The asymptotic stability of stable and time-autonomous discrete multidimensional behaviors

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Abstract: We generalize the important paper D. Napp-Avelli, P. Rapisarda, P. Rocha, 'Time-relevant stability of 2D systems', *Automatica* 47(2011), 2373-2382, to discrete time-autonomous (ta) (=time-relevant), but *not necessarily square-autonomous* behaviors in *arbitrary dimensions*. The discrete domain of the independent variables is the lattice of vectors of integers of arbitrary (but fixed) length whose first component is a natural number and interpreted as a discrete time instant. The stability of an autonomous behavior is defined by algebraic conditions on its characteristic variety. Under suitable additional conditions a discrete stable and time-autonomous behavior is asymptotically stable in the sense that its trajectories converge to zero when the time tends to infinity. We derive algorithms for the constructive verification of the assumptions of most of our results and in particular establish a constructive normal form of ta behaviors in arbitrary dimensions. The Fourier transform on finitely generated free abelian groups plays an important part in the derivations as it already did in the quoted paper for the group of integers. Stability and stabilization of multidimensional discrete behaviors were previously discussed by various colleagues, for instance by Bisiacco, Bose, Fornasini, Lin, Marchesini, Quadrat, Shankar, Sule, Valcher and Willems, but only partly from the analytic point of view.

Keywords: discrete multidimensional system, stability by spectral conditions, asymptotic stability, time-autonomy

1. INTRODUCTION

This conference paper without proofs is a short version of the regular paper Oberst and Scheicher [2012b] with the same title which has recently been submitted to *Math. Control Sign. Syst.*. *We generalize the important paper Napp, Rapisarda and Rocha [2011] to discrete time-autonomous (ta) (=time-relevant), but not necessarily square-autonomous behaviors in arbitrary dimensions and slightly correct [Napp, Rapisarda and Rocha, 2011, Thm. 10]. Whereas square autonomy is very restrictive in higher dimensions it can be assumed without loss of generality in dimension 2 as is done in Napp, Rapisarda and Rocha [2011]. As in the latter paper we define stability of an autonomous behavior by suitable spectral conditions, and show that under suitable additional conditions the trajectories of a stable and ta behavior are asymptotically stable in an analytic sense.*
Stability and stabilization of multidimensional discrete systems were studied by many authors, for instance by Bose [1985], Shankar and Sule [1992], Lin [2001], Quadrat [2003]. The survey article Oberst and Scheicher [2007] contains a comprehensive list of references. The analytic properties of trajectories of stable behaviors have been treated more rarely, for instance by Fornasini and Marchesini [1980], Bisiacco, Fornasini and Marchesini [1985], Bisiacco, Fornasini and Marchesini [1989], Valcher [2000] and Willems [2007]. The conference talk Willems [2007] dealt with the continuous case of partial differential equations , but strongly influenced the paper Napp, Rapisarda and Rocha [2011] and also the present paper.
The main results of the present paper are Theorems 1, 3, 5 and 8 whose formulation requires the data of Sections 2 and 3. Example 14 contains a whole class of examples for the illustration of Thm. 8. Example 15 implies a slight correction of [Napp, Rapisarda and Rocha, 2011, Thm. 10]. Section 5 shortly describes the constructivity of our results.

2. BASIC DATA AND TIME-AUTONOMOUS BEHAVIORS

For $n \geq 1$ we consider discrete $1+n$-dimensional behaviors over the lattice $N := \mathbb{N} \times \mathbb{Z}^n$ of independent variables $(t,\mu) = (t,\mu_1,\cdots,\mu_n)$ where the distinguished first component t is interpreted as discrete time. Due to its importance for the Fourier transform we use the base field $\mathbb{C}$ of complex numbers and the signal spaces

$$\begin{gathered} W_A := \mathbb{C}^{\mathbb{Z}^n} \text{ and } W_B := \mathbb{C}^{\mathbb{N}\times\mathbb{Z}^n} = W_A^{\mathbb{N}}, \\ W_B \ni w = (w(t,\mu))_{(t,\mu)\in N} = (w(0), w(1), \cdots), \\ w(t) \in W_A,\ w(t)(\mu) = w(t,\mu). \end{gathered} \tag{1}$$

The signals $w \in W_B$ are thus interpreted as time series of signals $w(t) \in \mathbb{C}^{\mathbb{Z}^n}$ at the discrete time instance $t \in \mathbb{N}$. Let $(s_0, s) =$

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$(s_0, s_1, \cdots, s_n)$ be a list of indeterminates. The associated operator $\mathbb{C}$-algebras are

$$A = \mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[s, s^{-1}] = \oplus_{\mu \in \mathbb{Z}^n} \mathbb{C}s^\mu,$$
$$s^{-1} := (s_1^{-1}, \cdots, s_n^{-1}), \text{ and} \tag{2}$$
$$B := \mathbb{C}[N] = A[s_0] = \mathbb{C}[s_0, s, s^{-1}] = \oplus_{t \in \mathbb{N}, \mu \in \mathbb{Z}^n} \mathbb{C}s_0^t s^\mu.$$

So A and B are Laurent polynomial algebras which act on W_A resp. W_B by the standard shift actions $\circ$. There are the canonical identifications

$$A^* := \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C}) = W_A \ni v,\ v(s^\mu) = v(\mu), \tag{3}$$

and also $B^* = W_B$. The modules W_A resp. W_B are large injective cogenerators in the category Mod_A of A-modules resp. in Mod_B and give rise to the standard strong duality between finitely generated (f.g.) modules and behaviors, compare, for instance, [Oberst and Scheicher, 2012a, Section 2].

We need W_B- and W_A-behaviors. A W_B-behavior $\mathscr{B} \subseteq W_B^\ell$ is defined by a matrix and its associated B-modules

$$R \in B^{k \times \ell},\ U = B^{1\times k}R := \sum_{i=1}^{k} BR_{i-} \subseteq B^{1\times\ell}$$
$$\text{and } M := B^{1\times\ell}/U:$$
$$\mathrm{sol}_{W_B}(M) \underset{\text{Malgrange 1962}}{:=} D(M) := \mathrm{Hom}_B(M, W_B) \cong \tag{4}$$
$$\underset{\text{Malgrange 1962}}{\cong} \mathscr{B} := \left\{ w \in W_B^\ell;\ U \circ w = 0 \right\} =$$
$$\left\{ w \in W_B^\ell;\ R \circ w = 0 \right\}.$$

Here $B^{1\times\ell}$ resp. $W_B^\ell := W_B^{\ell\times 1}$ consist of row resp. column vectors and the action $\circ$ of B on W_B is extended to matrix actions

$$B^{k\times\ell} \times W_B^{\ell\times m} \to W_B^{k\times m},\ (R, X) \mapsto R \circ X,$$
$$(R\circ X)_{ij} := \sum_{q=1}^{\ell} R_{iq} \circ X_{qj}. \tag{5}$$

For purposes of algebraic geometry we need the set

$$\Lambda_N := \mathbb{C} \times (\mathbb{C}\setminus\{0\})^n = \{(\lambda, \omega);\ \lambda \in \mathbb{C},\ \omega \in (\mathbb{C}\setminus\{0\})^n\} \tag{6}$$

of $1+n$-dimensional complex vectors which can be substituted into Laurent polynomials $f(s_0, s) \in B$. The vanishing set of an ideal $\mathfrak{b} \subseteq B$ is

$$V_{\Lambda_N}(\mathfrak{b}) := \{(\lambda, \omega) \in \Lambda_N;\ \forall f \in \mathfrak{b}:\ f(\lambda, \omega) = 0\}. \tag{7}$$

If $\mathscr{B}$ from (4) is autonomous or, equivalently, $\mathrm{rank}(R) = \ell$ or M is a torsion module with its nonzero annihilator

$$\mathfrak{b}_M := \mathrm{ann}_B(M) := \{f \in B;\ fM = 0\} = \mathrm{ann}_B(\mathscr{B}) \tag{8}$$

we consider its *characteristic variety* [Scheicher and Oberst, 2012, (51)]

$$\mathrm{char}(\mathscr{B}) := \mathrm{char}(M) :=$$
$$\{(\lambda, \omega) \in \Lambda_N;\ \mathrm{rank}(R(\lambda, \omega)) < \ell\} = V_{\Lambda_N}(\mathfrak{b}_M). \tag{9}$$

We make also use of the polydiscs and tori

$$\mathbb{D}^1 := \{z \in \mathbb{C};\ |z| < 1\},\ \mathbb{T}^1 := \{z \in \mathbb{C}; |z| = 1\} = \partial(\mathbb{D}^1),$$
$$\mathbb{D}^n := \left(\mathbb{D}^1\right)^n,\ \mathbb{T}^n := \left(\mathbb{T}^1\right)^n, \text{ hence}$$
$$\mathbb{R}/\mathbb{Z} \cong \mathbb{T}^1,\ x + \mathbb{Z} \mapsto \exp(2\pi i x), \text{ and } \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n \tag{10}$$

and the *stability decomposition*

$$\Lambda_N = \Lambda_1 \uplus \Lambda_2,\ \Lambda_2 := \{z \in \mathbb{C};\ |z| \geq 1\} \times \mathbb{T}^n \tag{11}$$

where Λ_1 resp. Λ_2 are considered as the *stable resp. unstable region*. The autonomous behavior $\mathscr{B}$ from (4) is called Λ_1-*stable* if the characteristic variety $\mathrm{char}(\mathscr{B})$ is contained in Λ_1 or $\mathrm{char}(\mathscr{B}) \bigcap \Lambda_2 = \emptyset$. This signifies (cf. [Napp, Rapisarda and Rocha, 2011, Thm. 10]) that

$$(\lambda, \omega) \in \mathrm{char}(\mathscr{B}) \text{ and } \omega \in \mathbb{T}^n \text{ imply } \lambda \in \mathbb{D}^1. \tag{12}$$

The W_B-behavior $\mathscr{B} \subset W_B^\ell$ is called *time-autonomous* (ta) (cf. Oberst and Scheicher [2012a] and its references for the history of this term and its principal contributors) or *time-relevant* Napp, Rapisarda and Rocha [2011] if there is a time instant d such that the projection

$$\mathrm{proj}: \mathscr{B} \to \left(W_A^\ell\right)^d = W_A^{\ell d},\ w \mapsto (w(0), \cdots, w(d-1))^\top,$$
$$\text{is injective and hence } \mathrm{proj}: \mathscr{B} \cong \mathscr{B}_d := \mathrm{proj}(\mathscr{B}). \tag{13}$$

Here the image $\mathscr{B}_d := \mathrm{proj}(\mathscr{B}) \subseteq W_A^{\ell d}$ is a computable W_A-behavior. We give a constructive characterization of time-autonomous behaviors:

Theorem 1. The following properties of an autonomous behavior (4) are equivalent:

(i) The behavior $\mathscr{B}$ is time-autonomous.
(ii) The finitely generated (f.g.) B-module M of $\mathscr{B}$ is also f.g. as A-module.
(iii) There is a monic (in s_0) polynomial
$$f = s_0^d + f_{d-1}(s)s_0^{d-1} + \cdots f_0(s) \in B = A[s_0]$$
which annihilates $\mathscr{B}$ or, equivalently, M.
(iv) The behavior $\mathscr{B}$ from (4) is isomorphic to a behavior $\mathscr{B}' \subset W_B^{\ell'}$ defined by a matrix
$$R' = \begin{pmatrix} R_0 \\ s_0\,\mathrm{id}_{\ell'} - E \end{pmatrix} \text{ where } R_0 \in A^{k'\times\ell'},$$
$$E \in A^{\ell'\times\ell'},\ \exists X \in A^{k'\times k'} \text{ with } R_0 E = X R_0:$$
$$\mathscr{B}' = \left\{ w' \in W_B^{\ell'};\ R' \circ w' = 0 \right\} = \tag{14}$$
$$\left\{ w';\ R_0 \circ w'(0) = 0,\ \forall t \in \mathbb{N}:\ w'(t) = E^t \circ w'(0) \right\}$$
$$\mathscr{B}' \cong \mathscr{B}'_1 = \left\{ v' \in W_A^{\ell'};\ R_0 \circ v' = 0 \right\},\ w' \mapsto w'(0).$$

Via the Gröbner basis algorithm all items are fully constructive.

Theorem 1 substantially improves [Oberst and Scheicher, 2012a, Thm. 3.7]. The new proof of this result, given in Oberst and Scheicher [2012b], but not in this conference paper, simplifies that of [loc.cit.].

Consider any autonomous behavior $\mathscr{B}$ as in (4), for instance that of Thm. 1,(iv). The next corollary permits to decide constructively whether $\mathscr{B}$ is square-autonomous which, in contrast to the case $n = 1$ of Napp, Rapisarda and Rocha [2011], is rarely the case for $n > 1$. Let $R_1 \in B^{k_1\times k}$ be a universal left annihilator of R so that

$$B^{1\times k_1} \xrightarrow{\circ R_1} B^{1\times k} \xrightarrow{\circ R} B^{1\times\ell} \xrightarrow{\text{can}} M \to 0 \text{ is exact, hence}$$
$$B^{1\times k}/B^{1\times k_1}R_1 \cong U = B^{1\times k}R,\ x + B^{1\times k_1}R_1 \mapsto xR. \tag{15}$$

The Quillen-Suslin theorem for modules over the Laurent algebra B implies that a f.g. projective B-module U is free.

Corollary 2. In the situation of (15) the following assertions are equivalent:

(i) The behavior $\mathscr{B}$ is square-autonomous.
(ii) The module U is free, necessarily of dimension ℓ since $\mathrm{rank}(R) = \ell$, or, equivalently, projective or M has projective dimension at most 1.
(iii) There is a matrix $S_1 \in B^{k\times k_1}$ such that $R_1 S_1 R_1 = R_1$.

These conditions imply: $E_1 := S_1 R_1 = E_1^2$, $R_1 E_1 = R_1$,

$$B^{1\times k}E = B^{1\times k_1}R_1,\; B^{1\times k} = B^{1\times k}E \oplus B^{1\times k}(\mathrm{id}_k - E) \text{ and}$$
$$B^{1\times k}(\mathrm{id}_k - E) \cong B^{1\times k}/B^{1\times k}E = B^{1\times k}/B^{1\times k_1}R_1 \cong U.$$

The universal left annihilator R_1 can be computed with all Computer Algebra systems. The existence of S_1 and thus the projectivity of U can be checked by means of [Oberst and Scheicher, 2007, Alg. 8.2,(4)].

3. APPLICATION OF THE FOURIER TRANSFORM

Let

$$\mathrm{L}^0(\mathbb{T}^n) \supset \mathrm{C}^0(\mathbb{T}^n) \supset \mathscr{D}(\mathbb{T}^n) \tag{16}$$

denote the $\mathbb{C}$-algebras of (Borel) measurable resp. continuous resp. smooth functions on $\mathbb{T}^n$ where two measurable functions are identified if their difference is zero almost everywhere. The measure on $\mathbb{T}^n$ is the unique Haar measure of total mass 1 which for continuous functions $f \in \mathrm{C}^0(\mathbb{T}^n)$ is given by the integral

$$\begin{aligned} &\int_{\mathbb{T}^n} f(\omega)d\omega = \int_{[0,1]^n} f(e^{2\pi i x})dx, \\ &e^{2\pi i x} := (e^{2\pi i x_1}, \cdots, e^{2\pi i x_n}),\; dx = dx_1 \cdots dx_n. \end{aligned} \tag{17}$$

A measurable function $f: \mathbb{T}^n \to \mathbb{C}$ is invertible in $\mathrm{L}^0(\mathbb{T}^n)$ if its zero set $V_{\mathbb{T}^n}(f) := \{\omega \in \mathbb{T}^n; f(\omega) = 0\}$ has measure 0. Its inverse is the function

$$g: \mathbb{T}^n \to \mathbb{C},\; g(\omega) := \begin{cases} f(\omega)^{-1} & \text{if } \omega \in \mathbb{T}^n \setminus V_{\mathbb{T}^n}(f) \\ 0 & \text{if } f(\omega) = 0 \end{cases}.$$

Since $\mathbb{T}^n$ is a product of infinite subsets of $\mathbb{C}$ the map

$$A \to \mathscr{D}(\mathbb{T}^n),\; f \mapsto (\omega \mapsto f(\omega)), \tag{18}$$

from a (Laurent) polynomial to its polynomial function is injective and therefore we may and do consider A as subalgebra of $\mathscr{D}(\mathbb{T}^n)$.

Theorem 3. (i) A nonzero Laurent polynomial $f \in A \subset \mathrm{L}^0(\mathbb{T}^n)$ has a zero set $V_{\mathbb{T}^n}(f)$ of measure 0 and is therefore invertible in $\mathrm{L}^0(\mathbb{T}^n)$. This implies that the quotient field $\mathrm{quot}(A) = \mathbb{C}(s)$ of rational functions in s is contained in $\mathrm{L}^0(\mathbb{T}^n)$.
(ii) A matrix $P \in A^{k\times p}$ has a left inverse in $\mathrm{L}^0(\mathbb{T}^n)^{p\times k}$ resp. in $\mathscr{D}(\mathbb{T}^n)^{p\times k}$ if and only if $\mathrm{rank}(P) = p$ resp. $\mathrm{rank}(P(\omega)) = p$ for all $\omega \in \mathbb{T}^n$.

The following facts concerning Fourier series are taken from [Schwartz, 1966, Section VII.1] and Bourbaki [1967]: In this context we need the function spaces

$$\begin{aligned} &\mathscr{S}'(\mathbb{Z}^n) := \left\{ v \in W_A;\; \exists a > 0 \exists k \in \mathbb{N} \text{ with } |v(\mu)| \le a|\mu|^k \right\} \\ &\mathrm{L}^2(\mathbb{Z}^n) := \left\{ v \in W_A;\; \|v\|_2^2 := \sum_{\mu\in\mathbb{Z}^n} |v(\mu)|^2 < \infty \right\} \subset \mathscr{S}'(\mathbb{Z}^n) \end{aligned} \tag{19}$$

where $|\mu| := |\mu_1| + \cdots + |\mu_n|$. These spaces are A-submodules of W_A and $\mathrm{L}^2(\mathbb{Z}^n)$ is a complex Hilbert space with the standard inner product. The space $\mathscr{D}(\mathbb{T}^n)$ is a Fréchet space in which $\lim_{t\to\infty} \varphi_t = \varphi$ if the φ_t and all their derivatives converge to φ, uniformly in $\omega \in \mathbb{T}^n$. The dual space of continuous $\mathbb{C}$-linear functions on $\mathscr{D}(\mathbb{T}^n)$ is the space $\mathscr{D}'(\mathbb{T}^n)$ of distributions on $\mathbb{T}^n$. It is a $\mathscr{D}(\mathbb{T}^n)$-module with the action

$$(\varphi F)(\psi) = F(\varphi\psi),\; \varphi, \psi \in \mathscr{D}(\mathbb{T}^n),\; F \in \mathscr{D}'(\mathbb{T}^n),$$

and contains the $\mathscr{D}(\mathbb{T}^n)$-submodule

$$\begin{aligned} &\mathrm{L}^2(\mathbb{T}^n) := \left\{ f \in \mathrm{L}^0(\mathbb{T}^n);\; \int_{\mathbb{T}^n} |f(\omega)|^2 d\omega < \infty \right\} \\ &\quad \text{with the canonical } \mathscr{D}(\mathbb{T}^n)\text{-linear inclusion} \\ &\mathrm{L}^2(\mathbb{T}^n) \to \mathscr{D}'(\mathbb{T}^n),\; f \mapsto \left(\varphi \mapsto \int_{\mathbb{T}^n} f(\omega)\varphi(\omega)d\omega \right). \end{aligned} \tag{20}$$

Again $\mathrm{L}^2(\mathbb{T}^n)$ is a Hilbert space with the standard inner product. The *Fourier transform on* $\mathbb{Z}^n$ is the A-linear isomorphism

$$\mathscr{F}_{\mathbb{Z}^n}: \mathscr{S}'(\mathbb{Z}^n) \cong \mathscr{D}'(\mathbb{T}^n),\; v \mapsto \widehat{v} := \sum_{\mu\in\mathbb{Z}^n} v(\mu)\omega^{-\mu}. \tag{21}$$

Here $\omega^{-\mu} \in \mathscr{D}(\mathbb{T}^n) \subset \mathscr{D}'(\mathbb{T}^n)$ and the sum converges in the sense of distributions. The inverse of $\mathscr{F}_{\mathbb{Z}^n}$ is *Fourier's cotransform on* $\mathbb{T}^n$, viz.

$$\overline{\mathscr{F}}_{\mathbb{T}^n}: \mathscr{D}'(\mathbb{T}^n) \cong \mathscr{S}'(\mathbb{Z}^n),\; F \mapsto \widehat{F},\; \widehat{F}(\mu) = F(\omega^{\mu}). \tag{22}$$

The Fourier transforms (21) and (22) induce inverse *isometries*

$$\mathscr{F}_{\mathbb{Z}^n}: \mathrm{L}^2(\mathbb{Z}^n) \cong \mathrm{L}^2(\mathbb{T}^n): \overline{\mathscr{F}}_{\mathbb{T}^n}. \tag{23}$$

The Fourier transform and cotransform are extended componentwise to vector signals.

Corollary 4. Assume

$$P \in A^{k\times p},\; y \in \begin{cases} \mathrm{L}^2(\mathbb{Z}^n)^p \\ \mathscr{S}'(\mathbb{Z}^n)^p \end{cases} \text{ and } P \circ y = 0. \text{ If}$$
$$\begin{cases} \mathrm{rank}(P) = p \\ \mathrm{rank}(P(\omega)) = p \text{ for all } \omega \in \mathbb{T}^n \end{cases} \text{ then } y = 0.$$

This follows from Thm. 3, (ii). Recall that

$$\left\{ y \in W_A^p;\; P \circ y = 0 \right\} = 0$$

if and only if P has a left inverse in $A^{p\times k}$.

Theorem 5. Consider an input/output behavior

$$\begin{aligned} &\mathscr{B} := \left\{ \binom{y}{u} \in W_A^{p+m};\; P \circ y = Q \circ u \right\} \\ &(P, -Q) \in A^{k\times(p+m)},\; \mathrm{rank}(P, -Q) = \mathrm{rank}(P) = p, \\ &PH = Q,\; H \in \mathbb{C}(s)^{p\times m}. \end{aligned}$$

Assume that H is continuous, i.e., $H \in \mathrm{C}^0(\mathbb{T}^n)^{p\times m}$. Then there are the isomorphisms

$$\begin{aligned} &\mathscr{B} \bigcap \mathrm{L}^2(\mathbb{Z}^n)^{p+m} \cong \mathrm{L}^2(\mathbb{Z}^n)^m \cong \\ &\left\{ \binom{\widehat{y}}{\widehat{u}} \in \mathrm{L}^2(\mathbb{T}^n)^{p+m};\; P\widehat{y} = Q\widehat{u} \right\} \cong \mathrm{L}^2(\mathbb{T}^n)^m \\ &\binom{y}{u} \leftrightarrow u \leftrightarrow \binom{\widehat{y}}{\widehat{u}} \leftrightarrow \widehat{u},\; \widehat{y} = \mathscr{F}_{\mathbb{Z}^n}(y) = H\widehat{u},\; \widehat{u} = \mathscr{F}_{\mathbb{Z}^n}(u). \end{aligned}$$

If even $\mathrm{rank}(P(\omega)) = p$ for all $\omega \in \mathbb{T}^n$ and thus $H \in \mathscr{D}(\mathbb{T}^n)^{p\times m}$ then there are also the isomorphisms

$$\begin{aligned} &\mathscr{B} \bigcap \mathscr{S}'(\mathbb{Z}^n)^{p+m} \cong \mathscr{S}'(\mathbb{Z}^n)^m \cong \\ &\left\{ \binom{\widehat{y}}{\widehat{u}} \in \mathscr{D}'(\mathbb{T}^n)^{p+m};\; P\widehat{y} = Q\widehat{u} \right\} \cong \mathscr{D}'(\mathbb{T}^n)^m. \end{aligned}$$

Since $\mathrm{L}^2(\mathbb{T}^n)$ resp. $\mathscr{D}'(\mathbb{T}^n)$ are modules over $\mathrm{C}^0(\mathbb{T}^n)$ resp. $\mathscr{D}(\mathbb{T}^n)$ the expression $H\widehat{u}$ is defined in both cases.

The equations $\widehat{P \circ y} = P\widehat{y} = Q\widehat{u} = PH\widehat{u}$ and $\widehat{y} = H\widehat{u}$ are equivalent since P can be canceled according to Thm. 3,(ii), and Cor. 4.

Corollary 6. A W_A-behavior $\mathscr{B} \subseteq W_A^{\ell}$ is autonomous if and only if $\mathscr{B} \bigcap \mathrm{L}^2(\mathbb{Z}^n)^{\ell} = 0$. In contrast, the autonomous behavior $\left\{ v \in \mathbb{C}^{\mathbb{Z}};\; (s_1 - 1) \circ v = 0 \right\} = \mathbb{C}(\cdots, 1, 1, \cdots)$ is contained in $\mathscr{S}'(\mathbb{Z})$.

Corollary 7. Under the assumptions of Thm. 5 let in addition $\ell := p = m$ and consider the W_B-behavior

$$\mathscr{B}_B := \left\{ w \in W_B^\ell = (W_A^\ell)^{\mathbb{N}};\ (Ps_0 - Q)\circ w = 0 \right\} =$$
$$\left\{ w \in W_B^\ell = (W_A^\ell)^{\mathbb{N}};\ \forall t \in \mathbb{N}:\ P\circ w(t+1) = Q\circ w(t) \right\}.$$

Then there are the isomorphisms

$$\mathscr{B}_B \bigcap \left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \cong \mathrm{L}^2(\mathbb{Z}^n)^\ell,$$
$$w \mapsto w(0),\ \widehat{w(t)} = H^t \widehat{w(0)},$$

resp.

$$\mathscr{B}_B \bigcap \left(\mathscr{S}'(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \cong \mathscr{S}'(\mathbb{Z}^n)^\ell,$$
$$w \mapsto w(0),\ \widehat{w(t)} = H^t \widehat{w(0)},$$

4. ASYMPTOTIC STABILITY

The main result of the paper is the following

Theorem 8. Assume that the behavior $\mathscr{B}$ from (4) is Λ_1-stable and time-autonomous (cf. Theorem 1) with

$$\mathrm{proj}: \mathscr{B} \cong \mathscr{B}_d,\ w \mapsto (w(0),\cdots,w(d-1))^\top,$$
$$\mathscr{B}_d \subseteq (W_A^\ell)^d = W_A^{\ell d}.$$

Assume in addition any input/output representation of $\mathscr{B}_d$:

$$\begin{gathered}\mathscr{B}_d = \left\{ \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in W_A^{p+m};\ P\circ y = Q\circ u \right\},\ \ell d = p+m, \\ (P,-Q) \in A^{k\times(p+m)},\ \mathrm{rank}(P) = \mathrm{rank}(P,-Q) = p, \\ PH = Q,\ H \in \mathbb{C}(s)^{p\times m}.\end{gathered} \tag{24}$$

If

$$\begin{gathered} w \in \mathscr{B} \text{ and } w(0),\cdots,w(d-1) \in \begin{cases} \mathrm{L}^2(\mathbb{Z}^n)^\ell \\ \mathscr{S}'(\mathbb{Z}^n)^\ell \end{cases} \\ \text{and if} \begin{cases} H \text{ is continuous} \\ \mathrm{rank}(P(\omega)) = p \text{ for } \omega \in \mathbb{T}^n \end{cases} \text{ then} \\ \forall t \in \mathbb{N}:\ w(t) \in \begin{cases} \mathrm{L}^2(\mathbb{Z}^n)^\ell \\ \mathscr{S}'(\mathbb{Z}^n)^\ell \end{cases} \text{ and} \\ \begin{cases} \lim_{t\to\infty} w(t) = 0, \text{ i.e., } \lim_{t\to\infty} \|w(t)\|_2 = 0. \\ \forall \mu \in \mathbb{Z}^n: \lim_{t\to\infty} w(t,\mu) = 0 \end{cases}. \end{gathered} \tag{25}$$

If without loss of generality already $d = 1$ and $\mathscr{B}_1$ is given by the data in Thm. 1,(iv), then the following canonical isomorphisms hold:

$$\begin{gathered} \mathscr{B} \bigcap \left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \cong \mathscr{B}_1 \bigcap \mathrm{L}^2(\mathbb{Z}^n)^\ell \cong \mathrm{L}^2(\mathbb{Z}^n)^m \text{ resp.} \\ \mathscr{B} \bigcap \left(\mathscr{S}'(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \cong \mathscr{B}_1 \bigcap \mathscr{S}'(\mathbb{Z}^n)^\ell \cong \mathscr{S}'(\mathbb{Z}^n)^m \\ w \leftrightarrow w(0) = \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \leftrightarrow u,\ w(t) = E^t \circ w(0),\ \widehat{y} = H\widehat{u}. \end{gathered} \tag{26}$$

We call these asymptotic stability properties of $\mathscr{B}$ L^2-stability resp. *weak* stability.

Remark 9. The Λ_1-stability and time-autonomy in Thm. 8 are needed for the proof of the asymptotic stability of the trajectories in $\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}$. The input/output condition on $\mathscr{B}_d$ makes sure that for $m > 0$ the intersection $\mathscr{B}_d \bigcap \left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^d$ is nonzero. If, for instance, $\mathscr{B}_d$ is autonomous and thus $m = 0$ this intersection is zero by Cor. 6, hence also $\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} = 0$ by (26) and the asymptotic stability from (25) holds for the zero trajectory only.

Corollary 10. Assume that $\mathscr{B}$ from (4) is Λ_1-stable and time-autonomous with $\mathscr{B} \cong \mathscr{B}_d$. Assume in addition that

(i) (Napp, Rapisarda and Rocha [2011]) $\mathscr{B}$ is square-autonomous, i.e., of the form
$$\mathscr{B} := \left\{ w \in W_B^\ell;\ R\circ w = 0 \right\},\ R \in B^{\ell\times\ell},\ \det(R) \neq 0,$$

(ii) or that $\mathscr{B}_d$ is *strictly controllable*, i.e., $\mathscr{B}_d \cong W_A^m$ for some m.

Then $\mathscr{B}$ is L^2-stable and weakly stable, i.e., both assertions of (25) hold.

Theorem 11. (cf. [Willems, 2007, Section VII]) Under the assumptions of Cor. 7 assume in addition that $\mathscr{B}_B$ is Λ_1-stable. If

$$\begin{gathered} w \in \begin{cases} \mathscr{B}_B \bigcap \left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \text{ and } H \text{ is continuous} \\ \mathscr{B}_B \bigcap \left(\mathscr{S}'(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \text{ and } \mathrm{rank}(P(\omega)) = p \text{ for } \omega \in \mathbb{T}^n \end{cases} \\ \text{then} \begin{cases} \lim_{t\to\infty} w(t) = 0 \in \mathrm{L}^2(\mathbb{Z}^n)^\ell \\ \forall \mu \in \mathbb{Z}^n: \lim_{t\to\infty} w(t,\mu) = 0 \end{cases}. \end{gathered} \tag{27}$$

Conjecture 12. Any Λ_1-stable and time-autonomous behavior is L^2- and weakly stable.

Theorem 13. (cf. [Willems, 2007, Section VII]) Assume the data of Cor. 7 with continuous H.

(i) If the behavior $\mathscr{B}_B$ is L^2-stable, i.e.,

$$\forall w \in \mathscr{B}_B \bigcap \left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}:\ \lim_{t\to\infty} w(t) = 0$$

then there is a subset $\Omega \subset \mathbb{T}^n$ of measure 1 such that

$$\mathrm{char}(\mathscr{B}_B) \bigcap (\mathbb{C}\times\Omega) \bigcap \Lambda_2 = \emptyset.$$

Recall that Λ_1-stability of $\mathscr{B}_B$ signifies that $\mathrm{char}(\mathscr{B}_B)\bigcap\Lambda_2 = \emptyset$.

(ii) For $\ell = 1$ the necessary condition of (i) for L^2-stability is also sufficient, but for $\ell > 1$ it is not.

Example 14. We construct a whole family of examples for Thm. 8 in the case $n = 2$ and $\ell = 2$. Consider nonzero Laurent polynomials $P, Q \in A = \mathbb{C}[s_1, s_1^{-1}, s_2, s_2^{-1}]$ such that $P(\omega) \neq 0$ for $\omega \in \mathbb{T}^2$, hence $P^{-1} \in \mathscr{D}(\mathbb{T}^2)$, for instance $P = s_1 - 2$ and $Q = s_2 - 1$. Define $R_0 := (P, -Q) \in A^{1\times 2}$. The ring A is factorial, so the greatest common divisor $g := \gcd(P,Q) \in A$ of P and Q in A exists. The factoriality of A also implies that

$$\ker\left((P,-Q)\circ : A^2 \to A\right) = A\left(g^{-1}(Q,P)\right)^\top. \tag{28}$$

This enables to determine all matrices $E \in A^{2\times 2}$ as needed in Thm. 1,(iv), viz.

$$\begin{gathered} \left\{ E \in A^{2\times 2};\ \exists a \in A \text{ with } R_0 E = aR_0 \right\} = \\ \left\{ E(a,b,c) := \begin{pmatrix} a+bQg^{-1} & cQg^{-1} \\ bPg^{-1} & a+cPg^{-1} \end{pmatrix};\ a,b,c \in A \right\}. \end{gathered} \tag{29}$$

According to Thm. 1,(iv), any $E = E(a,b,c)$ as in (29) gives rise to the matrix

$$R := R(a,b,c) := \begin{pmatrix} (P,-Q) \\ s_0\,\mathrm{id}_2 - E(a,b,c) \end{pmatrix} \in B^{3\times 2} \tag{30}$$

and the time-autonomous behavior

$$\begin{gathered} \mathscr{B} := \left\{ w \in W_B^2;\ R\circ w = 0 \right\},\ W_A := \mathbb{C}^{\mathbb{Z}^2},\ W_B := W_A^{\mathbb{N}}, \text{ with} \\ \mathscr{B}_1 = \mathrm{im}\left(\mathscr{B} \to W_A^2,\ w \mapsto w(0)\right) = \left\{ \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in W_A^2;\ P\circ y = Q\circ u \right\} \\ \text{and transfer function } H(\omega) = P(\omega)^{-1}Q(\omega) \in \mathscr{D}(\mathbb{T}^2). \end{gathered} \tag{31}$$

The three 2×2-subdeterminants of R are

$$d_1 := \det\begin{pmatrix} P & -Q \\ s_0-a-bQg^{-1} & -cQg^{-1}\end{pmatrix} = Q\det\begin{pmatrix} P & -1 \\ s_0-a-bQg^{-1} & -cg^{-1}\end{pmatrix} = Q\left(s_0-a-(bQ+cP)g^{-1}\right),$$
$$d_2 := \det\begin{pmatrix} P & -Q \\ -bPg^{-1} & s_0-a-cPg^{-1}\end{pmatrix} = P\det\begin{pmatrix} 1 & -Q \\ -bg^{-1} & s_0-a-cPg^{-1}\end{pmatrix} = P\left(s_0-a-(bQ+cP)g^{-1}\right),$$
$$d_3 := \det(s_0\,\mathrm{id}_2 - E). \tag{32}$$

Hence the characteristic variety of $\mathscr{B}$ is

$$\mathrm{char}(\mathscr{B}) = \{(\lambda,\omega)\in\Lambda_N;\ \mathrm{rank}(R(\lambda,\omega))<2\} = \{(\lambda,\omega)\in\Lambda_N;\ \forall i=1,2,3:\ d_i(\lambda,\omega)=0\}. \tag{33}$$

To check the Λ_1-stability condition $\mathrm{char}(\mathscr{B})\bigcap\Lambda_2=\emptyset$ choose $(\lambda,\omega)\in\mathrm{char}(\mathscr{B})\bigcap\left(\mathbb{C}\times\mathbb{T}^2\right)$. The conditions $d_i(\lambda,\omega)=0$ for $i=1,2$ and the assumption $P(\omega)\neq 0$ imply

$$\lambda = a+(bQ+cP)g^{-1} \text{ with } a:=a(\omega), b:=b(\omega),\dots \text{ and}$$
$$d_3(\lambda,\omega) = \det\begin{pmatrix} \lambda-a-bQg^{-1} & -cQg^{-1} \\ -bPg^{-1} & \lambda-a-cPg^{-1}\end{pmatrix} = \det\begin{pmatrix} cPg^{-1} & -cQg^{-1} \\ -bPg^{-1} & bQg^{-1}\end{pmatrix} = bc\det\begin{pmatrix} Pg^{-1} & -Qg^{-1} \\ -Pg^{-1} & Qg^{-1}\end{pmatrix} = 0. \tag{34}$$

We conclude that

$$\mathrm{char}(\mathscr{B})\bigcap\left(\mathbb{C}\times\mathbb{T}^2\right) = \left\{(a(\omega)+b(\omega)(Q/g)(\omega)+c(\omega)(P/g)(\omega),\omega);\ \omega\in\mathbb{T}^2\right\}.$$
$$\text{Hence } \mathrm{char}(\mathscr{B})\bigcap\Lambda_2=\emptyset \iff \rho:=\rho(a,b,c):= \max_{\omega\in\mathbb{T}^n}\left(|a(\omega)+b(\omega)(Q/g)(\omega)+c(\omega)(P/g)(\omega)|\right)<1. \tag{35}$$

Thus the time-autonomous system $\mathscr{B}$ from (31) is Λ_1-stable if and only if $\rho<1$. Since $H(\omega)$ is smooth Thm. 8 furnishes the L^2- and weak stability of $\mathscr{B}$, and moreover the following canonical isomorphisms

$$\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^2)^2\right)^{\mathbb{N}} \cong \mathscr{B}_1\bigcap\mathrm{L}^2(\mathbb{Z}^n)^2\cong\mathrm{L}^2(\mathbb{Z}^n) \text{ resp.}$$
$$\mathscr{B}\bigcap\left(\mathscr{S}'(\mathbb{Z}^2)^2\right)^{\mathbb{N}} \cong \mathscr{B}_1\bigcap\mathscr{S}'(\mathbb{Z}^n)^2\cong\mathscr{S}'(\mathbb{Z}^n) \tag{36}$$
$$w\leftrightarrow w(0)=\begin{pmatrix} y\\ u\end{pmatrix}\leftrightarrow u,\ w(t)=E^t\circ w(0),\ \widehat{y}=H(\omega)\widehat{u},$$
$$H(\omega)=P(\omega)^{-1}Q(\omega).$$

Since $d_3(\lambda,\omega)=\det(\lambda\,\mathrm{id}_2-E(\omega))=0$ the value $\lambda=a+b(Q/g)+c(P/g)$ with $a:=a(\omega)$ etc is one eigenvalue of $E(\omega)$. The second eigenvalue λ' of $E(\omega)$ satisfies

$$\lambda+\lambda' = a+b(Q/g)+c(P/g)+\lambda' = \mathrm{trace}(E(\omega)) = $$
$$= 2a+b(Q/g)+c(P/g), \text{ hence } \lambda'=a, \text{ indeed } d_3(a,\omega)=0. \tag{37}$$

If $\rho(a,b,c)\geq 1$ for the chosen (a,b,c) any $r>\rho(a,b,c)$ gives rise to the vector $r^{-1}(a,b,c)$ with

$$\rho\left(r^{-1}(a,b,c)\right)<1 \text{ and thus to the matrix}$$
$$R:=R(r^{-1}(a,b,c)):=\begin{pmatrix} (P,-Q) \\ s_0\,\mathrm{id}_2-E(r^{-1}(a,b,c))\end{pmatrix} \tag{38}$$

whose associated time-autonomous behavior $\mathscr{B}$ from (31) is Λ_1-, L^2- and weakly stable. A special example is obtained by

$$P:=s_1-2,\ Q:=s_2-1,\ g=\gcd(P,Q)=1,$$
$$r:=5/2<a:=3,\ b:=c:=1,$$
$$\rho:=\max_{\omega\in\mathbb{T}^2}\left(|3+(\omega_2-1)+(\omega_1-2)|\right)=2<r<a,$$
$$E=\frac{2}{5}\begin{pmatrix} s_2+2 & s_2-1 \\ s_1-2 & s_1+1\end{pmatrix},\ R:=\begin{pmatrix} s_1-2 & -(s_2-1) \\ s_0-(2/5)(s_2+2) & -(2/5)(s_2-1) \\ -(2/5)(s_1-2) & s_0-(2/5)(s_1+1)\end{pmatrix}. \tag{39}$$

The spectrum of the corresponding $E(\omega)$ is

$$\mathrm{spec}(E(\omega))=\left\{\lambda,\lambda'\right\} \text{ with}$$
$$\lambda=(2/5)(\omega_2+\omega_1),\ |\lambda|<1,\ \lambda'=a/r=6/5>1.$$

Thus no $E(\omega)$ is asymptotically stable and the time-autonomous behavior

$$\mathscr{B}' := \left\{w\in\left(\left(\mathbb{C}^{\mathbb{Z}^2}\right)^2\right)^{\mathbb{N}};\ (s_0\,\mathrm{id}_2-E)\circ w=0\right\} = \left\{w\in\left(\left(\mathbb{C}^{\mathbb{Z}^2}\right)^2\right)^{\mathbb{N}};\ w(t)=E^t\circ w(0)\right\} \tag{40}$$

is neither Λ_1-, nor L^2-nor weakly stable whereas its time-autonomous subbehavior $\mathscr{B}$ has all these properties.
Assume that $g=\gcd(P,Q)=1$. Then the sequence

$$0\to A\overset{\circ(P,-Q)}{\longrightarrow}A^{1\times 2}\overset{\begin{pmatrix} Q\\ P\end{pmatrix}}{\longrightarrow}A \text{ is exact}, \tag{41}$$

hence the system module $M_1:=A^{1\times 2}/A(P,-Q)$ of the behavior $\mathscr{B}_1$ is contained in A (up to isomorphism). This implies that M_1 is torsionfree and that thus $\mathscr{B}_1$ is controllable.
The A-module M_1 is projective and thus free (Theorem of Quillen-Suslin for Laurent polynomials) if and only if the matrix $(P,-Q)$ has a right inverse or, equivalently, $A=AP+AQ$, i.e., if P and Q are coprime. If M_1 is not free or, equivalently, $\mathscr{B}_1$ is not strictly controllable, for instance for $P=s_1-2$ and $Q=s_2+1$, then Cor. 10,(ii), is not applicable.
The method of Cor. 2 implies that the behavior $\mathscr{B}$ defined by the data in (39) is not square-autonomous and that hence Cor. 10,(i), (cf. Napp, Rapisarda and Rocha [2011]) is not applicable.

Example 15. In the situation of Cor. 7 assume $n=1$, hence $B=\mathbb{C}[s_0,s_1,s_1^{-1}]$ (cf. Napp, Rapisarda and Rocha [2011]).
(i) Let

$$\ell=1,\ p(s_1):=1,\ q(s_1)=\frac{1}{2}(s_1+1)$$
$$\mathscr{B}_B:=\left\{w\in\left(\mathbb{C}^{\mathbb{Z}}\right)^{\mathbb{N}};\ (s_0-q(s_1))\circ w=0\right\} = \left\{w\in\left(\mathbb{C}^{\mathbb{Z}}\right)^{\mathbb{N}};\ w(t)=\left(\frac{1}{2}(s_1+1)\right)^t\circ w(0)\right\}.$$

This $\mathscr{B}_B$ is ta and its characteristic variety is

$$\mathrm{char}(\mathscr{B}_B):=\{(\lambda,\omega)\in\mathbb{C}\times(\mathbb{C}\setminus\{0\});\ \lambda=q(\omega)\},$$
$$\text{hence } (1,1)\in\mathrm{char}(\mathscr{B})\bigcap\Lambda_2.$$

Thus $\mathscr{B}_B$ is not Λ_1-stable, but

$$\mathrm{char}(\mathscr{B}_B)\bigcap\left(\mathbb{C}\times(\mathbb{T}^1\setminus\{1\})\right)\bigcap\Lambda_2=\emptyset.$$

From Thm. 13,(ii), we infer that $\mathscr{B}_B$ is L^2-stable. Hence $\mathscr{B}_B$ is L^2-stable and not Λ_1-stable which represents a slight correction of [Napp, Rapisarda and Rocha, 2011, Thm. 10].
(ii) With the notations from Cor. 7 and (i) choose

$$k=\ell=2,\ P:=\mathrm{id}_2,\ Q:=\begin{pmatrix} q & 1\\ 0 & q\end{pmatrix},\ H:=P^{-1}Q=Q\Longrightarrow$$
$$\mathscr{B}_B:=\left\{w\in\left((\mathbb{C}^{\mathbb{Z}})^2\right)^{\mathbb{N}};\ (\mathrm{id}_2 s_0-Q(s_1))\circ w=0\right\} = \left\{w\in\left((\mathbb{C}^{\mathbb{Z}})^2\right)^{\mathbb{N}};\ w(t)=Q(s_1)^t\circ w(0)\right\},\ Q^t=\begin{pmatrix} q^t & tq^{t-1}\\ 0 & q^t\end{pmatrix}.$$

Again $\mathscr{B}$ is ta and $\mathrm{char}(\mathscr{B}_B)=\{(q(\omega),\omega);\ \omega\neq 0\}$, hence

$$\mathrm{char}(\mathscr{B}_B)\bigcap\left(\mathbb{C}\times(\mathbb{T}^1\setminus\{1\})\right)\bigcap\Lambda_2=\emptyset.$$

So the necessary condition of Thm. 13 for L^2-stability is satisfied. However, a simple calculation shows that the integral

$$\int_{\mathbb{T}^1} \mid \left(Q(\omega)^t\right)_{12} \mid^2 d\omega = \int_0^1 t^2 \left(\frac{1}{2}(1+\cos(2\pi x)\right)^{t-1} dx$$

with $\omega = e^{2\pi i x} \in \mathbb{T}^1$ converges to ∞ for $t \to \infty$. This implies that $\mathscr{B}_B$ is not L^2-stable.

5. CONSTRUCTIVITY

Consider the class $\mathfrak{C}(\Lambda_1)$ of all f.g. B-modules M with $\mathrm{char}(M) = V_{\Lambda_N}(\mathrm{ann}_B(M)) \subseteq \Lambda_1$, the multiplicatively closed set T_{ta} of monic (in s_0) polynomials in $B = A[s_0]$, viz.

$$T_{\mathrm{ta}} := \left\{ f = s_0^d + f_{d-1}s_0^{d-1} + \cdots + f_0 \in B;\ d \geq 0,\ f_i \in A \right\} \tag{42}$$

and the class of f.g. B-modules

$$\begin{aligned} \mathfrak{C}_{\mathrm{ta}} &:= \{{}_BC;\ {}_BC \text{ f.g.}, C_{T_{\mathrm{ta}}} = 0\} = \\ &\{{}_BC;\ {}_BC \text{ f.g.}, \exists t \in T_{\mathrm{ta}} \text{ with } tC = 0\}. \end{aligned} \tag{43}$$

The classes $\mathfrak{C}(\Lambda_1)$ and $\mathfrak{C}_{\mathrm{ta}}$ are *Serre categories*, i.e., are closed under isomorphisms, submodules, factor modules and extensions. A behavior $\mathscr{B} = D(M)$ is Λ_1-stable resp. time-autonomous (ta) if and only if $M \in \mathfrak{C}(\Lambda_1)$ resp. $M \in \mathfrak{C}_{\mathrm{ta}}$. Therefore a behavior $\mathscr{B} := D(M)$ is Λ_1-stable and ta as assumed in Thm. 8 if and only if M belongs to the Serre category $\mathfrak{C} := \mathfrak{C}(\Lambda_1) \bigcap \mathfrak{C}_{\mathrm{ta}}$. Since $\mathbb{T}^n$ and Λ_2 are semi-algebraic, i.e., defined by *real* equalities and inequalities in $\mathbb{C}^n = \mathbb{R}^{2n}$ resp. $\mathbb{C}^{1+n} = \mathbb{R}^{2n+2}$, one can decide constructively [Scheicher and Oberst, 2012, ,§7] whether a f.g. B-module belongs to $\mathfrak{C}(\Lambda_1)$. According to [Oberst and Scheicher, 2012a, Cor. 3.8] the inclusion $M \in \mathfrak{C}_{\mathrm{ta}}$ can also be decided constructively, and the same thus holds for $M \in \mathfrak{C}$. Hence the Λ_1-stability and time-autonomy of the behavior $\mathscr{B}$ in Thm. 8 can be verified constructively. Likewise the conditions $V_{(\mathbb{C}\setminus\{0\})^n}(\mathfrak{a}) \bigcap \mathbb{T}^n = \emptyset$, especially for $\mathfrak{a} = Af$, can be verified. For a matrix $P \in A^{k \times p}$ one can therefore constructively check the condition $\mathrm{rank}(P(\omega)) = p$ for all $\omega \in \mathbb{T}^n$ of Thm. 3 which implies $H = P^{-1}Q \in \mathscr{D}(\mathbb{T}^n)^{p \times m}$, i.e., the smoothness of H in Thm. 8. If f denotes the least common denominator of the entries of $H \in \mathbb{C}(s)^{p \times m}$ and if $V_{(\mathbb{C}\setminus\{0\})^n}(f) \bigcap \mathbb{T}^n = \emptyset$ then too H is smooth, and this can also be constructively verified. We do not yet have a test of the continuity or smoothness of H in general. Cor. 10,(i), especially implies a different constructive version of [Napp, Rapisarda and Rocha, 2011, Thm. 10, Sect. 5].

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