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REGULAR ALMOST INTERCONNECTION OF MULTIDIMENSIONAL BEHAVIORS*

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Abstract. Reading the doctoral thesis of Napp Avelli (2007) I realized that Gabriel's localization theory, which I applied in context with the stabilization of multidimensional input/output behaviors, can also be used for the construction of regular almost interconnections of behaviors in arbitrary dimensions and not only in two dimensions. In this paper I expose this theory in the language of quotient modules and derive an algorithm for arbitrary dimensions which has, however, not yet been implemented. Regular interconnections were introduced and discussed by Willems [*IEEE Trans. Automat. Control*, 36 (1991), pp. 259–294; 42 (1997), pp. 458–472] for one-dimensional behaviors. Their multidimensional counterparts have been treated by Rocha, Wood, Shankar, Zerz, Lomadze, Napp Avelli, and others since 1998. Two-dimensional almost direct sum decompositions and regular almost interconnections have been considered by Valcher and Bisiacco since 2000 and are also the main subject of Napp Avelli's thesis and his recent submitted paper. Roughly, two behaviors are almost equal if they differ by finite-dimensional behaviors only; so the latter are considered negligible in this context. Two-dimensional behaviors have special properties which are not discussed in the present paper. I also briefly discuss other variants of regular almost interconnections where only stable autonomous behaviors are considered negligible.

Key words. regular almost interconnection, multidimensional behavior, Gabriel localization

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1. Introduction.

Regular interconnections were introduced and discussed by Willems in one dimension [23], [24], [16]. Soon thereafter Rocha and Wood [18], Shankar [20], and Zerz and Lomadze [26] extended the notions and partially the results to behaviors in higher dimensions. In context with the two-dimensional controllable-autonomous decomposition, almost direct sum decompositions have been discussed by Valcher and Bisiacco since 2000 [22], [1], [2]. Roughly, two behaviors are almost equal if they differ by finite-dimensional behaviors only; so the latter are considered negligible in this context. For such identifications the theory of quotient categories and modules was developed fifty years ago by Serre, Gabriel et al. That two-dimensional behaviors have various properties which do not hold in higher dimensions can be inferred from [4, Chap. VII, sec. 4, Thms. 2, 4, 5, and 6 and Ex. 3] and was also observed in [11, Thms. 7.42 and 7.74]. The starting point of the present paper was Chapter 5 of Napp Avelli's thesis [9], [10], where, in particular, the basic theory for arbitrary dimensions and an algorithm for two-dimensional regular almost interconnection are presented. Reading this thesis I realized that Gabriel localization of polynomial modules, which I applied in [14] for the stabilization of multidimensional input/output systems, can also be used to sharpen Napp Avelli's general theory and to extend his two-dimensional algorithm [10, Cor. 22] to arbitrary higher dimensions. In the present paper those results, for instance, [10, Thm. 17], which hold in two dimensions only are not discussed.

The regular almost interconnection problem is the following: Let $\mathcal{F}$ be one of the standard, discrete or continuous, multidimensional F-signal spaces over a field

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F and let $\mathcal{B} \subseteq \mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{F}^l$ be behaviors defined as the solution spaces of linear systems of partial difference or differential equations with constant coefficients. Then $\mathcal{B}$ is called a *regular almost interconnection* of $\mathcal{B}_1$ and $\mathcal{B}_2$ if $\mathcal{F}^l = \mathcal{B}_1 + \mathcal{B}_2$ and if the behavior $(\mathcal{B}_1 \bigcap \mathcal{B}_2)/\mathcal{B}$ is finite-dimensional over F [10, Problem 19]. In the *almost* theory the behaviors $\mathcal{B}$ and $\mathcal{B}_1 \bigcap \mathcal{B}_2$ are then identified. The problem is to decide for given $\mathcal{B} \subseteq \mathcal{B}_1$ whether such a $\mathcal{B}_2$ exists and, if so, to construct one such $\mathcal{B}_2$. If in addition $\mathcal{B} = \mathcal{B}_1 \bigcap \mathcal{B}_2$, then $\mathcal{B}$ is called a regular interconnection of $\mathcal{B}_1$ and $\mathcal{B}_2$. The behavior $\mathcal{B}_1$ is interpreted as that of a given plant which by means of the controller $\mathcal{B}_2$ is changed to a desired behavior $\mathcal{B}$.

The main results of this paper are Theorem 3.3 and the associated Algorithm 4.1 for the solution of the regular almost interconnection problem in arbitrary dimensions. The latter makes essential use of Zerz and Lomadze's method [26, sec. 3] to check the existence of regular interconnections. The algorithmic problems in context with the stabilization of transfer matrices [7], [25] or input/output behaviors [14] do not appear here. Theorem 3.1 characterizes more general almost direct sum decompositions of behaviors [10, Problem 15] by means of quotient modules. But in contrast to the two-dimensional case of [10, Thm. 17] the corresponding algorithms in the general multidimensional situation of the present paper are not yet complete. Section 2 presents a survey without proofs of Gabriel's localization theory after [21] and [14, sec. 3] and derives several results which are needed for Theorems 3.1 and 3.3 and Algorithm 4.1. The short last section discusses other forms of regular almost interconnections where only stable autonomous behaviors are considered negligible.

2. Generalized quotient modules. In the first part of this section we introduce the module categories which are relevant for the regular almost interconnection problem and give a survey of section 3 of [14], but refer to [21] and [14] for the details, the used (standard) terminology, and the notation. Corollary 2.1 describes the essential properties of the quotient module $Q(M)$ of a module M. In the second part we prove several results which are essential for the construction of a regular almost interconnection which itself is described in sections 3 and 4.

Let F be a field and $A := F[s] := F[s_1, \dots, s_r]$ the polynomial algebra in $r \geq 2$ indeterminates with its quotient field $K := F(s)$. The category with the A-modules as objects and the A-linear maps as morphisms is denoted by Mod_A. The ring A is a factorial noetherian integral domain as assumed in section 3 of [14]. Let $\mathcal{F}$ be one of the *large injective cogenerator* signal A-modules from [11, main examples in sec. 2 and Thm. 2.54], for instance, the multisequence space $\mathcal{F} = \mathbb{C}^{\mathbb{N}^r}$, respectively, the space $\mathcal{F} = \mathcal{D}'(\mathbb{R}^r, \mathbb{C})$, of distributions in the discrete, respectively, continuous, cases of linear partial difference, respectively, differential, equations with constant coefficients. Its scalar multiplication is denoted by $\circ$. By definition an injective cogenerator is large if every finitely generated A-module M is a submodule of some finite power $\mathcal{F}^l$, up to isomorphism.

As in [14, sec. 3] let $\mathrm{Spec}(A)$, respectively, $\mathrm{Max}(A)$, denote the set of prime, respectively, of maximal, ideals of A. The *associator* or set of associated prime ideals of an A-module M is denoted by $\mathrm{Ass}(M)$. A prime ideal $\mathfrak{p}$ belongs to $\mathrm{Ass}(M)$ if $A/\mathfrak{p}$ is a submodule of M, up to isomorphism. The injective module $\mathcal{F}$ is a cogenerator if and only if it contains all simple modules $A/\mathfrak{m}$, $\mathfrak{m} \in \mathrm{Max}(A)$, up to isomorphism, i.e., if $\mathrm{Max}(A) \subseteq \mathrm{Ass}(\mathcal{F})$. It is a *large* injective cogenerator if and only if all modules $A/\mathfrak{p}$, $\mathfrak{p} \in \mathrm{Spec}(A)$, are contained in $\mathcal{F}$, up to isomorphism, or, in other terms, if $\mathrm{Ass}(\mathcal{F}) = \mathrm{Spec}(A)$ [11, Lem. 2.52]. In the complex continuous case the module

$$\mathcal{F}_{\mathrm{lf}} := \oplus_{\lambda \in \mathbb{C}^r} \mathbb{C}[t] e^{\lambda \bullet t},\ t = (t_1, \dots, t_r) \in \mathbb{R}^r,\ \lambda \bullet t := \lambda_1 t_1 + \cdots + \lambda_r t_r,$$

of *locally finite distributions* or *polynomial-exponential functions* is the unique least injective cogenerator [12, Thm. 6.6] and satisfies

$$\mathrm{Ass}(\mathcal{F}_{\mathrm{lf}}) = \mathrm{Max}(A), \text{ hence no } A/\mathfrak{p},\ \mathfrak{p} \in \mathrm{Spec}(A) \setminus \mathrm{Max}(A),$$

is contained in $\mathcal{F}_{\mathrm{lf}}$ and therefore $\mathcal{F}_{\mathrm{lf}}$ is definitely not large.

As the disjoint decomposition of $\mathrm{Spec}(A) = \mathrm{Ass}(\mathcal{F})$ according to [14, eq. (40)] we choose

$$\mathrm{Spec}(A) = \mathcal{P}_1 \uplus \mathcal{P}_2,\ \mathcal{P}_1 := \mathrm{Max}(A),\ \mathcal{P}_2 := \mathrm{Spec}(A) \setminus \mathrm{Max}(A). \tag{1}$$

Notice that any prime ideal $\mathfrak{p} = Ap$ of height one, where p is an irreducible polynomial, is not maximal in dimension $r \geq 2$ and therefore contained in $\mathcal{P}_2$. Since A is factorial we have

$$A = \bigcap_p \{A_{Ap};\ p \text{ irreducible}\} \subset K = F(s), \text{ hence also } A = \bigcap_{\mathfrak{p} \in \mathcal{P}_2} A_{\mathfrak{p}}. \tag{2}$$

A module M is called *locally finite* if its finitely generated submodules M' or, equivalently, its cyclic submodules Ax have finite F-dimension $[M' : F] := \dim_F(M')$. According to [4, Chap. IV, sec. 2.5, Prop. 7] or [13, Thm. 28] a module M is locally finite if and only if its associator $\mathrm{Ass}(M)$ or, equivalently, its support $\mathrm{supp}(M) := \{\mathfrak{p} \in \mathrm{Spec}(A);\ M_{\mathfrak{p}} \neq 0\}$ is contained in $\mathrm{Max}(A)$. The disjoint decomposition (1) gives rise to a direct sum decomposition

$$\begin{gathered} \mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \text{ with} \\ \mathrm{Ass}(\mathcal{F}_1) = \mathcal{P}_1 = \mathrm{Max}(A),\ \mathrm{Ass}(\mathcal{F}_2) = \mathcal{P}_2 = \mathrm{Spec}(A) \setminus \mathrm{Max}(A). \end{gathered} \tag{3}$$

Of course, both $\mathcal{F}_1$ and $\mathcal{F}_2$ are injective. For general decompositions $\mathrm{Spec}(A) = \mathcal{P}_1 \uplus \mathcal{P}_2$ as in [14, eq. (40)] the associated decomposition (3) is not unique. However, for the special decomposition of (1), $\mathcal{F}_1$ is uniquely determined and coincides with the locally finite part of $\mathcal{F}$. Indeed, according to [12, Thm. 1.14] we have

$$\begin{gathered} \mathcal{F}_1 = \mathcal{F}_{\mathrm{lf}} := \{y \in \mathcal{F};\ [A \circ y : F] < \infty\} = \bigoplus_{\mathfrak{m} \in \mathrm{Max}(A)} \mathcal{F}(\mathfrak{m}), \text{ where} \\ \mathcal{F}(\mathfrak{m}) = \bigcup_{k=0}^{\infty} \mathrm{ann}_{\mathcal{F}}(\mathfrak{m}^k),\ \mathrm{ann}_{\mathcal{F}}(\mathfrak{m}^k) := \{y \in \mathcal{F};\ \mathfrak{m}^k \circ y = 0\}. \end{gathered} \tag{4}$$

The module $\mathcal{F}(\mathfrak{m})$ is the indecomposable injective envelope of $A/\mathfrak{m}$ and even the least injective cogenerator over the local ring $A_{\mathfrak{m}}$. The injective module $\mathcal{F}_{\mathrm{lf}} = \mathcal{F}_1$ is the least injective cogenerator over $A = F[s]$. For many standard signal spaces $\mathcal{F}$ like $\mathbb{C}^{\mathbb{N}^r}$ or $\mathcal{D}'(\mathbb{R}^r, \mathbb{C})$ it coincides with the polynomial-exponential functions or sequences as derived in [12, Thms. 1.25, 5.26, 6.6, and 6.10]. While the direct complement $\mathcal{F}_2$ of $\mathcal{F}_{\mathrm{lf}}$ is not unique, the direct decomposition (3) implies the isomorphism of injective modules

$$\mathcal{F}_2 \cong \mathcal{F}/\mathcal{F}_{\mathrm{lf}}, \tag{5}$$

and therefore $\mathcal{F}_2$ is unique up to isomorphism. Notice, however, that in general no *constructive* description of a special $\mathcal{F}_2$ is available.

According to [14, eq. (50)] the decomposition (1) and the associated module $\mathcal{F}_2 \cong \mathcal{F}/\mathcal{F}_{\mathrm{lf}}$ give rise to a localization theory introduced by Gabriel [5]. More specifically we obtain the full *localizing or Serre subcategory* or *hereditary torsion class* $\mathfrak{C}$ of Mod_A and a *Gabriel topology* $\mathfrak{T}$ [21, Thm. VI.5.1] where

$$\begin{aligned}\mathfrak{C} &:= \{C \in \mathrm{Mod}_A;\ \mathrm{Hom}_A(C, \mathcal{F}_2) = 0\}\\ &= \{C \in \mathrm{Mod}_A;\ \forall \mathfrak{p} \in \mathcal{P}_2 :\ C_{\mathfrak{p}} = 0\},\\ \mathfrak{T} &:= \{\mathfrak{a} \subseteq A;\ A/\mathfrak{a} \in \mathfrak{C}\}\,.\end{aligned} \tag{6}$$

The modules in $\mathfrak{C}$ are called $\mathfrak{T}$-*torsion modules*. The second equality in (6) implies that

$$\mathfrak{C} := \{C;\ \mathrm{supp}(C) \subseteq \mathcal{P}_1 = \mathrm{Max}(A)\}. \tag{7}$$

This signifies that the $\mathfrak{T}$-torsion modules are exactly the locally finite ones. Either directly or by means of the injectivity of $\mathcal{F}_2$ one sees that the class $\mathfrak{C}$ is closed under taking submodules, factor modules, extensions, and direct sums, in particular

$$\mathfrak{C} = \{C \in \mathrm{Mod}_A;\ \forall x \in C :\ \mathrm{ann}_A(x) := \{a \in A; ax = 0\} \in \mathfrak{T}\}\,. \tag{8}$$

The largest submodule of M in $\mathfrak{C}$ or largest locally finite submodule M_{lf} of M is also called the $\mathfrak{T}$-*torsion radical* of M and denoted by $\mathrm{tor}_{\mathfrak{T}}(M) := M_{\mathrm{lf}}$. If $\mathrm{tor}_{\mathfrak{T}}(M) = 0$, the module is called $\mathfrak{T}$-*torsion free*. Due to (8) a $\mathfrak{T}$-torsion module is a torsion module in the usual sense, and hence a torsion free module is $\mathfrak{T}$-torsion free. The Chinese remainder theorem or [4, Chap. IV, sec. 2.5, Prop. 8] implies

$$M_{\mathrm{lf}} = \mathrm{tor}_{\mathfrak{T}}(M) = \oplus_{\mathfrak{m}\in \mathrm{Ass}(M)} M(\mathfrak{m}) \text{ with } M(\mathfrak{m}) := \bigcup_{k=0}^{\infty} \mathrm{ann}_M(\mathfrak{m}^k). \tag{9}$$

If ${}_AM$ is finitely generated, then $\mathrm{Ass}(M)$ is finite and the increasing sequence of annihilators $\mathrm{ann}_M(\mathfrak{m}^k)$ becomes stationary. If $e(\mathfrak{m})$ is the least index k such that $\mathrm{ann}_M(\mathfrak{m}^k) = \mathrm{ann}_M(\mathfrak{m}^{k+1})$, we obtain

$$\begin{gathered}M_{\mathrm{lf}} = \mathrm{tor}_{\mathfrak{T}}(M) = \oplus_{\mathfrak{m}\in \mathrm{Ass}(M)} M(\mathfrak{m}) \text{ and}\\ 0 = \mathrm{ann}_M(\mathfrak{m}^0) \subsetneq \mathrm{ann}_M(\mathfrak{m}^1) \subsetneq \cdots \subsetneq \mathrm{ann}_M(\mathfrak{m}^{e(\mathfrak{m})}) = M(\mathfrak{m}),\ e(\mathfrak{m}) > 0.\end{gathered} \tag{10}$$

An A-module N is called $\mathfrak{T}$-*closed* if for every ideal $\mathfrak{a} \in \mathfrak{T}$ the canonical map

$$\mathrm{Hom}(\mathrm{inj}, N) : N = \mathrm{Hom}_A(A, N) \to \mathrm{Hom}_A(\mathfrak{a}, N),\ x \mapsto (a \mapsto ax) \tag{11}$$

is an isomorphism or, in other words, if for every linear map $f : \mathfrak{a} \to N$ there is a unique $x \in N$ with $f(a) = ax$ for all $a \in \mathfrak{a}$. The full additive subcategory of Mod_A of all $\mathfrak{T}$-closed submodules is denoted by $\mathrm{Mod}_{A,\mathfrak{T}}$. Its properties are described in [21, Chap. X, sec. 1]. It is obviously closed under taking arbitrary inverse limits or, equivalently, under products and kernels and therefore the kernels, products, and limits in $\mathrm{Mod}_{A,\mathfrak{T}}$ coincide with the standard ones in Mod_A. The category $\mathrm{Mod}_{A,\mathfrak{T}}$ also admits arbitrary colimits or, equivalently, coproducts or direct sums and cokernels and is indeed abelian. According to [21, pp. 195–200, 213–216] the inclusion functor $\mathrm{inj} : \mathrm{Mod}_{A,\mathfrak{T}} \subset \mathrm{Mod}_A$ has an exact left adjoint *quotient module or localization* functor

$$Q : \mathrm{Mod}_A \to \mathrm{Mod}_{A,\mathfrak{T}},\ M \mapsto Q(M), \tag{12}$$

with the functorial adjunction morphism

$$\eta_M : M \to Q(M), \text{ i.e., } \mathrm{Hom}_A(Q(M), N) \cong \mathrm{Hom}_A(M, N), \; g \mapsto g\eta_M, \tag{13}$$

for $M \in \mathrm{Mod}_A$ and $N \in \mathrm{Mod}_{A,\mathfrak{T}}$. In [14, sec. 3] we used the notation $M_{\mathfrak{T}} := Q(M)$. The functor Q is unique up to functorial isomorphism. Concrete representations of $Q(M)$ in various cases will be given below.

The inclusion functor $\mathrm{inj} : \mathrm{Mod}_{A,\mathfrak{T}} \to \mathrm{Mod}_A$ preserves all limits as a right adjoint functor and is left exact in particular, but not right exact. If N_1 and N_2 are $\mathfrak{T}$-closed and $g : N_1 \to N_2$ is A-linear, the cokernel of g in the category $\mathrm{Mod}_{A,\mathfrak{T}}$ is given as

$$\mathrm{cok}_{\mathfrak{T}}(g : N_1 \to N_2) = Q\left(N_2/g(N_1)\right). \tag{14}$$

The kernel of the canonical map $\eta_M : M \to Q(M)$ is [21, Chap. IX, Lem. 1.2]

$$M_{\mathrm{lf}} = \mathrm{tor}_{\mathfrak{T}}(M) = \ker(\eta_M : M \to Q(M)). \tag{15}$$

Hence η_M is a monomorphism and then $M \subset Q(M)$ by identification if and only if M is $\mathfrak{T}$-torsionfree. The adjointness of Q and the properties of $\mathfrak{C}$ directly imply

$$\begin{gathered} Q(N) = N \text{ for all } N \in \mathrm{Mod}_{A,\mathfrak{T}} \text{ and} \\ Q(N) = 0 \iff N \in \mathfrak{C} \iff N \text{ is locally finite, hence} \\ Q(M) = Q(M/\mathrm{tor}_{\mathfrak{T}}(M)) \text{ for all } M \in \mathrm{Mod}_A. \end{gathered} \tag{16}$$

In particular, a $\mathfrak{T}$-closed module is $\mathfrak{T}$-torsionfree.

An arbitrary morphism $f : M_1 \to M_2$ in Mod_A gives rise to the exact sequences

$$\begin{gathered} 0 \to \ker(f) \xrightarrow{\mathrm{inj}} M_1 \xrightarrow{f} M_2 \xrightarrow{\mathrm{can}} \mathrm{cok}(f) = M_2/f(M_1) \to 0 \text{ in } \mathrm{Mod}_A \text{ and} \\ 0 \to Q(\ker(f)) \xrightarrow{\mathrm{inj}} Q(M_1) \xrightarrow{Q(f)} Q(M_2) \xrightarrow{Q(\mathrm{can})} Q(\mathrm{cok}(f)) \to 0 \text{ in } \mathrm{Mod}_{A,\mathfrak{T}}. \end{gathered} \tag{17}$$

Hence

$$Q(f) \text{ is } \begin{cases} \text{zero} \\ \text{a monomorphism} \\ \text{an epimorphism} \\ \text{an isomorphism} \end{cases} \iff \begin{cases} \mathrm{im}(f) \in \mathfrak{C} \\ \ker(f) \in \mathfrak{C} \\ \mathrm{cok}(f) \in \mathfrak{C} \\ \ker(f),\ \mathrm{cok}(f) \in \mathfrak{C}. \end{cases} \tag{18}$$

If $\mathrm{im}(f) \in \mathfrak{C}$, the map f is called $\mathfrak{T}$-*zero* or *almost zero* or *zero modulo* $\mathfrak{C}$. The *almost* terminology is due to Napp Avelli [9], [10]. Analogously we define $\mathfrak{T}$- or almost monomorphisms, epimorphisms, and isomorphisms. In dimension $r = 2$ these maps coincide with the *pseudozero, pseudoinjective maps*, etc., from [4, Chap. VII, sec. 4.4, Def. 3]. In particular,

$$\eta_M : M \to Q(M) \text{ with } Q(\eta_M) = \mathrm{id}_{Q(M)} : Q(M) \to Q(Q(M)) = Q(M) \tag{19}$$

is a $\mathfrak{T}$-isomorphism.

COROLLARY 2.1. *The quotient functor Q is characterized by the the following equivalent properties, i.e., every additive functor $Q_1 : \mathrm{Mod}_A \to \mathrm{Mod}_{A,\mathfrak{T}}$ with these properties coincides with Q up to a functorial isomorphism:*

1. *The adjointness relation* (13) *holds for Q_1.*

2. $Q_1 : \mathrm{Mod}_A \to \mathrm{Mod}_{A,\mathfrak{T}}$ *is exact and* (16) *holds for* Q_1.
3. $Q_1(N) = N$ *for each* $\mathfrak{T}$*-closed module* N, *and* $Q_1(f)$ *is an isomorphism if* f *is a* $\mathfrak{T}$*-isomorphism.*

In particular, a module M *is annihilated by* Q, *i.e.,* $Q(M) = 0$, *if and only if it is locally finite. Thus application of* Q *signifies to ignore locally finite and especially finite-dimensional* A*-modules. Most of the subsequent derivations except* (20) *and* (21) *use the preceding properties only.*

According to [14, Lems. 3.2 and 3.4] and (2) the quotient module $Q(A)$ of A is

$$Q(A) = \bigcap_{\mathfrak{p}\in\mathcal{P}_2} A_{\mathfrak{p}} = A \subset K = F(s) \tag{20}$$

and thus coincides with A. If, more generally, U is a finitely generated torsionfree module and thus a submodule of some $A^{1\times m}$, the quotient $Q(U)$ is given [14, Lem. 3.6] as

$$U \subseteq Q(U) = \bigcap_{\mathfrak{p}\in\mathcal{P}_2} U_{\mathfrak{p}} \subseteq Q(A^{1\times m}) = A^{1\times m}. \tag{21}$$

In particular, $Q(U)$ is itself a finitely generated A-module. The canonical Gel'fand map

$$U \to U^{**} := \mathrm{Hom}_A(\mathrm{Hom}_A(U, A), A),\ u \mapsto (\alpha \mapsto \alpha(u)),$$

is injective and U^{**} is naturally identified with its bidual lattice in $A^{1\times m}$ [4, Chap. VII, p. 517]. Then [4, Chap. VII, sec. 4.2, Thm. 2]

$$U \subseteq Q(U) = \bigcap_{\mathfrak{p}\in\mathcal{P}_2} U_{\mathfrak{p}} \subseteq \bigcap_{p \text{ irreducible}} U_{Ap} = U^{**} \subseteq A^{1\times m}. \tag{22}$$

Therefore, if U is reflexive, i.e., if the Gel'fand map is an isomorphism or $U = U^{**}$, we get that $Q(U) = U$ and U is $\mathfrak{T}$-closed. In particular, the dual module $M^* = (M/\operatorname{tor}(M))^*$ and the bidual module M^{**} are reflexive and thus $\mathfrak{T}$-closed for any finitely generated A-module M. If M is a finitely generated torsion module, the quotient $Q(M) = Q(M/M_{lf})$ is not finitely generated in general, and this creates problems for explicit computations.

Remark 2.2. In dimension $r = 2$ we have

$$\mathcal{P}_2 = \mathrm{Spec}(A) \setminus \mathrm{Max}(A) = \{Ap;\ p \text{ irreducible}\} \cup \{0\},$$

and therefore $Q(U) = U^{**}$ for a finitely generated torsionfree U. From the general theory we conclude that $U^{**}/U = Q(U)/U$ is finite-dimensional. The module $U^+ := U^{**}$ is free and plays an important part in Napp Avelli's two-dimensional theory [10, p. 7, Thm. 12, Lem. 21].

The module $\mathcal{F}_2 \cong \mathcal{F}/\mathcal{F}_{\mathrm{lf}}$ is an injective cogenerator of $\mathrm{Mod}_{A,\mathfrak{T}}$ [21, Chap. X, Prop. 1.9].

Lemma 2.3 (cf. [21, Chap. IX, Prop. 2.1]). *Let* N *be a* $\mathfrak{T}$*-closed module and* $f : M_1 \to M_2$ *a* $\mathfrak{T}$*-isomorphism, i.e., a linear map with* $\ker(f)$, $\operatorname{cok}(f) \in \mathfrak{C}$.

1. *The canonical map*

$$\mathrm{Hom}(f, N) : \mathrm{Hom}_A(M_2, N) \to \mathrm{Hom}_A(M_1, N),\ g_2 \mapsto g_1 := g_2 f,$$

is an isomorphism or, equivalently, any linear map $g_1 : M_1 \to N$ *can be uniquely extended to* $g_2 : M_2 \to N$ *with* $g_2 f = g_1$.

2. *If $M_1 \subseteq M_2$, $M_2/M_1 \in \mathfrak{C}$, and $\mathrm{tor}_{\mathfrak{T}}(M_2) = 0$, the inclusion $M_1 \subseteq M_2$ is essential; i.e., for each nonzero submodule U of M_2 also $U \bigcap M_1$ is nonzero. In particular, the inclusion $M_2 \subseteq Q(M_2)$ is essential.*

Proof.

1. The adjointness isomorphism $\mathrm{Hom}_A(Q(M), N) \cong \mathrm{Hom}_A(M, N)$ implies

$$\mathrm{Hom}_A(M_2, N) \cong \mathrm{Hom}_A(Q(M_2), N) \overset{\mathrm{Hom}(Q(f),N)}{\cong},$$
$$\mathrm{Hom}_A(Q(M_1), N) \cong \mathrm{Hom}_A(M_1, N).$$

2. Let $0 \neq x \in U$. Since $\overline{x} \in M_2/M_1 \in \mathfrak{C}$ the annihilator $\mathfrak{a} := \mathrm{ann}_A(\overline{x})$ belongs to $\mathfrak{T}$ and $\mathfrak{a}x \subseteq M_1$. The condition

$$\mathrm{tor}_{\mathfrak{T}}(M_2) = 0 \quad \text{implies} \quad 0 \neq \mathfrak{a}x \subseteq U \bigcap M_1$$

and thus the assertion. □

THEOREM 2.4. *If M is a submodule of the $\mathfrak{T}$-closed module N, then*

$$M \subseteq Q(M) \subseteq N \text{ and } Q(M)/M = \mathrm{tor}_{\mathfrak{T}}(N/M).$$

In other words, $Q(M)$ is the largest submodule V with $M \subseteq V \subseteq N$ and locally finite V/M, and hence

$$Q(M) = \{y \in N;\ \exists \mathfrak{a} \in \mathfrak{T} \text{ with } \mathfrak{a}y \subseteq M\}$$
$$= \{y \in N;\ \dim_F((Ay + M)/M) < \infty\}.$$

Notice here that Q is unique up to a functorial isomorphism only. But as submodule of N the quotient module $Q(M)$ is uniquely determined as the largest submodule with locally finite factor module.

Proof.

1. Since N is $\mathfrak{T}$-closed its $\mathfrak{T}$-torsion submodule is zero, and the same holds for M. We infer $M \subseteq Q(M)$ and $Q(M)/M \in \mathfrak{C}$. By item 1 of Lemma 2.3 the injection $\mathrm{inj} : M \to N$ has a unique extension

$$g : Q(M) \to N \text{ with } \ker(g) \bigcap M = \ker(\mathrm{inj}) = 0, \text{ hence } \ker(g) = 0$$

according to item 2 of the lemma. Thus g is a monomorphism and

$$Q(M) \cong g(Q(M)) \text{ and } M = g(M) \subseteq g(Q(M)) \subseteq N,\ g(Q(M))/M \in \mathfrak{C}.$$

Without loss of generality we therefore assume

$$M \subseteq Q(M) \subseteq N \text{ and } Q(M)/M \subseteq \mathrm{tor}_{\mathfrak{T}}(N/M).$$

2. Let $U \supseteq M$ be the unique submodule of N with $U/M = \mathrm{tor}_{\mathfrak{T}}(N/M)$, and hence $Q(M) \subseteq U$. Since U/M belongs to $\mathfrak{C}$ so does $U/Q(M)$. Since $Q(M)$ is $\mathfrak{T}$-closed the identity map $\mathrm{id}_{Q(M)}$ has a unique extension $h : U \to Q(M)$, again by Lemma 2.3, item 1, and then

$$U = Q(M) \oplus \ker(h) \text{ and } \ker(h) \bigcap Q(M) = 0.$$

Since the inclusion $Q(M) \subseteq U$ is essential according to item 2 of the lemma we infer $\ker(h) = 0$ and $U = Q(M)$ as asserted. □

THEOREM 2.5. *Let M be a finitely generated A-module with*

$$M_{\mathrm{lf}} = \mathrm{tor}_{\mathfrak{T}}(M) = 0 \text{ or } \mathrm{Ass}(M) \subseteq \mathcal{P}_2 = \mathrm{Spec}(A) \setminus \mathrm{Max}(A).$$

1. *The set $S := \bigcap_{\mathfrak{p}\in\mathrm{Ass}(M)}(A\setminus\mathfrak{p}) \subset A$ is multiplicatively closed and the quotient ring*

$$A_S = \bigcap_{\mathfrak{p}\in\mathrm{Ass}(M)} A_{\mathfrak{p}} \subseteq K = F(s) \text{ is semilocal with}$$
$$\mathrm{Max}(A_S) = \{\mathfrak{p}_S;\ \mathfrak{p} \text{ is maximal in } \mathrm{Ass}(M)\}.$$

2. *The canonical map $M \to M_S$ is injective.*
3. *The module M_S is $\mathfrak{T}$-closed as A-module.*

By Theorem 2.4 *this implies*

$$M \subseteq Q(M) \subseteq M_S,\ Q(M)/M = \mathrm{tor}_{\mathfrak{T}}(M_S/M) \text{ and}$$
$$Q(M) = \{y \in M_S;\ \dim_F((Ay+M)/M) < \infty\}. \tag{23}$$

Proof.
1. The first assertion follows from [4, Chap. II, sec. 3.5, Prop. 17].
2. Corollary 2 from [4, Chap. IV, sec. 1.2] implies

$$S := \bigcap_{\mathfrak{p}\in\mathrm{Ass}(M)} (A\setminus\mathfrak{p}) = \{s \in A;\ s\circ : M \to M \text{ is injective}\} \text{ and thus also}$$
$$\ker\left(\mathrm{can}: M \to M_S,\ x \mapsto \frac{x}{1}\right) = \{x \in M;\ \exists s \in S \text{ with } sx = 0\} = 0.$$

Therefore we can and do identify $M \subseteq M_S,\ x = \frac{x}{1}$.
3. Let $\mathfrak{a} \in \mathfrak{T}$ and thus

$$\mathrm{supp}(A/\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(A);\ \mathfrak{a} \subseteq \mathfrak{p}\} \subseteq \mathrm{Max}(A).$$

We show that $\mathfrak{a} \cap S \neq \emptyset$ and therefore $\mathfrak{a}_S = A_S$. If

$$\mathfrak{a} \cap S = \emptyset, \text{ then } \mathfrak{a} \subseteq \bigcup_{\mathfrak{p}\in\mathrm{Ass}(M)} \mathfrak{p} \text{ and } \mathfrak{a} \subseteq \mathfrak{p}_0$$

for some $\mathfrak{p}_0 \in \mathrm{Ass}(M)$, the last implication following from [4, Chap. II, sec. 1.2, Prop. 2]. Since $\mathfrak{a} \in \mathfrak{T}$ this prime ideal $\mathfrak{p}_0$ is maximal, and since $\mathrm{Ass}(M) \subset \mathrm{Spec}(A) \setminus \mathrm{Max}(A)$ it is not. This is a contradiction and therefore $\mathfrak{a}_S = A_S$. But then

$$M_S \cong \mathrm{Hom}_{A_S}(A_S, M_S) = \mathrm{Hom}_{A_S}(\mathfrak{a}_S, M_S) \cong \mathrm{Hom}_A(\mathfrak{a}, M_S) \text{ for all } \mathfrak{a} \in \mathfrak{T}.$$

This signifies that M_S is $\mathfrak{T}$-closed. □

3. Regular almost interconnections. The assumptions are the same as in the preceding section. The next theorem characterizes the almost direct sum decomposition of modules and behaviors [10, Problem 15, Thm. 17] by means of quotient modules, but does *not* contain the two-dimensional constructive part of [10, Thm. 17].

THEOREM 3.1. *For a finitely generated A-module M and a submodule M_1 the following assertions are equivalent:*

1. *There is a submodule* M_2 *of* M *such that the canonical map*

$$+ : M_1 \times M_2 \xrightarrow{+} M,\ (x_1, x_2) \mapsto x_1 + x_2,\ \textit{ is a } \mathfrak{T}\textit{-isomorphism,}$$

i.e., has F*-finite dimensional kernel, isomorphic to* $M_1 \bigcap M_2$*, and cokernel* $M/(M_1 + M_2)$.

2. $Q(M_1)$ *is a direct summand of* $Q(M) = Q(M/\operatorname{tor}_{\mathfrak{T}}(M))$.

If these conditions are satisfied and if M *is* $\mathfrak{T}$*-torsionfree, i.e.,* $M_{\mathrm{lf}} = \operatorname{tor}_{\mathfrak{T}}(M) = 0$*, then the map* $+$ *is injective, i.e.,* $M_1 \bigcap M_2 = 0$ *and* $M_1 + M_2 = M_1 \oplus M_2$.

Proof. $1 \implies 2$: Recall that Q is exact and that thus $Q(M_1)$ is a submodule of $Q(M)$. The $\mathfrak{T}$-isomorphism implies the isomorphism

$$+ = Q(+) : Q(M_1 \times M_2) = Q(M_1) \times Q(M_2) \cong Q(M), \text{ and hence}$$
$$Q(M_1) \oplus Q(M_2) = Q(M).$$

$2 \implies 1$: Assume $Q(M_1) \oplus V = Q(M)$. As a direct summand of a $\mathfrak{T}$-closed module V is also $\mathfrak{T}$-closed. Let $g := \eta_M : M \to Q(M)$ be the universal map which is a $\mathfrak{T}$-isomorphism and hence has locally finite kernel and cokernel. Define $M_2 := g^{-1}(V) \subseteq M$ and the restriction $g|M_2 : M_2 \to V$. With g also its restriction has locally finite kernel and cokernel, is therefore a $\mathfrak{T}$-isomorphism too, and induces the isomorphism $Q(g|M_2) : Q(M_2) \cong Q(V) = V$. Summing up we obtain commutative diagrams

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{+} & M \\ \downarrow \eta_{M_1} \times g|M_2 & & \downarrow g \\ Q(M_1) \times V & \xrightarrow{+} & Q(M) \end{array} \quad \text{and} \quad \begin{array}{ccc} Q(M_1) \times Q(M_2) & \xrightarrow{+} & Q(M) \\ \downarrow Q(\eta_{M_1}) \times Q(g|M_2) & & \downarrow Q(g) \\ Q(M_1) \times V & \xrightarrow{+} & Q(M) \end{array},$$

where the right diagram is obtained from the left one by application of Q, where $Q(N) = N$ for each $\mathfrak{T}$-closed module, and where the vertical maps and the lower horizontal map in the right diagram are bijective by construction, respectively, due to $Q(M_1) \oplus V = Q(M)$. Therefore all maps in the right diagram are isomorphisms and hence all morphisms in the left diagram and especially $+ : M_1 \times M_2 \to M$ are $\mathfrak{T}$-isomorphisms as asserted.

By construction $M_1 \bigcap M_2 \cong \ker(+)$ is F-finite-dimensional and thus a $\mathfrak{T}$-torsion submodule of M whose largest $\mathfrak{T}$-torsion submodule is $\operatorname{tor}_{\mathfrak{T}}(M)$ and zero by assumption. This implies $M_1 \bigcap M_2 = 0$. ◻

The behavioral interpretation of the preceding theorem is the following: Let

$$M = A^{1\times l}/U \supset M_i = U_i/U,\ i = 1, 2, \text{ and hence}$$

$$\ker(M_1 \times M_2 \xrightarrow{+} M) \cong M_1 \bigcap M_2 = \left(U_1 \bigcap U_2\right)/U, \tag{24}$$

$$\operatorname{cok}(M_1 \times M_2 \xrightarrow{+} M) = M/(M_1 + M_2) \cong A^{1\times l}/(U_1 + U_2).$$

The modules in the last two rows are F-finite-dimensional by Theorem 3.1. Let $D := \operatorname{Hom}_A(-, \mathcal{F})$ denote the duality functor. For the standard injective cogenerators from [11] the module M is finite-dimensional if and only if $D(M)$ has this property [13, Thm. 17], and then

$$\dim_F(M) = \dim_F(D(M)). \tag{25}$$

The preceding modules give rise to the behaviors [11, Cor. 2.48]

$$
\begin{aligned}
&\mathcal{B} := U^\perp := \{w \in \mathcal{F}^l;\ U \circ w = 0\} \cong D(M) = \mathrm{Hom}_A(A^{1\times l}/U, \mathcal{F}) \text{ with}\\
&\qquad U = \mathcal{B}^\perp := \{\xi \in A^{1\times l};\ \xi \circ \mathcal{B} = 0\},\\
&\mathcal{B}_i := U_i^\perp \subseteq \mathcal{B} = U^\perp,\ \mathcal{B}_1 \bigcap \mathcal{B}_2 = (U_1+U_2)^\perp,\ \mathcal{B}_1+\mathcal{B}_2 = \left(U_1 \bigcap U_2\right)^\perp,\\
&\mathcal{B}/(\mathcal{B}_1+\mathcal{B}_2) \cong D\left(\left(U_1\bigcap U_2\right)/U\right),\ \mathcal{B}_1\bigcap\mathcal{B}_2 \cong D\left(A^{1\times l}/(U_1+U_2)\right).
\end{aligned} \tag{26}
$$

Comparison of (24) and (26) by means of (25) furnishes the following.

Corollary and Definition 3.2 ($\mathfrak{T}$- or almost direct sum decomposition). *For behaviors $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B} \subseteq \mathcal{F}^l$ the behaviors $\mathcal{B}_1 \bigcap \mathcal{B}_2$ and $\mathcal{B}/(\mathcal{B}_1+\mathcal{B}_2)$ are finite-dimensional if and only if this holds for the modules*

$$
\begin{aligned}
&A^{1\times l}/\left(\mathcal{B}_1^\perp + \mathcal{B}_2^\perp\right) \text{ and } \left(\mathcal{B}_1^\perp \bigcap \mathcal{B}_2^\perp\right)/\mathcal{B}^\perp, \text{ and then}\\
&\dim_F\left(\mathcal{B}_1\bigcap\mathcal{B}_2\right) = \dim_F\left(A^{1\times l}/\left(\mathcal{B}_1^\perp+\mathcal{B}_2^\perp\right)\right) \text{ and}\\
&\dim_F\left(\mathcal{B}/\left(\mathcal{B}_1+\mathcal{B}_2\right)\right) = \dim_F\left(\left(\mathcal{B}_1^\perp\bigcap\mathcal{B}_2^\perp\right)/\mathcal{B}^\perp\right).
\end{aligned}
$$

Under these equivalent conditions $\mathcal{B}$ is called the $\mathfrak{T}$- or almost direct sum *of the two subbehaviors $\mathcal{B}_1$ and $\mathcal{B}_2$* [10, Problem 15]. *For $r = 2$ Theorem* 17 *of* [10] *characterizes these decompositions constructively.*

If $A^{1\times l}/\mathcal{B}^\perp$ is $\mathfrak{T}$-torsionfree, i.e., if $\mathcal{B}$ has no finite-dimensional factor behavior, then the equality $\mathcal{B} = \mathcal{B}_1+\mathcal{B}_2$ holds. The paper [2] *discusses almost direct sum decompositions for $r = 2$ with this additional property $\mathcal{B}_1+\mathcal{B}_2 = \mathcal{B}$.*

For the regular almost interconnection problem we assume that

$$
\begin{aligned}
&\mathcal{B} \subseteq \mathcal{B}_1,\ \mathcal{B}_2 \subseteq \mathcal{F}^l,\ \mathcal{B} := U^\perp,\ \mathcal{B}_i := U_i^\perp, \text{ and hence } U = \mathcal{B}^\perp,\ U_i = \mathcal{B}_i^\perp,\\
&\mathcal{B}_1\bigcap\mathcal{B}_2 = (U_1+U_2)^\perp,\ \left(\mathcal{B}_1\bigcap\mathcal{B}_2\right)/\mathcal{B} \cong D\left(U/(U_1+U_2)\right),\\
&\mathcal{B}_1+\mathcal{B}_2 = \left(U_1\bigcap U_2\right)^\perp \cong D\left(A^{1\times l}/\left(U_1\bigcap U_2\right)\right) \subseteq \mathcal{F}^l = D(A^{1\times l}).
\end{aligned} \tag{27}
$$

Theorem 3.3 (regular $\mathfrak{T}$- or almost interconnection). *For subbehaviors $\mathcal{B} \subseteq \mathcal{B}_1 \subseteq \mathcal{F}^l$ and the just introduced notations, the following assertions are equivalent:*

1. *There is a submodule U_2 of U such that the canonical map $U_1 \times U_2 \xrightarrow{+} U$ is a $\mathfrak{T}$-isomorphism, and then $U_1 \bigcap U_2 = 0$.*
2. *$Q(U_1)$ is a direct summand of $Q(U) \subseteq A^{1\times l}$.*
3. *There is a subbehavior $\mathcal{B}_2$ of $\mathcal{F}^l$ which also contains $\mathcal{B}$ such that $\mathcal{B}_1+\mathcal{B}_2 = \mathcal{F}^l$ and $(\mathcal{B}_1\bigcap\mathcal{B}_2)/\mathcal{B}$ is finite-dimensional.*
4. *There is a subbehavior $\mathcal{B}_2$ of $\mathcal{F}^l$ which also contains $\mathcal{B}$ such that*

$$\mathcal{B}_1+\mathcal{B}_2 = \mathcal{F}^l \text{ and } \mathcal{B}_1\bigcap\mathcal{B}_2\bigcap\mathcal{F}_2^l = \mathcal{B}\bigcap\mathcal{F}_2^l.$$

If the preceding equivalent conditions are satisfied, the behavior $\mathcal{B}$ is called a regular $\mathfrak{T}$- or almost interconnection *of $\mathcal{B}_1$ and $\mathcal{B}_2$. Item 4 is of limited value since a direct summand $\mathcal{F}_2$ cannot be determined constructively.*

In dimension $r = 2$ we have $Q(U_1) = U_1^+$ and $Q(U) = U^+$ according to Remark 2.2. *Napp Avelli's criterion* [10, Lem. 21] *thus coincides with item* 2.

Proof. 1 $\iff$ 2: Theorem 3.1. Moreover, since U is torsionfree and $U_1 \bigcap U_2 \cong \ker(+)$ is finite-dimensional the identity $U_1 \bigcap U_2 = 0$ follows.

2 $\iff$ 3: This follows like Corollary 3.2 from (27) and the equivalence

$$U_1 \bigcap U_2 = 0 \iff \mathcal{B}_1 + \mathcal{B}_2 = \left(U_1 \bigcap U_2\right)^{\perp} = 0^{\perp} = \mathcal{F}^l.$$

3 $\implies$ 4: The decomposition $\mathcal{F} = \mathcal{F}_{\mathrm{lf}} \oplus \mathcal{F}_2$ from (3) induces decompositions

$$\mathcal{B} \cong \mathrm{Hom}_A(A^{1\times l}/U, \mathcal{F}) \cong \mathrm{Hom}_A(A^{1\times l}/U, \mathcal{F}_{\mathrm{lf}}) \oplus \mathrm{Hom}_A(A^{1\times l}/U, \mathcal{F}_2),$$

$$\text{and hence } \mathcal{B} = \left(\mathcal{B} \bigcap \mathcal{F}_{\mathrm{lf}}^l\right) \oplus \left(\mathcal{B} \bigcap \mathcal{F}_2^l\right) \text{ with } \mathcal{B} \bigcap \mathcal{F}_2^l \cong \mathrm{Hom}_A(A^{1\times l}/U, \mathcal{F}_2)$$

and analogous decompositions for all other behaviors. The $\mathfrak{T}$-torsion modules C are characterized by $\mathrm{Hom}_A(C, \mathcal{F}_2) = 0$ according to (6), in particular $\mathrm{Hom}_A(U/(U_1 + U_2), \mathcal{F}_2) = 0$. Since $\mathcal{F}_2$ is injective the functor $D_2 := \mathrm{Hom}_A(-, \mathcal{F}_2)$ is exact. Therefore the canonical exact sequence

$$0 \to U/(U_1 + U_2) \to A^{1\times l}/(U_1 + U_2) \to A^{1\times l}/U \to 0$$

induces the exact sequence

$$0 \to D_2(A^{1\times l}/U) \to D_2(A^{1\times l}/(U_1 + U_2)) \to D_2(U/(U_1 + U_2) = 0,$$

$$\text{and hence } \mathcal{B} \bigcap \mathcal{F}_2^l = \mathcal{B}_1 \bigcap \mathcal{B}_2 \bigcap \mathcal{F}_2^l.$$

4 $\implies$ 3: analogous. ∎

4. Algorithms. In this section we prove Algorithm 4.1, which makes Theorem 3.3 constructive, and thereby extend the algorithm of Napp Avelli [10, Cor. 22] to higher dimensions than two, but with a completely different method. The algorithm makes essential use of the algorithm of Zerz and Lomadze for regular interconnection [26, sec. 3]. More generally, we describe how to check in Theorem 3.1 whether $Q(M_1)$ is a direct summand of $Q(M)$ and how to construct M_2 if this is the case. The latter algorithm is not complete if the finitely generated module ${}_AM$ is not torsionfree since then $Q(M)$ is not A-finitely generated in general. For the torsionfree modules $U_1 \subseteq U$ in Theorem 3.3 this problem does not arise. We leave it to the future and the younger generation to really implement the algorithms which are only described in words here.

Ideals and, more generally, submodules U of $A^{1\times l}$ are given by finitely many generators. Arbitrary finitely generated A-modules are described by generators and relations in the form $M = A^{1\times l}/U$. Via Gröbner bases also Hom-modules $\mathrm{Hom}_A(M_1, M_2)$ and the annihilators $\mathrm{ann}_A(M)$, $\mathrm{ann}_M(\mathfrak{a})$, $\mathfrak{a} \subseteq A$, and especially $\mathrm{ann}_M(\mathfrak{m}^k)$, $\mathfrak{m} \in \mathrm{Max}(A)$, can be computed. The finite associator $\mathrm{Ass}(M)$ and then also

$$(28) \qquad M_{\mathrm{lf}} = \mathrm{tor}_{\mathfrak{T}}(M) = \oplus_{\mathfrak{m}\in \mathrm{Ass}(M)\bigcap \mathrm{Max}(A)} M(\mathfrak{m}),\ M(\mathfrak{m}) := \bigcup_{k=0}^{\infty} \mathrm{ann}_M(\mathfrak{m}^k),$$

and M/M_{lf} can be computed since the ascending sequences of the $\mathrm{ann}_M(\mathfrak{m}^k)$ become stationary. Since $Q(M) = Q(M/M_{\mathrm{lf}})$ one may assume for constructive purposes that $M_{\mathrm{lf}} = \mathrm{tor}_{\mathfrak{T}}(M) = 0$ and $M \subseteq Q(M) \subseteq M_S$ as in Theorem 2.5. In what follows we therefore assume

$$(29) \qquad \begin{gathered} M_1 \subseteq M \subseteq Q(M) \subseteq M_S,\ S := \bigcap_{\mathfrak{p}\in \mathrm{Ass}(M)} (A \setminus \mathfrak{p}), \\ Q(M_1) \subseteq Q(M),\ Q(M)/M = \mathrm{tor}_{\mathfrak{T}}(M_S/M). \end{gathered}$$

ALGORITHM 4.1. *The following algorithm checks for arbitrary dimension $r \geq 2$ whether a subbehavior $\mathcal{B} \subseteq \mathcal{B}_1$ admits a regular almost interconnection and, if so, constructs it. In the situation of Theorem* 3.3 *we assume $U_1 \subseteq U \subseteq A^{1\times l}$ and that U and U_1 are nonzero. We are going to compute $Q(U) = \bigcap_{\mathfrak{p}\in\mathcal{P}_2} U_\mathfrak{p} \subseteq A^{1\times l}$; cf.* (21). *It is clear that infinite intersections cannot, in general, be calculated. The zero ideal is the only prime ideal associated with U since it is torsionfree. Hence $S = A \setminus \{0\}$ in* (29) *and*

$$U \subseteq Q(U) \subset U_S = KU \subseteq K^{1\times l},\ K = F(s),\ Q(U)/U = \mathrm{tor}_\mathfrak{T}(KU/U). \tag{30}$$

As submodule of $A^{1\times l} = Q(A^{1\times l})$ the module $Q(U)$ is finitely generated.

We choose a free A-module V with

$$U \subseteq V \subset KU = KV \subseteq KA^{1\times l} = K^{1\times l} \tag{31}$$

and obtain the exact sequence

$$\begin{gathered} 0 \to V/U \xrightarrow{\text{inj}} KU/U \xrightarrow{\text{can}} KU/V \to 0,\ \textit{and hence} \\ \mathrm{Ass}(KU/U) \subseteq \mathrm{Ass}(V/U) \bigcup \mathrm{Ass}(KU/V)\ [4, \text{Chap. IV, sec. 1.2, Prop. 3}]. \end{gathered} \tag{32}$$

If $m := \mathrm{rank}(U) := \dim_K(KU)$, then

$$\begin{gathered} V \cong A^{1\times m},\ KV \cong K^{1\times m}\ \textit{and}\ KU/V = KV/V \cong (K/A)^{1\times m},\ \textit{and hence} \\ \mathrm{Ass}(KU/V) = \mathrm{Ass}(K/A) = \{Ap; p\ \textit{irreducible}\} \subseteq \mathcal{P}_2. \end{gathered} \tag{33}$$

From the preceding two equations we infer

$$\mathcal{M} := \mathrm{Ass}(KU/U) \bigcap \mathrm{Max}(A) = \mathrm{Ass}(V/U) \bigcap \mathrm{Max}(A). \tag{34}$$

Since V/U is finitely generated, the set $\mathcal{M}$ in (34) *is finite and can be computed. From* (9) *and* (30) *we infer*

$$\begin{gathered} Q(U)/U = \mathrm{tor}_\mathfrak{T}(KU/U) = \oplus_{\mathfrak{m}\in\mathcal{M}}(KU/U)(\mathfrak{m})\ \textit{with} \\ (KU/U)(\mathfrak{m}) := \bigcup_{k=0}^{\infty} \mathrm{ann}_{KU/U}(\mathfrak{m}^k). \end{gathered} \tag{35}$$

Notice that for a nonzero ideal $\mathfrak{a}$ and a nonzero element $a \in \mathfrak{a}$ the annihilator

$$\begin{aligned} \mathrm{ann}_{KU/U}(\mathfrak{a}) &:= \left\{\overline{\xi} \in KU/U;\ \mathfrak{a}\xi \subseteq U\right\} = \left\{\overline{\xi} \in a^{-1}U/U;\ \mathfrak{a}\xi \subseteq U\right\} \\ &\cong \{\overline{\eta} \in U/aU;\ \mathfrak{a}\eta \subseteq aU\},\ \overline{\xi} \mapsto \overline{a\xi}, \end{aligned} \tag{36}$$

and especially $\mathrm{ann}_{KU/U}(\mathfrak{m}^k)$ can be computed. Since $Q(U)/U$ is finitely generated there are exponents $e(\mathfrak{m}) \in \mathbb{N}$ with

$$0 = \mathrm{ann}_{KU/U}(\mathfrak{m}^0) \subsetneq \cdots \subsetneq \mathrm{ann}_{KU/U}(\mathfrak{m}^{e(\mathfrak{m})}) = (KU/U)(\mathfrak{m}). \tag{37}$$

These exponents and $(KU/U)(\mathfrak{m})$ can be calculated too. This implies that also generators of

$$Q(U)/U = \mathrm{tor}_\mathfrak{T}(KU/U)\ \textit{and finally of}\ Q(U) \tag{38}$$

can be constructively determined. In the same fashion generators of $Q(U_1)$ *can be computed. Assume now that matrices*

$$R \in A^{k\times l} \text{ and } R_1 \in A^{k_1\times l} \text{ with } Q(U_1) = A^{1\times k_1}R_1 \subseteq Q(U) = A^{1\times k}R \tag{39}$$

have been determined. Compute a matrix $R_2 \in A^{g\times k}$ *such that*

$$A^{1\times g}R_2 = \ker\left(A^{1\times k} \overset{\circ R}{\to} Q(U)/Q(U_1) = A^{1\times k}R/A^{1\times k_1}R_1,\ \xi \mapsto \overline{\xi R}\right). \tag{40}$$

According to [26, p. 1077] *the submodule* $Q(U_1)$ *is a direct summand of* $Q(U)$ *if and only if the inhomogeneous linear system*

$$\begin{gathered} R_2R = R_2XR_1 \text{ has a solution } X \in A^{k\times k_1} \text{ and then} \\ Q(U)/Q(U_1) \to Q(U),\ \overline{\xi R} \mapsto \xi(R - XR_1), \end{gathered} \tag{41}$$

is the associated section of the canonical map. If X *exists, we infer*

$$Q(U) = Q(U_1) \oplus A^{1\times k}(R - XR_1), \tag{42}$$

and finally, by Theorems 3.1 *and* 3.3*, the* $\mathfrak{T}$*-isomorphism*

$$+ : U_1 \times U_2 \to U,\ (u_1, u_2) \mapsto u_1 + u_2,\ \text{with } U_2 := U \bigcap A^{1\times k}(R - XR_1) \tag{43}$$

and the regular almost interconnection

$$\begin{gathered} \mathcal{B} := U^\perp \subseteq \mathcal{B}_1 \bigcap \mathcal{B}_2,\ \mathcal{B}_i := U_i^\perp,\ \text{with} \\ \mathcal{B}_2 = \left(U \bigcap A^{1\times k}(R - XR_1)\right)^\perp = \mathcal{B} + \{w \in \mathcal{F}^l;\ (R - XR_1)\circ w = 0\}. \end{gathered} \tag{44}$$

Corollary 4.2. *The preceding Algorithm* 4.1 *can be applied to the computation of* $Q(M)$ *and* $Q(M_1)$ *in the situation of Theorem* 3.1 *and Corollary* 3.2 *if* M *and hence also* M_1 *are torsionfree. It thus furnishes a constructive almost direct sum decomposition of a controllable behavior* $\mathcal{B}$ *if this exists.*

We proceed with the situation of Theorem 3.1 and use (29) too. The next theorem gives another equivalent condition to the two equivalent conditions from Theorem 3.1, but the new condition is not fully constructive.

Theorem 4.3. *For the data from* (29) *the following properties are equivalent:*

1. $Q(M_1)$ *is a direct summand of* $Q(M)$ *or there is a submodule* M_2 *of* M *such that* $+ : M_1 \times M_2 \to M$ *is a* $\mathfrak{T}$*-isomorphism.*
2. *There is an* $s \in S$ *and a linear map* $f : M \to M_1$ *such that* $f(x_1) = sx_1$ *for all* $x_1 \in M_1$*, and hence* $f(M_1) = sM_1$*, and* $f(M)/sM_1 \in \mathfrak{C}$*.*

Then also $M_{1,S}$ *is a direct summand of* M_S*.*

For fixed s *the existence of* f *in condition* 2 *can be checked via Gröbner bases again. There is no algorithm, however, which does this for all infinitely many* s*. The method of Zerz and Lomadze* [26] *also enables us to check whether* $M_{1,S}$ *is a direct summand of* M_S *and, if so, to construct a direct complement. However, in general this direct decomposition does not imply a direct decomposition* $Q(M) = Q(M_1) \oplus V$ *as in Algorithm* 4.1.

Proof. 2 $\implies$ 1: The map $s\circ : M_S \mapsto M_S$ is an isomorphism. Thus f induces the map

$$g := s^{-1}f : M \to s^{-1}M_1 \text{ with } g(x_1) = x_1 \text{ for } x_1 \in M_1 \text{ and}$$
$$g(M)/M_1 \subseteq \mathrm{tor}_{\mathfrak{T}}(s^{-1}M_1/M_1) \subseteq \mathrm{tor}_{\mathfrak{T}}(M_{1,S}/M_1) = Q(M_1)/M_1, \text{ and hence}$$
$$g : M \to Q(M_1) \text{ and } g|M_1 = \mathrm{inj} : M_1 \to Q(M_1).$$

Since $Q(M_1)$ is $\mathfrak{T}$-closed and since $Q(M)/M \in \mathfrak{C}$ the map g admits a unique extension $h : Q(M) \to Q(M_1)$ which also satisfies $h|M_1 = \mathrm{inj}$ and therefore $h|Q(M_1) = \mathrm{id}_{Q(M_1)}$. The preceding arguments hold by means of Lemma 2.3. We conclude $Q(M) = Q(M_1) \oplus \ker(h)$ as asserted. The map $g : M \to s^{-1}M_1 \subseteq M_{1,S}$ with $g|M_1 = \mathrm{inj}$ also induces the map

$$g_S : M_S \to M_{1,S} \text{ with } g_S|M_{1,S} = \mathrm{id}_{M_{1,S}}, \text{ and hence } M_S = M_{1,S} \oplus \ker(g_S).$$

1 $\implies$ 2: Let $g : Q(M) \to Q(M_1)$ be a retraction of the injection, i.e., with $g(x_1) = x_1$ for all $x_1 \in Q(M_1)$. This induces

$$g(M) \subseteq Q(M_1) \subseteq M_{1,S} = \bigcup_{s\in S} s^{-1}M_1 \text{ and } g(M)/M_1 \subseteq \mathrm{tor}_{\mathfrak{T}}(M_{1,S}/M_1).$$

Since M is finitely generated and since the submodules $s^{-1}M_1$ of $M_{1,S}$ are directed upwards, there is an $s \in S$ such that

$$g(M) \subseteq s^{-1}M_1,\ g(x_1) = x_1 \text{ for } x_1 \in M_1 \text{ and } g(M)/M_1 \subseteq \mathrm{tor}_{\mathfrak{T}}(s^{-1}M_1/M_1).$$

The map $f := sg : M \to M_1$ then has the asserted properties. ∎

COROLLARY 4.4. *With the notation from* (29) *the quotient module* $Q(M)$ *of a finitely generated module* M *with* $M_{\mathrm{lf}} = 0$ *admits the representation*

$$Q(M) = \bigcup_{s\in S} U(s),\ M \subseteq U(s) \subseteq s^{-1}M,\ U(s)/M = \mathrm{tor}_{\mathfrak{T}}(s^{-1}M/M).$$

Each single $U(s)$ *can be computed, but the infinite directed union is not finitely generated in general and cannot be calculated.*

5. Stable finite-dimensional systems. In this section we are going to talk about stable systems and therefore assume that F is the field $\mathbb{C}$ of complex numbers. To motivate the following considerations consider the $\mathbb{C}$-one-dimensional, unstable differential behavior

$$\mathcal{B} := \{y \in \mathrm{C}^\infty(\mathbb{R}, \mathbb{C}); (s-1)\circ y = 0\} = \mathbb{C}e^t.$$

This example shows that the consideration of a behavior up to a $\mathbb{C}$-finite-dimensional one as in the preceding sections may be misleading if this finite-dimensional part is unstable. In stability and stabilization theory it is therefore customary to neglect only those autonomous systems which are stable in a suitable sense, for instance, asymptotically stable in the standard one-dimensional theory. Feedback stabilization of multidimensional input/output systems was treated with this philosophy in [14]. In this short section we describe how the theory of the preceding sections can be extended to the case where only stable finite-dimensional behaviors are neglected. It turns out that the localization theory is very versatile and can be easily adapted to this situation.

The assumptions of section 2 remain in force. As in [14, sec. 5, eq. (70)] we additionally choose a disjoint decomposition

$$\mathbb{C}^r := \Lambda_1 \uplus \Lambda_2 \tag{45}$$

into a *stable part* Λ_1 and an *unstable part* Λ_2. The standard choice in the continuous case is $\Lambda_2 := \{z \in \mathbb{C};\ \Re(z) \geq 0\}^r$. For $r = 1$ the set Λ_1 consists of the complex numbers with negative real part which is customarily used for continuous stabilization theory. Multidimensional stability and stabilization with respect to (45) have been discussed in [14]. In the complex continuous case the decomposition (45) implies the direct sum decomposition

$$\mathcal{F}_1 = \mathcal{F}_{\rm lf} = \mathcal{F}_{{\rm lf},1} \oplus \mathcal{F}_{{\rm lf},2},\ \mathcal{F}_{{\rm lf},i} = \oplus_{\lambda\in\Lambda_i}\mathbb{C}[t]e^{\lambda\bullet t},\ i = 1, 2, \tag{46}$$

of the space of polynomial-exponential functions. For the real and the discrete cases analogous decompositions hold. These decompositions, in turn, furnish the direct decomposition

$$\mathcal{F} = \mathcal{F}_{\rm lf} \oplus \mathcal{F}_2 = \mathcal{F}_1^\Lambda \oplus \mathcal{F}_2^\Lambda,\ \mathcal{F}_1^\Lambda := \mathcal{F}_{{\rm lf},1},\ \mathcal{F}_2^\Lambda = \mathcal{F}_{{\rm lf},2} \oplus \mathcal{F}_2. \tag{47}$$

The map

$$\mathbb{C}^r \cong \mathrm{Max}(A),\ \lambda \mapsto \mathfrak{m}_\lambda := \sum_{k=1}^{r} A(s_k - \lambda_k), \tag{48}$$

is bijective. The decomposition $\mathcal{F} := \mathcal{F}_1^\Lambda \oplus \mathcal{F}_2^\Lambda$ then induces the disjoint decomposition

$$\begin{gathered}\mathrm{Spec}(A) = \mathcal{P}_1^\Lambda \uplus \mathcal{P}_2^\Lambda \text{ with } \mathcal{P}_1^\Lambda := \mathrm{Ass}(\mathcal{F}_1^\Lambda) = \{\mathfrak{m}_\lambda;\ \lambda \in \Lambda_1\} \subseteq \mathrm{Max}(A)\\ \text{and } \mathcal{P}_2^\Lambda := \mathrm{Ass}(\mathcal{F}_2^\Lambda) = (\mathrm{Spec}(A) \setminus \mathrm{Max}(A)) \uplus \{\mathfrak{m}_\lambda;\ \lambda \in \Lambda_2\}.\end{gathered} \tag{49}$$

A locally finite (torsion) module M is called *stable* with respect to the decomposition (45) if $\mathrm{Ass}(M) \subseteq \mathcal{P}_1^\Lambda$. If $M = A^{1\times p}/U$ is stable and $\mathbb{C}$-finite-dimensional, its associated autonomous behavior $\mathcal{B} \cong \mathrm{Hom}_A(M, \mathcal{F})$ consists of polynomial-exponential trajectories with exponents in Λ_1 [13, eq. (38)]; indeed

$$\mathcal{B} = \oplus_{\lambda\in\Lambda_1,\ \mathfrak{m}_\lambda\in\mathrm{Ass}(M)}\mathcal{B}(\mathfrak{m}_\lambda),\ \mathcal{B}(\mathfrak{m}_\lambda) \subseteq \mathbb{C}[t]^p e^{\lambda\bullet t}. \tag{50}$$

For $r = 1$ and $\Lambda_1 = \{z \in \mathbb{C}; \Re(z) < 0\}$ this signifies that $\mathcal{B}$ is asymptotically stable.

The Serre subcategory $\mathfrak{C}$ of locally finite modules and its associated Gabriel topology is now replaced by the subcategory and Gabriel topology

$$\begin{gathered}\mathfrak{C}^\Lambda := \{C \in \mathrm{Mod}_A;\ \mathrm{Hom}_A(C, \mathcal{F}_2^\Lambda) = 0\} = \{C \in \mathrm{Mod}_A;\ \forall \mathfrak{p} \in \mathcal{P}_2^\Lambda : C_\mathfrak{p} = 0\} \subseteq \mathfrak{C},\\ \mathfrak{T}^\Lambda := \{\mathfrak{a} \subseteq A;\ A/\mathfrak{a} \in \mathfrak{C}^\Lambda\} \subseteq \mathfrak{T},\\ \mathfrak{C}^\Lambda = \{C \in \mathrm{Mod}_A;\ \forall x \in C : \mathrm{ann}_A(x) \in \mathfrak{T}^\Lambda\} = \{C \in \mathrm{Mod}_A;\ \mathrm{Ass}(C) \subseteq \mathcal{P}_1^\Lambda\}.\end{gathered} \tag{51}$$

$\mathfrak{T}^\Lambda$-torsion, $\mathfrak{T}^\Lambda$-torsionfree, and $\mathfrak{T}^\Lambda$-closed modules, the category $\mathrm{Mod}_{A,\mathfrak{T}^\Lambda}$, and the quotient functor $Q^\Lambda : \mathrm{Mod}_A \to \mathrm{Mod}_{A,\mathfrak{T}^\Lambda}$ are defined in analogy to the case of the topology $\mathfrak{T}$ and have the corresponding properties. The $\mathfrak{T}^\Lambda$-torsion radical $\mathrm{tor}_{\mathfrak{T}^\Lambda}(M)$ is the largest $\mathfrak{T}^\Lambda$-torsion submodule of M and contained in $M_{\mathrm{lf}} = \mathrm{tor}_{\mathfrak{T}}(M)$. Every $\mathfrak{T}$-closed module is also $\mathfrak{T}^\Lambda$-closed since $\mathfrak{T}^\Lambda \subseteq \mathfrak{T}$. Therefore every A-module M gives rise to the commutative diagram with exact rows,

$$\begin{array}{ccccccc} 0 \to & \mathrm{tor}_{\mathfrak{T}^\Lambda}(M) & \xrightarrow{\mathrm{inj}} & M & \xrightarrow{\eta^\Lambda} & Q^\Lambda(M) = Q^\Lambda(M/\mathrm{tor}_{\mathfrak{T}^\Lambda}(M)) \\ & \bigcap & & \| & & \downarrow g = Q^\Lambda(\eta) \\ 0 \to & \mathrm{tor}_{\mathfrak{T}}(M) & \xrightarrow{\mathrm{inj}} & M & \xrightarrow{\eta} & Q(M) = Q^\Lambda Q(M), \end{array} \tag{52}$$

where $g = Q^\Lambda(\eta)$ is the unique A-linear map with $g\eta^\Lambda = \eta$.

Mutatis mutandis equations (14)–(22), Lemma 2.3, Theorems 2.4, 2.5, 3.1, and 3.3, and Algorithm 4.1 hold when $\mathcal{F}_1, \mathcal{F}_2, \mathcal{P}_1, \mathcal{P}_2$ are replaced by $\mathcal{F}_1^\Lambda, \mathcal{F}_2^\Lambda, \mathcal{P}_1^\Lambda, \mathcal{P}_2^\Lambda$. We thus obtain a theory of behaviors up to $\mathbb{C}$-finite-dimensional, Λ-stable ones.

Remark 5.1. The localization theory can be applied to more general decompositions $\mathrm{Spec}(A) = \mathcal{Q}_1 \uplus \mathcal{Q}_2$ with the property that $\mathfrak{p}_1 \subset \mathfrak{p}_2 \in \mathcal{Q}_2$ implies $\mathfrak{p}_1 \in \mathcal{Q}_2$, for instance, for given n with $0 < n \le r$,

$$\mathcal{Q}_1 := \{\mathfrak{p} \in \mathrm{Spec}(A);\ \dim(A/\mathfrak{p}) < n\},\ \mathcal{Q}_2 := \{\mathfrak{p} \in \mathrm{Spec}(A);\ \dim(A_\mathfrak{p}) \le r - n\}.$$

Here we used $\dim(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) = r$. In particular,

$$\mathcal{P}_1 = \mathrm{Max}(A) = \{\mathfrak{p} \in \mathrm{Spec}(A);\ \dim(A/\mathfrak{p}) < 1\},$$
$$\mathcal{P}_2 = \mathrm{Spec}(A) \setminus \mathrm{Max}(A) = \{\mathfrak{p};\ \dim(A_\mathfrak{p}) \le r - 1\}.$$

The choice

$$\mathcal{Q}_1 := \{\mathfrak{p} \in \mathrm{Spec}(A);\ \dim(A/\mathfrak{p}) < r - 1\},$$
$$\mathcal{Q}_2 := \{\mathfrak{p} \in \mathrm{Spec}(A);\ \dim(A_\mathfrak{p}) \le 1\}$$

leads to the pseudozero modules and pseudoisomorphisms from [4, Chap. VII, sec. 4.4, Defs. 2 and 3].

Remark 5.2. In [14] the localization theory was applied to the Serre subcategory and Gabriel topology

$$\begin{gathered} \mathfrak{C}^{\mathrm{IO},\Lambda} := \{C \in \mathrm{Mod}_A;\ \forall \lambda \in \Lambda_2 : C_{\mathfrak{m}_\lambda} = 0\} \supset \\ \mathfrak{C}^\Lambda = \left\{C \in \mathfrak{C}^{\mathrm{IO},\Lambda};\ C \text{ locally finite}\right)\right\} \text{ and } \mathfrak{T}^{\mathrm{IO},\Lambda} := \left\{\mathfrak{a} \subseteq A;\ A/\mathfrak{a} \in \mathfrak{C}^{\mathrm{IO},\Lambda}\right\}. \end{gathered} \tag{53}$$

The corresponding localization functor $Q^{\mathrm{IO},\Lambda}$ satisfies [14, Lem. 3.4]

$$Q^{\mathrm{IO},\Lambda}(A) = A_T \text{ with } T := \{t \in A;\ \forall \lambda \in \Lambda_2 : t(\lambda) \neq 0\}. \tag{54}$$

A finitely generated (torsion) module $M \in \mathfrak{C}^{\mathrm{IO},\Lambda}$ gives rise to an autonomous behavior $\mathcal{B}^0 := \mathrm{Hom}_A(M, \mathcal{F})$ which we call Λ-stable or T-stable. In [14] we mainly discussed its part $\mathrm{Hom}_A(M, \mathcal{F}_{\mathrm{lf}})$ of polynomial-exponential trajectories. Theorems 3.1 and 3.3 also hold for this Serre subcategory and associated quotient module and give rise to a theory of regular $\mathfrak{T}^{\mathrm{IO},\Lambda}$-interconnections up to T-stable autonomous behaviors. Since $\mathfrak{C}^{\mathrm{IO},\Lambda}$ is much bigger than $\mathfrak{C}^\Lambda$ from (51) and contains many nonlocally finite modules, there are many more associated regular $\mathfrak{T}^{\mathrm{IO},\Lambda}$- than $\mathfrak{T}^\Lambda$-interconnections.

The torsion radical $\mathrm{tor}_{\mathfrak{T}^{\mathrm{IO},\Lambda}}(M)$ is given by the infinite intersection [14, Lemma 3.5]

$$\begin{aligned}\mathrm{tor}_{\mathfrak{T}^{\mathrm{IO},\Lambda}}(M) &= \bigcap_{\lambda\in\Lambda_2} \ker\left(M \xrightarrow{\mathrm{can}} M_{\mathfrak{m}_\lambda},\ x \mapsto \frac{x}{1}\right)\\ &= \{x \in M;\ \forall\lambda \in \Lambda_2 \exists t_\lambda \in A \text{ with } t_\lambda(\lambda) \neq 0 \text{ and } t_\lambda x = 0\}\end{aligned} \tag{55}$$

and cannot be easily computed in general. The quotient module $Q^{\mathrm{IO},\Lambda}(U)$ of a finitely generated torsionfree module $U \subseteq A^{1\times l}$ is also given by the infinite intersection [14, Lem. 3.6]

$$Q^{\mathrm{IO},\Lambda}(U) = A_T^{1\times l} \bigcap \bigcap_{\lambda\in\Lambda_2} U_{\mathfrak{m}_\lambda} \tag{56}$$

and is also hard to compute in general. Define, however, $M := A^{1\times l}/U$ and assume that all modules $M_{\mathfrak{m}_\lambda} = A^{1\times l}_{\mathfrak{m}_\lambda}/U_{\mathfrak{m}_\lambda}$, $\lambda \in \Lambda_2$, are torsionfree. Then $Q^{\mathrm{IO},\Lambda}(U)$ can be computed as [14, Thm. 5.14]

$$Q^{\mathrm{IO},\Lambda}(U) = U_{\mathrm{cont},T} \subseteq A_T^{1\times l},\ \text{where } U_{\mathrm{cont}}/U = \mathrm{tor}(A^{1\times l}/U) \tag{57}$$

is the torsion submodule of M. If even M is torsionfree, i.e., if the behavior $\mathrm{Hom}_A(M,\mathcal{F})$ is controllable, then $Q^{\mathrm{IO},\Lambda}(U) = U_T$ is the usual quotient module and its A_T-generators are the A-generators of U.

Hence, if in Theorem 3.3 the given behaviors $\mathcal{B} \subseteq \mathcal{B}_1$ are controllable, matrices

$$\begin{gathered} R \in A^{k\times l},\ R_1 \in A^{k_1\times l} \text{ with } U = A^{1\times k}R,\ U_1 = A^{1\times k_1}R_1 \text{ and}\\ Q(U) = U_T = A_T^{1\times k}R,\ Q(U_1) = U_{1,T} = A_T^{1\times k_1}R_1 \end{gathered} \tag{58}$$

are given. Hence $\mathcal{B} \subseteq \mathcal{B}_1$ admit a regular $\mathfrak{T}^{\mathrm{IO},\Lambda}$-interconnection if and only if equations (41) have a solution $X \in A_T^{k\times k_1}$. But there is yet no algorithm which solves *inhomogeneous* systems of linear equations in the quotient ring

$$A_T,\ T = \{t \in A;\ \forall\lambda \in \Lambda_2 : t(\lambda) \neq 0\},$$

as pointed out in [14, Rem. 5.10]; cf. also [7] and [25].

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