A constructive test for exponential stability of linear time-varying discrete-time systems

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Abstract

We complete the stability results of the paper H. Bourlès, B. Marinescu, U. Oberst, 'Exponentially stable linear time-varying discrete behaviors', SIAM J. Control Optim. 53(2015), 2725-2761, and for this purpose use the linear time-varying (LTV) discrete-time behaviors and the exponential stability (e.s.) of this paper. In the main theorem we characterize the e.s. of an autonomous LTV system by standard spectral properties of a complex matrix connected with the system. We extend the theory of discrete-time LTV behaviors, developed in the quoted publication, from the coefficient field of rational functions to that of locally convergent Laurent series or even of Puiseux series. The stability test can and has to be applied in connection with the construction of stabilizing compensators.

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1 Introduction

We complete the stability results of [4] and use the notions, in particular the linear time-varying (LTV) discrete-time behaviors and the exponential stability (e.s.), of this paper and of [5]. In the main Thm. 1.1 we characterize e.s. of an autonomous LTV system by standard spectral properties of a complex matrix connected with the system. Due to, for instance, [6], [15], Thm. 1.1 furnishes a *constructive* test for e.s.. This test can and has to be applied in connection with the construction of stabilizing compensators; cf. [12] in the case of differential LTV systems. The proof of Thm. 1.1 is contained in Section 3. In Section 2 we generalize the theory of discrete-time LTV behaviors, developed in [4], from the coefficient field of *rational functions* to that of *locally convergent Laurent series* by exposing the necessary modifications only. These more general behaviors are used in Thm. 1.1. In Section 4 we shortly extend the results to the still larger field of *locally convergent Puiseux series* (cf. [5, §3.1]). The latter field seems to be the largest coefficient field for which a reasonable stability theory for *general LTV systems* can be developed. We refer to the books [16, pp. 423-461] and [8, pp. 193-368] for comprehensive surveys of exponential stability of *state space systems*. Part II of the book [3] contains a detailed theory of general LTV behaviors and their stability that was modified in the papers [4] and [5]. We also refer to the recent papers [10], [13],

[7], [1], [2].
The main result Thm. 1.1 requires some preliminary explanations: Let $\mathbb{C} < z >$ denote the local principal ideal domain of *locally convergent power series* in the variable z and $\mathbf{K} := \mathbb{C} << z >>$ its quotient field of locally convergent *Laurent series*. The ring $\mathbb{C} < z >$ has the unique prime element z up to units. Every nonzero element $a \in \mathbf{K}$ has a unique representation

$$\begin{aligned} a(z) = z^k b(z),\ k \in \mathbb{Z},\ b(z) := \sum_{i=0}^{\infty} b_i z^i,\ b_0 \neq 0, \\ \sigma(a) := \sigma(b) := \limsup_{i \in \mathbb{N}} \sqrt[i]{|b_i|} < \infty. \end{aligned} \tag{1}$$

The series $b(z)$ with $b_0 \neq 0$ is a unit of $\mathbb{C} < z >$. The inverse $\rho(a) := \sigma(a)^{-1}$ is the convergence radius of a and hence $a(z)$ is a holomorphic function in the pointed open disc

$$\mathbf{D}(\rho(a)) \setminus \{0\},\ \mathbf{D}(\rho) := \{z \in \mathbb{C};\ |z| < \rho\},\ \rho > 0,\ b(0) = b_0, \tag{2}$$

and even in the whole disc $\mathbf{D}(\rho(a))$ if $k \geq 0$ and a is a power series. In particular, the function $a(t^{-1}) = t^{-k} b(t^{-1})$ is a smooth function on the real open interval $(\sigma(a), \infty) := \{t \in \mathbb{R};\ t > \sigma(a)\}$, hence

$$a(t^{-1}) \in \mathrm{C}^{\infty}(\sigma(a), \infty) := \{f : (\sigma(a), \infty) \to \mathbb{C};\ f \text{ smooth}\}. \tag{3}$$

The sequences $a(t^{-1}),\ t \in \mathbb{N},\ t > \sigma(a)$, are the time-varying coefficients of the difference equations of the present paper that in [4] were used for $a \in \mathbb{C}(z) = \mathbb{C}(z^{-1})$.

Exponential stability (e.s.) is a property of autonomous behaviors [4, Def. 1.7]. Every autonomous behavior $\mathcal{B}$ is isomorphic to a *Kalman state space behavior* $\mathcal{B}(A, t_0)$ defined by a matrix $A = (A_{\mu\nu})_{\mu,\nu} \in \mathbb{C} << z >>^{q \times q}$ and $t_0 \in \mathbb{N}$, cf. (7), and $\mathcal{B}$ is e.s. if and only if $\mathcal{B}(A, t_0)$ is e.s. [4, Lemma 3.7]. With

$$\sigma(A) := \max\{\sigma(A_{\mu\nu});\ 1 \leq \mu, \nu \leq q\},\ \rho(A) := \sigma(A)^{-1}, \tag{4}$$

the function $A(t^{-1})$ is a smooth matrix function on the real interval $(\sigma(A), \infty)$. For $t_0 \in \mathbb{N}$ we consider the signal space

$$W(t_0) := \mathbb{C}^{t_0 + \mathbb{N}} := \{(w(t))_{t \geq t_0};\ t \in \mathbb{N},\ w(t) \in \mathbb{C}\} \tag{5}$$

of complex sequences or discrete signals starting at the initial time t_0. For $q \in \mathbb{N}$ we use the column spaces $\mathbb{C}^q$ and $W(t_0)^q$ and identify

$$W(t_0)^q = (\mathbb{C}^q)^{t_0 + \mathbb{N}} \ni (w_1, \cdots, w_q)^\top = (w(t_0), w(t_0 + 1), \cdots),\ w_i(t) = w(t)_i. \tag{6}$$

If $t_0 > \sigma(A),\ t_0 \in \mathbb{N}$, the matrix A gives rise to the *state space equation* resp. the *behavior* or solution space

$$\begin{aligned} & x(t+1) = A(t^{-1}) x(t),\ t \geq t_0, \text{ resp.} \\ & \mathcal{B}(A, t_0) := \left\{x \in W(t_0)^q;\ \forall t \geq t_0 :\ x(t+1) = A(t^{-1}) x(t)\right\}. \end{aligned} \tag{7}$$

The *transition matrix* [16, p. 392] associated to (7) is

$$\Phi_A(t, t_0) := A((t-1)^{-1}) * \cdots * A(t_0^{-1}),\ t \geq t_0 > \sigma(A),\ \Phi_A(t_0, t_0) = \mathrm{id}_q. \tag{8}$$

There is the obvious isomorphism

$$\mathcal{B}(A,t_0)\cong\mathbb{C}^q,\ x\mapsto x(t_0),\ x(t)=\Phi_A(t,t_0)x(t_0). \tag{9}$$

For $\xi=(\xi_1,\cdots,\xi_q)^\top\in\mathbb{C}^q$ and $M\in\mathbb{C}^{q\times q}$ we use the maximum norms

$$\|\xi\|:=\max_i|\xi_i| \text{ and } \|M\|:=\max\left\{\|M\xi\|;\ \xi\in\mathbb{C}^q,\ \|\xi\|=1\right\}. \tag{10}$$

The *spectrum* $\mathrm{spec}(M)$ of M is the set of its eigenvalues. Recall that M is nilpotent, i.e., $M^m=0$ for some $m\in\mathbb{N}$, if and only if $\mathrm{spec}(M)=\{0\}$.
W.l.o.g. we assume $A\neq 0$. Then A admits a unique representation

$$A(z)=z^{-k}B(z),\ k\in\mathbb{Z},\ B(z)=\sum_{i=0}^{\infty}B_iz^i\in\mathbb{C}<z>^{q\times q},\ B_i\in\mathbb{C}^{q\times q},\ B_0\neq 0. \tag{11}$$

The coefficient functions $a(t^{-1}),\ a\in\mathbf{K}$, are of at most polynomial growth (p.g.f.) on each closed interval $[\sigma_1,\infty),\ \sigma_1>\sigma(a)$, i.e., there are $c>0$ and $m\in\mathbb{N}$ such that $|a(t^{-1})|\le ct^m$ for $t\ge\sigma_1$ [5, (29)]. For nonzero a there is $\sigma_2>\sigma(a)$ such that $a(t^{-1})\neq 0$ for $t\ge\sigma_2$. According to [4], [5] the smoothness, at most polynomial growth and no zeros for large t are the decisive properties of the time-varying coefficients $a(t^{-1})$. An analogous definition applies to sequences in in $\mathbb{C}^{t_0+\mathbb{N}}$. The system, i.e., the matrix A and the equation and behavior from (7), are called *exponentially stable* (e.s.) [4, Def. 1.7, Cor. 3.3] if

$$\begin{aligned}&\exists t_0>\sigma(A)\exists\alpha>0\exists \text{ p.g.f. } \varphi\in\mathbb{C}^{t_0+\mathbb{N}},\ \varphi>0,\\ &\quad\forall t\ge t_1\ge t_0:\ \|\Phi_A(t,t_1)\|\le\varphi(t_1)e^{-\alpha(t-t_1)}.\end{aligned} \tag{12}$$

In [4] we used the notation $\rho:=e^{-\alpha}<1$ and $\rho^{t-t_1}=e^{-\alpha(t-t_1)}$. In particular, e.s. implies asymptotic stability, i.e., $\lim_{t\to\infty}\Phi_A(t,t_0)=0$. The system is called *uniformly e.s.* (u.e.s.) [16, Def. 22.5] if the p.g.f. φ in (12) can be chosen constant. In [16] the author considers LTV state space equations $x(t+1)=F(t)x(t),\ t\ge 0$, with an arbitrary sequence of complex matrices $F=(F(0),F(1),\cdots)\in\left(\mathbb{C}^{q\times q}\right)^{\mathbb{N}}=\left(\mathbb{C}^{\mathbb{N}}\right)^{q\times q}$. All stability results in [16, Chs. 22-24] require additional properties of F. Our choice in [4] and in the present paper is

$$F(t):=A(t^{-1}),\ A\in\mathbb{C}<<z>>^{q\times q}\supset\mathbb{C}(z)^{q\times q},\ t>\sigma(A). \tag{13}$$

Theorem 1.1. *Consider a nonzero matrix $A(z)\in\mathbb{C}<<z>>^{q\times q}$ and the state space system defined by the data from* (7) *and* (8).

(i) If $A=\sum_{i=0}^{\infty}A_iz^i$ is a power series then the system is e.s. if and only if $\mathrm{spec}(A_0)\subset\mathbf{D}(1)=\{\lambda\in\mathbb{C},\ |\lambda|<1\}$.

(ii) If $k>0$ in the representation $A=z^{-k}(B_0+B_1z+\cdots)$ from (11) *and if the matrix B_0 is not nilpotent then the system is not e.s. and indeed*

$$\exists t_0>\sigma(A)\forall t_1\ge t_0:\ \sup_{t\ge t_1}\|\Phi_A(t,t_1)\|=\infty, \tag{14}$$

i.e., the system is not stable.

(iii) Assume $k>0$ in the representation $A=z^{-k}(B_0+B_1z+\cdots)$ from (11) *and*

$$\det(B(z))=b_\ell z^\ell c(z)\neq 0,\ c(z)=1+c_1z+c_2z^2+\cdots\in\mathbb{C}<z>. \tag{15}$$

If $kq>\ell$ then (14) *holds and the system is not e.s..*

Example 1.2. That B_0 in item (ii) of Thm. 1.1 is not nilpotent cannot be omitted. To see this consider the nilpotent matrix $B_0 := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$. For $0 \neq \lambda \in \mathbb{C}$ and $\rho := |\lambda| = e^{-\alpha} > 0,\ \alpha \in \mathbb{R}$, define

$$A(z) := z^{-1}B(z),\ B(z) := B_0 + z\lambda\,\mathrm{id}_2 \Longrightarrow A(t^{-1}) = B_0 t + \lambda\,\mathrm{id}_2 = \left(\begin{smallmatrix} \lambda & t \\ 0 & \lambda \end{smallmatrix}\right)$$
$$\Longrightarrow \Phi_A(t,t_0) = \begin{pmatrix} \lambda^{t-t_0} & \lambda^{t-t_0-1}\sum_{i=t_0}^{t-1} i \\ 0 & \lambda^{t-t_0} \end{pmatrix}, \det(B(z)) = \lambda^2 z^2,\ \det(A(t^{-1})) = \lambda^2. \tag{16}$$

For $\rho \geq 1$ the sequence $\Phi_A(t,t_0)$ does not converge to zero and therefore the system is not e.s.. The sum $\sum_{i=t_0}^{t-1} i$ grows polynomially. If $\rho = e^{-\alpha} < 1$ or $\alpha > 0$ the transition matrix $\Phi_A(t,t_0)$ decreases exponentially with a decay factor $e^{-\alpha'(t-t_0)}$ for every α' with $0 < \alpha' < \alpha$.

Remark 1.3. If $\mathcal{B}$ is an arbitrary autonomous behavior, cf. (52), (53), [4, §2.5], the matrix A and an explicit isomorphism $\mathcal{B} \cong \mathcal{B}(A,t_0)$ can be computed with the *Ore-Modules* package [6], [15]. Since $\mathcal{B}$ is e.s. if and only if $\mathcal{B}(A,t_0)$ is e.s. Thm. 1.1 establishes a constructive test for e.s. of almost all autonomous systems. The exception occurs with systems for which the assumptions of Thm. 1.1,(ii),(iii), are not satisfied.

Notations and abbreviations: $\mathbf{D}(\rho) := \{z \in \mathbb{C};\ |z| < \rho\},\ \rho > 0$, e.s.= exponentially stable, exponential stability, f.g.=finitely generated, LTV=linear time-varying, p.g.f.= polynomial growth function, resp.=respectively, $\mathrm{spec}(M)$:=set of eigenvalues of a square complex matrix M, u.e.s.= uniformly e.s., w.e.s.= weakly e.s., w.l.o.g.=without loss of generality, $X^{p\times q}$=set of $p \times q$-matrices with entries in X, $X^{1\times q}$=rows, $X^q := X^{q\times 1}$=columns, $X^{\bullet\times\bullet} := \bigcup_{p,q\geq 0} X^{p\times q}$.

2 Laurent coefficients

We explain the basic notions of a variant of the theory from [4] since we use the difference field $\mathbf{K} = \mathbb{C} << z >>$ instead of the field $\mathbb{C}(z) \subset \mathbf{K}$ of rational functions in [4]. Recall $W(t_0) = \mathbb{C}^{t_0+\mathbb{N}}$ for $t_0 \in \mathbb{N}$ from (5). The space $\mathbb{C}^{t_0+\mathbb{N}} = W(t_0)$ is also a difference $\mathbb{C}$-algebra with the componentwise multiplication and the shift algebra homomorphism

$$\Phi_d : \mathbb{C}^{t_0+\mathbb{N}} \to \mathbb{C}^{t_0+\mathbb{N}},\ c \mapsto \Phi_d(c),\ \Phi_d(c)(t) = c(t+1),\ t \geq t_0. \tag{17}$$

It gives rise to the *noncommutative skew-polynomial algebra of difference operators* [9, Section 1.2.3], [4, (20)]

$$\mathbf{B}(t_0) := \mathbb{C}^{t_0+\mathbb{N}}[s;\Phi_d] := \bigoplus_{j=0}^{\infty} \mathbb{C}^{t_0+\mathbb{N}} s^j,\ sc = \Phi_d(c)s,\ c \in \mathbb{C}^{t_0+\mathbb{N}}. \tag{18}$$

The space $W(t_0)$ is a left $\mathbf{B}(t_0)$-module with the action $f \circ w$ for $f = \sum_{j=0}^{\infty} f_j s^j \in \mathbf{B}(t_0)$ and $w \in W(t_0)$ [4, (21)], defined by

$$(f \circ w)(t) := \sum_j f_j(t) w(t+j),\ t \geq t_0. \tag{19}$$

As usual this is extended to the action $R\circ w$ [4, (22)] of a matrix

$$R=\sum_j R_j s^j\in\mathbf{B}(t_0)^{p\times q},\ R_j\in\left(\mathbb{C}^{t_0+\mathbb{N}}\right)^{p\times q},\ \text{on } w\in W(t_0)^q \text{ by}$$
$$(R\circ w)(t):=\sum_j R_j(t)w(t+j),\ t\geq t_0. \tag{20}$$

The *behavior or solution space* defined by R is

$$\mathcal{B}(R,t_0):=\left\{w\in W(t_0)^q;\ R\circ w=0\right\}. \tag{21}$$

For $\sigma>0$ the algebra $\mathrm{C}^\infty(\sigma,\infty)$ is also a difference algebra with the algebra endomorphism

$$\Phi_s:\ \mathrm{C}^\infty(\sigma,\infty)\to\mathrm{C}^\infty(\sigma,\infty),\ \Phi_s(f)(t):=f(t+1). \tag{22}$$

It gives rise to the skew-polynomial algebra

$$\mathbf{A}_s(\sigma):=\mathrm{C}^\infty(\sigma,\infty)[s;\Phi_s]:=\oplus_{j\in\mathbb{N}}\mathrm{C}^\infty(\sigma,\infty)s^j,\ sf=\Phi_s(f)s,\ f\in\mathrm{C}^\infty(\sigma,\infty). \tag{23}$$

For $t_0>\sigma$ the map

$$\Psi_s:\mathrm{C}^\infty(\sigma,\infty)\to\mathbb{C}^{t_0+\mathbb{N}},\ f\mapsto(f(t))_{t\geq t_0}, \tag{24}$$

is a difference algebra homomorphism since $\Phi_d\Psi_s=\Psi_s\Phi_s$ and therefore its extension

$$\Psi_s:\mathbf{A}_s(\sigma)=\mathrm{C}^\infty(\sigma,\infty)[s;\Phi_2]\to\mathbf{B}(t_0)=\mathbb{C}^{t_0+\mathbb{N}}[s;\Phi_d],$$
$$\Psi_s\left(\sum_j f_j s^j\right):=\sum_j\Psi_s(f_j)s^j, \tag{25}$$

(denoted with the same letter) is an algebra homomorphism. The algebras $\mathrm{C}^\infty(\sigma,\infty)$ and $\mathbb{C}^{t_0+\mathbb{N}}$ are not noetherian and have many zero-divisors and therefore very little is known about the rings of difference operators from (18) and (23) and their modules. Therefore we restrict the time-varying coefficients of discrete difference equations to sequences

$$a(t^{-1}),\ t\geq t_0>\sigma(a)=\sigma(b),\ a=z^k b,\ b=\sum_{i=0}^\infty b_i z^i\in\mathbb{C}<z>,$$
$$a(t^{-1})=t^{-k}\sum_{i=0}^\infty b_i t^{-i}. \tag{26}$$

We finally consider $\mathbf{K}$ as a difference field with a field automorphism $\Phi:\mathbf{K}\xrightarrow{\cong}\mathbf{K}$ [14] and define the corresponding operator domain

$$\mathbf{A}:=\mathbf{K}[s;\Phi]=\oplus_{j\in\mathbb{N}}\mathbf{K}s^j,\ sa=\Phi(a)s. \tag{27}$$

The $\mathbb{C}$-algebra $\mathbf{A}$ is a noncommutative euclidean domain [9, §1.2], especially principal, and the f.g. left $\mathbf{A}$-modules are precisely known [9, Thm. 1.2.9, §5.7., Cor. 5.7.19]. The definition of Φ and the connection of $\mathbf{A}$ with $\mathbf{A}_s$ require a preparation:

For an open set $U\subseteq\mathbb{C}$ let $\mathcal{O}(U)$ denote the $\mathbb{C}$-algebra of holomorphic functions in

U. So any $a \in \mathbb{C} < z >$ defines the holomorphic function $a(z) \in \mathcal{O}\left(\mathbf{D}(\rho(a)) \setminus \{0\}\right)$. In general, this can be extended to larger connected open sets. For $\rho > 0$ we define the subset $\mathbf{K}(\rho) \subset \mathbf{K}$ as follows: An element $a \in \mathbf{K}$ belongs to $\mathbf{K}(\rho)$ if there is an open connected neighborhood $U(a)$ of 0 and a holomorphic function $f \in \mathcal{O}\left(U(a) \setminus \{0\}\right)$ such that $[0, \rho) \subset U(a)$ and $f(z) = a(z)$ for $z \neq 0$ near 0. In other terms, the germ of f at 0 is a. Since $U(a) \setminus \{0\}$ is connected the function f is unique with these properties.

Lemma 2.1. *(i) The value $a(t^{-1}) := f(t^{-1})$ for $t > \rho^{-1}$ is independent of the choice of the extension $f \in \mathcal{O}\left(U \setminus \{0\}\right)$ of a.*
(ii) The set $\mathbf{K}(\rho),\ \rho > 0,$ is a subalgebra of $\mathbf{K}$, i.e., additively and multiplicatively closed, and the map

$$\Psi_\rho :\ \mathbf{K}(\rho) \to \mathrm{C}^\infty(\rho^{-1}, \infty),\ a \mapsto a(t^{-1}), \tag{28}$$

is an algebra monomorphism

Proof. (i) Let $f_i \in \mathcal{O}\left(U_i \setminus \{0\}\right),\ i = 1, 2,$ be two such extensions. The (open) connected component U_3 of $U_1 \bigcap U_2$ containing 0 also contains $[0, \rho)$. Since f_1 and f_2 are holomorphic on $U_3 \setminus \{0\}$ and extend a and since $U_3 \setminus \{0\}$ is connected we conclude $f_1|_{U_3 \setminus \{0\}} = f_2|_{U_3 \setminus \{0\}}$ and hence $f_1(t^{-1}) = f_2(t^{-1})$ for $t > \rho^{-1}$.
(ii) For $a_1, a_2 \in \mathbf{K}(\rho)$ the intersection $U(a_1) \bigcap U(a_2)$ is an open neighborhood of 0 and contains $[0, \rho)$. Let f_i denote the holomorphic extensions of the a_i to $U(a_i) \setminus \{0\}$ and U_3 the (open) connected component of $U(a_1) \bigcap U(a_2)$ containing 0 and hence $[0, \rho)$. The function $f_1 + / * f_2$ is holomorphic on $(U(a_1) \bigcap U(a_2)) \setminus \{0\}$ and hence on $U_3 \setminus \{0\}$ and obviously coincides with $a_1 + / * a_2$ near zero, hence $a_1 + / * a_2 \in \mathbf{K}(\rho)$. For $t > \rho^{-1}$ this implies

$$(a_1 + / * a_2)(t^{-1}) := (f_1 + / * f_2)(t^{-1}) = f_1(t^{-1}) + / * f_2(t^{-1}) = a_1(t^{-1}) + / * a_2(t^{-1}).$$

(iii) Assume $a = a_0 + a_1 z + \cdots \in \mathbb{C} < z >$ w.l.o.g. Since 0 is an accumulation point of $(0, \rho)$ the condition $a|_{(0,\rho)} = 0$ implies $a = 0$. □

The equation $(t+1)^{-1} = t^{-1}(t^{-1}+1)^{-1}$ implies

$$\begin{aligned} \forall t > \sigma(a) :\ & a((t+1)^{-1}) = a\left(t^{-1}(t^{-1}+1)^{-1}\right) \\ & = (t+1)^{-k} \sum_{i=0}^{\infty} b_i (1+t)^{-i} = \left(\frac{t^{-1}}{t^{-1}+1}\right)^k \sum_{i=0}^{\infty} b_i \left(\frac{t^{-1}}{t^{-1}+1}\right)^i . \end{aligned} \tag{29}$$

This suggests to make $\mathbb{C} << z >>$ a difference field [14, Ex. 1.2], [4, §4.7] via the field automorphism

$$\begin{aligned} & \Phi : \mathbb{C} << z >> \xrightarrow{\cong} \mathbb{C} << z >>,\ \Phi(z) = z(1+z)^{-1},\ \Phi(z^{-1}) = z^{-1} + 1 \\ & \Phi\left(z^k \sum_{i=0}^{\infty} b_i z^i\right) = \left(\frac{z}{1+z}\right)^k \sum_{i=0}^{\infty} b_i \left(\frac{z}{1+z}\right)^i, \\ & (1+z)^{-i} = \sum_{j=0}^{\infty} \binom{-i}{j} z^j \in \mathbb{C} < z > . \end{aligned} \tag{30}$$

Lemma 2.2. *For $a(z) = z^k b(z) \in \mathbf{K},\ b(z) \in \mathbb{C} < z >$, and $m \in \mathbb{N}$ the following assertions hold:*

1. $\Phi^m(a) = a\left(z(1+mz)^{-1}\right)$.
2. $\sigma(\Phi^m(a)) \leq \sigma(a) + m,\ \rho(\Phi^m(a)) \geq (\rho(a)^{-1} + m)^{-1}$.
3. *If $\rho > 0$ and $a \in \mathbf{K}(\rho)$ then also $\Phi^m(a) \in \mathbf{K}(\rho)$ and $\Phi^m(a)(t^{-1}) = a\left((t+m)^{-1}\right)$ for $t > \rho^{-1}$. With Lemma 2.1 this implies that $\mathbf{K}(\rho)$ is a difference subalgebra of $\mathbf{K}$.*

Proof. 1. The equation follows by induction from

$$\begin{aligned}&\Phi^{m+1}(a) = \Phi^m(a)(z(1+z)^{-1})\\&= a\left(z(1+z)^{-1}(1+mz(1+z)^{-1})^{-1}\right) = a\left(z(1+(m+1)z)^{-1}\right).\end{aligned}$$

2. For $m = 0$ the assertion is obvious. So assume $m > 0$ and

$$\begin{aligned}&|z| < (\sigma(a)+m)^{-1}(<1) \Longrightarrow |z^{-1}+m| \geq |z|^{-1} - m > \sigma(a)\\&\Longrightarrow |\frac{z}{1+mz}| = |\frac{1}{z^{-1}+m}| \leq \frac{1}{|z|^{-1}-m} < \rho(a)\\&\Longrightarrow \rho(\Phi^m(a)) = \rho\left(a(z(1+mz)^{-1})\right) \geq (\sigma(a)+m)^{-1}\\&\Longrightarrow \sigma(\Phi^m(a)) \leq \sigma(a) + m.\end{aligned} \tag{31}$$

3. W.l.o.g. we assume $0 \neq a = b \in \mathbb{C} < z >$. Let $f \in \mathcal{O}(U(a))$ be a holomorphic extension of a with $U(a) \supset [0, \rho)$. Consider the projective line $\overline{\mathbb{C}} = \mathbb{C} \uplus \{\infty\}$. For $m \geq 0$ there are the inverse biholomorphic maps

$$\overline{\mathbb{C}} \cong \overline{\mathbb{C}},\ z = w(1-mw)^{-1} \leftrightarrow w = z(1+mz)^{-1}. \tag{32}$$

They induce the inverse biholomorphic maps

$$\begin{aligned}\overline{\mathbb{C}} \setminus \{-1/m, \infty\} = \mathbb{C} \setminus \{-1/m\} \cong \overline{\mathbb{C}} \setminus \{1/m, \infty\} = \mathbb{C} \setminus \{1/m\} \text{ and}\\ V := \left\{z \in \mathbb{C};\ z(1+mz)^{-1} \in U(a)\right\} \cong U(a) \setminus \{1/m\}.\end{aligned} \tag{33}$$

Since $U(a)$ and $U(a) \setminus \{1/m\}$ are open and connected the set V is a connected open neighborhood of 0 and contains $[0, \rho)$ since $U(a)$ does. The function $f\left(z(1+mz)^{-1}\right)$ is holomorphic on V and coincides with $\Phi^m(a) = a\left(z(1+mz)^{-1}\right)$ for small z. This means that it is a holomorphic extension of $\Phi^m(a)$ and thus $\Phi^m(a) \in \mathbf{K}(\rho)$. Finally

$$\begin{aligned}\forall t > \rho^{-1}:\ \Phi^m(a)(t^{-1}) &= f\left(t^{-1}(1+mt^{-1})^{-1}\right)\\&= f\left((t+m)^{-1}\right) \underset{(t+m)^{-1} \in (0,\rho) \subset U(a)}{=} a\left((t+m)^{-1}\right).\end{aligned}$$

□

Example 2.3. Let

$$\begin{aligned}&a(z) := (1-z^2)^{-1} = \sum_{j=0}^{\infty} z^{2j} \in \mathbb{C} < z > \Longrightarrow \rho(a) = 1,\ U(a) = \mathbb{C} \setminus \{1, -1\}\\&\Phi(a) = a(z(1+z)) = (1+z)^2(1+2z)^{-1},\ U(\Phi(a)) = \mathbb{C} \setminus \{-1/2\}\\&\rho(\Phi(a)) = 1/2,\ (0, \rho(a)) \subset U(\Phi(a)).\end{aligned} \tag{34}$$

Corollary and Definition 2.4. *According to Lemmas 2.1 and 2.2,(3),* $\mathbf{K}(\rho)$ *is a difference subalgebra of* $\mathbf{K}$ *and the map* Ψ_ρ *from* (28) *is a monomorphism of difference algebras, i.e.,* $\Phi_s\Psi_\rho = \Psi_\rho\Phi$*, cf.* (22)*. This obviously implies that*

$$\mathbf{A}(\rho) := \mathbf{K}(\rho)[s;\Phi] \subset \mathbf{A} = \mathbf{K}[s;\Phi] \tag{35}$$

is a subalgebra of $\mathbf{A}$ *and that the extended map*

$$\Psi_\rho : \mathbf{A}(\rho) = \mathbf{K}(\rho)[s;\Phi] \to \mathbf{A}_s(\rho^{-1}) = \mathrm{C}^\infty(\rho^{-1},\infty)[s;\Phi_s],\ a(z)s^j \mapsto a(t^{-1})s^j, \tag{36}$$

cf. (23)*, is an algebra monomorphism.*

We combine Cor. 2.4 with (25). For $\rho > 0$ and $t_0 > \rho^{-1}$ we define the composed algebra monomorphism

$$\Psi := \Psi_s\Psi_\rho : \quad \begin{array}{ccccc} \mathbf{A}(\rho) & \xrightarrow{\Psi_\rho} & \mathbf{A}_s(\rho^{-1}) & \xrightarrow{\Psi_s} & \mathbf{B}(t_0) \\ a(z)s^j & \mapsto & a(t^{-1})s^j & \mapsto & (a(t^{-1}))_{t\geq t_0}s^j \end{array}. \tag{37}$$

Here the map Ψ is injective on $\mathbf{K}(\rho)$ and hence also on $\mathbf{A}(\rho)$.

Corollary 2.5. *For* $\rho > 0$ *and* $t_0 > \rho^{-1}$ *the algebra monomorphism* Ψ *and the left* $\mathbf{B}(t_0)$*- module structure of* $W(t_0) = \mathbb{C}^{t_0+\mathbb{N}}$ *from* (19) *imply the action* $\circ$ *of* $\mathbf{A}(\rho)$ *on* $W(t_0)$ *defined by*

$$\begin{aligned} & f \circ w := \Psi(f)\circ w,\ f = \sum_j a_j s^j \in \mathbf{A}(\rho),\ w \in W(t_0), \\ & \forall t \geq t_0 > \rho^{-1} :\ (f\circ w)(t) = \sum_j a_j(t^{-1})w(t+j). \end{aligned} \tag{38}$$

This action makes $W(t_0)$ *a left* $\mathbf{A}(\rho)$*-module.*

The maps Ψ, Φ, Φ_d are extended to matrices componentwise. We also canonically extend the definitions of $\rho(a)$, $\sigma(a)$ and $U(a)$ for $a \in \mathbf{K}$ to operators resp. matrices

$$\begin{aligned} & f = \sum_j f_j s^j \in \mathbf{A},\ A = (A_{\mu\nu})_{\mu\nu} \in \mathbf{K}^{p\times q}, \\ & R = \sum_j R_j s^j \in \mathbf{A}^{p\times q},\ R_j \in \mathbf{K}^{p\times q} : \\ & \rho(f) = \min_j \rho(f_j),\ \rho(A) := \min_{\mu,\nu}\rho(A_{\mu\nu}),\ \rho(R) = \min_j \rho(R_j) \\ & \sigma(f) := \rho(f)^{-1},\ \sigma(A) := \rho(A)^{-1},\ \sigma(R) := \rho(R)^{-1} \\ & U(f) := \text{ connected component of } \bigcap_j U(f_j) \text{ containing } 0. \end{aligned} \tag{39}$$

The connected open neighborhoods $U(A), U(R)$ are defined analogously. A matrix $R = \sum_j R_j s^j \in \mathbf{A}(\rho)^{p\times q} \subset \mathbf{A}^{p\times q}$ with $R_j \in \mathbf{K}(\rho)^{p\times q}$ and $t_0 > \rho^{-1}$ give rise to the behavior

$$\begin{aligned} \mathcal{B}(R,t_0) &:= \{w \in W(t_0)^q;\ R\circ w = 0\} \\ &= \left\{ w \in W(t_0)^q;\ \forall t \geq t_0 :\ \sum_j R_j(t^{-1})w(t+j) = 0 \right\}. \end{aligned} \tag{40}$$

For $A \in \mathbf{K}(\rho)^{q\times q}$ the state space behavior from (7) now obtains the form

$$\mathcal{B}(s\,\mathrm{id}_q - A, t_0) = \mathcal{B}(A, t_0) = \left\{x \in W(t_0)^q;\ \forall t \geq t_0:\ x(t+1) = A(t^{-1})x(t)\right\}. \tag{41}$$

Remark 2.6. Notice that $W(t_0)$ is not an $\mathbf{A}$-, but only an $\mathbf{A}(\rho)$-left module for $t_0 > \rho^{-1}$. In contrast to the well-known algebraic structure of $\mathbf{A}$ that of $\mathbf{A}(\rho)$ and its f.g. left modules is unknown. To enable a module-behavior duality between f.g. $\mathbf{A}$-left modules and behaviors the behaviors from (40) have to be modified as in [4, (7), (9)].

The subspace $\mathbb{C}^{(\mathbb{N})} \subset \mathbb{C}^{\mathbb{N}}$ consists of the signals $w \in \mathbb{C}^{\mathbb{N}}$ with finite support $\mathrm{supp}(w) := \{t \in \mathbb{N};\ w(t) \neq 0\}$. The factor space $W(\infty) := \mathbb{C}^{\mathbb{N}}/\mathbb{C}^{(\mathbb{N})}$ is an $\mathbf{A}$-left module with the action $f \circ (w + \mathbb{C}^{(\mathbb{N})}) := v + \mathbb{C}^{(\mathbb{N})}$ where

$$f = \sum_j f_j s^j \in \mathbf{A}(\rho) \subset \mathbf{A}, v(t) := \begin{cases} \sum_j f_j(t^{-1})w(t+j) & \text{if } t > \rho^{-1} \\ 0 & \text{if } t \leq \rho^{-1} \end{cases}. \tag{42}$$

This signal space $W(\infty)$ was already defined in [14, p. 5]. In [3, Thm. 839] it was shown for the coefficient field $\mathbb{C}(z)$ instead of $\mathbb{C} << z >>$ here that it is a *large injective* $\mathbf{A}$*-cogenerator* and thus enables a module-behavior duality. This signal module $W(\infty)$ is unsuitable for the stability theory of LTV systems since the signals $w + \mathbb{C}^{(\mathbb{N})}$ do not have well-defined values $w(t) \in \mathbb{C}$ and in particular no initial value $w(t_0)$. So (uniform) exponential stability of state space behaviors as in (12) or of general behaviors [4, Def. 1.7] cannot be defined.

Two behavior families

$$\mathcal{B}_i := \left(\mathcal{B}(R_i, t_0)\right)_{t_0 > \rho_i^{-1}},\ R_i \in \mathbf{A}(\rho_i)^{p_i\times q} \subset \mathbf{A}^{p_i\times q},\ \ i = 1, 2, \tag{43}$$

are called *equivalent*, in signs $\mathcal{B}_1 \equiv \mathcal{B}_2$, if and only if

$$\exists t_1 > \max(\rho_1^{-1}, \rho_2^{-1})\forall t_2 \geq t_1:\ \mathcal{B}(R_1, t_2) = \mathcal{B}(R_2, t_2). \tag{44}$$

Since t_1 can be chosen large one may always assume that $\rho_i = \rho(R_i)$ for $i = 1, 2$. The equivalence class of $\mathcal{B}_1$ is denoted by $\mathrm{cl}(\mathcal{B}_1)$. The study of this class means to study the behaviors $\mathcal{B}(R_1, t_2)$ for large t_2. This is appropriate for stability questions where the trajectories $w(t)$ of a behavior are studied for $t \to \infty$.

If M is a f.g. $\mathbf{A}$-module with a given list $\mathbf{w} := (\mathbf{w}_1, \cdots, \mathbf{w}_q)^\top$ of generators there is the canonical isomorphism

$$\begin{gathered}\mathbf{A}^{1\times q}/U \cong M,\ \xi + U \mapsto \xi\mathbf{w} = \sum_i \xi_i \mathbf{w}_i \text{ where} \\ \xi = (\xi_1, \cdots, \xi_q) \in \mathbf{A}^{1\times q},\ U := \left\{\xi \in \mathbf{A}^{1\times q};\ \xi\mathbf{w} = 0\right\}.\end{gathered} \tag{45}$$

Since $\mathbf{A}$ is noetherian the submodule U is f.g. and thus generated by the rows of some matrix $R \in \mathbf{A}^{p\times q}$, i.e., $U = \mathbf{A}^{1\times p}R$. Since $\mathbf{A}$ is even a principal ideal domain U is free and one may assume that $\dim_{\mathbf{A}}(U) = \mathrm{rank}(R) = p$. The matrix R gives rise to behaviors

$$\mathcal{B}(R, t_0),\ t_0 > \sigma(R),\ \text{and their class}\ \mathrm{cl}\left(\left(\mathcal{B}(R, t_0)\right)_{t_0 > \sigma(R)}\right). \tag{46}$$

It turns out that this class depends on U only and not on the special choice of R [4, Lemma 2.5], is denoted by

$$\mathcal{B}(U) := \mathrm{cl}\left(\left(\mathcal{B}(R, t_0)\right)_{t_0 > \sigma(R)}\right),\ U = \mathbf{A}^{1\times p}R \subseteq \mathbf{A}^{1\times q}, \tag{47}$$

and called *the behavior defined by* U. These behaviors $\mathcal{B}(U)$ were introduced in [4, (9)] for the coefficient field $\mathbb{C}(z)=\mathbb{C}(z^{-1})$ of rational functions instead of $\mathbf{K}\supset\mathbb{C}(z)$.

Remark 2.7. The following three properties of the coefficient sequences $\left(a(t^{-1})\right)_{t>\sigma(a)}$, $a\in\mathbf{K}$, are decisive:

1. Any nonzero $a\in\mathbb{C}<z>$ can be written as $a=a_0+zc(z)$ where $c(z)$ is bounded for $|z|\leq\rho<\rho(a)$ and hence $a(t^{-1})=a_0+t^{-1}c(t^{-1})$ with bounded $c(t^{-1})$ for $t\geq t_0>\sigma(a)$. For large t the term $t^{-1}c(t^{-1})$ is a small disturbance of the constant a_0 and thus the perturbation results [16, Thm. 24.7], [4, Lemma 3.15] are applicable.

2. The sequences are p.g.f., indeed

$$\begin{aligned} &a=z^{-m}b,\ m\geq 0,\ b\in\mathbb{C}<z>\\ &\Longrightarrow \exists c>0\forall t\geq t_0>\sigma(a):\ |a(t^{-1})|\leq ct^m. \end{aligned} \tag{48}$$

3. The sequences have no zeros for large t, i.e., $\exists t_1>\sigma(a)\forall t\geq t_1:\ a(t^{-1})\neq 0$.

Result 2.8. *(*Meta-theorem*) With the obvious necessary modifications all essential notions and results from [4] hold for the coefficient field* $\mathbf{K}$ *and the behaviors* $\mathcal{B}(U)$ *defined in* (43)*.*

For the preceding result one checks that the proofs of [4] use the properties of Remark 2.7 only. In particular, there is a canonical definition of behavior morphisms such that the behaviors $\mathcal{B}(U)$ with these morphisms form an abelian category and the assignment $M=\mathbf{A}^{1\times q}/U\mapsto\mathcal{B}(U)$ is a categorical duality from f.g. $\mathbf{A}$-left modules with a given finite list of generators to behaviors [4, Cor. 2.7,Thm. 1.6]. If $M_i=\mathbf{A}^{1\times q_i}/U_i,\ U_i=\mathbf{A}^{1\times p_i}R_i,\ R_i\in\mathbf{A}^{p_i\times q_i},\ i=1,2$, are two modules any $\mathbf{A}$- linear map $\varphi:M_1\to M_2$ has the form

$$\begin{aligned} &\varphi=(\circ P)_{\mathrm{ind}}:M_1\to M_2,\ \xi+U_1\mapsto\xi P+U_2,\ \text{where}\\ &P\in\mathbf{A}^{q_1\times q_2},\ \exists X\in\mathbf{A}^{p_1\times p_2}\ \text{with}\ R_1P=XR_2. \end{aligned} \tag{49}$$

If $R_1,R_2,P,X\in\mathbf{A}(\rho)^{\bullet\times\bullet}$, for instance if $\rho<\min(\rho(R_1),\rho(R_2),\rho(P),\rho(X))$, and if $t_0>\rho^{-1}$ Cor. 2.5 implies

$$\begin{aligned} &\forall w\in W(t_0)^{q_2}:\ R_1\circ(P\circ w)=X\circ(R_2\circ w)\\ &\Longrightarrow P\circ:\ \mathcal{B}(R_2,t_0)\to\mathcal{B}(R_1,t_0),\ w_2\mapsto P\circ w_2. \end{aligned} \tag{50}$$

The equivalence class $\mathcal{B}(\varphi):=\mathrm{cl}\left((P\circ:\mathcal{B}(R_2,t_0)\to\mathcal{B}(R_1,t_0))_{t_0>\rho^{-1}}\right)$ is defined as in (44), cf. [4, (33),(50)], and defines the behavior morphism

$$\mathcal{B}(\varphi):\mathcal{B}(U_2)\to\mathcal{B}(U_1). \tag{51}$$

The map φ is an isomorphism if and only if $\mathcal{B}(\varphi)$ is one, i.e., if $P\circ:\mathcal{B}(R_2,t_1)\to\mathcal{B}(R_1,t_1)$ is an isomorphism for sufficiently large t_1.
A f.g. module $M=\mathbf{A}^{1\times q_1}/U_1$ is a torsion module, cf. [4, §2.5], if and only if $n:=\dim_{\mathbf{K}}(M)<\infty$ or if and only if it is isomorphic to a module in state space form, i.e.,

$$\begin{aligned} &M=\mathbf{A}^{1\times q_1}/U_1\cong M_2=\mathbf{A}^{1\times n}/U_2,\ U_2=\mathbf{A}^{1\times n}(s\,\mathrm{id}_n-A),\ A\in\mathbf{K}^{n\times n}\\ &\mathcal{B}(U_2)=\mathrm{cl}\left((\mathcal{B}(s\,\mathrm{id}_n-A,t_0))_{t_0>\sigma(A)}\right)\cong\mathcal{B}(U_1). \end{aligned} \tag{52}$$

The behavior

$$\mathcal{B}(U) = \mathrm{cl}\left((\mathcal{B}(R,t_0))_{t_0>\sigma(R)}\right) \text{ with } R \in \mathbf{A}^{p\times q},\ U = \mathbf{A}^{1\times p}R,\ M = \mathbf{A}^{1\times q}/U \tag{53}$$

is called *autonomous* if and only if there is $t_1 > \sigma(R)$ such all trajectories $w \in \mathcal{B}(R,t_2),\ t_2 \geq t_1$, are determined by the initial vector $(w(t_2),\cdots,w(t_2+d)$ of some fixed length d. This is the case if and only if the module $M = \mathbf{A}^{1\times q}/U$ is a torsion module.
Exponential stability (e.s.) of an autonomous behavior $\mathcal{B}(U)$ and of its torsion module $M = \mathbf{A}^{1\times q}$ [4, Def. 1.7] is defined by a more general version of (12) and preserved by behavior isomorphisms [4, §3.2]. The e.s. behaviors and modules form Serre categories, i.e., are closed under subobjects, factor objects and extensions [4, Thms. 1.8, 3.11].

3 The proof of Thm. 1.1

(i) Since A is a power series we can write

$$\begin{aligned} &A(z) = A_0 + zC(z) \in \mathbb{C} < z >^{q\times q},\ C(z) = A_1 + A_2 z + \cdots \in \mathbb{C} < z >^{q\times q} \\ &\Longrightarrow A(t^{-1}) = A_0 + t^{-1}C(t^{-1}),\ t \geq t_0 > \sigma(A). \end{aligned} \tag{54}$$

The function $C(t^{-1})$ is bounded for $t \geq t_0$ and therefore $t^{-1}C(t^{-1})$ is a disturbance term that is small for large t.
(a) If $\mathrm{spec}(A_0) \subset \mathbf{D}(1)$ the system $x(t+1) = A_0 x(t),\ t \geq t_0$, is uniformly exponentially stable (u.e.s.). According to [16, Thm. 24.7], [4, Cor. 3.17] the equation $x(t+1) = A(t^{-1})x(t),\ t \geq t_0$, is also u.e.s. and therefore e.s..
(b) Assume that A_0 has an eigenvalue λ with $|\lambda| > 1$. According to [4, Thm. 3.21] the system is exponentially unstable and, in particular,

$$\exists t_0 > \sigma(A) \exists \rho > 1 \forall t \geq t_1 \geq t_0 :\ \|\Phi_A(t,t_1)\| \geq \rho^{t-t_1} \Longrightarrow \sup_{t\geq t_1} \|\Phi_A(t,t_1)\| = \infty. \tag{55}$$

This implies that the system $x(t+1) = A(t^{-1})x(t)$ is not e.s..
(c) Assume that A_0 has an eigenvalue λ with $|\lambda| = 1$ and that the system $x(t+1) = A(t^{-1})x(t),\ t > \sigma(A)$, is e.s. By (12)

$$\begin{aligned} &\exists t_0 > \sigma(A) \exists \alpha > 0,\ \rho := e^{-\alpha} < 1, \exists \text{ p.g.f. } \varphi \in \mathbb{C}^{t_0+\mathbb{N}},\ \varphi > 0, \\ &\forall t \geq t_1 \geq t_0 :\ \|\Phi_A(t,t_1)\| \leq \varphi(t_1)\rho^{t-t_1}. \end{aligned} \tag{56}$$

Now consider the modified system

$$\begin{aligned} &y(t+1) = e^{\alpha}A(t^{-1})y(t) = \rho^{-1}A(t^{-1})y(t),\ t > \sigma(A), \text{ with} \\ &\rho^{-1}A(z) = (\rho^{-1}A_0) + z(\rho^{-1}C(z)). \end{aligned} \tag{57}$$

The matrix $\rho^{-1}A_0$ has the eigenvalue $\rho^{-1}\lambda$ with $|\rho^{-1}\lambda| = \rho^{-1} = e^{\alpha} > 1$. From (b) we infer

$$\begin{aligned} &\exists t_2 \geq t_1 \forall t \geq t_3 \geq t_2 :\ \sup_{t\geq t_3} \|\Phi_{\rho^{-1}A}(t,t_3)\| = \infty. \text{ But} \\ &\Phi_{\rho^{-1}A}(t,t_3) = \rho^{-(t-t_3)}\Phi_A(t,t_3) \\ &\Longrightarrow \|\Phi_{\rho^{-1}A}(t,t_3)\| = \rho^{-(t-t_3)}\|\Phi_A(t,t_3)\| \leq \rho^{-(t-t_3)}(\varphi(t_3)\rho^{t-t_3}) = \varphi(t_3). \end{aligned} \tag{58}$$

The first and the last line of (58) are in contradiction and therefore $x(t+1)=A(t^{-1})x(t)$ cannot be e.s..
This completes the proof of part (i) of the theorem.
(ii) Assume $A(z)=z^{-k}B(z),\ k>0,\ B(z)=B_0+B_1z+\cdots\in\mathbb{C}<z>^{q\times q}$. Also assume that B_0 is not nilpotent and thus has a nonzero eigenvalue λ. Choose $\rho>|\lambda|^{-1}$ so that $|\rho\lambda|>1$ for the eigenvalue $\rho\lambda$ of the matrix ρB_0. According to (i)(b) ρB is not stable and

$$\begin{aligned}
&\exists t_0>\sigma(A)=\sigma(\rho B)\forall t\geq t_1\geq t_0:\ \sup_{t\geq t_1}\|\Phi_{\rho B}(t,t_1)\|=\infty.\text{ But}\\
&A(t^{-1})=t^k\rho^{-1}(\rho B)(t^{-1}),\ \Gamma(t,t_1):=(t-1)*\cdots*t_1\\
&\Longrightarrow\forall t\geq t_1\geq t_0:\ \Phi_A(t,t_1)=\Gamma(t,t_1)^k\rho^{-(t-t_1)}\Phi_{\rho B}(t,t_1)\\
&\Longrightarrow\|\Phi_A(t,t_1)\|=\Gamma(t,t_1)^k\rho^{-(t-t_1)}\|\Phi_{\rho B}(t,t_1)\|.\text{ Further}\\
&\Gamma(t,t_1)^k\rho^{-(t-t_1)}=\frac{(t-1)^k}{\rho}*\cdots*\frac{t_1^k}{\rho}\underset{t\to\infty}{\longrightarrow}\infty\\
&\Longrightarrow\forall t_1\geq t_0:\ \sup_{t\geq t_1}\|\Phi_A(t,t_1)\|=\infty.
\end{aligned}\tag{59}$$

(iii) Under the condition of Thm. 1.1,(iii), choose $t_0>\sigma(A)$ such that $|c(t^{-1})|\geq 1/2$ for $t\geq t_0$. The determinant of

$$\begin{aligned}
&A(z)=z^{-k}B(z)\text{ is }\det(A(z))=z^{-kq}\det(B(z))=b_\ell z^{-(kq-\ell)}c(z)\\
&\Longrightarrow\forall t\geq t_0:\ |\det(A(t^{-1}))|=|b_\ell|t^{kq-\ell}|c(t^{-1})|\geq(|b_\ell|/2)t^{kq-\ell}\\
&\Longrightarrow\forall t\geq t_1\geq t_0:\ |\det(\Phi_A(t,t_1))|\geq\Gamma(t,t_1)^{kq-\ell}(|b_\ell|/2)^{t-t_1}\underset{t\to\infty}{\longrightarrow}\infty
\end{aligned}\tag{60}$$

where the last implication follows as in (59) due to $kq-\ell>0$. If the sequence $\|\Phi_A(t,t_1)\|,\ t\geq t_1$, was bounded so would be the sequence of determinants $|\det(\Phi_A(t,t_1))|$.

4 Puiseux series and weak exponential stability

To a large extent Thm. 1.1 can be extended to the difference field $\mathbf{P}$ of *locally convergent Puiseux series*, cf. [5, §3.1], i.e.,

$$\mathbf{P}:=\bigcup_{m\geq 1}\mathbb{C}<<z^{1/m}>>;\Phi(z)=z(1+z)^{-1},\ \Phi(z^{1/m})=z^{1/m}(1+z)^{-1/m}.\tag{61}$$

and its Φ-invariant Bézout and valuation subdomain

$$\mathbf{L}:=\bigcup_{m\geq 1}\mathbb{C}<z^{1/m}>.\tag{62}$$

The field $\mathbf{P}$ is the algebraic closure of $\mathbf{K}=\mathbb{C}<<z>>$ [11]. If m_1 divides m_2 then $z^{1/m_1}=(z^{1/m_2})^{m_2/m_1}$ and hence $\mathbb{C}<z^{1/m_1}>\subseteq\mathbb{C}<z^{1/m_2}>$. The nonzero elements of $\mathbb{C}<<z^{1/m}>>$ have the unique form

$$\begin{aligned}
&a(z^{1/m})=z^{k/m}b(z^{1/m}),\ a=z^kb\in\mathbb{C}<<z>>,\\
&k\in\mathbb{Z},\ b=\sum_i b_iz^i\in\mathbb{C}<z>,\ b_0\neq 0.
\end{aligned}\tag{63}$$

Such an $a(z^{1/m})$ induces the smooth function $a(t^{-1/m}) = t^{-k/m}\sum_i b_i t^{-i/m}$ on the real interval $(\sigma(a)^m, \infty)$. The coefficient rings give rise to the operator domains

$$\mathbf{L}[s;\Phi] \subseteq \mathbf{B} := \mathbf{P}[s;\Phi]; = \oplus_{j\in\mathbb{N}}\mathbf{P}s^j. \tag{64}$$

Again $\mathbf{B}$ is a left and right euclidean domain. A matrix

$$\begin{aligned} &R = \sum_j R_j(z^{1/m})s^j \in \mathbf{B}^{p\times q} \text{ with} \\ &R_j(z^{1/m}) = \Big(R_{j\mu\nu}(z^{1/m})\Big)_{\mu,\nu} \in \mathbb{C} << z^{1/m} >>^{p\times q}, \\ &\sigma(R) := \max\nolimits_{j,\mu,\nu} \sigma(R_{j\mu\nu}(z))^m. \end{aligned} \tag{65}$$

acts on $w \in W(t_0)^q = (\mathbb{C}^q)^{t_0+\mathbb{N}}$ via

$$\begin{aligned} &(R\circ w)(t) := \sum_j R_j(t^{-1/m})w(t+j),\ t_0 > \sigma(R), \\ &\mathcal{B}(R,t_0) := \{w \in W(t_0)^q;\ R\circ w = 0\}. \end{aligned} \tag{66}$$

In particular, a matrix $A(z^{1/m}) \in \mathbf{P}^{q\times q}$ induces the state space equation and behaviors

$$\begin{aligned} &x(t+1) = A(t^{-1/m})x(t),\ t \ge t_0 > \sigma(A)^m, \\ &\mathcal{B}(A(z^{1/m}),t_0) := \mathcal{B}(s\,\mathrm{id}_q - A(z^{1/m}),t_0) \\ &\qquad = \Big\{x \in W(t_0)^q;\ x(t+1) = A(t^{-1/m})x(t)\Big\} \end{aligned} \tag{67}$$

Result 4.1. *Since $\mathbf{P}$ satisfies the conditions of Remark 2.7 the notions and theorems of [4] also hold for $\mathbf{P}$ like for $\mathbb{C}(z)$ in [4] and for $\mathbf{K} = \mathbb{C} << z >>$ in Section 2.*

Thm. 1.1 holds with the following modification.

Corollary 4.2. *Consider $A(z^{1/m}) \in \mathbb{C} << z^{1/m} >>^{q\times q}$ and the system $x(t+1) = A(t^{-1/m})x(t),\ t > \sigma(A)^m$.*

(i) If $A(z^{1/m}) = A_0 + A_1 z^{1/m} + \cdots \in \mathbb{C} < z^{1/m} >^{q\times q}$ the system is e.s. if and only if $\mathrm{spec}(A_0) \subset \mathbf{D}(1)$.

(ii) If $A(z^{1/m}) = z^{-k/m}B(z^{1/m}),\ B(z^{1/m}) = \sum_{i=0}^{\infty} B_i z^{i/m}$, $k > 0$ and B_0 is not nilpotent then

$$\exists t_0 > \sigma(A)^m \forall t_1 \ge t_0 :\ \sup_{t\ge t_1} \|\Phi_{A(t^{-1/m})}(t,t_1)\| = \infty \tag{68}$$

and the system is not e.s..

(iii) If for $A(z^{1/m})$ as in (ii) the determinant of $\det(B(z))$ has the form $\det(B(z)) = b_\ell z^\ell c(z) \neq 0$ where $c(z) = 1 + \sum_{i=1}^{\infty} c_i z^i$ and $kq - \ell > 0$ then (68) holds and the system is not e.s..

One can weaken the definition of exponential stability to *weak exponential stability* (w.e.s.) as in [5, Def. 2.4]. This is also preserved by behavior isomorphisms. For the state space system (67) w.e.s. holds if and only if

$$\exists t_0 > \sigma(A)^m \exists \rho = e^{-\alpha} < 1\ (\alpha > 0) \exists \mu > 0 \exists \text{ p.g.f. } \varphi \in \mathbb{C}^{t_0+\mathbb{N}},\ \varphi > 0, \\ \forall t_1 \geq t_0:\ \|\Phi_{A(t^{-1/m})}(t, t_1)\| \leq \varphi(t_1)\rho^{t^\mu - t_1^\mu} = \varphi(t_1)e^{-\alpha(t^\mu - t_0^\mu)}. \tag{69}$$

The difference to e.s. is the exponent $\mu > 0$. If $\mu < 1$ the factor $\exp(-\alpha t^\mu)$ decreases more slowly than $\exp(-\alpha t)$. Cor. 4.2 obtains a slightly weaker form.

Corollary 4.3. *Cor. 4.2 remains true for w.e.s. instead of e.s. with the following exception in part (i): If $A(z^{1/m}) = \sum_{i=0}^{\infty} A_i z^{i/m} \in \mathbb{C} < z^{1/m} >^{q\times q}$ then the system is e.s. and hence w.e.s. resp. unstable if $\mathrm{spec}(A_0) \subset \mathbf{D}(1)$ resp. if $|\lambda| > 1$ for at least one eigenvalue of A_0. If $|\lambda| \leq 1$ for all eigenvalues of A_0 and $|\lambda| = 1$ for at least one the system may be w.e.s. in contrast to Cor. 4.2,(i), cf. Example 4.4.*

Example 4.4. (Cf. [14, §6.1,§7.1]) Let $\alpha > 0$, $\rho := e^{-\alpha}$, $\mu = k_0/m,\ 0 < k_0 < m.$ The signal $x(t) := e^{-\alpha t^\mu} = \rho^{t^\mu}$ satisfies

$$\begin{aligned}
&x(t+1)/x(t) = \exp\left(-\alpha((t+1)^\mu - t^\mu)\right). \text{ But} \\
&(t+1)^\mu - t^\mu = t^\mu\left((1+t^{-1})^\mu - 1\right) = \left(z^{-\mu}\left((1+z)^\mu - 1\right)\right)(t^{-1}) \\
&z^{-\mu}\left((1+z)^\mu - 1\right) = z^{-\mu}\sum_{k=1}^{\infty}\binom{\mu}{k}z^k = \sum_{k=1}^{\infty}\binom{\mu}{k}\left(z^{1/m}\right)^{km-k_0} \in \mathbb{C} < z^{1/m} > z^{1/m} \\
&\Longrightarrow a(z^{1/m}) := \exp\left(-\alpha z^{-\mu}\left((1+z)^\mu - 1\right)\right) = 1 - \alpha\mu z^{1-\mu} + \cdots \in \mathbb{C} < z^{1/m} > \\
&\Longrightarrow a(t^{-1/m}) = \exp\left(-\alpha((t+1)^\mu - t^\mu)\right),\ x(t+1) = a(t^{-1})x(t), \\
&\Longrightarrow x(t) = \exp\left(-\alpha(t^\mu - t_0^\mu)\right)x(t_0),\ |x(t)| = \exp\left(-\alpha(t^\mu - t_0^\mu)\right)|x(t_0)|.
\end{aligned} \tag{70}$$

This equation is w.e.s. with exponent μ. But $a(0) = 1$ and the equation is not e.s by Cor. 4.2.

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