbb{N}^r = N \uplus \Gamma = [\biguplus_{\alpha \in A_N} (\alpha + \mathbb{N}^{S(\alpha)})] \uplus [\biguplus_{\alpha \in A_\Gamma} (\alpha + \mathbb{N}^{S(\alpha)})] \\ \text{with a finite set } A := A_N \uplus A_\Gamma \text{ of vertices in } \mathbb{N}^r \\ \text{and subsets } S(\alpha) \subset [r] = \{1, \cdots, r\}, \text{ for } \alpha \in A.\end{gathered} \tag{31}$$

Algorithm 9. *Induction beginning*: $r = 1,\ N = D + \mathbb{N}$: If $N = D = \emptyset$, we choose $A_N := \emptyset$, $A_\Gamma = \{0\}$ and $S(0) := \{1\}$. If $N \neq \emptyset$ and $d := \min(D)$ we define $A_N := \{d\}$, $S(d) := \{1\}$, $A_\Gamma := \{0, \cdots, d-1\}$ and $S(\alpha) := \emptyset$ for $\alpha \in A_\Gamma$ and obtain, with $\mathbb{N}^\emptyset = \{0\}$, the disjoint decomposition

$$\mathbb{N} = \mathbb{N}^{\{1\}} = (d + \mathbb{N}^{\{1\}}) \uplus (0 + \mathbb{N}^\emptyset) \uplus \cdots \uplus (d - 1 + \mathbb{N}^\emptyset)$$

as asserted.

Induction step: $r > 1$. Assume that the decomposition has been constructed for subsets of $\mathbb{N}^{r-1}$. We identify

$\mathbb{N}^r = \mathbb{N} \times \mathbb{N}^{r-1} \ni \alpha = (\alpha_1, \alpha_{II}),\ \alpha_{II} := (\alpha_2, \cdots, \alpha_r)$ and obtain $N_{II} := \mathrm{proj}_{II}(N) := \{\alpha_{II};\ \alpha = (\alpha_1, \alpha_{II}) \in$

The representation $N = D + \mathbb{N}^r$ implies

$$N = \biguplus_{i=0}^{\infty} \big(\{i\} \times N_{II}(i)\big)$$
$$\text{with} \tag{32}$$
$$N_{II}(i) := \{\alpha_{II};\ \alpha = (i, \alpha_{II}) \in N\} = D_{II}(i) + \mathbb{N}^{r-1} \text{ and}$$
$$D_{II}(i) := \{\alpha_{II};\ \alpha = (\alpha_1, \alpha_{II}) \in D \text{ for some } \alpha_1 \le i\}.$$

The $N_{II}(i)$ and $D_{II}(i)$ are increasing, i.e.

$$\begin{aligned} N_{II}(0) \subseteq N_{II}(1) \subseteq \cdots \subseteq N_{II} &= \bigcup_{i=0}^{\infty} N_{II}(i) \\ D_{II}(0) \subseteq D_{II}(1) \subseteq \cdots \subseteq D_{II} &= \bigcup_{i=0}^{\infty} D_{II}(i). \end{aligned} \tag{33}$$

Since D and hence D_{II} are finite the sequences of the $D_{II}(i)$ and hence of the $N_{II}(i) = D_{II}(i) + \mathbb{N}^r$ become stationary and we define

$$\begin{aligned} k := \min\{i \in \mathbb{N};\ N_{II}(i) = N_{II} \text{ or } \min_{cw}(D_{II}(i)) = \min_{cw}(D_{II})\}, \text{ hence} \\ \min_{cw}(D_{II}(i)) = \min_{cw}(D_{II}) \text{ and } N_{II}(i) = N_{II} \text{ for } k \le i. \end{aligned} \tag{34}$$

A simple argument shows that, like in [11, p.133], k can also be determined as

$$k = \max(\mathrm{proj}_I(\min_{cw}(N))) := \max\{\alpha_1 \in \mathbb{N};\ \exists(\alpha_1, \alpha_{II}) \in \min_{cw}(N)\} \tag{35}$$

We conclude

$$\begin{aligned} \mathbb{N}^r = \biguplus_{i=0}^{\infty} \big(\{i\} \times \mathbb{N}^{r-1}\big) = \biguplus_{i=0}^{k-1} \big(\{i\} \times N_{II}(i)\big) \uplus \biguplus_{i=0}^{k-1} \big(\{i\} \times (\mathbb{N}^{r-1} \setminus N_{II}(i))\big) \uplus \\ \big((k + \mathbb{N}) \times N_{II}\big) \uplus \big((k + \mathbb{N}) \times (\mathbb{N}^{r-1} \setminus N_{II})\big). \end{aligned} \tag{36}$$

By induction applied to $N_{II}(i) \subset \mathbb{N}^{r-1} = \mathbb{N}^{\{2,\cdots,r\}},\ 0 \le i \le k$, we obtain

$$\begin{aligned} \mathbb{N}^{r-1} = N_{II}(i) \uplus \Gamma_{II}(i) = \\ [\biguplus_{\alpha_{II} \in B_{II}(i)} (\alpha_{II} + \mathbb{N}^{S_{II}(i,\alpha_{II})})] \uplus [\biguplus_{\alpha_{II} \in C_{II}(i)} (\alpha_{II} + \mathbb{N}^{S_{II}(i,\alpha_{II})})] \end{aligned} \tag{37}$$

where

$$\begin{aligned} B_{II}(i) := A_{N_{II}(i)},\ C_{II}(i) := A_{\Gamma_{II}(i)},\ A_{II}(i) := B_{II}(i) \uplus C_{II}(i), \\ S_{II}(i, \alpha_{II}) \subset \{2, \cdots, r\} \text{ for } \alpha_{II} \in A_{II}(i). \end{aligned} \tag{38}$$

Hence

$$\{i\} \times N_{II}(i) = \biguplus_{\alpha_{II} \in B_{II}(i)} (i, \alpha_{II}) + \mathbb{N}^{S_{II}(i,\alpha_{II})} \text{ for } i \le k-1 \{i\} \times (\mathbb{N}^{r-1} \setminus N_{II}(i)) = \biguplus_{\alpha_{II} \in C_{II}(i)} (i, \alpha_{II}) + \mathbb{N}^{S_{II}(i,\alpha_{II})} \text{ f}$$

We define

$$A_N := \{(i, \alpha_{II});\ 0 \le i \le k,\ \alpha_{II} \in B_{II}(i)\} A_\Gamma := \{(i, \alpha_{II});\ 0 \le i \le k,\ \alpha_{II} \in C_{II}(i)\} A := A_N \uplus A_\Gamma S(i, \alpha_{II}) :=$$

Equations (36)-(39) imply the desired disjoint decomposition

$$\mathbb{N}^r = N \uplus \Gamma = [\biguplus_{\alpha \in A_N} (\alpha + \mathbb{N}^{S(\alpha)})] \uplus [\biguplus_{\alpha \in A_\Gamma} (\alpha + \mathbb{N}^{S(\alpha)})]. \tag{39}$$

Notice that the decomposition (39) depends on N only and not on the special choice of the finite set D.

Remark 10. (i) For every permutation $\sigma \in S_r$ one obtains a decomposition like (31) which, in general, differs from (31). Instead of (32) one starts with the decomposition $\mathbb{N}^r = \mathbb{N}^{\{\sigma(1)\}} \times \mathbb{N}^{\{\sigma(2),\cdots,\sigma(r)\}}$ and proceeds in steps given by σ.
(ii) Riquier [11, ch.V, §81] considers the decomposition $\Gamma = \biguplus_{\alpha \in A_\Gamma} (\alpha + \mathbb{N}^{S(\alpha)})$ only. Ritt [12, ch.IX, §100,(7)] considers a decomposition of all of $\mathbb{N}^r$, but uses $\bar{k} := \min\{i;\ D_{II}(i) = D_{II}\}$ instead of (34) and (35). His decomposition has many more terms than that of (39), depends on the choice of D and not only on $N = D + \mathbb{N}^r$ as already the case $r = 1$ shows and is therefore less suitable.
(iii) The preceding algorithm furnishes a disjoint decomposition which sharpens [2, ch.9, §2, theorem 3].

Example 11. $r = 2$, $D := \{(1,1),(2,0)\}$, $N := D + \mathbb{N}^2$. Then

$$\begin{gathered} N = \{0\} \times \emptyset \uplus \{1\} \times N_{II}(1) \uplus (2 + \mathbb{N}) \times N_{II} \text{ with} \\ N_{II}(0) = \emptyset,\ N_{II}(1) = D_{II}(1) + \mathbb{N} = 1 + \mathbb{N}, \\ N_{II} = N_{II}(2) = D_{II}(2) + \mathbb{N} = \{0,1\} + \mathbb{N} = \mathbb{N},\ \text{hence} \\ \mathbb{N}^2 = N \uplus \Gamma = \\ [\{1\} \times (1 + \mathbb{N}) \uplus (2 + \mathbb{N}) \times \mathbb{N}] \uplus [\{0\} \times \mathbb{N} \uplus \{1\} \times \{0\}] = \\ [(1,1) + \mathbb{N}^{\{2\}} \uplus (2,0) + \mathbb{N}^{\{1,2\}}] \uplus [(0,0) + \mathbb{N}^{\{2\}} \uplus (1,0) + \mathbb{N}^{\emptyset}]. \end{gathered}$$

Figure 2 exhibits the disjoint decomposition of $\mathbb{N}^2$.

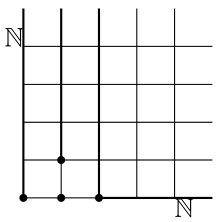

Figure 2: The decomposition of $\mathbb{N}^2$ with respect $N = \{(1,1),(2,0)\} + \mathbb{N}^2$

Application of algorithm 9 to the degree sets $\deg(U)_j$ from (22) gives

$$\begin{gathered} N(j) := \deg(U)_j \\ \mathbb{N}^r = N(j) \uplus \Gamma(j) = \biguplus_{\alpha \in A_{j,N(j)}} (\alpha + \mathbb{N}^{S_j(\alpha)}) \uplus \biguplus_{\alpha \in A_{j,\Gamma(j)}} (\alpha + \mathbb{N}^{S_j(\alpha)}) \end{gathered} \tag{40}$$

We insert these decompositions into the trivial decomposition

$$[\ell] \times \mathbb{N}^r = \biguplus_{j=1}^{\ell} [\{j\} \times N(j) \uplus \{j\} \times \Gamma(j)]$$

to obtain

Corollary 12. *Let U be a $F[s]$-submodule of $F[s]^\ell$ and let $\leq$ be a term-order on $[\ell]\times\mathbb{N}^r$. The algorithm 9 furnishes the disjoint decomposition*

$$[\ell]\times\mathbb{N}^r = \deg(U)\uplus\Gamma =$$
$$[\biguplus_{j=1}^{\ell}\biguplus_{\alpha\in A_{j,N(j)}}(\{j\}\times(\alpha+\mathbb{N}^{S_j(\alpha)}))]\uplus[\biguplus_{j=1}^{\ell}\biguplus_{\alpha\in A_{j,\Gamma(j)}}(\{j\}\times(\alpha+\mathbb{N}^{S_j(\alpha)}))].$$

Definition 13. For any subset $S\subset[r]$ let $S' := [r]\setminus S$ and let $(-)\mid S$ be the F-algebra homomorphism

$$\begin{aligned}(-)\mid S : F^{\mathbb{N}^r} = F[[z]] &\to F^{\mathbb{N}^S} = F[[z\mid S]]\\ (a(\nu))_{\nu\in\mathbb{N}^r} = \sum_{\nu\in\mathbb{N}^r}a(\nu)z^\nu &\mapsto a\mid S := (a(\nu))_{\nu\in\mathbb{N}^S} = \sum_{\nu\in\mathbb{N}^S}a(\nu)z^\nu\ .\end{aligned}\tag{41}$$

Then $z_\rho\mid S = \begin{cases} z_\rho & \text{if } \rho\in S\\ 0 & \text{if } \rho\in S'\end{cases}$.

Lemma 14. *If $S\subset[r]$ and*

$$h = g + z^\alpha f\in F^{(\alpha+\mathbb{N}^S)'}\oplus F^{\alpha+\mathbb{N}^S} = F^{(\alpha+\mathbb{N}^S)'}\oplus z^\alpha F[[z\mid S]] = F[[z]] \text{ then}$$
$$(s^\alpha\circ h)\mid S = f\ .$$

Proof. Let $g=\sum_{\nu\notin\alpha+\mathbb{N}^S}g(\nu)z^\nu$. Since

$$s^\alpha\circ z^\nu = \begin{cases} z^{\nu-\alpha} & \text{if } \nu-\alpha\in\mathbb{N}^r\\ 0 & \text{otherwise}\end{cases},$$

we have

$$s^\alpha\circ g = \sum_{\nu-\alpha\in\mathbb{N}^r\setminus\mathbb{N}^S}g(\nu)z^{\nu-\alpha} \text{ and } s^\alpha\circ(z^\alpha f) = f\ .$$

Any $z^{\nu-\alpha}$ with $\nu-\alpha\in\mathbb{N}^r\setminus\mathbb{N}^S$ contains a factor $z_\rho,\ \rho\in S'$, hence

$$z_\rho\mid S = 0,\ z^{\nu-\alpha}\mid S = 0 \text{ and } (s^\alpha\circ g)\mid S = 0.$$

Now $h = g+z^\alpha f$ implies the asserted equality

$$(s^\alpha\circ h)\mid S = (s^\alpha\circ g)\mid S + (s^\alpha\circ(z^\alpha f))\mid S = 0 + f\mid S = f.$$

□

Theorem 15. (Compare [11, §81], [12, §100]) *Let U be a $F[s]$-submodule of $F[s]^{1\times\ell}$, let $\leq$ be a term-order on $[\ell]\times\mathbb{N}^r$ and $\Gamma := \big([\ell]\times\mathbb{N}^r\big)\setminus\deg(U)$. Then the state space F^Γ admits the direct sum decomposition*

$$F^\Gamma = \bigoplus_{j=1}^{\ell}\bigoplus_{\alpha\in A_{j,\Gamma(j)}}z^\alpha F[[z\mid S_j(\alpha)]]\delta_j\subset F[[z]]^\ell = \bigoplus_{j=1}^{\ell}F[[z]]\delta_j,$$

i.e. every initial data

$$x = \sum_{j=1}^{\ell}x_j\delta_j\in F^\Gamma\subset F^{[\ell]\times\mathbb{N}^r} = F[[z]]^\ell,\ x_j = \sum_{\nu\in\Gamma(j)}x_j(\nu)z^\nu$$

admits a unique sum representation

$$x = \sum_{j=1}^{\ell} x_j \delta_j,\ x_j = \sum_{\alpha \in A_{j,\Gamma(j)}} z^\alpha f_j^\alpha,\ f_j^\alpha \in F[[z \mid S_j(\alpha)]].$$

Moreover $(s^\alpha \circ x_j) \mid S_j(\alpha) = f_j^\alpha$ *for all* $j = 1, \cdots, l$ *and* $\alpha \in A_{j,\Gamma(j)}$.
In the language of [11] and [12]: The solutions $w \in F[[z]]^\ell$ *of the system* $R \circ w = v$ *depend on the arbitrary "functions", i.e. power series* f_j^α, $j = 1, \ldots, \ell$, $\alpha \in A_{j,\Gamma(j)}$, *which give rise to the initial data* $x = \sum_{j=1}^{\ell} \sum_{\alpha \in A_{j,\Gamma(j)}} z^\alpha f_j^\alpha \delta_j$ *and the unique solution* w *with* $w \mid \Gamma = x$.

Proof. Corollary 12 and

$$F^\Gamma = F^{\biguplus_{j=1}^{\ell} \biguplus_{\alpha \in A_{j,\Gamma(j)}} \{j\} \times (\alpha + \mathbb{N}^{S_j(\alpha)})} =$$

$$\bigoplus_{j=1}^{\ell} \bigoplus_{\alpha \in A_{j,\Gamma(j)}} F^{\{j\} \times (\alpha + \mathbb{N}^{S_j(\alpha)})} = \biguplus_{j=1}^{\ell} \biguplus_{\alpha \in A_{j,\Gamma(j)}} z^\alpha F[[z \mid S_j(\alpha)]] \delta_j\ .$$

□

2.3 Partial differential equations

The results for partial *difference* equations can, in the case $\mathcal{M} = \mathbb{N}^r$ and $\mathcal{D} = F[s]$, be trivially translated into those for linear systems of partial *differential* equations with constant coefficients. For this purpose we assume that the field F has *characteristic zero* and let

$$\partial_\rho : F[[z]] \to F[[z]],\ b \mapsto \partial_\rho b = \partial b / \partial z_\rho,\ \rho = 1, \cdots, r \tag{42}$$

denote the partial derivatives. Then $F[s]$ acts on $F[[z]]$ by partial differential operators via

$$\begin{gathered} s^\mu \bullet b := \partial^\mu(b),\ s^\mu \bullet z^\nu = (\nu)_\mu z^{\nu - \mu} \text{ where} \\ (\nu)_\mu := \mu! \binom{\nu}{\mu} = \prod_{\rho=1}^{r} \nu_\rho * (\nu_\rho - 1) * \cdots * (\nu_\rho - \mu_\rho + 1) \\ \mu! = \prod_{\rho=1}^{r} \mu_\rho!,\ \binom{\nu}{\mu} = \prod_{\rho=1}^{r} \binom{\nu_\rho}{\mu_\rho}. \end{gathered} \tag{43}$$

It is obvious that the *Borel isomorphism*

$$\begin{gathered} \widehat{(-)} : (F[[z]], \circ) \cong (F[[z]], \bullet) \\ a = (a(\nu))_{\nu \in \mathbb{N}^r} = \sum_{\nu \in \mathbb{N}^r} a(\nu) z^\nu \mapsto \hat{a} := \sum_{\nu \in \mathbb{N}^r} \frac{a(\nu)}{\nu!} z^\nu,\ \widehat{z^\nu} = \frac{z^\nu}{\nu!} \end{gathered} \tag{44}$$

is an $F[s]-$isomorphism, i.e. $\widehat{s^\mu \circ a} = \partial^\mu(\hat{a})$. By means of this isomorphism the results of theorems 5 and 8 can be transferred to $(F[[z]], \bullet)$. We obtain the following theorems which replace theorems 5, 8 and 15 for the case of partial differential equations.

Theorem 16. (The Cauchy problem for partial differential equations) *Let F be a field of characteristic zero. The data are those of theorems 5 and 8.*
(i) For an arbitrary right side

$$\hat{v} = \sum_{(i,\mu)\in[k]\times\mathbb{N}^r} \frac{(\partial^\mu \hat{v}_i)(0)}{\mu!} z^\mu \delta_i \in F[[z]]^k$$

and arbitrary initial data

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \cdots \\ \hat{x}_l \end{pmatrix} = \sum_{(j,\nu)\in\Gamma} \frac{(\partial^\nu \hat{x}_j)(0)}{\nu!} z^\nu \delta_j \in F^\Gamma \subset F^{[\ell]\times\mathbb{N}^r} = F[[z]]^\ell$$

the Cauchy problem

$$R(\partial)(\hat{w}) = \hat{v},\ \hat{w} \mid \Gamma = \hat{x} \text{ or } (\partial^\nu \hat{w}_j)(0) = (\partial^\nu \hat{x}_j)(0) \text{ for } (j,\nu)\in\Gamma$$

has a unique solution $\hat{w} = \begin{pmatrix} \hat{w}_1 \\ \cdots \\ \hat{w}_l \end{pmatrix} \in F[[z]]^\ell$ *if and only if $\hat{v}$ satisfies the integrability condition $L(\partial)\hat{v} = 0$.*
(ii) In the situation of theorem 8 the functions

$$\begin{gathered} \mathcal{H}_{P,s}\widehat{(-,(i,\mu))} := \\ \sum_{(j,\nu)\in[p]\times\mathbb{N}^r} \frac{\mathcal{H}_{P,s}((j,\nu),(i,\mu))}{\nu!} z^\nu \delta_j \in F[[z]]^p,\ (i,\mu)\in\Gamma_P \\ \text{resp.} \\ \mathcal{H}_P\widehat{(-,(i,\mu))} = \\ \sum_{(j,\nu)\in[p]\times\mathbb{N}^r} \frac{\mathcal{H}_P((j,\nu),(i,\mu))}{\nu!} z^\nu \delta_j \in F[[z]]^p,\ (i,\mu)\in[m]\times\mathbb{N}^r \end{gathered} \tag{45}$$

are the unique solutions of the Cauchy problems

$$\begin{gathered} P(\partial)\hat{y} = 0,\ (\partial^\nu \hat{y}_j)(0) = \delta_{(j,\nu),(i,\mu)},\ (j,\nu)\in\Gamma_P, \\ \text{resp.} \\ P(\partial)\hat{y} = Q(\partial)(\frac{z^\mu}{\mu!}\delta_i),\ (\partial^\nu \hat{y}_j)(0) = 0,\ (j,\nu)\in\Gamma_P. \end{gathered}$$

The *integral*

$$I_j : F[[z]] \to F[[z]],\ b = \sum_{\nu\in\mathbb{N}^r} b(\nu) z^\nu \mapsto I_j(b) := \sum_{\nu\in\mathbb{N}^r} \frac{b(\nu)}{\nu_j+1} z^\nu z_j. \tag{46}$$

is a right inverse of the partial derivative ∂_j. A simple variant of theorem 15 is the following result.

Theorem 17. (Initial data for partial differential equations, compare [11, §81]) *In the situation of theorem 15 any initial data*

$$\hat{x} = \sum_{(j,\nu)\in\Gamma} \frac{(\partial^\nu \hat{x}_j)(0)}{\nu!} z^\nu \delta_j \in F^\Gamma \subset F^{[\ell]\times\mathbb{N}^r} = F[[z]]^\ell$$

can also be uniquely expressed as

$$\hat{x} = \sum_{j=1}^{\ell} \sum_{\alpha \in A_{j,\Gamma(j)}} z^\alpha F_j^\alpha \delta_j \text{ with } F_j^\alpha \in F[[z \mid S_j(\alpha)]].$$

The connection of the $\hat{x}_j$ *with the* F_j^α *is given by*

$$z^\alpha F_j^\alpha = I^\alpha[(\partial^\alpha \hat{x}_j) \mid S_j(\alpha)] \tag{47}$$

The proof is essentially the same as that of theorem 15. The last equation for $z^\alpha F_j^\alpha$ replaces the simpler equation $s^\alpha \circ x_j \mid S_j(\alpha) = f_j^\alpha$ in theorem 15.

2.4 Injective cogenerators and duality

We finally explain the connection of the preceding work with the algebraic notion of an injective cogenerator, the duality between modules and behaviours and Fuhrmann's notion of a (shift) model.

Definition 18. A $\mathcal{D}-$ module I is called *injective* if the functor $\mathrm{Hom}_\mathcal{D}(-, I)$ preserves exactness or is exact. If, in addition, this functor reflects exactness or, equivalently, if $M \neq 0$ implies $\mathrm{Hom}_D(M, I) \neq 0$ then I is called an injective cogenerator. The latter property is also equivalent with the existence of a $\mathcal{D}-$linear embedding $f : M \to I^\Lambda$ for every $\mathcal{D}-$module M and some index set Λ. If every *finitely generated* $\mathcal{D}-$module M admits a $\mathcal{D}-$linear embedding $f : M \to I^k,\ k \in \mathbb{N}$, into some *finite* product of copies of I then it is called a *large injective cogenerator.*

Let M be a $\mathcal{D}$-module. Then there is the $\mathcal{D}$-linear isomorphism

$$\mathrm{Hom}_F(M, F) \cong \mathrm{Hom}_\mathcal{D}(M, \mathrm{Hom}_F(\mathcal{D}, F)) \ , \ \varphi \leftrightarrow \Phi \ ,$$

where $\Phi(x)(d) = \varphi(dx)$, for all $x \in M$, $d \in \mathcal{D}$. Hence by lemma 1 and equation (10) the $\mathcal{D}$-module $\mathcal{A} = F^\mathcal{M}$ is an injective cogenerator. This implies that the $F[s]-$modules $(F[[z]], \circ)$ and $(F[[z]], \bullet)$ (in characteristic zero) are injective cogenerators too.

Remark 19. The $\mathcal{D}$-module $\mathcal{A} = F^\mathcal{M}$ is even a large injective cogenerator as was shown in [8, §3, theorem 15] in a lengthy proof. The large injective cogenerator property of a $\mathcal{D}-$module I implies that there is a full categorical duality $M \leftrightarrow \mathcal{B}$ between finitely generated $\mathcal{D}-$modules M and $I-$behaviours $\mathcal{B}$ given by the correspondence

$$M = \mathcal{D}^{1\times l}/\mathcal{D}^{1\times k}R \cong \mathrm{Hom}_{\mathrm{End}_\mathcal{D}(I)}(\mathcal{B}, I) \text{ and}$$
$$\mathcal{B} := \{w \in I^\ell;\ R \circ w = 0\} \cong \mathrm{Hom}_\mathcal{D}(M, I) \text{ for } R \in \mathcal{D}^{k\times l}$$

where a behaviour is defined as a finitely generated $\mathrm{End}_\mathcal{D}(I)-$submodule of some $I^k,\ k \in \mathbb{N}$. This duality introduced in [8] permits the transformation of theorems in commutative algebra into theorems on multidimensional systems and has become a valuable tool in multidimensional systems theory.

Remark 20. The large injective cogenerator property of $(F[[z]], \circ)$, i.e. the existence of $F[s]-$linear embeddings $f: M \to F[[z]]^k$ and the ensuing $F[s]-$isomorphisms $M \cong (f(M), \circ)$ for finitely generated $F[s]-$ modules M shows that the shift action $\circ$ is the universal scalar multiplication or $F[s]-$module structure for finitely generated modules. In the language of P. Fuhrmann [4] $(f(M), \circ)$ could be called a *power series* or *shift model* of M. Fuhrmann's polynomial, rational and finite dimensional models in [4, Def.5.1.1, p.112, Th.6.3.1] have multi-dimensional counter-parts, but are much too special for $r \geq 2$. Even $F-$finite-dimensional $F[s]-$modules cannot be classified (a problem of *wild* representation type) and no structure theorem like the Smith form and its consequences in dimension one exists for $r \geq 2$.

3 Convergent solutions

3.1 Convergent solutions for the lattice $\mathbb{N}^r$

In this section we assume that $\mathcal{M} = \mathbb{N}^r$ and that the base field F is the field $\mathbb{C}$ of complex numbers. See [6, ch.II] or [8, ch.4] for a short introduction into convergent power series in several variables. An open *polydisc* in $\mathbb{C}^r$ is a set of the form

$$P(\varepsilon) := \prod_{j=1}^{r} \{\zeta_j \in \mathbb{C};\ |\zeta_j| < \varepsilon_j\},\ \varepsilon = (\varepsilon_j)_{j=1,\cdots,r} \in \mathbb{R}^r_{>0}, \tag{48}$$

where $\mathbb{R}_{>0} := \{\lambda \in \mathbb{R};\ \lambda > 0\}$. A formal power series $b = \sum_{\mu \in \mathbb{N}^r} b(\mu) z^\mu$ is called *convergent* if $\sum_{\mu \in \mathbb{N}^r} b(\mu)\zeta^\mu$ converges for some $\zeta \in \mathbb{C}^r$ with non-zero components $\zeta_1, \ldots, \zeta_r$. With

$$\begin{gathered} C := \sup_{\mu \in \mathbb{N}^r} |b(\mu)||\zeta|^\mu,\ |\zeta| := (|\zeta_1|, \cdots, |\zeta_r|) \in \mathbb{R}^r_{>0}, \text{and} \\ \rho := |\zeta|^{-1} := (|\zeta_1|^{-1}, \cdots, |\zeta_r|^{-1}) \end{gathered} \tag{49}$$

we get

$$\begin{gathered} |b(\mu)| \leq C\rho^\mu \\ \text{and for } \eta \in P(|\zeta|) \\ |\sum_{\mu \in \mathbb{N}^r} b(\mu)\eta^\mu| \leq \sum_{\mu \in \mathbb{N}^r} |b(\mu)||\eta|^\mu \leq C \sum_{\mu \in \mathbb{N}^r} |\eta|^\mu \rho^\mu = C \prod_{j=1}^{r} \frac{1}{1 - \frac{|\eta_j|}{\zeta_j}} < \infty. \end{gathered} \tag{50}$$

This implies that the series $b = \sum_{\mu \in \mathbb{N}^r} b(\mu) z^\mu$ converges compactly or uniformly on compact subsets in the polydisc $P(|\zeta|)$. The subspace

$$\begin{gathered} \mathbb{C}\langle z \rangle := \\ \{b \in \mathbb{C}[[z]];\ \sum_{\mu \in \mathbb{N}^r} b(\mu)\zeta^\mu \text{ converges for some } \zeta \in \left(\mathbb{C} \setminus \{0\}\right)^r\} = \\ \{b \in \mathbb{C}[[z]];\ b = \sum_{\mu \in \mathbb{N}^r} b(\mu) z^\mu \text{ converges in some polydisc } P(\varepsilon),\ \varepsilon \in \mathbb{R}^r_{>0}\} = \\ \{b = (b(\mu))_{\mu \in \mathbb{N}^r} \in \mathbb{C}^{\mathbb{N}^r}; \exists C > 0, \exists \rho \in \mathbb{R}^r_{>0} \text{ such that } |b(\mu)| \leq C\rho^\mu\} \end{gathered} \tag{51}$$

of $\mathbb{C}[[z]]$ is called the space of convergent power series. It is well-known that it is a subalgebra of $\mathbb{C}[[z]]$ with respect to the convolution multiplication and a $\mathbb{C}[s]-$submodule with respect to the shift module structure.
If $b = \sum_{\mu\in\mathbb{N}^r} b(\mu)z^\mu$ converges in the polydisc $P(\varepsilon)$, then b is a holomorphic (=olotope in [11]) function on $P(\varepsilon)$. A power series which converges everywhere is just a holomorphic function on $\mathbb{C}^r$ and called an *entire* function. The subspace

$$O(\mathbb{C}^r) := \{b \in \mathbb{C}\langle z\rangle;\ b \text{ is entire}\} \tag{52}$$

of all entire functions is also a subalgebra and a $\mathbb{C}[s]-$ submodule of $\mathbb{C}\langle z\rangle$. Finally the space

$$O(\mathbb{C}^r;\ \exp) := \\ = \{b \in O(\mathbb{C}^r);\ \exists C > 0, \exists \rho \in \mathbb{R}^r_{>0} \text{ such that } |b(\mu)| \le C\exp(\langle \rho, |\mu|\rangle)\} \tag{53}$$

of all entire functions of *exponential type* with $\langle \rho, |\mu|\rangle := \sum_{j=1}^r \rho_j|\mu_j|$ is also a subalgebra and a $\mathbb{C}[s]-$ submodule of $\mathbb{C}\langle z\rangle$. The Borel isomorphism (44) induces the $\mathbb{C}[s]-$isomorphism

$$(\mathbb{C}\langle z\rangle, \circ) \cong (O(\mathbb{C}^r;\ \exp), \bullet). \tag{54}$$

Now assume the situation of theorem 5. We are going to show that the unique solution of the Cauchy problem $R \circ w = v,\ w \mid \Gamma = x,\ L \circ v = 0$ is convergent if the initial data x and the right side v have this property.
First we define a well order on $[\ell] \times \mathbb{N}^r$ according to [11, ch.VII, §104] or [12, ch.IX, §106].

Assumption 21. We choose matrices $W \in \mathbb{R}^{\ell\times s}$ and $U \in \mathbb{R}^{r\times s}_{\ge 0}$, $s > 0$, such that the map

$$\omega : [\ell] \times \mathbb{N}^r \to \mathbb{R}^{1\times s},\ (j, \nu) \mapsto \omega(j, \nu) := W_{j-} + \nu U$$

is injective. In particular, the map $\mathbb{N}^r \to \mathbb{R}^{1\times s},\ \nu \mapsto \nu U$, is then injective and the rows of U are non-zero. On $\mathbb{R}^{1\times s}$ we choose the lexicographic order and on $[\ell] \times \mathbb{N}^r$ the induced one with

$$(i, \mu) < (j, \nu) : \quad \Leftrightarrow \quad \omega(i, \mu) = W_{i-} + \mu U < \omega(j, \nu) = W_{j-} + \nu U\ .$$

A simple check shows that this order on $[\ell] \times \mathbb{N}^r$ is a term order. All standard term orders can be defined in this fashion. The indices $\omega(j, \nu)$ are called “cotes” by Riquier and “marks” by Ritt. In the following theorems we always consider a term order of this type.
By $\le_{cw}$ we denote the componentwise order on $\mathbb{R}^s$, i.e. $\mu \le_{cw} \nu$ if and only if $\mu_i \le \nu_i$ for all $i \in [s]$.

Lemma 22. (Compare [12, ch.IX, §107])
(i) Let $\varepsilon \in \mathbb{R}_{>0}$, $\bar{\Theta} \in \mathbb{R}^s_{>0}$, G a finite set, and let $(\Lambda^g)_{g\in G}$ be a finite family of finite subsets of $\mathbb{R}^s$. Let $\lambda^g(0) := \max(\Lambda^g)$ be the maximal element of Λ^g with respect to the lexicographic order. Then there is $\bar{S} \in \mathbb{R}^r$ such that

$$\bar{S} \ge_{cw} \bar{\Theta} \text{ and } \bar{S}^\lambda \le \varepsilon \bar{S}^{\lambda^g(0)},$$

for all $g \in G$ and $\lambda \in \Lambda^g \setminus \{\lambda^g(0)\}$.
(ii) Let $\varepsilon \in \mathbb{R}_{>0}$, $\Theta \in \mathbb{R}^s_{>0}$, G a finite set, and let $(\Lambda^g)_{g\in G}$ be a finite family of

finite subsets of $[\ell]\times\mathbb{N}^r$. *Let* $\lambda^g(0) := (j^g(0), \nu^g(0)) := \max(\Lambda^g)$ *be the maximal element of* Λ^g *with respect to the term order of assumption 21. Then there are* $S \in \mathbb{R}^r$ *and* $T_j > 0,\ j = 1, \cdots, \ell$, *such that*

$$S \geq_{cw} \Theta \ \text{and}\ S^\nu T_j \leq \varepsilon S^{\nu^g(0)} T_{j^g(0)},$$

for all $g \in G$ *and* $(j,\nu) \in \Lambda^g \setminus \{\lambda^g(0)\}$.

Proof. (i) For $g \in G$ and $\sigma = 1, \cdots, s$, we define

$$\Lambda^g(\sigma) := \{\lambda \in \Lambda^g \setminus \{\lambda^g(0)\};\ \min\{i \in [s];\ \lambda_i \neq \lambda_i^g(0)\} = \sigma\}\ .$$

If $\lambda \in \Lambda^g(\sigma)$ then

$$\lambda_i = \lambda_i^g(0) \text{ for } i < \sigma,\ \lambda_\sigma < \lambda_\sigma^g(0) \text{ and } \bar{S}^{\lambda-\lambda^g(0)} = \bar{S}_\sigma^{\lambda_\sigma-\lambda_\sigma^g(0)} \prod_{i=\sigma+1}^{s} \bar{S}_i^{\lambda_i-\lambda_i^g(0)} \tag{55}$$

for arbitrary $\bar{S} \in \mathbb{R}^s_{>0}$. We construct the $\bar{S}_\sigma$ by recursion on $\sigma = s, \cdots, 1$, and assume that for $\sigma+1 \leq i \leq s$ the numbers $\bar{S}_i \geq \bar{\Theta}_i$ have been constructed such that

$$\bar{S}^{\lambda-\lambda^g(0)} \leq \varepsilon \text{ for all } g \in G \text{ and } \lambda \in \Lambda^g(\sigma+1) \uplus \cdots \uplus \Lambda^g(s). \tag{56}$$

Since $\lambda_\sigma < \lambda_\sigma^g(0)$ for all $g \in G$ and $\lambda \in \Lambda^g(\sigma)$ we can choose $\bar{S}_\sigma \geq \bar{\Theta}_\sigma$ such that

$$\bar{S}_\sigma^{\lambda_\sigma-\lambda_\sigma^g(0)} \leq \varepsilon[\prod_{i=\sigma+1}^{s} \bar{S}_i^{\lambda_i-\lambda_i^g(0)}]^{-1} \text{ for all } g \in G \text{ and } \lambda \in \Lambda^g(\sigma)$$

and hence, by (55), equation (56) is satisfied for all $g \in G$ and $\lambda \in \Lambda^g(\sigma) \uplus \cdots \uplus \Lambda^g(s)$. For $\sigma = 1$ this gives the asserted inequalities.
(ii) For $(j,\nu) \in [\ell] \times \mathbb{N}^r$ we have

$$\omega(j,\nu) = W_{j-} + \nu U = W_{j-} + \sum_{i=1}^{r} \nu_i U_{i-} \text{ and } \bar{S}^{\omega(j,\nu)} = \bar{S}^{W_{j-}} \prod_{i=1}^{r} (\bar{S}^{U_{i-}})^{\nu_i}$$

for arbitrary $\bar{S}$. Since the rows of U are non-zero we can choose

$$\bar{\Theta} \in \mathbb{R}^s_{>0} \text{ such that } \Theta_i \leq \bar{\Theta}^{U_{i-}} \text{ for } i = 1, \cdots, r.$$

By part (i) applied to $\omega(\Lambda^g) \subset \mathbb{R}^s,\ g \in G$, there is $\bar{S} \geq_{cw} \bar{\Theta}$ such that

$$\bar{S}^{\omega(j,\nu)} \leq \varepsilon \bar{S}^{\omega(j^g(0),\nu^g(0))} \text{ for all } g \in G \text{ and } (j,\nu) \in \Lambda^g \setminus \{(j^g(0), \nu^g(0))\},\ \text{hence}$$

$$\bar{S}^{W_{j-}} \prod_{i=1}^{r} (\bar{S}^{U_{i-}})^{\nu_i} \leq \varepsilon \bar{S}^{W_{j^g(0)-}} \prod_{i=1}^{r} (\bar{S}^{U_{i-}})^{\nu_i^g(0)}\ .$$

This inequality yields the asserted one with

$$S_i := \bar{S}^{U_{i-}} \geq \bar{\Theta}^{U_{i-}} \geq \Theta_i \text{ for } i = 1, \cdots, r, \text{ and } T_j := \bar{S}^{W_{j-}}.$$

□

Lemma 23. *(Recursive solution of the Cauchy problem) Let* $w \in \mathcal{A}^\ell$ *be the solution of the Cauchy problem*

$$R \circ w = v \ , \ \ w|\Gamma = x, \ L \circ v = 0$$

as in theorem 5 and let G *be a Gröbner basis of* $U := \mathcal{D}^{1\times k}R$. *For* $g \in G$ *let*

$$\begin{gathered}(j(g), d(g)) := \deg(g), \ \ g := s^{d(g)}\delta_{j(g)} + \sum_{(i,\mu)<(j(g),d(g))} g_i(\mu)s^\mu\delta_i \ , \\ X_{g-} \in \mathcal{D}^{1\times k} \ \text{such that} \ g = X_{g-}R \\ \text{and} \ v^g := X_{g-} \circ v \ . \end{gathered} \tag{57}$$

If $(j,\nu) \in \deg(U) = \bigcup_{g\in G}(\deg(g) + \mathbb{N}^r)$, $g \in G$ *and* $\lambda \in \mathbb{N}^r$ *such that* $j(g) = j$ *and* $\nu = d(g) + \lambda$, *then*

$$w_j(\nu) = -\sum_{i,\mu} g_i(\mu)w_i(\lambda + \mu) + v^g(\lambda)$$

and

$$(i, \mu + \lambda) < (j, \nu) \ \text{for all} \ (i, \mu) \ \text{with} \ g_i(\mu) \neq 0 \ .$$

Proof. From $R \circ w = v$ we infer

$$g \circ w = X_{g-}R \circ w = X_{g-} \circ R \circ w = X_{g-} \circ v = v^g \ .$$

Equation (57) implies

$$s^\nu\delta_j = s^\lambda g - \sum_{i,\mu} g_i(\mu)s^{\lambda+\mu}\delta_i$$

and

$$\begin{gathered}w_j(\nu) = \big(s^\nu\delta_j \circ w\big)(0) = \\ = \big(s^\lambda g \circ w\big)(0) - \big(\sum_{i,\mu} g_i(\mu)s^{\lambda+\mu}\delta_i \circ w\big)(0) = v^g(\lambda) - \sum_{i,\mu} g_i(\mu)w_i(\lambda+\mu) \ .\end{gathered}$$

Moreover $(i,\mu) < (j(g), d(g))$ implies $(i, \mu+\lambda) < (j, d(g)+\lambda) = (j,\nu)$. ☐

Theorem 24. (Convergent solutions of the Cauchy problem)
(i) In the situation of theorem 5 let $w \in (\mathbb{C}^{\mathbb{N}^r})^\ell = \mathbb{C}\,[[z]]^\ell$ *be the unique solution of the Cauchy problem*

$$R \circ w = v, \ w \mid \Gamma = x, \ L \circ v = 0$$

for given initial data $x \in \mathbb{C}^\Gamma$ *and right side* $v \in \mathbb{C}^{[k]\times\mathbb{N}^r} = \mathbb{C}\,[[z]]^k$. *If* x *and* v *are convergent, then so is* w, *i.e.* $x \in \mathbb{C}^\Gamma \cap \mathbb{C}\,\langle z\rangle^\ell$ *and* $v \in \mathbb{C}\,\langle z\rangle^k$ *imply* $w \in \mathbb{C}\,\langle z\rangle^\ell$.
(ii) The $\mathbb{C}\,[s]-$*module* $\mathbb{C}\,\langle z\rangle$ *is a large injective cogenerator.*

Proof. (i) We use the notations of the preceding lemma. Since v is convergent, $X_g \circ v = v^g$, and $\mathbb{C}\,\langle z\rangle^\ell$ is shift-invariant, v^g is convergent too. Since x and v^g are convergent there are $C^1 \in \mathbb{R}_{>0}$ and $\Theta^1 \in \mathbb{R}^r_{>0}$ such that

$$\begin{gathered}|x_j(\nu)| \leq C^1(\Theta^1)^\nu \ \text{for all} \ (j,\nu) \in \Gamma, \\ \text{and} \ |v^g(\nu)| \leq C^1(\Theta^1)^\nu \ \text{for all} \ g \in G \ \text{and} \ \nu \in \mathbb{N}^r.\end{gathered} \tag{58}$$

We choose $\varepsilon > 0$ such that

$$\varepsilon * \sum_{(i,\mu)<(j(g),d(g))} |g_i(\mu)| < 1 \text{ for all } g \in G \ .$$

Lemma 22 applied to $\Lambda^g := \{(i,\mu); g_i(\mu) \neq 0\}$, $g \in G$, with $\max(\Lambda^g) = (j(g), d(g))$ furnishes $\Theta \geq_{cw} \Theta^1$ such that

$$\Theta^\mu T_i \leq \varepsilon \Theta^{d(g)} T_{j(g)} \text{ for all } g \in G \text{ and } (i,\mu) < (j(g), d(g)) \text{ with } g_i(\mu) \neq 0 \ . \tag{59}$$

We choose $C > 0$ such that

$$C^1 \Theta^{-d(g)} T_{j(g)}^{-1} \leq C[1 - \varepsilon * \sum_{(i,\mu)\in\Lambda^g\setminus\{(j(g),d(g))\}} |g_i(\mu)|] \text{ for all } g \in G \quad \text{and} \quad C \geq \frac{C^1}{T_i} \text{ for all } i = 1, \cdots, \ell \ . \tag{60}$$

We claim that

$$|w_j(\nu)| \leq C\Theta^\nu T_j \text{ for all } (j,\nu) \in [\ell] \times \mathbb{N}^r \tag{61}$$

and hence $w \in \mathbb{C}\langle z\rangle^\ell$ as asserted.
Assume that this is false and let (j,ν) be minimal with $|w_j(\nu)| > C\Theta^\nu T_j$. Since $C \geq \frac{C^1}{T_i}$ for all $i = 1, \ldots, \ell$ we get

$$|w_i(\mu)| = |x_i(\mu)| \leq C^1(\Theta^1)^\mu \leq C\Theta^\mu T_i \text{ for all } (i,\mu) \in \Gamma$$

and hence $(j,\nu) \in \deg(U)$. By Lemma 23 we have

$$w_j(\nu) = -\sum_{i,\mu} g_i(\mu) w_i(\lambda + \mu) + v^g(\lambda)$$

and $(i,\mu) \in \Lambda^g \setminus \{(j(g), d(g))\}$ implies $(i, \mu + \lambda) < (j, d(g) + \lambda) = (j,\nu)$. Hence

$$\begin{aligned} |w_j(\nu)| &\leq \sum_{(i,\mu)\in\Lambda^g\setminus\{(j(g),d(g))\}} |g_i(\mu)||w_i(\lambda+\mu)| + |v^g(\lambda)| \leq \\ &\sum_{(i,\mu)\in\Lambda^g\setminus\{(j(g),d(g))\}} |g_i(\mu)| C\Theta^{\lambda+\mu} T_i + C^1\Theta^\lambda \leq \\ &\sum_{(i,\mu)\in\Lambda^g\setminus\{(j(g),d(g))\}} |g_i(\mu)| C\varepsilon\Theta^{d(g)+\lambda} T_j + C^1\Theta^{-d(g)} T_j^{-1} \Theta^{d(g)+\lambda} T_j = \\ &[C(\varepsilon * \sum_{(i,\mu)\in\Lambda^g\setminus\{(j(g),d(g))\}} |g_i(\mu)|) + C^1\Theta^{-d(g)} T_j^{-1}]\Theta^\nu T_j \leq \\ &(\text{by (60)}) \ \leq C\Theta^\nu T_j \ , \end{aligned} \tag{62}$$

which contradicts the assumption $|w_j(\nu)| > C\Theta^\nu T_j$.
(ii) Since $\mathbb{C}[s]$ is noetherian a standard argument (see, for instance, [8, ch.2, lemma 31]) shows that $\mathbb{C}\langle z\rangle$ is injective if and only if the exactness of a sequence (13)

$$\mathbb{C}[s]^{1\times g} \xrightarrow{\circ L} \mathbb{C}[s]^{1\times k} \xrightarrow{\circ R} \mathbb{C}[s]^{1\times l}$$

implies that of the dual sequence

$$\mathbb{C}\langle z\rangle^g = \mathrm{Hom}_{\mathbb{C}[s]}(\mathbb{C}[s]^g, \mathbb{C}\langle z\rangle) \xleftarrow{L\circ} \mathbb{C}\langle z\rangle^k \xleftarrow{R\circ} \mathbb{C}\langle z\rangle^\ell$$

or $\mathrm{im}(R\circ) = \ker(L\circ)$. The inclusion $\mathrm{im}(R\circ) \subset \ker(L\circ)$ follows from $LR = 0$. If, conversely, $v \in \mathbb{C}\langle z\rangle^k$ satisfies the integrability condition $L \circ v = 0$, the Cauchy problem $R \circ w = v,\ w \mid \Gamma = 0$, has a convergent solution $w \in \mathbb{C}\langle z\rangle^\ell$ according to part (i) of this theorem and hence $v \in \mathrm{im}(R\circ)$ and $\mathrm{im}(R\circ) \supset \ker(L\circ)$.
The unique solvability of the Cauchy problem implies the isomorphism

$$\mathrm{Hom}_{\mathbb{C}[s]}(M, (\mathbb{C}\langle z\rangle, \circ)) \cong \{w \in \mathbb{C}\langle z\rangle^\ell;\ R \circ w = 0\} \cong \mathbb{C}^\Gamma.$$

(See Remark 19). But $M = \mathbb{C}[s]^{1\times\ell}/\mathbb{C}[s]^{1\times k}R \neq 0$ implies that $\deg(\mathbb{C}[s]^{1\times k}R) \subset [\ell] \times \mathbb{N}^r$ and $\Gamma \neq \emptyset$ and hence the cogenerator property $\mathrm{Hom}_{\mathbb{C}[s]}(M, (\mathbb{C}\langle z\rangle, \circ)) \neq 0$. That $(\mathbb{C}\langle z\rangle, \circ))$ is even a *large* injective cogenerator will be shown in connection with corollary 26. □

Corollary 25. *In the situation of theorem 8 let* $y = \mathcal{H}_{P,s}x_P + \mathcal{H}_P u \in \mathbb{C}[[z]]^p$ *be the unique solution of the Cauchy problem*

$$P \circ y = Q \circ u,\ y \mid \Gamma_P = x_P,$$

where the input $u \in \mathbb{C}\langle z\rangle^m$ *and the initial data* $x_P \in \mathbb{C}^{\Gamma_P} \cap \mathbb{C}\langle z\rangle^p$ *are assumed convergent. Then the solution* y *is convergent too. In particular, this holds for the columns*

$$y_s^{(i,\mu)} := \mathcal{H}_{P,s}(-,(i,\mu)) = \sum_{(j,\nu)\in[p]\times\mathbb{N}^r} \mathcal{H}_{P,s}((j,\nu),(i,\mu))z^\nu\delta_j \in \mathbb{C}\langle z\rangle^p,\ (i,\mu) \in \Gamma_P$$

$(u := 0,\ x_P := \delta_{-,(i,\mu)})$ *and for*

$$y^{(i,\mu)} := \mathcal{H}_P(-,(i,\mu)) = \sum_{(j,\nu)\in[p]\times\mathbb{N}^r} \mathcal{H}_P((j,\nu),(i,\mu))z^\nu\delta_j,\ (i,\mu) \in [m] \times \mathbb{N}^r$$

$(u := \delta_{-,(i,\mu)},\ x_P := 0).$

3.2 Partial differential equations

Theorem 26. *(i) In the situation of theorem 17 the initial data*

$$\hat{x} = \sum_{(j,\nu)\in\Gamma} \frac{(\partial^\nu \hat{x}_j)(0)}{\nu!} z^\nu \delta_j = \sum_{j=1}^{\ell} \sum_{\alpha\in A_{j,\Gamma(j)}} z^\alpha F_j^\alpha \delta_j,\ F_j^\alpha \in \mathbb{C}[[z \mid S_j(\alpha)]]$$

are entire of exponential type if and only if the functions $F_j^\alpha,\ j = 1,\cdots,l,\ \alpha \in A_{j,\Gamma(j)}$*, have this property, i.e.* $\hat{x} \in \mathbb{C}^\Gamma \cap O(\mathbb{C}^r;\exp)^\ell$ *if and only if* $F_j^\alpha \in O(\mathbb{C}^{S_j(\alpha)};\exp)$ *for all* $j = 1,\ldots,\ell$ *and* $\alpha \in A_{j,\Gamma(j)}$.
(ii) Let, in the situation of theorem 16, the right side $\hat{v}$ *and the initial data* $\hat{x}$ *be entire of exponential type, ie.*

$$\hat{v} = \sum_{(i,\mu)\in[k]\times\in\mathbb{N}^r} \frac{(\partial^\mu \hat{v}_i)(0)}{\mu!} z^\mu \delta_i \in O(\mathbb{C}^r;\exp)^k \text{ and}$$

$$\hat{x} = \sum_{(j,\nu)\in\Gamma} \frac{(\partial^\nu \hat{x}_j)(0)}{\nu!} z^\nu \delta_j \in \mathbb{C}^\Gamma \cap O(\mathbb{C}^r;\exp)^\ell.$$

Then the unique solution $\hat{w}$ *of the Cauchy problem*

$$R(\partial)(\hat{y}) = \hat{v},\ \hat{y} \mid \Gamma = \hat{x} \text{ or } (\partial^\nu \hat{y}_j)(0) = (\partial^\nu \hat{x}_j)(0) \text{ for } (j,\nu) \in \Gamma,\ L(\partial)\hat{v} = 0$$

has the same property or $\hat{w} \in O(\mathbb{C}^r; \exp)^\ell$. *This holds, in particular, for the unique solutions* $\hat{y}_s^{(i,\mu)}$ *and* $\hat{y}^{(i,\mu)}$ *of the Cauchy problems*

$$P(\partial)\hat{y} = 0,\ (\partial^\nu \hat{y}_j)(0) = \delta_{(j,\nu),(i,\mu)},\ (j,\nu) \in \Gamma_P,\ (i,\mu) \in \Gamma_P,\ \text{resp.}$$
$$P(\partial)\hat{y} = Q(\partial)(\frac{z^\mu}{\mu!}\delta_i),\ (\partial^\nu \hat{y}_j)(0) = 0,\ (j,\nu) \in \Gamma,\ (i,\mu) \in [m] \times \mathbb{N}^r.$$

(iii) The $\mathbb{C}[s]-$*module* $(O(\mathbb{C}^r; \exp); \bullet)$ *is a large injective cogenerator.*

The proof follows from theorem 24 and the Borel isomorphism (54). That $(O(\mathbb{C}^r, \exp), \bullet) \cong (\mathbb{C}\langle z\rangle, \circ)$ is even a *large* injective cogenerator was shown in [8, §4, theorem 54]

Remark 27. (i) The injectivity of $(O(\mathbb{C}^r; \exp); \bullet)$ or rather the validity of the fundamental principle for it was already shown by L. Ehrenpreis [3, ch.V, example 2, p.138] by a functional analytic method. It seems to us that our proof of this fact is much simpler than Ehrenpreis' one.
(ii) The holomorphy of the functions $\hat{y}_s^{(i,\mu)}$ was conjectured by P.S. Pedersen [10].

We finally show that theorem 26 is also valid for the $\mathbb{C}[s]-$ module $(\mathbb{C}\langle z\rangle, \bullet)$ instead of $(O(\mathbb{C}^r; \exp), \bullet)$. In the following considerations we systematically use the Borel isomorphism (44)

$$(\mathbb{C}[[z]], \circ) \cong (\mathbb{C}[[z]], \bullet),\ a \leftrightarrow \hat{a},$$
$$a = (a_\mu)_{\mu \in \mathbb{N}^r} = \sum_{\mu \in \mathbb{N}^r} a_\mu z^\mu,\ \hat{a} = \sum_{\mu \in \mathbb{N}^r} \frac{a_\mu}{\mu!} z^\mu,\ a_\mu = (\partial^\mu \hat{a})(0). \tag{63}$$

Assumption 28. Like Riquier [11, ch.VII, §104] we use a term order as in assumption 21 with the additional property that the first columns of the matrices W and U satisfy $W_{-1} \in \mathbb{N}^\ell \subset \mathbb{R}^\ell_{\geq 0}$ and $U_{-1} = (1, \cdots, 1)^\top$. In other terms

$$\omega(i,\mu) = (\omega_1(i,\mu), \cdots, \omega_s(i,\mu)) \in \mathbb{R}^{1\times s}$$
$$\omega_1(i,\mu) = W_{i1} + \mu_1 + \cdots + \mu_r = W_{i1} + |\mu| \in \mathbb{N} \text{ where} \tag{64}$$
$$|\mu| := \mu_1 + \cdots + \mu_r.$$

With respect to this term order the intervals

$$\{(i,\mu) \in [\ell] \times \mathbb{N}^r;\ (i,\mu) \leq (j,\nu)\} \subset$$
$$\{(i,\mu);\ \omega_1(i,\mu) = W_{i1} + \mu_1 + \cdots + \mu_r \leq \omega_1(j,\nu) = W_{j1} + \nu_1 + \cdots + \nu_r\}$$

are finite. Hence $[\ell] \times \mathbb{N}^r$ is order isomorphic to $\mathbb{N}$ and therefore admits standard induction proofs.

Theorem 29. *The notations are those from theorem 16 and assumption 28.*
(i) If the right side $\hat{v}$ *and the initial data* $\hat{x}$ *are convergent, i.e.* $\hat{v} \in \mathbb{C}\langle z\rangle^k$ *and* $\hat{x} \in \mathbb{C}^\Gamma \cap \mathbb{C}\langle z\rangle^\ell$, *then the unique solution* $\hat{w}$ *of the Cauchy problem*

$$R \bullet \hat{w} = R(\partial)\hat{w} = \hat{v},\ \hat{w} \mid \Gamma = \hat{x} \text{ or } w_j(\nu) = x_j(\nu) \text{ for } (j,\nu) \in \Gamma,\ L(\partial)\hat{v} = 0 \tag{65}$$

is convergent too or $\hat{w} \in \mathbb{C}\langle z\rangle^\ell$.
(ii) The $\mathbb{C}[s]-$ *module* $(\mathbb{C}\langle z\rangle, \bullet)$ *is a large injective cogenerator.*

The proof of this result was inspired by that of Riquier's existence theorem [11, ch.VII, §115] as exposed in [11, ch.VII, pp.190-254] or by Ritt [12, ch.IX]. Theorem 29 differs from Riquier's result (see remark 37), however, and our proof is shorter and formulated in today's mathematical language. The details of the quoted proofs are hard to understand. Proposition (ii) follows from proposition (i) as in theorem 24. Since $(O(\mathbb{C}^r;\exp)) \subset \mathbb{C}\langle z\rangle$ and since $(O(\mathbb{C}^r;\exp),\bullet)$ is a large cogenerator so is $(\mathbb{C}\langle z\rangle,\bullet)$. In a series of reduction steps the problem is reduced to the case of ordinary differential equations.
If $a=\sum_{\mu\in\mathbb{N}^r} a_\mu z^\mu$ and $b=\sum_{\mu\in\mathbb{N}^r} b_\mu z^\mu$ are formal power series and if b has non-negative coefficients b_μ we say that b *dominates* or is a *majorant* of a if and only if

$$\text{for all } \mu\in\mathbb{N}^r:\ |a_\mu|\le b_\mu. \tag{66}$$

If b is convergent and dominates a then a is obviously convergent too. For $\mu\in\mathbb{N}^r$ let $((\mu)) := \frac{|\mu|}{\mu_1!\cdots\mu_r!}$ denote the multinomial coefficient.

Lemma 30. *If $a=\sum_{\mu\in\mathbb{N}^r} a_\mu z^\mu$ is convergent there is a convergent power series $b=\sum_{k=0}^\infty b_k t^k$ in one variable t with non-negative coefficients b_k such that $b(z_1+\cdots+z_r)$ dominates a.*

Proof. By assumption there are $C>0$ and $\rho\in\mathbb{R}^r_{>0}$ such that $|a_\mu|\le C\rho^\mu$ for all μ. Define

$$\sigma := \max_j \rho_j \text{ and } b_k := \max_{\mu,\ |\mu|=k}\frac{|a_\mu|}{((\mu))} \text{ for } k\in\mathbb{N}.$$

These definitions imply $b_k=\frac{|a_\mu|}{((\mu))}$ for some μ with $|\mu|=k$ and then

$$0\le b_k=\frac{|a_\mu|}{((\mu))}\le C\rho^\mu/((\mu))\le C\sigma^{|\mu|}=C\sigma^k.$$

This shows that $b:=\sum_{k=0}^\infty b_k t^k\in\mathbb{C}\langle t\rangle$ with non-negative coefficients. Moreover

$$b(z_1+\cdots+z_r)=\sum_k b_k(z_1+\cdots+z_r)^k=$$

$$=\sum_k\sum_{\mu,\ |\mu|=k} b_k((\mu))z^\mu=\sum_\mu b_{|\mu|}((\mu))z^\mu$$

Since $|a_\mu|\le b_{|\mu|}((\mu))$ by construction the series $b(z_1+\cdots+z_r)$ dominates a. □

The following two lemmas serve as reduction procedures in the proof of theorem 29 and have the following form: If theorem 29 holds under a stronger assumption A then it holds under a weaker assumption B.

Lemma 31. *Let G be a Gröbner basis of U and let $\varepsilon>0$ be arbitrary. It suffices to prove theorem 29 for the case that, with the notations of lemma 23,*

$$|g_i(\lambda)|\le\varepsilon \text{ for all } g\in G,\ (i,\lambda)<(j(g),d(g)).$$

Proof. Assume that the theorem has been proven under these additional conditions and that $\hat{w}$ is the unique solution of (65). Choose $\varepsilon_1>0$ such that

$$\varepsilon_1 * |g_i(\lambda)|\le\varepsilon \text{ for all } g\in G,\ (i,\lambda)<(j(g),d(g)).$$

Due to lemma 22 there are $\Theta \in \mathbb{R}^r_{>0}$ and $T_1 > 0, \cdots, T_l > 0$ such that

$$\Theta^\lambda T_i \leq \varepsilon_1 * \Theta^{d(g)T_{j(g)}} \text{ for all } g \in G,\ (i,\lambda) < (j(g), d(g)),\ g_i(\lambda) \neq 0.$$

Now consider the ring automorphism

$$\varphi : \mathbb{C}[s] \cong \mathbb{C}[s],\ s_j \mapsto \Theta_j s_j,\ s^\mu \mapsto \Theta^\mu s^\mu,$$

and the $\varphi-$ semi-linear automorphism

$$\tau : \mathbb{C}[s]^{1\times\ell} \cong \mathbb{C}[s]^{1\times\ell},\ s^\mu \delta_i \mapsto \Theta^\mu s^\mu T_i \delta_i.$$

Obviously τ preserves degrees and generators, hence

$$\tau(U) = \sum_{g\in G} \varphi(\mathbb{C}[s])\tau(g) = \sum_{g\in G} \mathbb{C}[s]\tau(g)$$

$$\deg(U) = \deg(\tau(U)) = \bigcup_{g\in G} \deg(g) + \mathbb{N}^r = \bigcup_{g\in G} \deg(\tau(g)) + \mathbb{N}^r.$$

But this implies that $\{\tau(g) \mid g \in G\}$ is a Gröbner basis of $\tau(U)$.
The representation

$$g = s^{d(g)}\delta_{j(g)} + \sum_{(i,\lambda)<(j(g),d(g))} g_i(\lambda)s^\lambda \delta_i \text{ implies}$$

$$\tau(g) = \Theta^{d(g)} s^{d(g)} T_{j(g)}\delta_{j(g)} + \sum_{(i,\lambda)<(j(g),d(g))} g_i(\lambda)\Theta^\lambda s^\lambda T_i \delta_i \text{ or, normalized,}$$

$$\Theta^{-d(g)} T_{j(g)}^{-1} \tau(g) = s^{d(g)}\delta_{j(g)} + \sum_{(i,\lambda)<(j(g),d(g))} g_i(\lambda) \frac{\Theta^\lambda T_i}{\Theta^{d(g)} T_{j(g)}} s^\lambda \delta_i.$$

By construction the coefficients of this normalized Gröbner basis of $\tau(U)$ satisfy

$$|g_i(\lambda)| \frac{\Theta^\lambda T_i}{\Theta^{d(g)} T_{j(g)}} \leq |g_i(\lambda)| * \varepsilon_1 \leq \varepsilon, \text{ for all } g \in G,\ (i,\lambda) < (j(g), d(g)). \quad (67)$$

Finally consider the map τ^* dual to τ or

$$\tau^* : \mathbb{C}[[z]]^\ell \cong \mathbb{C}[[z]]^\ell,\ \hat{w} = (\hat{w}_j)_{j=1,\cdots,l} \mapsto (T_j^{-1}(\hat{w}_j(z/\Theta))_{j=1,\cdots,l}$$

where $z/\Theta = (\frac{z_1}{\Theta_1}, \cdots, \frac{z_r}{\Theta_r})$. But

$$\partial^\mu(\hat{w}_i(z/\Theta)) = \Theta^{-\mu}(\partial^\mu \hat{w}_i)(z/\Theta), \text{ hence}$$

$$\tau(s^\mu \delta_i) \bullet \tau^*(\hat{w}) = \Theta^\mu T_i \partial^\mu (T_i^{-1} \hat{w}_i(z/\Theta)) = (\partial^\mu \hat{w}_i)(z/\Theta)$$

and finally

$$\tau(g) \bullet \tau^*(\hat{w}) = (g \bullet \hat{w})(z/\Theta) = \hat{v}^g(z/\Theta),\ g \in G,\ \tau^*(\hat{w}) \mid \Gamma = \tau^*(\hat{x}). \quad (68)$$

With $\hat{v}^g$ and $\hat{x}$ the series $\hat{v}^g(z/\Theta)$ and $\tau^*(\hat{x})$ are also convergent. Hence $\tau^*(\hat{w})$ is the unique solution of the Cauchy problem (68) which, by equation (67), satisfies the additional condition under which we assume that theorem 29 has been proven. Hence $\tau^*(\hat{w})$ and then also $\hat{w}$ are convergent. □

Lemma 32. *It suffices to prove theorem 29 for the case that, given any finite subset* Δ *of* $[\ell]\times\mathbb{N}^r$*, the solution* $\hat{w}$ *satisfies the initial condition* $\hat{w}\mid\Gamma\bigcup\Delta=0$

Proof. So assume that the theorem has been proven under this additional condition, and let $\hat{w}$ denote the unique solution in the general case. From $R\bullet\hat{w}=\hat{v}$ we derive

$$R\bullet(\hat{w}-\hat{w}\mid\Gamma\bigcup\Delta)=\hat{v}-R\bullet(\hat{w}\mid\Gamma\bigcup\Delta),\ (\hat{w}-\hat{w}\mid\Gamma\bigcup\Delta)\mid\Gamma\bigcup\Delta=0. \quad (69)$$

But, by assumption, $\hat{w}\mid\Gamma$ is convergent and therefore

$$\hat{w}\mid\Gamma\bigcup\Delta=\hat{w}\mid\Gamma+\sum_{(j,\nu)\in\Delta\setminus\Gamma}\frac{w_j(\nu)}{\nu!}z^\nu\delta_j$$

is convergent too since $\Delta\setminus\Gamma$ is finite. Therefore the Cauchy problem (69) is of the restricted type for which we assume theorem 29 proven. Therefore

$$\hat{w}-\hat{w}\mid\Gamma\bigcup\Delta \text{ and then also } \hat{w}=(\hat{w}-\hat{w}\mid\Gamma\bigcup\Delta)+\hat{w}\mid\Gamma\bigcup\Delta$$

are convergent as asserted. □

According to assumption 28 the numbers

$$\omega_1(j,\nu)=W_{j1}+|\nu|=W_{j1}+\nu_1+\cdots+\nu_r$$

are non-negative integers. We define

$$\begin{gathered}N:=\deg(U),\ N_k:=\{(j,\nu)\in N;\ \omega_1(j,\nu)=W_{j1}+|\nu|=k\}\text{ for }k\in\mathbb{N}\\ \text{hence } N=\biguplus_{k=0}^{\infty}N_k\\ p:=\max\{W_{j(g)1}+|d(g)|,W_{j1};\ g\in G,\ j=1,\cdots,\ell\}\\ e(j):=p-W_{j1},\ j=1,\cdots,\ell.\end{gathered} \quad (70)$$

We have to include the W_{j1} in the definition of the maximum p since it may happen that not every j is a $j(g)$. If, however, $\operatorname{rank}(R)=l$ this cannot happen and, according to theorem 8, we could assume this. We define the finite set of polynomial vectors

$$\begin{gathered}H:=\{h:=s^\mu g;\ g\in G,\ |\mu|=p-\omega_1(\deg(g))\}\\ h=s^\mu g=s^{d(g)+\mu}\delta_{j(g)}+\sum_{(i,\lambda)<(j(g),d(g))}g_i(\lambda)s^{\lambda+\mu}\delta_i=:\\ s^{d(h)}\delta_{j(h)}+\sum_{(i,\lambda)<(j(h),d(h))}h_i(\lambda)s^\lambda\delta_i\\ (j(h),d(h)):=\deg(h)=(j(g),d(g)+\mu),\ \omega_1(\deg(h))=p\\ |d(h)|=p-W_{j(h)1}.\end{gathered} \quad (71)$$

Moreover the coefficients of the vectors $h\in H$ coincide with those of the vectors $g\in G$ and can hence, according to lemma 31, be assumed of small absolute value.

Lemma 33. *For the data from above*

$$\{(j,\nu) \in N;\ \omega_1(j,\nu) \geq p\} = \biguplus_{k=p}^{\infty} N_k = \bigcup_{h\in H} \deg(h) + \mathbb{N}^r.$$

Proof. $\supseteq$: $\quad \omega_1(\deg(h)+\mu) = \omega_1(\deg(h)) + |\mu| = p + |\mu| \geq p.$
$\subseteq$: Let

$$(j,\nu) \in N_k \subset N = \bigcup_{g\in G} \deg(g) + \mathbb{N}^r,\ k \geq p. \text{ Then}$$

$$(j,\nu) = \deg(g) + \mu,\ p \leq \omega_1(j,\nu) = \omega_1(\deg(g)) + |\mu|,\ \omega_1(\deg(g)) \leq p.$$

Hence there is a λ such that

$$\lambda \leq_{cw} \mu \text{ and } \omega_1(\deg(g)) + |\lambda| = \omega_1(\deg(s^\lambda g)) = \omega_1(\deg(h)) = p,\ h := s^\lambda g \in H$$
$$(j,\nu) = \deg(g) + \mu = \deg(g) + \lambda + (\mu - \lambda) = \deg(h) + (\mu-\lambda) \in \deg(h) + \mathbb{N}^r.$$

□

Corollary 34. *Let $\hat{w}$ be the solution of the Cauchy problem*

$$R \bullet \hat{w} = \hat{v} \text{ or } g \bullet \hat{w} = \hat{v}^g \text{ for all } g \in G$$

with the additional property that

$$\hat{w} \mid (\Gamma \bigcup N_0 \bigcup \cdots \bigcup N_{p-1}) = 0.$$

For $h = s^\mu g \in H$ we define

$$\hat{v}^h := s^\mu \bullet \hat{v}^g = (s^\mu g) \bullet \hat{w} = h \bullet \hat{w}.$$

In analogy to lemma 23 the coefficients $w_j(\nu)$ can be computed inductively in the following fashion.
1.case*: $(j,\nu) \in \Gamma \bigcup N_0 \bigcup \cdots \bigcup N_{p-1}$. Then $w_j(\nu) = 0$ by assumption.*
2.case*: $\omega_1(j,\nu) \geq p$. By the preceding lemma there is a representation*

$$(j,\nu) = \deg(h) + \mu,\ h = s^{d(h)}\delta_{j(h)} + \sum_{(i,\lambda)<(j(h),d(h))} h_i(\lambda) s^\lambda \delta_i \in H,$$

which gives rise to the recursion formula

$$w_j(\nu) = -\sum_{(i,\lambda)<(j(h),d(h))} h_i(\lambda) w_i(\lambda+\mu) + v^h(\mu) \tag{72}$$

The next considerations concern the case of ordinary differential equations. We call t the independent variable and $D := d/dt$. We also interpret D as an indeterminate in the polynomial algebra $\mathbb{C}[D]$. With the $e(j)$ from (70) we consider a system

$$D^{e(j)}\hat{\phi}_j - \sum_{i=1}^{\ell} a(j,i) D^{e(i)}\hat{\phi}_i - \sum_{i=1}^{\ell}\sum_{\alpha<e(i)} a(j,i,\alpha) D^\alpha \hat{\phi}_i = \hat{\psi}_j,\ j = 1,\cdots,\ell, \tag{73}$$

where the $\hat{\psi}_j$ are given convergent power series. We want to find a convergent solution $\hat{\phi} = (\hat{\phi}_j)_{j=1,\cdots,\ell}$.

Lemma 35. *Assume a system* (73) *with the following additional properties: The coefficients* $a(j,i)$ *and* $a(j,i,\alpha)$ *and* $\psi_j(\alpha) = (D^\alpha \hat{\psi}_j)(0),\ j = 1,\cdots,l,\ \alpha \in \mathbb{N}$, *are non-negative. The matrix* $A := (a(j,i))_{j,i} \in \mathbb{R}^{l\times l}$ *has a norm less than one with respect to any matrix norm. Then the system* (73) *with the zero initial condition* $(D^\alpha \hat{\phi}_i)(0) = 0$ *for* $i = 1,\cdots,l,\ 0 \le \alpha < e(i)$, *has a unique solution, and this is convergent and has non-negative coefficients* $\phi_i(\alpha) = (D^\alpha \hat{\phi}_i)(0)$ *for all* $i = 1,\cdots,l,\ \alpha \in \mathbb{N}$.

Proof. (i) We first assume that the matrix A is zero so that the system (73) is

$$D^{e(j)}\hat{\phi}_j - \sum_{i=1}^{\ell}\sum_{\alpha<e(i)} a(j,i,\alpha)D^\alpha\hat{\phi}_i = \hat{\psi}_j,\ j = 1,\cdots,l. \tag{74}$$

This system is solved for the highest derivatives and can therefore be transformed into a first order system. Indeed, pose

$$\hat{\phi}_i^\alpha := D^\alpha\hat{\phi}_i \text{ for } i = 1,\cdots,l,\ 0 \le \alpha < e(i),$$
$$\widetilde{\phi} := (\hat{\phi}_i^\alpha)_{i,\alpha},\ \widetilde{\psi} := (\hat{\psi}_i^\alpha)_{i,\alpha},\ \hat{\psi}_i^\alpha := \begin{cases}\hat{\psi}_i & \text{if } \alpha = e(i)-1\\ 0 & \text{if } 0 \le \alpha \le e(i)-2.\end{cases}$$

Then (74) obtains the form

$$D\hat{\phi}_j^\beta = \begin{cases}\sum_{i=1}^{\ell}\sum_{\alpha<e(i)} a(j,i,\alpha)\hat{\phi}_i^\alpha + \hat{\psi}_j^\beta & \text{if } \beta = e(j)-1\\ \hat{\phi}_j^{\beta+1} & \text{if } 0 \le \beta < e(j)-1\end{cases} \quad \text{or}$$
$$D\widetilde{\phi} = \widetilde{A}\widetilde{\phi} + \widetilde{\psi}.$$

with an obvious matrix $\widetilde{A} \in \mathbb{C}^{e\times e},\ e := e(1) + \cdots e(r)$. The unique solution of this first order system with zero initial condition is

$$\widetilde{\phi} = \int_0^t e^{(t-\tau)\widetilde{A}}\widetilde{\psi}(\tau)d\tau$$

and is obviously holomorphic near zero since $\widetilde{\psi}$ has this property. Therefore the system (74) with zero initial condition has the unique solution $\hat{\phi} = (\hat{\phi}_i^0)_{i=1,\cdots,l}$ and this is convergent. Moreover the associated system of difference equations

$$D^{e(j)}\circ\phi_j - \sum_{i=1}^{\ell}\sum_{\alpha<e(i)} a(j,i,\alpha)D^\alpha\circ\phi_i = \psi_j,\ j = 1,\cdots,l,$$

implies, as in lemma 23, the recursion formulas

$$\phi_j(\beta) = \begin{cases}0 & \text{if } \beta < e(j)\\ \sum_{i=0}^{\ell}\sum_{0\le\alpha<e(i)} a(j,i,\alpha)\phi_i(\alpha+\beta-e(j)) + \psi_j(\beta-e(j)) & \text{if } \beta \ge e(j).\end{cases}$$

Since the $a(j,i,\alpha)$ are non-negative so are the $\phi_j(\beta)$ as a simple induction on the preceding recursion proves.
(ii) In the general case the system (73) can be written as

$$(I_l - A)\begin{pmatrix}D^{e(1)}\hat{\phi}_1\\ \cdots\\ D^{e(l)}\hat{\phi}_l\end{pmatrix} + \hat{\Theta} = \hat{\psi} \text{ with } \hat{\Theta}_j := \sum_{i,\alpha<e(i)} a(j,i,\alpha)D^\alpha\hat{\phi}_i,\ j = 1,\cdots,l. \tag{75}$$

Since the norm of A is less than 1 the matrix $I_l - A$ is invertible and $(I_l - A)^{-1} = \sum_{k=0}^{\infty} A^k \geq_{cw} 0$. But then the system (75) has the same solutions as

$$\begin{pmatrix} D^{e(1)}\hat{\phi}_1 \\ \cdots \\ D^{e(l)}\hat{\phi}_l \end{pmatrix} + (I_l - A)^{-1}\hat{\Theta} = (I_l - A)^{-1}\hat{\psi}$$

and this has the form discussed in part (i) since $(I_l - A)^{-1} \geq_{cw} 0$. □

Assumption 36. For the remainder of the proof of theorem 29 we choose all coefficients $a(j,i)$ and $a(j,i,\alpha)$ in (73) positive and such that the matrix $(a(j,i))_{j,i} \in \mathbb{R}^{l\times l}$ has a (matrix) norm less than one and assume that all coefficients $h_i(\lambda)$ have small absolute value. Due to the reduction lemma 31 we may assume this without loss of generality. The maximum suitable size will be specified in the following proof.

Proof. (of theorem 29) The data are those which have been introduced in the preceding lemmas.
(i) Without loss of generality according to lemma 32 we assume that $\hat{w}$ is a solution of (65) with $\hat{w} \mid \Gamma \bigcup N_0 \bigcup \cdots \bigcup N_{p-1} = 0$. We are going to show that $\hat{w}$ is convergent by constructing a convergent majorant. The series $\hat{w}$ satisfies the system

$$h \bullet \hat{w} = \hat{v}^h \in \mathbb{C}\langle z\rangle,\ h \in H,\ \hat{w} \mid \Gamma \bigcup N_0 \bigcup \cdots \bigcup N_{p-1} = 0.$$

(ii) According to lemma 30 we choose convergent power series $\hat{\psi}_j \in \mathbb{C}\langle t\rangle$ with non-negative coefficients such that $\hat{\psi}_j(z_1 + \cdots + z_r)$ dominates all $\hat{v}^h(z)$ with $j(h) = j$.
(iii) Let $\hat{\phi}$ be the unique solution of the system (73) with these $\hat{\psi}_j$ and zero initial data. By lemma 35 the solution $\hat{\phi}$ is convergent and has non-negative coefficients. The same then holds for the series

$$\hat{w}_j^0(z) := \hat{\phi}_j(z_1 + \cdots z_r),\ j = 1, \cdots, l,\ \text{with } \partial^\mu \hat{w}_j^0(z) = (D^{|\mu|}\hat{\phi}_j)(z_1 + \cdots z_r).$$

(iv) For

$$h = s^{d(h)}\delta_{j(h)} + \sum_{(i,\lambda)<\deg(h)=(j(h),d(h))} h_i(\lambda)s^\lambda\delta_i \in H$$

consider the expression

$$\begin{gathered}(s^{d(h)}\delta_{j(h)} - \sum_{(i,\lambda)<\deg(h)=(j(h),d(h))} |h_i(\lambda)|s^\lambda\delta_i) \bullet \hat{w}^0 = \\ D^{|d(h)|}\hat{\phi}_{j(h)} - \sum_{(i,\lambda)<\deg(h)=(j(h),d(h))} |h_i(\lambda)|D^{|\lambda|}\hat{\phi}_i = \\ D^{e(j(h))}\hat{\phi}_{j(h)} - \sum_{i=1}^{\ell} b(h,i)D^{e(i)}\hat{\phi}_i - \sum_{i=1}^{\ell}\sum_{\alpha<e(i)} b(h,i,\alpha)D^\alpha\hat{\phi}_i \text{ where} \\ b(h,i) := \sum\{|h_i(\lambda)|;\ (i,\lambda) < \deg(h),\ \omega_1(i,\lambda) = p \text{ or } |\lambda| = e(i)\} \\ b(h,i,\alpha) := \sum\{|h_i(\lambda)|;\ (i,\lambda) < \deg(h),\ |\lambda| = \alpha\}.\end{gathered} \tag{76}$$

Now we choose the absolute values of the $h_i(\lambda)$ so small that

$$b(h,i) \leq a(j(h),i),\ b(h,i,\alpha) \leq a(j(h),i,\alpha)$$
$$\text{for } h \in H,\ i = 1,\cdots,l,\ 0 \leq \alpha < e(i)\ .$$

Then equation (76) implies

$$(s^{d(h)}\delta_{j(h)} - \sum_{(i,\lambda)<\deg(h)=(j(h),d(h))} |h_i(\lambda)|s^\lambda\delta_i) \bullet \hat{w}^0 =$$
$$D^{e(j(h))}\hat{\phi}_{j(h)} - \sum_{i=1}^{\ell} a(j(h),i)D^{e(i)}\hat{\phi}_i - \sum_{i=1}^{\ell}\sum_{\alpha<e(i)} a(j(h),i,\alpha)D^\alpha\hat{\phi}_i +$$
$$\sum_{i=1}^{\ell}(a(j(h),i)-b(h,i))D^{e(i)}\hat{\phi}_i + \sum_{i=1}^{\ell}\sum_{\alpha<e(i)}(a(j(h),i,\alpha)-b(h,i,\alpha))D^\alpha\hat{\phi}_i =$$
$$\hat{\psi}_{j(h)} + \sum_{i=1}^{\ell}(a(j(h),i)-b(h,i))D^{e(i)}\hat{\phi}_i +$$
$$\sum_{i=1}^{\ell}\sum_{\alpha<e(i)}(a(j(h),i,\alpha)-b(h,i,\alpha))D^\alpha\hat{\phi}_i =: \hat{v}^{0,h},\ h \in H, \tag{77}$$

or

$$(s^{d(h)}\delta_{j(h)} - \sum_{(i,\lambda)<\deg(h)=(j(h),d(h))} |h_i(\lambda)|s^\lambda\delta_i) \bullet \hat{w}^0 = \hat{v}^{0,h},\ h \in H.$$

Since $\hat{\psi}_{j(h)}(z_1 + \cdots + z_r)$ dominates $\hat{v}^h$ by construction, since the numbers $a(j(h),i) - b(h,i)$ and $a(j(h),i,\alpha) - b(h,i,\alpha)$ are non-negative as are the coefficients of the $\hat{\phi}_i$ the series $\hat{v}^{0,h}$ also dominates $\hat{v}^h$ for each $h \in H$ and has non-negative coefficients. The associated difference equations

$$h \circ w = (s^{d(h)}\delta_{j(h)} + \sum_{(i,\lambda)<\deg(h)=(j(h),d(h))} h_i(\lambda)s^\lambda\delta_i) \circ w = v^h \text{ and}$$
$$(s^{d(h)}\delta_{j(h)} - \sum_{(i,\lambda)<\deg(h)=(j(h),d(h))} |h_i(\lambda)|s^\lambda\delta_i) \circ w^0 = v^{0,h},\ h \in H,$$

give rise to the recursive equations

$$0 = w_j(\nu) \leq w_j^0(\nu) \text{ for } (j,\nu) \in N_0 \uplus \cdots \uplus N_{p-1},$$

and, for

$$(j,\nu) = \deg(h) + \mu = (j,d(h)) + \mu \in \biguplus_{k=p}^{\infty} N_k = \bigcup_{h\in H} \deg(h) + \mathbb{N}^r,$$
$$w_j(\nu) = -\sum_{(i,\lambda)<(j,d(h))} h_i(\lambda)w_i(\lambda+\mu) + v^h(\mu),\ \text{hence}$$
$$|w_j(\nu)| \leq \sum_{(i,\lambda)<(j,d(h))} |h_i(\lambda)||w_i(\lambda+\mu)| + |v^h(\mu)| \text{ and}$$
$$w_j^0(\nu) = \sum_{(i,\lambda)<(j,d(h))} |h_i(\lambda)|w_i^0(\lambda+\mu) + v^{0,h}(\mu).$$

Since $(i,\lambda+\mu)<(j,\nu)$ for $(i,\lambda)<(j,d(h))$ we assume inductively that $|w_i(\lambda+\mu)|\le w_i^0(\lambda+\mu)$. Since $\hat{v}^{0,h}$ dominates $\hat{v}^h$ and has non-negative coefficients also $|v^h(\mu)|\le v^{0,h}(\mu)$ for all $h\in H$ and $\mu\in\mathbb{N}^r$. The preceding recursive equations then imply $|w_j(\nu)|\le w_j^0(\nu)$ and, by induction, that the last inequality holds for all $(j,\nu)\in[\ell]\times\mathbb{N}^r$. This signifies that $\hat{w}^0$ dominates $\hat{w}$. Since $\hat{w}^0$ is convergent so is $\hat{w}$, and this was to be shown. □

Remark 37. (*Orthonomic systems* according to [11, ch.VII, §104] and [12, ch.IX, §108]) Riquier considers so-called *orthonomic*, possibly non-linear systems of partial differential equations. In the linear, constant coefficient case such a system has the form

$$h(\partial)w=v^h,\ h\in H\subset\mathbb{C}[s]^{1\times\ell} \tag{78}$$

where H is a finite set of polynomial vectors with the properties from definition 38 below (It has nothing to do with the H from the last proof!). With

$$R:=(h)_{h\in H}\in\mathbb{C}[s]^{H\times\ell},\ v:=(v^h)_{h\in H}\in\mathbb{C}[[z]]^H$$

the system (78) has the standard form $R(\partial)w=v$ as in theorem 16. Without loss of generality we assume that the $h\in H$ have leading coefficient 1. Riquier chooses an order as in assumption 28. Let

$$\widetilde{N}:=\bigcup_{h\in H}(\deg(h)+\mathbb{N}^r),\ \widetilde{\Gamma}:=([\ell]\times\mathbb{N}^r)\setminus\widetilde{N}$$
$$U:=\mathbb{C}[s]^{1\times H}R=\sum_{h\in H}\mathbb{C}[s]h,\ \Gamma:=([\ell]\times\mathbb{N}^r)\setminus\deg(U),\ \text{ hence}$$
$$\widetilde{N}\subset\deg(U),\ \Gamma\subset\widetilde{\Gamma}.$$

Definition 38. The system (78) is called *orthonomic* if
(i) the degrees $\deg(h),\ h\in H$, are mutually distinct and
(ii) $h-lt(h)\in\oplus_{(i,\mu)\in\widetilde{\Gamma}}\mathbb{C}\,s^\mu\delta_i$ for all $h\in H$ where $lt(h)$ denotes the leading term of h.

If H is the reduced Gröbner basis of U the system is orthonomic and moreover $\Gamma=\widetilde{\Gamma}$. In general, however, Γ is a proper subset of $\widetilde{\Gamma}$ as the following example shows. Riquier calls the derivatives $\partial^\nu w_j$ *principal* resp. *parametric* according to whether $(j,\nu)\in\widetilde{N}$ resp. $(j,\nu)\in\widetilde{\Gamma}$ and $w\mid\widetilde{\Gamma}$ the *initial determination* of w. In contrast to theorem 16 the Cauchy problem

$$R(\partial)w=v,\ w\mid\widetilde{\Gamma}=\widetilde{x}\in\mathbb{C}^{\widetilde{\Gamma}},\ L(\partial)v=0 \tag{79}$$

does not always have a solution. If w^Γ denotes the unique solution of the Cauchy problem

$$R(\partial)w^\Gamma=v,\ w^\Gamma\mid\Gamma=\widetilde{x}\mid\Gamma,\ L(\partial)v=0$$

according to theorem 16 then (79) has a solution w which moreover is unique and coincides with w^Γ if and only if the initial determination $\widetilde{x}$ satisfies the compatibility condition

$$\widetilde{x}\mid(\widetilde{\Gamma}\setminus\Gamma)=w^\Gamma\mid(\widetilde{\Gamma}\setminus\Gamma). \tag{80}$$

whereas its part $\widetilde{x} \mid \Gamma$ can be chosen freely. Riquier derives other compatibility conditions which he also calls integrability conditions and calls the system *passive* if they are satisfied. The disadvantage of Riquier's approach is that the set $\widetilde{\Gamma}$ is too large and therefore integrability conditions are also needed for the initial determination. On the other hand, Riquier's existence theorem applies to passive orthonomic *non-linear* systems of the form

$$\partial^{d(h)} y_{j(h)} = f_h(z, \partial^\alpha y_i;\ (i,\alpha) < (j(h), d(h)),\ (i,\alpha) \in \widetilde{\Gamma}),\ h \in H.$$

with locally holomorphic functions f_h. These systems are "solved for the highest derivatives" and therefore admit unique *convergent* solutions for convergent initial determinations, passivity being, of course, required. Arbitrary implicit linear systems even of ordinary differential equations with *variable* coefficients, for instance, are not orthonomic and may not admit any convergent, but only hyper-function solutions.

Example 39. Consider for the graded lexicographic order on $\mathbb{N}^2$ and $\ell = 1$ the simple system

$$\begin{gathered} \partial_1^2 w + \partial_2 w = 0,\ \partial_1\partial_2 w + \partial_1 w = 0, \\ H = \{h_1, h_2\},\ h_1 = s_1^2 + s_2,\ h_2 = s_1 s_2 + s_1, \\ \widetilde{N} = \{(2,0), (1,1)\} + \mathbb{N}^2,\ \widetilde{\Gamma} = 0 \times \mathbb{N} \uplus \{(1,0\}\,. \end{gathered}$$

Obviously the system is orthonomic. The reduced Gröbner basis of
$U = \mathbb{C}[s]h_1 + \mathbb{C}[s]h_2$ is $G = \{h_1, h_2, h_3\}$ with $h_3 = s_2^2 + s_2$, hence
$\deg(U) = \{(2,0), (1,1), (0,2)\} + \mathbb{N}^2$ and $\Gamma = \{(0,0), ((1,0), (0,1)\}$.
Hence Γ is finite and the initial data which can be freely prescribed have the form $x = x(0,0) + x(1,0)z_1 + x(0,1)z_2$. The initial determinations according to Riquier have the form $\widetilde{x} = z_1 f^{1,0} + f^{0,0}(z_2)$ where $f^{1,0} \in \mathbb{C}$ and $f^{0,0} \in \mathbb{C}[[z_2]]$.

3.3 Convergent solutions for the lattice $\mathbb{Z}^r$

In the remainder of this paper we prove an analogue of theorem 24 for the monoid

$$\mathcal{M} := \mathbb{Z}^r \tag{81}$$

by reduction to the case of the monoid

$$\widetilde{\mathcal{M}} := \mathbb{N}^r \times \mathbb{N}^r = \mathbb{N}^{2r} \tag{82}$$

for which theorem 24 is directly applicable. This reduction method was introduced in [15] and further developed in [7] and is explained below as far as necessary. In fact this result could be easily extended to the monoids $\mathcal{M} = \mathbb{N}^{r_1} \times \mathbb{Z}^{r_2}$, however, for the sake of notational simplicity we restrict the considerations to $\mathcal{M} = \mathbb{Z}^r$.

Definition 40. For $\mu \in \mathbb{Z}^r$ let $|\mu| := (|\mu_1|, \ldots, |\mu_r|)$.
A signal $b \in \mathcal{A} = \mathbb{C}^{\mathcal{M}}$ is *convergent* if and only if there are $C \in \mathbb{R}_{>0}$ and $\rho \in \mathbb{R}^r_{>0}$ such that $|b(\mu)| \leq C\rho^{|\mu|}$ for all $\mu \in \mathcal{M}$.

Example 41. Consider the simplest case $r=1$ and $\mathcal{M}=\mathbb{Z}$. A sequence $a=(a(\mu))_{\mu\in\mathbb{Z}}$ is convergent if there are $C>0$ and $\rho>0$ such that $|a(\mu)|\leq C*\rho^{|\mu|}$ for all integers μ or, equivalently, that both power series

$$a_+ := \sum_{\mu=0}^{\infty} a(\mu)z^\mu \text{ and } a_- := \sum_{\mu=0}^{\infty} a(-\mu)z^\mu$$

are convergent (near z=0). The convergence of the second series also signifies that the series $\sum_{\mu=0}^{-\infty} a(\mu)z^\mu$ converges near $z=\infty$. Convergence does *not*, however, signify that the Laurent series $\overline{a} := \sum_{\mu=-\infty}^{\infty} a(\mu)z^\mu$ converges in $U\setminus\{0\}$ for some open neighbourhood U of zero so that $\overline{a}$ would be a holomorphic function with zero as an isolated singularity.

Let

$$\begin{gathered}\mathcal{D} := \mathbb{C}\left[s,s^{-1}\right] := \mathbb{C}\left[s_1,\ldots,s_r,s_1^{-1},\ldots,s_r^{-1}\right] \\ \text{and} \\ \widetilde{\mathcal{D}} := \mathbb{C}\left[s,t\right] := \mathbb{C}\left[s_1,\ldots,s_r,t_1,\ldots,t_r\right].\end{gathered} \tag{83}$$

The associated signal spaces are the

$$\begin{gathered}\mathcal{D}-\text{module } \mathcal{A} := \mathbb{C}^{\mathcal{M}} = \mathcal{D}^* \text{ resp. the} \\ \widetilde{\mathcal{D}}-\text{module } \widetilde{\mathcal{A}} := \mathbb{C}^{\widetilde{\mathcal{M}}} = \widetilde{\mathcal{D}}^* .\end{gathered} \tag{84}$$

The map

$$\tau : \widetilde{\mathcal{M}} \longrightarrow \mathcal{M}\ ,\ (\mu,\nu) \longmapsto \mu-\nu\ ,$$

is a surjective monoid-homomorphism. For $\nu\in\mathcal{M}=\mathbb{Z}^r$ we define

$$\begin{gathered}\nu_+ := (\max(\nu_i,0))_{1\leq i\leq r} \in \mathbb{N}^r \text{ and } \nu_- := (-\nu)_+\ , \\ \text{hence } \nu=\nu_+-\nu_- \text{ and } |\nu|=\nu_++\nu_-.\end{gathered} \tag{85}$$

The map

$$\sigma : \mathcal{M} \longmapsto \widetilde{\mathcal{M}}\ ,\ \mu \longmapsto (\mu_+,\mu_-)\ ,$$

is a (non-homomorphic) section of τ, i.e. $\tau\sigma=\mathrm{id}_{\mathcal{M}}$. These maps induce inverse bijections

$$\begin{gathered}\mathrm{im}(\sigma) \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \mathcal{M} \text{ where} \\ \mathrm{im}(\sigma) = \{(\mu,\nu)\in\widetilde{\mathcal{M}};\ \mu_i\cdot\nu_i=0,\ 1\leq i\leq r\}.\end{gathered} \tag{86}$$

The maps σ and τ are extended to the maps

$$[\ell]\times\widetilde{\mathcal{M}} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} [\ell]\times\mathcal{M},\ \tau(j,(\mu,\nu)) := (j,\tau((\mu,\nu))),\ \sigma(j,\mu) := (j,\sigma(\mu)), \tag{87}$$

with the same names and again we have $\tau\sigma=\mathrm{id}_{[\ell]\times\mathcal{M}}$. These maps induce the $\mathbb{C}$ −linear maps

$$\begin{gathered}\widetilde{\mathcal{A}}^\ell = \mathbb{C}^{[\ell]\times\widetilde{\mathcal{M}}} \underset{\mathbb{C}^\sigma}{\overset{\mathbb{C}^\tau}{\leftrightarrows}} \mathcal{A}^\ell = \mathbb{C}^{[\ell]\times\mathcal{M}},\ \mathbb{C}^\tau(a) = a\circ\tau,\ \mathbb{C}^\sigma(\widetilde{a}) = \widetilde{a}\circ\sigma \\ (a\circ\tau)(\mu,\nu) = a(\mu-\nu),\ (\widetilde{a}\circ\sigma)(\mu) = \widetilde{a}(\mu_+,\mu_-) \\ \text{with} \\ \mathbb{C}^\sigma\circ\mathbb{C}^\tau = \mathrm{id}_{\mathcal{A}^\ell},\end{gathered} \tag{88}$$

i.e. $\mathbb{C}^{\tau}$ is a section of the surjective linear map $\mathbb{C}^{\sigma}$.
Let φ be the surjective algebra homomorphism

$$\varphi : \widetilde{\mathcal{D}} \to \mathcal{D},\ s_i \mapsto s_i,\ t_i \mapsto s_i^{-1},\ 1 \le i \le r\,. \tag{89}$$

Its kernel is generated by the polynomials $s_i t_i - 1,\ 1 \le i \le r$. Let

$$\varphi^* : \mathcal{D}^* = \mathcal{A} \to \widetilde{\mathcal{D}}^* = \widetilde{\mathcal{A}}$$

be the dual of φ. It induces the isomorphism

$$\varphi^*_{ind} : \mathcal{A} \cong \{\widetilde{a} \in \widetilde{\mathcal{A}};\ (s_i t_i - 1) \circ \widetilde{a} = 0 \text{ for } i = 1, \cdots, r\ \} =: \widetilde{\mathcal{A}}_{inv}\ . \tag{90}$$

The signals in $\widetilde{\mathcal{A}}_{inv}$ are called *invariant* for obvious reasons. It is easily checked that

$$\varphi^* = \mathbb{C}^{\tau} \text{ and that } \varphi^*[\varphi(\widetilde{d}) \circ a] = \widetilde{d} \circ \varphi^*(a),\ \widetilde{d} \in \widetilde{\mathcal{D}},\ a \in \mathcal{A}, \tag{91}$$

the latter equation meaning that φ^* is $\varphi-$ semi-linear. The map φ is extended componentwise to matrices, i.e.

$$\varphi(\widetilde{R}) := (\varphi(\widetilde{R}_{ij}))_{i,j} \text{ for } \widetilde{R} \in \widetilde{\mathcal{D}}^{k \times \ell},$$

and again the relations

$$\begin{gathered}\varphi^* = \mathbb{C}^{\tau} : \mathcal{A}^{\ell} = (\mathcal{D}^{1\times l})^* = \mathbb{C}^{[\ell]\times \mathcal{M}} \to \widetilde{\mathcal{A}}^{\ell} = (\widetilde{\mathcal{D}}^{1\times l})^* = \mathbb{C}^{[\ell]\times \widetilde{\mathcal{M}}} \\ \text{and} \\ \varphi^*[\varphi(\widetilde{R}) \circ w] = \widetilde{R} \circ \varphi^*(w),\ \widetilde{R} \in \widetilde{\mathcal{D}}^{k\times l},\ w \in \mathcal{A}^{\ell},\end{gathered} \tag{92}$$

hold. Since $\mathbb{C}^{\sigma} \circ \mathbb{C}^{\tau} = \mathbb{C}^{\sigma} \circ \varphi^* = \mathrm{id}_{\mathcal{A}^{\ell}}$ equations (90) and (92) induce the inverse isomorphisms

$$\widetilde{\mathcal{A}}^{\ell}_{inv} \cong \mathcal{A}^{\ell},\ \widetilde{w} = \varphi^*(w) = w \circ \tau \leftrightarrow w = \widetilde{w} \circ \sigma. \tag{93}$$

Lemma 42. *Let $\mathcal{C}$ resp. $\widetilde{\mathcal{C}}$ be the subspaces of convergent signals in $\mathcal{A}$ resp. $\widetilde{\mathcal{A}}$. The inverse isomorphisms* (90) *and* (93) *induce the isomorphisms*

$$\widetilde{\mathcal{C}} \bigcap \widetilde{\mathcal{A}}_{inv} \cong \mathcal{C} \text{ and } \widetilde{\mathcal{C}}^{\ell} \bigcap \widetilde{\mathcal{A}}^{\ell}_{inv} \cong \mathcal{C}^{\ell}.$$

Proof. We show the result for $\ell = 1$.
(i) Let $a \in \mathcal{C}$, hence

$$\widetilde{a} := \varphi^*(a) = a \circ \tau,\ \widetilde{a}(\mu, \nu) = a(\mu - \nu).$$

Since a is convergent there are $C > 0,\ \rho \in \mathbb{R}^{r}_{>0}$ and such that

$$|a(\mu - \nu)| \le C\rho^{|\mu-\nu|} \text{ for all } (\mu, \nu) \in \widetilde{\mathcal{M}}.$$

We choose $\rho \ge_{cw} (1, 1, \cdots, 1)$ and use $|\mu - \nu| \le \mu + \nu$ to obtain

$$|\widetilde{a}(\mu, \nu)| \le C * \rho^{\mu} * \rho^{\nu} \text{ for all } (\mu, \nu) \in \widetilde{\mathcal{M}},$$

and hence $\widetilde{a} \in \widetilde{\mathcal{C}} \bigcap \widetilde{\mathcal{A}}_{inv}$.
(ii) Let, conversely, $\widetilde{a} \in \widetilde{\mathcal{A}}$ be invariant and convergent. Equation (90) implies the unique representation

$$\widetilde{a} = \varphi^*(a) = a \circ \tau,\ a = \widetilde{a} \circ \sigma,\ a \in \mathcal{A}.$$

The convergence of $\widetilde{a}$ implies the existence of $C \in \mathbb{R}_{>0}$ and $\rho \in \mathbb{R}^r_{>0}$ with such that

$$|\widetilde{a}(\mu,\nu)| \leq C * \rho^\mu * \rho^\nu \text{ for all } (\mu,\nu) \in \widetilde{\mathcal{M}} \text{ and hence}$$
$$|a(\mu)| = |\widetilde{a}(\mu_+,\mu_-)| \leq C * \rho^{\mu_+} * \rho^{\mu_-} = C * \rho^{|\mu|}$$

since $|\mu| = \mu_+ + \mu_-$. But this signifies that a is convergent. □

For the next two theorems we use a term order on $[\ell] \times \widetilde{\mathcal{M}}$ as in assumption 21 and on $[\ell] \times \mathcal{M}$ the well-order induced by the injection σ, i.e.

$$(i,\mu) < (j,\nu) : \quad \Leftrightarrow \quad (i,\mu_+,\mu_-) < (j,\nu_+,\nu_-). \tag{94}$$

Theorem 43.

(1) The order from (94) is a generalized term order with respect to the conic decomposition $\left(\mathbb{N}^T \oplus (-\mathbb{N}^{T'})\right)_{T\subseteq[r]}$ of $\mathbb{Z}^r$, where $T' := [r] \setminus T$ and $\mathbb{N}^T \oplus (-\mathbb{N}^{T'}) = \{\nu \in \mathbb{Z}^r \mid \nu_j \geq 0 \text{ for } j \in T,\ \nu_j \leq 0 \text{ for } j \in T'\}$.

(2) Let U be a $\mathcal{D}$-submodule of $\mathcal{D}^{1\times\ell}$, $\widetilde{U} := \varphi^{-1}(U)$, $\widetilde{\Gamma} := \left([\ell] \times \widetilde{\mathcal{M}}\right) \setminus \deg(\widetilde{U})$ and $\Gamma := \left([\ell] \times \mathcal{M}\right) \setminus \deg(U)$.
Then

$$\sigma(\Gamma) = \widetilde{\Gamma} \text{ and } \tau(\widetilde{\Gamma}) = \Gamma \ .$$

(3) If $\widetilde{G} \subset \widetilde{\mathcal{D}}^{1\times\ell}$ is the reduced Gröbner basis of $\widetilde{U}$ then

$$G := \{\varphi(\widetilde{g}) \mid \widetilde{g} \in \widetilde{G},\ \deg(\widetilde{g}) \in \operatorname{im}(\sigma)\}$$

is a Gröbner basis of U.

Proof. (1) Let $\mu \in \mathbb{Z}^r$, $\nu,\lambda \in \mathbb{N}^T \oplus (-\mathbb{N}^{T'})$ and $(i,\mu) \leq (i',\nu)$, i.e. $(i,\mu_+,\mu_-) \leq (i',\nu_+,\nu_-)$. We have to show that

$$(i,(\mu+\lambda)_+,(\mu+\lambda)_-) \leq (i',(\nu+\lambda)_+,(\nu+\lambda)_-) \ .$$

For $\nu,\lambda \in \mathbb{N}^T \oplus (-\mathbb{N}^{T'})$ we have $\nu_+ = \nu|T \in \mathbb{N}^T \subseteq \mathbb{N}^r$ and $\nu_- = -\nu|T'$, hence $(\nu_+,\nu_-) = (\nu|T, -\nu|T')$ and

$$((\nu+\lambda)_+,(\nu+\lambda)_-) = (\nu|T + \lambda|T, -\nu|T' - \lambda|T') = (\nu_+,\nu_-) + (\lambda|T, -\lambda|T') \ .$$

For $\rho \in [r]$ the inequality

$$((\mu+\lambda)_+)_\rho = \max(0,\mu_\rho+\lambda_\rho) \leq \begin{cases} \max(0,\mu_\rho)+\lambda_\rho & \text{if } \rho \in T(\lambda_\rho \geq 0) \\ \max(0,\mu_\rho) & \text{if } \rho \in T'(\lambda_\rho \leq 0) \end{cases} \tag{95}$$
$$= (\mu_+)_\rho + (\lambda|T)_\rho = (\mu_+ + \lambda|T)_\rho$$

holds and hence

$$(\mu+\lambda)_+ \leq_{cw} \mu_+ + \lambda|T,\ (\mu+\lambda)_+ \leq \mu_+ + \lambda|T.$$

Likewise

$$(\mu+\lambda)_- \leq_{cw} \mu_- + (-\lambda|T'),\ (\mu+\lambda)_- \leq \mu_- + (-\lambda|T').$$

Hence

$$(i,(\mu+\lambda)_+,(\mu+\lambda)_-) \le (i,\mu_+ + \lambda|T,\mu_- + (-\lambda)|T') = (i,\mu_+,\mu_-) + (\lambda|T,-\lambda|T') \le$$

$$\le (i',\nu_+,\nu_-) + (\lambda|T,-\lambda|T') = (i',(\nu+\lambda)_+,(\nu+\lambda)_-)$$

as needed.
(2) For simplicity we choose $\ell = 1$, the general case $\ell \ge 1$ being proven in the same fashion. The set $\{s_i t_i - 1 \mid 1 \le i \le r\}$ is a Gröbner basis of $\ker(\varphi)$ hence

$$\deg(\ker(\varphi)) = \bigcup_{1\le i\le r} \big(((0,\dots,0,\overset{i}{1},0,\dots,0),(0,\dots,0,\overset{i}{1},0,\dots,0)) + (\mathbb{N}^r \times \mathbb{N}^r)\big).$$

Recall that $\mathrm{im}(\sigma) = \{(\mu,\nu)\in \mathbb{N}^r\times\mathbb{N}^r \mid \mu_i\nu_i = 0,\ 1\le i\le r\}$. Hence

$$\mathbb{N}^r \times \mathbb{N}^r = \mathrm{im}(\sigma) \uplus \deg(\ker(\varphi))\ .$$

The $\mathbb{C}$-linear map

$$\psi : \mathcal{D} \longrightarrow \widetilde{\mathcal{D}}\ ,\ s^\mu \longmapsto s^{\mu_+}t^{\mu_-}\ ,$$

is a section of φ and satisfies $\deg(\psi(f)) = \sigma(\deg(f))$ by the definition of the well-order on $\mathbb{Z}^r$, hence

$$\widetilde{\mathcal{D}} = \mathrm{im}(\psi)\oplus\ker(\varphi),\quad \widetilde{U} = \psi(U)\oplus\ker(\varphi)$$

and

$$\deg(\psi(U)) = \sigma(\deg(U)) \subseteq \mathrm{im}(\sigma)\ .$$

From

$$\deg(\psi(U))\cap\deg(\ker(\varphi)) \subseteq \mathrm{im}(\sigma)\cap\deg(\ker(\varphi)) = \emptyset$$

we infer

$$\deg(\widetilde{U}) = \deg(\psi(U)\oplus\ker(\varphi)) =$$

$$= \deg(\psi(U))\uplus\deg(\ker(\varphi)) = \sigma(\deg(U))\uplus\deg(\ker(\varphi))$$

and finally

$$\widetilde{\Gamma} = \mathbb{N}^{2r}\setminus\deg(\widetilde{U}) = [\mathrm{im}(\sigma)\uplus\deg(\ker(\varphi))]\setminus[\sigma(\deg(U))\uplus\deg(\ker(\varphi))] =$$

$$= \mathrm{im}(\sigma)\setminus\sigma(\deg(U)) = \sigma(\mathbb{Z}^r\setminus\deg(U)) = \sigma(\Gamma)$$

since σ is injective. This and the identity $\tau\sigma = \mathrm{id}$ imply $\tau(\widetilde{\Gamma}) = \Gamma$.
(3) Again we choose $\ell = 1$ for notational simplicity. Since $\widetilde{G}$ is the reduced Gröbner basis of $\widetilde{U}$ its elements have the form

$$\widetilde{g} = (s,t)^{\widetilde{d}} + \sum_{\widetilde{\lambda}<\widetilde{d},\,\widetilde{\lambda}\in\widetilde{\Gamma}} \widetilde{g}(\widetilde{\lambda})(s,t)^{\widetilde{\lambda}}\ .$$

Since $\widetilde{\Gamma}\subseteq\mathrm{im}(\sigma)$ there are unique $\lambda\in\Gamma$ such that $\widetilde{\lambda} = \sigma(\lambda)$, hence

$$\widetilde{g} = (s,t)^{\widetilde{d}} + \sum_{\lambda\in\Gamma,\,\sigma(\lambda)<\widetilde{d}} \widetilde{g}(\sigma(\lambda))(s,t)^{\sigma(\lambda)}\ .$$

Now let

$$\nu \in \deg(U)\cap(\mathbb{N}^T\oplus(-\mathbb{N})^{T'})\ .$$

Then

$$\sigma(\nu) = (\nu_+, \nu_-) = (\nu|T, -\nu|T') \in \mathbb{N}^T \times \mathbb{N}^{T'} \subseteq \operatorname{im}(\sigma) \subseteq \mathbb{N}^r \times \mathbb{N}^r$$

and

$$\sigma(\nu) \in \sigma(\deg(U)) \subseteq \deg(\widetilde{U}) = \bigcup_{\widetilde{g}\in\widetilde{G}} \deg(\widetilde{g} + \mathbb{N}^{2r}) \ .$$

Therefore there are $\widetilde{g} \in \widetilde{G}$ and $\widetilde{\alpha} \in \mathbb{N}^{2r}$ with

$$\sigma(\nu) = \widetilde{d} + \widetilde{\alpha} \in \mathbb{N}^T \times \mathbb{N}^{T'},$$

where $\widetilde{d} := \deg(\widetilde{g})$. Hence $\widetilde{d}, \widetilde{\alpha} \in \mathbb{N}^T \times \mathbb{N}^{T'}$ and there are unique elements $d, \alpha \in \mathbb{N}^T \oplus (-\mathbb{N})^{T'} \subseteq \mathbb{Z}^r$ such that $\widetilde{d} = \sigma(d)$, $\widetilde{\alpha} = \sigma(\alpha)$. Thus

$$\widetilde{g} = (s,t)^{\sigma(d)} + \sum_{\lambda\in\Gamma,\, \sigma(\lambda)<\widetilde{d}} \widetilde{g}(\sigma(\lambda))(s,t)^{\sigma(\lambda)} \ ,$$

$$g := \varphi(\widetilde{g}) = s^d + \sum_{\lambda\in\Gamma,\, \lambda<d} \widetilde{g}(\sigma(\lambda))s^\lambda$$

and

$$s^\alpha g = s^{d+\alpha} + \sum_{\lambda\in\Gamma,\, \lambda<d} \widetilde{g}(\sigma(\lambda))s^{\lambda+\alpha} \ .$$

But the order on $\mathbb{Z}^r$ is a generalized term order, thus $\lambda < d$ and $d, \alpha \in \mathbb{N}^T \oplus (-\mathbb{N})^{T'}$ imply $\lambda+\alpha < d+\alpha$. Hence $\deg(s^\alpha g) = d+\alpha = \nu$. Therefore

$$\deg(U) = \{\deg(s^\alpha g) \mid g \in G,\ \alpha \in \mathbb{Z}^r\}$$

which signifies that G is indeed a Gröbner basis of U. □

Theorem 44. *(i) In the situation of theorem 5 (with respect to the term order from* (94)*) let $w \in \mathcal{A}^\ell$ be the unique solution of the Cauchy problem*

$$R \circ w = v,\ w \mid \Gamma = x,\ L \circ v = 0,$$

for given initial data $x \in \mathbb{C}^\Gamma$ and $v \in \mathbb{C}[[z]]^\ell$. If v and x are convergent then so is w.
(ii) The module $(\mathcal{C}, \circ)$ is large injective cogenerator.

Proof. (i) For the proof of (i) we choose a matrix $\widehat{R} \in \widetilde{\mathcal{D}}^{k\times l}$ such that $\varphi(\widehat{R}) = R$. According to (92) we obtain

$$\widehat{R} \circ \varphi^*(w) = \varphi^*[\varphi(\widehat{R}) \circ w] = \varphi^*[R \circ w] = \varphi^*(v).$$

Since $\varphi^*(w)$ is invariant by (92) it satisfies the equations

$$(s_i t_i - 1)\delta_j \circ \varphi^*(w) = 0,\ j = 1, \cdots, \ell,\ i = 1, \cdots, r, \text{ and hence}$$

$$\widetilde{R} \circ \varphi^*(w) = \begin{pmatrix} \varphi^*(v) \\ 0 \end{pmatrix} \text{ where}$$

$$\widetilde{R} := \begin{pmatrix} \widehat{R} \\ (s_i t_i - 1)\delta_j,\ 1 \le j \le \ell,\ 1 \le i \le r \end{pmatrix} \in \widetilde{\mathcal{D}}^{(k+\ell r)\times\ell}.$$

Moreover

$$\widetilde{U} := \widetilde{\mathcal{D}}^{1\times(k+\ell r)}\widetilde{R} = \widetilde{\mathcal{D}}^{1\times k}\widehat{R} + \sum_{j,i}\widetilde{\mathcal{D}}(s_it_i-1)\delta_j = \varphi^{-1}(U),\ U := \mathcal{D}^{1\times k}R.$$

Theorem 43,(2), and equation (88) imply the inverse bijections

$$\widetilde{\Gamma} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \Gamma \text{ and } \mathbb{C}^{\widetilde{\Gamma}} \underset{\mathbb{C}^{\sigma}}{\overset{\mathbb{C}^{\tau}}{\leftrightarrows}} \mathbb{C}^{\Gamma},\ \widetilde{x} = x\circ\tau \leftrightarrow x = \widetilde{x}\circ\sigma.$$

Since the initial data are assumed convergent, i.e. $x \in \mathbb{C}^{\Gamma}\bigcap\mathcal{C}^{\ell}$, so is $\widetilde{x} := \varphi^*(x) = x\circ\tau$ according to lemma 42. Moreover the initial condition $w \mid \Gamma = x$ implies

$$\varphi^*(w) \mid \widetilde{\Gamma} = (w\circ\tau) \mid \widetilde{\Gamma} = (w \mid \Gamma)\circ\tau = x\circ\tau = \widetilde{x}.$$

Summing up we obtain that $\varphi^*(w) = w\circ\tau$ is the solution of the the Cauchy problem

$$\widetilde{R}\circ\varphi^*(w) = \begin{pmatrix}\varphi^*(v)\\ 0\end{pmatrix},\ \varphi^*(w) \mid \widetilde{\Gamma} = \widetilde{x}$$

with convergent right side $\binom{\varphi^*(v)}{0}$ and convergent initial data $\widetilde{x}$. From theorem 24 we infer that $\varphi^*(w)$ is convergent too, and hence so is $w = \varphi^*(w)\circ\sigma$ again by lemma 42.

(ii) As before the unique solvability of the Cauchy problem implies that $(C,\circ)$ is an injective cogenerator. Assume now that $M = \mathcal{D}^{1\times\ell}/U$ is a finitely generated $\mathcal{D}$-module. The $\widetilde{\mathcal{D}}$-epimorphism $\varphi : \widetilde{\mathcal{D}}^{1\times\ell} \longrightarrow \mathcal{D}^{1\times\ell}$ induces the $\widetilde{\mathcal{D}}$- and $\mathcal{D}$-isomorphism

$$\varphi_{ind} : \widetilde{M} := \widetilde{\mathcal{D}}^{1\times\ell}/\varphi^{-1}(U) \cong M = \mathcal{D}^{1\times\ell}/U\ .$$

Since $(\widetilde{\mathcal{C}},\circ)$ is a large injective $\widetilde{\mathcal{D}}$-cogenerator according to theorem 24 there is a $\widetilde{\mathcal{D}}$-monomorphism $f : \widetilde{M} \longrightarrow \widetilde{\mathcal{C}}^m$ for some $m \in \mathbb{N}$. Since

$$(s_it_i-1)\widetilde{\mathcal{D}}^{1\times\ell} \subseteq \ker(\varphi) \subseteq \varphi^{-1}(U)$$

for all $i = 1,\ldots,r$ we get

$$(s_it_i-1)\circ f(\widetilde{M}) = 0,\ i = 1,\ldots,r, \quad \text{i.e. } f(\widetilde{M}) \subseteq \widetilde{\mathcal{C}}^m_{inv} \cong \mathcal{C}^m\ .$$

The resulting monomorphism

$$M \cong \widetilde{M} \xrightarrow{f} \widetilde{\mathcal{C}}^m_{inv} \cong \mathcal{C}^m$$

is the desired $\mathcal{D}$-monomorphism from M into some finite power of $\mathcal{C}$. □

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