Acta Applicandae Mathematicae **20**: 1–175, 1990.

Multidimensional Constant Linear Systems

ULRICH OBERST*

Institut für Mathematik der Universität Innsbruck, Technikerstrasse 25, A-6020 Innsbruck, Austria

(Received: 12 July 1990)

Abstract. A continuous resp. discrete r-dimensional ($r \geq 1$) system is the solution space of a system of linear partial differential resp. difference equations with constant coefficients for a vector of functions or distributions in r variables resp. of r-fold indexed sequences. Although such linear systems, both multidimensional and multivariable, have been used and studied in analysis and algebra for a long time, for instance by Ehrenpreis *et al.* thirty years ago, these systems have only recently been recognized as objects of special significance for system theory and for technical applications. Their introduction in this context in the discrete one-dimensional ($r = 1$) case is due to J. C. Willems. The main duality theorem of this paper establishes a categorical duality between these multidimensional systems and finitely generated modules over the polynomial algebra in r indeterminates by making use of deep results in the areas of partial differential equations, several complex variables and algebra. This duality theorem makes many notions and theorems from algebra available for system theoretic considerations. This strategy is pursued here in several directions and is similar to the use of polynomial algebra in the standard one-dimensional theory, but mathematically more difficult. The following subjects are treated: input-output structures of systems and their transfer matrix, signal flow spaces and graphs of systems and block diagrams, transfer equivalence and (minimal) realizations, controllability and observability, rank singularities and their connection with the integral respresentation theorem, invertible systems, the constructive solution of the Cauchy problem and convolutional transfer operators for discrete systems. Several constructions on the basis of the Gröbner basis algorithms are executed. The connections with other approaches to multidimensional systems are established as far as possible (to the author).

AMS subject classifications (1980). Primary: 93B25 Secondary: 35E20, 35N05, 39A10, 44A40, 44A55, 93B05, 93B07, 93B15, 93B20, 93C05, 93C20, 94C99.

Key words. multidimensional system, injective cogenerator, duality, fundamental principle, input-output system, transfer matrix, signal flow, realization, Cauchy problem, convolutional transfer operator, operational calculus, invertible system, Gröbner basis.

0. INTRODUCTION

This paper is based on the three preprints [OB2].

Since the last century "function" modules **A** over the polynomial algebra $F[s]=F[s_1,\cdots,s_r]$ with coefficients in a field F have played an important role in the theory of differential or difference equations, the operation of F[s] on **A** being given by *r* pairwise commuting F-linear endomorphisms

$$L_i : \mathbf{A} \longrightarrow \mathbf{A}\ ,\ a \longmapsto L_i(a) = s_i a\ ,\ i=1,\cdots,r\ .$$

In the typical *continuous case* $\mathbf{A}:=C^\infty(\mathbb{R}^r)$ is the $\mathbb{C}$-vector space of complex-valued C^∞-functions $a=a(t_1,\cdots,t_r)$ on $\mathbb{R}^r$ with the operation $L_i(a)=\partial_i a := \partial a/\partial t_i$, $i=1,\cdots,r$. For a monomial $s^m = s_1^{m(1)}\cdots s_r^{m(r)} \in \mathbb{C}[s]$, $m=(m(1),\cdots,m(r)) \in \mathbb{N}^r$, this means

$$s^m a = \partial^{|m|} a/\partial t_1^{m(1)}\cdots\partial t_r^{m(r)}\ ,\ |m|:=m(1)+\cdots+m(r)\ .$$

In the typical *discrete case* $\mathbf{A}:=F^{\mathbb{N}^r}$ is the vector space of multiindexed sequences $a=(a(n);\ n\in\mathbb{N}^r)=(a(n(1),\ \cdots,n(r));\ n(i)\in\mathbb{N}\ ,i=1,\cdots,r)$ with entries in the field F , and the operation of F[s] on $F^{\mathbb{N}^r}$ is given by the *left shifts* $(s^m a)(n)=a(m+n)$, $m,n\in\mathbb{N}^r$.

Starting some thirty years ago R. Kalman , H. Rosenbrock and many others have used well-known properties of the polynomial ring in one variable ($r=1$) over a field to develop the *state space , the geometric and the polynomial approach* to one-dimensional system theory , the independent variable being usually *time* , by considering a *vector space* **A** of *signals* as a module over the polynomial ring . Also around 1960 Ehrenpreis , Malgrange and Palamodov (see [EH] , [MAL] , [PAL]) exploited the $\mathbb{C}[s]$-module structure of $C^\infty(\mathbb{R}^r)$, the space $\mathcal{D}'(\mathbb{R}^r)$ of distributions and other spaces to prove the fundamental existence theorems for systems of linear partial differential equations with constant coefficients . In this paper I develop multidimensional system theory ($r\geq 1$) in analogy to the "classical" models by making use of the results of Ehrenpreis et al. and of similar results in the discrete case.

For a F[s]-module **A** of signals a **A**-system is a subspace of some $\mathbf{A}^l$, $l\in\mathbb{N}$, of the form

(1) $$S=\{ w=(w_1 \cdots w_l)^T \in \mathbf{A}^l ;\ Rw=0\}$$

where $R \in F[s]^{k,l}$ is a polynomial matrix. With the entries $R_{ij}=\sum\{ R_{ij}(m)s^m ;\ m \in \mathbb{N}^r\} \in F[s]$ of R the defining equation $Rw=0$ of (1) means

$$\sum\{ R_{ij}(m)L_1^{m(1)} \cdots L_r^{m(r)}(w_j) ;\ j=1,\cdots,l,\ m \in \mathbb{N}^r\}=0 \text{ for } i=1,\cdots,k .$$

In the discrete case $\mathbf{A}=F^{\mathbb{N}^r}$ this signifies

$$\sum\{ R_{ij}(m)\, w_j(m+n) ;\ m \in \mathbb{N}^r ,\ j=1,\cdots,l\}=0 \text{ for all } i=1,\cdots,k \text{ and } n \in \mathbb{N}^r$$

whereas in the continuous case with $\mathbf{A}=C^\infty(\mathbb{R}^r)$ one obtains the system of partial differential equations

$$\sum\{ R_{ij}(m)\partial_1^{m(1)} \cdots \partial_r^{m(r)} w_j(t) ;\ j=1,\cdots,l ,\ m \in \mathbb{N}^r \}=0 ,\ i=1,\cdots,k ,$$

with constant coefficients $R_{ij}(m)$. In the "classical" case with $r=1$, $F=\mathbb{R}$ and $\mathbf{A}=\mathbb{R}^{\mathbb{N}}$ these systems were introduced by J.C.Willems in [WIL] under the name of *AR-systems* . Continuous systems of this type ($r=1$, $F=\mathbb{R}$, $\mathbf{A}=C^\infty(\mathbb{R})$) are treated as *differential input-output systems* in the book [BY] of Blomberg and Ylinen . There are several points of contact between this book and my paper in the one-dimensional continuous case. The defining matrix $R \in F[s]^{k,l}$ of S in (1) gives also rise to the finitely generated $F[s]$-module $M=F[s]^l/R^TF[s]^k$ where R^T is the transposed matrix of R and $R^TF[s]^k$ its column module in $F[s]^l$. The main result of this paper is that the correspondence

(2) $$M=F[s]^l/R^TF[s]^k \longmapsto S=\{ w \in \mathbf{A}^l ;\ Rw=0 \}$$

is one-one , more precisely a *categorical duality* between finitely generated $F[s]$-modules and $\mathbf{A}$-systems as introduced above . Hence the linear and commutative algebra over $F[s_1,\cdots,s_r]$ is applicable to multidimensional system theory as in the one-dimensional situation . An easy , but important observation for the duality $M \longleftrightarrow S$ is the isomorphism

(3) $$\mathrm{Hom}_{F[s]}(F[s]^l/R^TF[s]^k ,\mathbf{A}) \cong S ,\ f \longmapsto (f(\bar{e}_1),\cdots,f(\bar{e}_l))^T$$

where $e_1,\cdots,e_l$ denote the standard basis vectors in $F[s]^l$ and $\bar{e}_1,\cdots,\bar{e}_l$ their residue classes in M . For systems of partial differential equations this observation is due to Malgrange [MAL] . The isomorphism (3) leads to the question under which condition on $\mathbf{A}$ the contravariant Hom-functor $M \mapsto \mathrm{Hom}_{F[s]}(M ,\mathbf{A})$ induces a duality on finitely generated modules . The prototype of such a duality is the Pontrjagin duality between discrete and compact abelian groups where $\mathbf{A}:=S^1=\{z\in\mathbb{C}\,;\ |z|=1\}$ is the circle group . The general question was completely answered in [ROO] and [OB1] . The necessary and sufficient condition on $\mathbf{A}$ is that it is a *large injective cogenerator* (see §2 for this notion) . It is the main work of this paper to show that the F[s]-modules $\mathbf{A}$ from system theory like $F^{\mathbb{N}^r}$ and $C^\infty(\mathbb{R}^r)$ above are large injective cogenerators . This algebraization also suggests to investigate $\mathbf{A}$-systems for any large injective cogenerator $\mathbf{A}$ over an arbitrary commutative noetherian ring $\mathbf{D}$. This is done in §2 . In particular the discrete and the continuous cases can be treated at the same time. For one-dimensional system theory this analogy between discrete and continuous systems is well-known , but the explanations of this observation are often insufficient .

In §1 I introduce several F[s]-modules $\mathbf{A}$ which are interesting for system theory . Besides $F^{\mathbb{N}^r}$ and $C^\infty(\mathbb{R}^r)$ as above I consider *convergent power series* $\mathbb{C}\langle t_1,\cdots,t_r\rangle\subset\mathbb{C}\{t\}:=\mathbb{C}^{\mathbb{N}^r}$ in the discrete case and the $\mathbb{C}[s]$-modules of *distributions* $\mathcal{D}'(\mathbb{R}^r)$ and of *entire functions* $\mathcal{O}(\mathbb{C}^r)$ in the continuous case and also their *real models* . *Delay-differential* equations can be formulated in this frame-work , but my main theorems are not directly applicable in this case .

The main technical work of this paper is contained in the paragraphs 3 and 4 where I prove for the discrete and the continuous cases respectively that the F[s]-modules $\mathbf{A}$ from §1 are large injective cogenerators .The injectivity of $\mathbf{A}$ in the continuous cases is one of the main results of Ehrenpreis et al. ([EH] , [MAL] , [PAL]). The proof of the large cogenerator property in the continuous case makes heavy use of the theory of complex spaces and the valuable books [GR1] , [GR2] and [GR3] by Grauert and Remmert . In the discrete case I consider an arbitrary affine F-algebra $\mathbf{D}$ and its dual space $\mathbf{A}:=\mathrm{Hom}_F(\mathbf{D},F)$ as a $\mathbf{D}$-module . The case $\mathbf{D}=F[s_1,\cdots,s_r]$ and $\mathbf{A}=\mathrm{Hom}_F(F[s],F)=F\{t_1,\cdots,t_r\}=F^{\mathbb{N}^r}$ from above is of this type . In particular , I characterize systems $S\subset\mathbf{A}^l$ as $\mathbf{D}$-submodules of $\mathbf{A}^l$ which are also closed with respect to the canonical *linearly compact* topology on $\mathbf{A}$, thus generalizing the corresponding theorem of Willems in [WIL] . In §3,(59) , I explain the connection with Macaulay's inverse system of a modular system.

Some important results of abstract system theory from §2 are explained here for the polynomial algebra F[s] instead of an arbitrary commutative noetherian ring **D** as in §2 . These results are applicable , in particular , to all F[s]-modules **A** from §1 . A typical application of the injectivity of **A** in system theory is the following result (theorem 2.34) : If $S=\{w_1\in \mathbf{A}^{l(1)}\ ;\ R_1 w_1=0\}$ is a system and $P\in F[s]^{l(2),l(1)}$ is a matrix the image $PS=\{Pw_1\,;\,w_1\in S\}$ is a system too , and indeed $PS=\{w_2\in \mathbf{A}^{l(2)}\,;\,R_2w_2=0\}$ where R_2 and some Y are universal matrices with $R_2P=YR_1$ (see (2.28) for this notion). A particular case gives the abstract *Rosenbrock systems* (2.41).

The main duality theorem (2.56) yielding the 1-1-correspondence $M \longleftrightarrow S$ from (2) is verbally taken from [OB1] . The first main consequence of the *full* duality theorem is the *quasi-uniqueness* of the defining matrix (Cor. 2.63) : If a system $S=\{w\in \mathbf{A}^l;R_1w=0\}=\{w\in \mathbf{A}^l;R_2w=0\}$ is defined by two polynomial matrices R_1 and R_2 there are polynomial matrices X_1 and X_2 such that $R_2=X_1R_1$ and $R_1=X_2R_2$.

An important consequence of this is the existence of invariant *input-output structures* and their associated *transfer matrices* for a system $S=\{w\in \mathbf{A}^l;\,Rw=0\}$, $R\in F[s]^{k,l}$. As a matrix with coefficients in the *field* F(s) of rational functions the matrix has a rank p . Put $m := l-p$. These numbers depend on S only and not on the special defining matrix R (theorem 2.65) For any p-tupel of F(s) - linearly independent columns of R and after a corresponding column permutation this matrix can be written as

$$R=(-Q,P\ )\ ,\ Q\in F[s]^{k,m}\ ,\ P\in F[s]^{k,p}\ ,\ \mathrm{rank}(R)=\mathrm{rank}(P)=p\ .$$

With the corresponding decomposition $w=(u,y)\in \mathbf{A}^l=\mathbf{A}^{m+p}$ of the signal vector one obtains

$$S=\{(u\,y)\in \mathbf{A}^l\ ;\ Py=Qu\} \tag{4}$$

I call $w=(u\,y)$ a *IO-structure* and (4) a *IO-form* of S . The IO-form (4) is characterized by the fact that for every $u\in \mathbf{A}^m$ there is a $y\in \mathbf{A}^p$ such that $(u\,y)\in S$ (theorem 2.69) . Therefore u resp. y are called an *input* resp. *output vector* for S and m resp. p the *input* resp. *output dimension* . Moreover there is a unique rational matrix $H \in F(s)^{p,m}$ satisfying $Q=PH$, and H depends on S and its IO-structure $(u\, y)$, but not on the special choice of the matrices R, P, Q. This H is the *transfer matrix* of S with respect to the chosen IO-structure. In the one-dimensional case input-output structures are discussed in [BY] and [WIL]. Contrary to the one-dimensional case a system S does not admit a IO-representation (4) with a *proper* transfer matrix in general. Indeed, generically, the transfer matrix is not proper for $r>1$. Therefore the properness of H is not included in the definition of a IO-structure as it is done in [WIL] for the one-dimensional case. In §6 I define a *transfer operator* or *function* $H(R): \mathbf{A}^m \longrightarrow \mathbf{A}^p$ for systems with a *weakly proper* transfer matrix H by means of a suitable *operator calculus* and derive many of its standard properties known from the case $r=1$.

At the end of §2 I also consider the systems of linear equations with coefficients in the field $F(s)$

$$R\widehat{w}=0 \text{ resp. } P\widehat{y}=Q\widehat{u}\ ,\ \widehat{w}\in F(s)^l\ ,\ \widehat{y}\in F(s)^p\ ,\ \widehat{u}\in F(s)^m\ ,$$

and show that the assignment

$$S=\{w\in \mathbf{A}^l ; Rw=0\} \longmapsto \widehat{S}:=\{\widehat{w}\in F(s)^l ;\ R\widehat{w}=0\}$$

is an exact functor (theorem 2.91). The main assertion here is that $\widehat{S}$ depends on S only and not on the special choice of the matrix R. The $F(s)$-vector space $\widehat{S}$ contains the complete information on the transfer matrix H of S with respect to the IO-structure $(u\, y)$ (theorem 2.94). I call $\widehat{S}$ the *signal flow space* of S in generalization of the *signal flow* or *Mason* graph of a *block diagram* which is of great significance in the engineering literature (see [CH]). In § 8 of this paper I will give some graph theoretic consequences of the assignment $S \longmapsto \widehat{S}$ like *series* and *parallel connection, feedback constructions* etc. for *block diagrams* of systems.

Finally, system theory for a large injective cogenerator $\mathbf{A}$ over $\mathbf{D}$ induces the same for the $\mathbf{D}/\mathbf{J}$-module $\{a\in \mathbf{A}; pa=0 \text{ for all } p\in \mathbf{J}\}$ where $\mathbf{J}$ is any ideal of $\mathbf{D}$ (theorem 2.99). I give several interesting examples.

In §5 I formulate and solve the *canonical Cauchy problem* for discrete IO-systems. In this context I was influenced by ideas from J. Gregor [GRE].

The system is given as in (4), the signal space being $\mathbf{A}=F^{\mathbb{N}^r}$, F a field. Using the theory of standard or Gröbner bases (compare [BU] and [PAU]) I *canonically* construct finite subsets $D(j)\subset\mathbb{N}^r, D(j)\neq\emptyset$, $j=1,\cdots,p$, and the derived sets

$$G'=\{(j,d+n);\ j=1,\cdots,p, d\in D(j), n\in\mathbb{N}^r\},$$

$$G=([p]\times\mathbb{N}^r)\setminus G'=\{(j,n); j=1,\cdots, p,\ n\in\mathbb{N}^r,\ (j,n)\notin G'\}$$

(Corollary 5.33) such that the *initial value* or *Cauchy problem*

(5) $\quad P(L)(y)=Q(L)(u), y|G=x$, i.e. $y_j(n)=x_j(n)$ for all $(j,n)\in G$,

for arbitrarily given $u\in\mathbf{A}^m$ and $x=(x_j(n);(j,n)\in G)\in F^G$ has a unique solution $y=(y_j(n);\ j=1,\cdots,p\ ,n\in\mathbb{N}^r)\in(F^{\mathbb{N}^r})^p$ (theorem 5.41). Theorem 5.63 contains an *algorithm* for the calculation of each component $y_j(n)$ in *finitely* many steps. This algorithm depends on the *Gröbner basis algorithm* due to Buchberger [BU] and others. Contrary to the continuous case of partial differential equations where the initial value problem can be solved uniquely only for hyperbolic systems (compare [TR], Ch. II) theorem 5.41 proves that for discrete systems of partial difference equations the initial value or Cauchy problem can always be canonically defined and constructively solved. The solution of the Cauchy problem gives also rise to the isomorphism

(6) $\quad \ker(P(L))=\{y\in\mathbf{A}^p;\ P(L)(y)=0\}\cong F^G,\ y\mapsto y|G$

and the *transfer operators*

(7) $\quad \tilde{K}:\mathrm{im}(P(L))\longrightarrow\mathbf{A}^p,\ v\longmapsto\tilde{K}(v),\ \tilde{H}:=\tilde{K}Q(L):\mathbf{A}^m\longrightarrow\mathbf{A}^p,\ u\longmapsto\tilde{H}(u)$

given uniquely by the equations

$$P(L)(\tilde{K}(v))=v,\ P(L)(\tilde{H}(u))=Q(L)(u),\ \tilde{K}(v)|G=\tilde{H}(u)|G=0.$$

I call F^G the *canonical state space* of the system $S=\{(uy);Py=Qu\}$. With the same technique I prove in theorem 5.71 that for arbitrary large injective cogenerators $\mathbf{A}$ over $F[s]=F[s_1,\cdots,s_r]$ the IO-system $S\subset\mathbf{A}^{m+p}$ with given IO-structure (u,y) can be uniquely characterized by a pair (P^{rg},H) of matrices where $P^{rg}\in F[s]^{l,p}$, $l:=|(D(1))|+\cdots+|(D(p))|$, is the *reduced* Gröbner matrix of S and $H\in F(s)^{p,m}$ its transfer matrix satisfying $P^{rg}H\in F[s]^{l,n}$. In other words, (P^{rg},H) is a *complete system of invariants* for $S \subset \mathbf{A}^{m+p}$ with given IO-structure (u,y).

The questions of §6, but not the results were again inspired by [GRE]. The starting point is Willems' observation [WIL] that in the one-dimensional discrete case any system $S \subset (F^{\mathbb{N}})^{m+p} = (F\{t\})^{m+p}$ admits a IO-structure with proper transfer matrix H(s). This means that H(s) is a power series in $t := s^{-1}$, i.e. $H(s) \in (F(s) \cap F\{t\})^{p,m}$. Moreover it is then well-known that the transfer operator $\tilde{H}$ from (7) is given by convolution with H, i.e. $\tilde{H}(u) = H * u$ where $u = (\sum_n u_j(n) t^n;\ j = 1, \cdots, m) \in F\{t\}^m$ is also considered as a power series vector. Also if P is square and P^{-1} is proper then $\tilde{K}(v) = P^{-1} * v$. In §6 I investigate the same questions for the multidimensional discrete situation, i.e. under which circumstances the transfer operators $\tilde{H}$ and $\tilde{K}$ are *given by convolution with* H respectively $P(s)^{-1}$.

In theorem 6.86 I show for square matrices $P \in F[s]^{p,p}$ that the operator $\tilde{K}$ from (7) is given by convolution if and only if P is column reduced. Thus the main results (6.101), (6.106) and (6.109) of §6 concern systems with column reduced P. By several examples - the easiest is (6.118) - I motivate why it is appropriate to consider systems over rings instead of fields. In the main results I therefore consider systems

(8)
$$S = \{uy) \in \mathbf{A}^{m+p};\ P(L)(y) = Q(L)(u)\},\ \mathbf{A} = \mathbf{B}^{\mathbb{N}^r} = \mathbf{B}\{t_1, \cdots, t_r\},$$
$$P \in K[s_1, \cdots, s_r]^{p,p}, \det(P) \neq 0, Q \in K[s]^{p,m},\ PH = Q$$

where K is a noetherian, regular, factorial integral domain, for instance a polynomial ring over a field and where **B** is a faithful K-module. The case K=**B**=F leads back to the systems (4), the standard non-field example is given by $K = F[\sigma_1, \cdots, \sigma_\rho]$, $\mathbf{B} = F^{\mathbb{N}^\rho} = F\{\tau_1, \cdots, \tau_\rho\}$, $K[s] = F[\sigma_1, \cdots, \sigma_\rho, s_1, \cdots, s_r]$ and $\mathbf{A} = F\{\tau_1, \cdots, \tau_\rho, t_1, \cdots, t_r\}$.

The first part of §6 introduces and investigates *(weakly) proper rational* functions and matrices and develops an *operator calculus* for these (see 6.35, 6.40, 6.45). These results are then applied to the formulation and solution of various *Cauchy problems with transfer operators given by* *convolution* . Theorem 6.74 is a discrete analogue of the Cauchy-Kowalewskaja theorem. The main theorems (6.101) resp. (6.106) resp. (6.109) formulate and solve the Cauchy problem for *column reduced* resp. *row reduced* resp. both column and row reduced matrices $P \in K[s]^{p,p}$. Theorem 6.109 is then applied to generalized Roesser systems (6.113) and to systems introduced and investigated by E. Fornasini, G. Marchesini (see, for instance, [FM3]) and by L. Baratchart [BA]and others (see [TZ]) . In an appendix to §6 I indicate a multidimensional operator or operational calculus for arbitrary $F[s_1,\cdots,s_r]$-modules $\mathbf{A}$ instead of $F^{\mathbb{N}^r}$ as in §6 where I follow the one-dimensional model of [BE] and [PR] . This calculus is applicable to $\mathbf{A}=C^{\infty}(\mathbb{R}^r)$ over $\mathbb{C}[s_1,\cdots,s_r]$, but not to $\mathcal{D}'(\mathbb{R}^r)$.

The first part of §7 is devoted to transfer equivalence and realization theory . Here I explain the results for a large injective cogenerator over $F[s_1,\cdots,s_r]$, F a field . Systems S_1 and S_2 in $\mathbf{A}^l$ are *transfer equivalent* if their signal flow spaces $\widehat{S}_1$ and $\widehat{S}_2$ in $F(s)^l$ coincide . A *realization* of a $F(s)$-subspace V of $F(s)^l$ is a system $S \subset \mathbf{A}^l$ with $\widehat{S}=V$. Theorem 7.17 characterizes transfer equivalence and theorem 7.19 the lattice of all systems S which realize a given V . In (7.21) I show that every such V admits a *unique minimal* realization S_{min}, and that an arbitrary realization $S=\{w \in \mathbf{A}^l; Rw=0\}$, $R \in F[s]^{k,l}$, is minimal if and only if its module $M(S)=F[s]^l/R^T F[s]^k$ is *torsionfree* or if and only if R is *left factor prime* . In (7.24) and (7.25) I develop an algorithm for the construction of the unique minimai realization which uses the Gröbner basis algorithms , for instance from [PAU] . The realization of a transfer matrix $H \in F(s)^{p,m}$ is defined to be a IO-system with H as transfer matrix or , equivalently , with $\widehat{S}=\mathrm{graph}(H) \subset F(s)^{m+p}$. The preceding theory is applicable .

In the second section of §7 I characterize systems $S=\{w; Rw=0\}$ whose module $M=F[s]^l/R^T F[s]^k$ has a small projective dimension . That M has projective dimension ≤ 1 signifies that one or all IO-structures of S admit a representation

(9) $\quad S=\{(u\,y) \in \mathbf{A}^{m+p}; Py=Qu\}, P \in F[s]^{p,p}$, $\det(P) \neq 0$, $PH=Q$,

i.e. with *square* P and hence $H=P^{-1}Q$ (Theorem 7.39) . The latter equation is usually called a left MFD (*left matrix fraction description*) of H . Every matrix H trivially admits such a left MFD , but the minimal realization

$$S_{min} = \{ (u y) \in \mathbf{A}^{m+p} \ ; \ P_{min} y = Q_{min} u \} \ , \ P_{min} H = Q_{min} \ ,$$

of H is , in general , not of the type (9) . By (7.42) this is , however , the case for the polynomial ring $F[s_1, s_2]$ in two indeterminates and explains to a certain extent why multidimensional system theory has been mainly developed in the 2-d case . Theorems 7.52 resp. 7.53 characterize those systems whose module is torsionfree of projective dimension ≤ 1 resp. free and which are minimal in particular . In [ROC] these systems are called weakly resp. strongly *controllable* in the case $r=2$, $\mathbf{D}=\mathbb{R}[s_1, s_2, s_1^{-1}, s_2^{-1}]$ and $\mathbf{A}=\mathbb{R}^{\mathbb{Z}^2}$. These results generalize [WOL] , Th. 5.3.1 (a) . In theorem 7.63 I similarly define and characterize multidimensional *observable* Rosenbrock systems .

The last section of §7 is devoted to the discussion of *rank singularities* which are used in [BFM] for *stability* and *stabilization* problems in the 2-d case . For a IO-system

$$S=\{(u y) \in \mathbf{A}^{m+p} \ ; P(L)(y) = Q(L)(u)\} , P \in \mathbb{C}[s]^{k,p} , \mathrm{rank}(P)=p \ , \ PH=Q ,$$

in the continuous case of partial differential equations and for the torsion system module

$M:=\mathbb{C}[s]^p / P^T \mathbb{C}[s]^k$ of $\ker(P)=\{y \in \mathbf{A} ; P(L)(y)=0\} \cong S \cap (0 \times \mathbf{A}^p) , y \longleftrightarrow (0,y)$, the algebraic subset $RS(M):=\{\zeta \in \mathbb{C}^r ; \mathrm{rank}(P(\zeta)) < p\} \subset \mathbb{C}^r$ of rank singularities coincides with the support of M and is called the *characteristic variety* of M or P in [BJ], Ch.8 . In an appendix to §7 I take the opportunity to explain the important and difficult *integral representation theorem* from [EH], Ch.7, and [PAL], Ch.6 , in the presentation of [BJ], Ch.8, and its connection with RS(M): Any solution y of the "overdetermined" system $P(L)(y)=0$ is a finite sum of integral solutions

(10) $$\int a(\tau, t) e^{\tau \cdot t} d\mu(\tau) \ , \ \tau \in \mathbb{C}^r \ , \ t \in \mathbb{R}^r \ , \ \tau \cdot t := \tau_1 t_1 + \cdots + \tau_r t_r \ ,$$

where μ is a measure on $\mathbb{C}^r$ and where the

$$a(\tau,t) e^{\tau \cdot t} \ , \ a(\tau,t) \in \mathbb{C}[\tau, t]^p \ , \ \tau \in \mathrm{Supp}(\mu) = \text{support of } \mu \ ,$$

are so-called *exponential solutions* with $P(L_t)(a(\tau,t)e^{\tau \cdot t}) = 0$. In particular

$$\{\tau \in \mathbb{C}^r \ ; a(\tau, -) \neq 0 \ , \ \tau \in \mathrm{Supp}(\mu)\} \subset RS(M) \ ,$$

i.e. those τ which really contribute to the integral (10) lie in RS(M) . In the discrete cases one has to replace $e^{\tau \cdot t}$ by $\sum\{\tau^n t^n ; n \in \mathbb{N}^r\}$. I believe that the integral representation theorem will be of great use for multidimensional system theory , in particular for stability and stabilization questions as in [BOS2],Ch.3, and [BFM] and for the investigation of the "state space" ker(P) . But more work is required to realize this potential .

In the last paragraph 8 I introduce and investigate *multidimensional block diagrams* and their *associated systems* and give several applications . I took the 1-d models mainly from [KUO] , [CH] and [WOL] . Theorem 8.9 characterizes under which conditions the signals at the *initial nodes* of a block diagram can be chosen as the inputs of a IO-structure . In (8.12) , (8.14) and (8.18) I explain *parallel* , *series* and *feedback* compositions of systems and their transfer matrices . I use these results for the characterization of (*left*, *right*) *invertible systems* along the lines of [WOL], §5.5 , and finally treat the algebraic side of the problem of *exact model matching* .

Open problems : (i) This paper contains several algorithms , most of them based on the standard or Gröbner basis algorithms of Buchberger et al. , for instance (5.63) , (5.71) , (5.81) , (6.111-112) , (7.24-25) , (7.29) , (8.9) and (8.29). It is a major , but, modulo the available program packages , not too difficult task to write programs for these algorithms . (ii) The study of the integral representation theorem and its implications for multidimensional systems as explained above . (iii) The discrete theory applies to the large injective cogenerator $F^{\mathbb{Z}^r}$ over $F[s_1, s_1^{-1}, \cdots, s_r, s_r^{-1}]$. The results of §5 and §6 , in particular the Cauchy problem , however , were only obtained for $F^{\mathbb{N}^r}$ and should be formulated and proven for $\mathbb{Z}^r$ instead of $\mathbb{N}^r$ since , for instance , $\mathbb{Z}^2$ is a better model for the discrete coordinates of an image than $\mathbb{N}^2$ (compare [ROC] and [RW]) . ||

It is possible to understand the system theoretic results in the paragraphs 2 and 5 to 8 of this paper without going into the many technical details of the paragraphs 3 and 4 . Whereas the techniques used in this paper are new for system theory as far as I know, the basic notions and the formulation of many results are derived from the one-dimensional models as , hopefully , every system theorist can realize .

Notations: As usual , the letters $\mathbb{N},\mathbb{Z},\mathbb{R},\mathbb{C}$ denote the sets of natural numbers , integers , real resp. complex numbers . If X and I are sets the functions x from I to X or elements of X^I are written as $x=(x(i);i\in I)=(x_i)_{i\in I}$. The kernel , image resp. cokernel of a linear map $f:M\longrightarrow N$ between modules over some ring R are $\ker(f)$, $\mathrm{im}(f)$ resp. $\mathrm{cok}(f):=N/\mathrm{im}(f)$; $\mathrm{Hom}_R(M,N)$ is the group or even R-module (for commutative R) of all R-linear maps from M to N . The symbol $||$ denotes the end of a theorem or proof . Inside a paragraph theorems , lemmas , formulas etc. are numbered consecutively . The expressions (j) resp. (i.j) mean the j.th theorem etc. in the current resp. the i.th paragraph .

Acknowledgements : The starting point for this work was the paper [WIL] by J.C.Willems . I did part of the research for this paper while I was a visiting professor at the Department of Mathematics and the Center for Mathematical System Theory of the University of Florida in Gainesville . I am grateful to Professor Rudolf Kalman and the other colleagues of the department for their kind invitation and this valuable opportunity . I thank an anonymus referee for his careful reading of the §§ 1-4 and for his suggestions and corrections . My colleague Dr. Franz Pauer was very helpful with the theory of standard or Gröbner bases . I thank our secretary Gerda Nail for typing most of this paper in "Signum" and my colleague Dr. Hubert Herdlinger for his help with this text processing system .

Table of contents:

1. SYSTEMS OF PARTIAL DIFFERENTIAL OR DIFFERENCE EQUATIONS

Almost all mathematical models for real systems require a class of functions $a=a(t)$ which are admitted as "signals". In the one-dimensional case the independent variable t is usually *time*. As a mathematical model for the physical time one usually chooses the set $\mathbb{R}$ of real numbers in the so called *continuous* case respectively the sets $\mathbb{Z}:=\{\cdots -1,0,1,\cdots\}$ or $\mathbb{N}:=\{0,1,2\cdots\}$ of integers in the *discrete* case. Numerical calculations in the continuous case also lead to the discrete situation by "discretization". The values $a(t)$ of the signals a are customarily taken from the field $\mathbb{C}$ of complex numbers, for digital signals the binary field $\mathbb{Z}_2:=\mathbb{Z}/\mathbb{Z}2=\{0,1\}$ or another finite field are often more appropriate.

The *multidimensional* systems of this paper are distinguished by the fact that $t=(t_1,\cdots,t_r)$ consists of $r\geq 1$ independent variables $t_1,\cdots,t_r$ which are interpreted as time, space or generalized coordinates of some phase space. For $r=2$ a suitable and customary interpretation of $t=(t_1,t_2)$ is that as the two planar coordinates of an image. In analogy to the standard one dimensional case I take the sets $\mathbb{R}^r$ respectively $\mathbb{N}^r$ or $\mathbb{Z}^r$ as the (definition) domains of the signals and talk of the *continuous* respectively the *discrete* case.

(1) **Assumption and interpretation:** Let F be a field which, in most cases, is interpreted as the set of admissible values of the signals. The standard examples are the fields $\mathbb{R}$ respectively $\mathbb{C}$ of real resp. complex numbers or the finite fields. Let $\mathbf{A}$ be an F-vector space which is interpreted as the set of admissible signals. In most cases $\mathbf{A}$ will be a subspace of the function space

$$\mathbb{C}^{\mathbb{R}^r}:=\{a\ ;\ a:\mathbb{R}^r\longrightarrow \mathbb{C}\}\ ,\ F=\mathbb{C}\ ,$$

in the continuous case or of

$$F^{\mathbb{N}^r}:=\{\ a;\ a:\mathbb{N}^r\longrightarrow F\},\ F \text{ arbitrary},$$

in the discrete case. Since I consider *linear* systems only the vector space structure of $\mathbf{A}$ is justified and assumed as usual. ||

The systems considered in this paper like in the larger part of the literature are given by differential or difference equations, in this paper restricted to linear equations with constant coefficients. To formulate these equations we need the partial difference or derivative

(2) $$L_i: \mathbf{A} \longrightarrow \mathbf{A}\ ,\ i=1,\cdots,r\ ,$$

in the i.th direction. These operators commute and induce on $\mathbf{A}$ the structure of a module over the polynomial algebra $F[s]:=F[s_1,\cdots,s_r]$ in r indeterminates s_i via

(3) $$p\,a:=p(L_1,\cdots,L_r)(a),\ p\in F[s_1,\cdots,s_r],\ a\in\mathbf{A}$$ where

(4) $$F[s_1,\cdots,s_r] \longrightarrow \mathrm{End}_F(\mathbf{A}),\ p\longmapsto p(L_1,\cdots,L_r)\ ,$$

is the substitution homomorphism. Here $\mathrm{End}_F(\mathbf{A})$ denotes the F-algebra of F-linear maps from $\mathbf{A}$ in to itself. We are thus led to the following

(5) **Assumption and Definition:** Let

$$\mathbf{D}:=F[s]=F[s_1,\cdots,s_r]$$

be the polynomial algebra in r indeterminates over F. The vector space $\mathbf{A}$ of "admissible signals" is assumed to be a $\mathbf{D}$-module. The scalar multiplication with s_i is denoted by

(6) $$L_i : \mathbf{A} \longrightarrow \mathbf{A}\ ,\ a \longmapsto L_i\ (a):=s_i a.\ \ ||$$

The main examples

(7) **Formal power series:** Let

(8) $$\mathbf{A}=F^{\mathbb{N}^r}:=F\{t\}=F\{t_1,\cdots,t_r\}$$

be the F-vector space of formal power series

(9) $$a=(a(n);n\in\mathbb{N}^r)=\sum\{a(n)t^n;n\in\mathbb{N}^r\}=$$
$$=\sum\{a(n(1),\cdots,n(r))t_1^{n(1)}\cdots t_r^{n(r)};\ n(1),\cdots,n(r)\in\mathbb{N}\}$$

in r indeterminates $t_1,\cdots,t_r$. Such a power series a is nothing else than a multiindexed sequence $a=(a(n);n\in\mathbb{N}^r)$ of elements in F, i.e. an arbitrary function $a:\mathbb{N}^r\longrightarrow F$, $n\longmapsto a(n)$, from the lattice $\mathbb{N}^r$ to F . In some situations the notation

$$a_n=a_{n(1),\cdots,n(r)}:=a(n(1),\cdots,n(r)),\ n=(n(1),\cdots,n(r))\in\mathbb{N}^r,$$

is preferable. With the usual componentwise F-vector space structure and with convolution as multiplication this $\mathbf{A}:=F\{t\}$ is a commutative

F-algebra. The convolution, however, plays a role only in the later developments in context with the transfer function. The $\mathbf{D}=F[s_1,\cdots,s_r]$-module structure is given by the " *left shifts* "

(10) $L_i: F^{\mathbb{N}^r} \longrightarrow F^{\mathbb{N}^r}, a \mapsto L_i(a), i=1,\cdots,r,\ L_i(a)(n)=a(n(1),\cdots,n(i)+1,\cdots,n(r))$

via (3). Inductively one obtains

(11) $$(s^m a)(n)=L^m(a)(n)=a(m+n), m,n\in\mathbb{N}^r,$$

$$\text{where } s^m=s_1^{m(1)}\cdots s_r^{m(r)} \text{ and } L^m=L_1^{m(1)}\cdots L_r^{m(r)}.$$

This implies

(12) $$(pa)(n)=p(L)(a)(n)=\sum\{p_m a(m+n)\ ;\ m\in\mathbb{N}^r\}$$

for a polynomial $p=\sum p_m s^m$ in $F[s]$. Primarily this example gives rise to the terminology "*left shift*". For $r=1$ one obtains the usual space $\mathbf{A}=F^{\mathbb{N}}$ of sequences $a=(a(0)\ a(1)\ a(2)\ \cdots)$ with the left shift

$$L: a=(a(0)\ a(1)\ a(2)\cdots)\longmapsto(a(1)\ a(2)\ \cdots).$$

In the same fashion one can consider the space $F^{\mathbb{Z}^r}$ of functions *a* on the whole integral lattice $\mathbb{Z}^r$. The left shifts are given by (10) and are bijective. The convolution, however, is not defined on all of $F^{\mathbb{N}^r}$. ||

(13) **Convergent power series:** Let $F=\mathbb{C}$ be the field of complex numbers and

(14) $$\mathbb{C}\langle t\rangle\subset\mathbb{C}\{t\}$$

the subspace of $\mathbb{C}\{t\}$ of *convergent* power series (see [GR1] or (4.3)). A power series

$$a=(a_n; n\in\mathbb{N}^r)=\sum\{a_n t^n; n\in\mathbb{N}^r\}\in\mathbb{C}\{t\}$$

is called convergent if for some positive real numbers $R_1,\cdots,R_r$ the series

$$\sum |a_n| R^n := \sum |a_n| R_1^{n(1)}\cdots R_r^{n(r)} < \infty$$

converges (to a real number, not to ∞). The details are explained in § 4. The space $\mathbb{C}\langle t\rangle$ is trivially a $\mathbf{D}=\mathbb{C}[s]$-submodule of $\mathbb{C}\{t\}$ and thus itself a $\mathbb{C}[s]$-module as required in (5). The $\mathbb{C}[s]$-module $\mathbb{C}\langle t\rangle\subset\mathbb{C}\{t\}=\mathbb{C}^{\mathbb{N}^r}$ contains all those spaces of sequences usually investigated in system theory like the space

(15) $$l^\infty:=\{a\in\mathbb{C}\{t\}; \mathrm{Sup}_n|a_n|<\infty\}$$

of bounded or *stable* sequences, the space

(16) $$\{a\in\mathbb{C}\{t\};\ \lim_n a_n=0\}$$

of *asymptotically stable* or zero sequences and the spaces

(17) $$l^p := \{a = (a_n; n \in \mathbb{N}^r); \sum |a_n|^p < \infty\} \quad \text{where } 0 < p < \infty .$$

(13 real) Over the real field $\mathbb{R}$ there is the $\mathbb{R}[s]$-submodule $\mathbb{R}\langle t\rangle \subset \mathbb{C}\langle t\rangle$ of convergent power series with real coefficients such that $\mathbb{C}\langle t\rangle$ is the complexification of $\mathbb{R}\langle t\rangle$. This means the the canonical map $\mathrm{can}: \mathbb{C}\otimes_{\mathbb{R}} \mathbb{R}\langle t\rangle \xrightarrow{\cong} \mathbb{C}\langle t\rangle$ is a $\mathbb{C}[s] = \mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}[s]$ - isomorphism . In this fashion the main theorem for **A** in the complex case induces the same result for the real models $\mathbf{A}_{\mathbb{R}}$ over $\mathbb{R}[s]$. ||

(18) **Differentiable functions:** Let

$$C^\infty(\mathbb{R}^r) := \{a : \mathbb{R}^r \longrightarrow \mathbb{C};\ a \text{ is infinitely often differentiable}\}$$

be the space of complex valued C^∞-(=infinitely often differentiable) functions. This space becomes a $\mathbb{C}[s_1,\cdots,s_r]$-module via

(19) $$s_i a := L_i(a) := \partial_i(a) := \partial a/\partial t_i,\ i = 1,\cdots,r ,$$

where $a = a(t_i,\cdots,t_r) \in C^\infty(\mathbb{R}^r)$. This is the classical space in which one looks for solutions of partial differential equations. From (19) one derives

(20) $$s^m a = L^m(a) = \partial^m a = \partial^{|m|} a/\partial t_1{}^{m(1)}\cdots\partial t_r{}^{m(r)},\ m \in \mathbb{N}^r ,$$

where $|m| = m(1) + \cdots + m(r)$ and

(21) $$pa = p(\partial_1,\cdots,\partial_r)(a) = \sum\{p_m \partial^m a; m \in \mathbb{N}^r\}$$

for a polynomial $p = \sum p_m s^m \in \mathbb{C}[s]$.

(18 real) The real model of $\mathbf{A} := C^\infty(\mathbb{R}^r)$ is its $\mathbb{R}[s]$-submodule

$$\mathbf{A}_{\mathbb{R}} := C^\infty_{\mathbb{R}}(\mathbb{R}^r) := \{a : \mathbb{R}^r \longrightarrow \mathbb{R};\ a \text{ is } \infty \text{ - often differentiable}\}$$

which again induces the canonical $\mathbb{C}[s] = \mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}[s]$ - isomorphism $\mathbb{C}\otimes_{\mathbb{R}} C^\infty_{\mathbb{R}}(\mathbb{R}^r) \cong C^\infty(\mathbb{R}^r)$. ||

(22) **Distributions:** The solutions of partial differential equations may have singularities. Out of this and other reasons analysts prefer to work with distributions instead of C^∞-functions. As usual, let

(23) $$\mathcal{D}'(\mathbb{R}^r) = \{a;\ a \text{ is a distribution on } \mathbb{R}^r\}$$

be the $\mathbb{C}$- vector space of distributions on $\mathbb{R}^r$ (see [SCH], [HÖ1]) or any book on partial differential equations). The $\mathbb{C}[s]$-module structure on $\mathcal{D}'(\mathbb{R}^r)$ is given by differential operators as in (19) , (20) and (21). Other admissible $\mathbb{C}[s]$-modules of distributions are the subspaces

$\mathcal{D}'^F(\mathbb{R}^r) \subset \mathcal{D}'(\mathbb{R}^r)$ of distributions of *finite order* and the spaces $\mathcal{D}'(\Omega)$ or $\mathcal{D}'^F(\Omega)$ of distributions on an open convex subset Ω of $\mathbb{R}^r$. See § 4 for the details.

(22 real) The real-valued distributions are given by $\mathcal{D}'_{\mathbb{R}}(\mathbb{R}^r) := \widehat{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{D}_{\mathbb{R}}(\mathbb{R}^r), \mathbb{R})$ where $\mathcal{D}_{\mathbb{R}}(\mathbb{R}^r)$ is the $\mathbb{R}$-subspace of $C^{\infty}_{\mathbb{R}}(\mathbb{R}^r)$ of functions with compact support and where $\widehat{\mathrm{Hom}}$ denotes the continuous $\mathbb{R}$-linear functionals . Again $\mathcal{D}'(\mathbb{R}^r)$ is the complexification of $\mathcal{D}'_{\mathbb{R}}(\mathbb{R}^r)$ as $\mathcal{D}(\mathbb{R}^r)$ is that of $\mathcal{D}_{\mathbb{R}}(\mathbb{R}^r)$. || .

(24) **Entire functions:** As usual denote by $\mathcal{O}(\mathbb{C}^r) \subset \mathbb{C}\langle t\rangle \subset \mathbb{C}\{t\}$ the space of all entire functions $a=a(t)=a(t_1,\cdots,t_r)$, $t \in \mathbb{C}^r$, which are , by definition , holomorphic in the whole space $\mathbb{C}^r$ and thus representable as power series $a=\sum\{a_n t^n; n \in \mathbb{N}^r\} \in \mathbb{C}\langle t\rangle$ which converge everywhere. The $\mathbb{C}[s]$- structure is given by differentiation as in (19), (20) and (21). Thus $\mathcal{O}(\mathbb{C}^r)$ is not a $\mathbb{C}[s]$-submodule of $\mathbb{C}\langle t\rangle$ with the structure from (7); the easy connection between the two structures is discussed in §4 . By

(25) $$\mathcal{O}(\mathbb{C}^r;\exp) \subset \mathcal{O}(\mathbb{C}^r)$$

I denote the $\mathbb{C}[s]$-submodule of all entire functions of *exponential growth* . The details are discussed in (4.27). ||

For the examples (7) to (24) above I will prove the main theorem (2.54) of this paper. There are, however, F[s]-modules **A** of interest for system theory for which the main theorem of is not valid.

(26) **Delay-differential equations:** In the simplest case consider the $\mathbb{C}$-vector space $\mathbb{C}^{\infty}(\mathbb{R})$ with the $\mathbb{C}[s_1,s_2]$-module structure given by

(27) $$s_1 a = da/dt \text{ and } (s_2 a)(t) = a(t-1)$$

for $a=a(t) \in C^{\infty}(\mathbb{R})$. Such systems are often dealt with in the context of "systems over rings" (see, for instance, [KAM]). The theory of this paper is not directly applicable. ||

Systems of equations

(28) **Convention** : For any set X and natural numbers k,l the symbol $X^{k,l}$ denotes the set of $k \times l$-matrices with coefficients in X . In particular ,

$X^k := X^{k,1}$ is the set of *column* vectors whereas the set of *row* vectors is denoted by the complete symbol $X^{1,1}$. If X is a module over some ring so is $X^{k,1}$ with the componentwise addition and scalar multiplication . ||

(29) **Assumption** : Let **D** denote an arbitrary commutative ring , not necessarily a polynomial ring , and **A** a **D**-module . ||

The scalar multiplication of a **D**-module M induces **D**-bilinear maps

(30) $$\mathbf{D}^{1,m} \times M^{m,n} \longrightarrow M^{1,n} \ , \ (P,Q) \longmapsto PQ \ ,$$

with $(PQ)_{ij} = \sum \{ P_{ik} Q_{kj} \,; k=1,\cdots,m \}$. In particular , a matrix $R=(R_{ij})_{i,j}$ in $\mathbf{D}^{k,1}$ induces a **D**-linear map

(31) $$R : \mathbf{A}^1 \longrightarrow \mathbf{A}^k \ , \ w=(w_1,\cdots,w_1)^T \longmapsto Rw \ ,$$

which , in turn , implies the **D**-linear map

(32) $$\mathbf{D}^{k,1} \longrightarrow \mathrm{Hom}_{\mathbf{D}}(\mathbf{A}^1, \mathbf{A}^k) \ , \ R \longmapsto (w \longmapsto Rw) \ .$$

Remark that the map (32) is injective if and only if for k=1=1 the map

(33) $$\mathbf{D} \longrightarrow \mathrm{End}_{\mathbf{D}}(\mathbf{A}) \ , \ p \longmapsto (a \longmapsto pa)$$

is injective , i.e. if and only if **A** is a faithful **D**-module . This and much more is true for the F[s]-modules **A** from (7) to (24) . If **A** is a F[s]-module as in (5) the map ($p \longmapsto pa$) is given as $pa = p(L)(a)$ and hence (32) is given by

(34) $$R(L) : \mathbf{A}^1 \longrightarrow \mathbf{A}^k \ , \ w \longmapsto Rw = R(L)(w) \quad \text{where}$$
$$(Rw)_i = \sum_j R_{ij} w_j = \sum_j R_{ij}(L_1,\cdots,L_r)(w_j) \quad \text{for } i=1,\cdots,k \, .$$

Since (32) is **D**-linear its kernel $\{ w ; Rw=0 \}$ is a **D**-submodule of $\mathbf{A}^1$.

(35) **Definition** (System) : Assumption (29) . A system of linear equations in the **D**-module **A** is given as

(36) $Rw = u$, $R \in \mathbf{D}^{k,1}$, $w \in \mathbf{A}^1$, $u \in \mathbf{A}^k$, i.e. as $\sum \{ R_{ij} w_j \ ; \ j=1,\cdots,1 \} = u_i$ for $i=1,\cdots,k$.

For a F[s]-module **A** as in (5) a system (36) is more specifically given in the form $\sum \{ R_{ij}(L_1,\cdots,L_r)(w_j) \,; j=1,\cdots,1 \} = u_i$, $i=1,\cdots,k$, with $L_i(a) = s_i a$, $i=1,\cdots,k$.

The solution space $S := \{ w \in \mathbf{A}^1 ; \ Rw=0 \} \subset \mathbf{A}^1$ of the homogeneous system $Rw=0$ is a **D**-submodule of $\mathbf{A}^1$ and called a **A**- *system* in this paper . ||

I derived this system notion from [WIL] . The general theory of linear systems of partial differential equations with constant coefficients is mainly due to Ehrenpreis , Malgrange and Palamodov (see [EH], [MAL] and [PAL]) . The theory of such systems with *variable* coefficients is much more difficult and the subject of much active research (see , for instance , [KAS] , [KKK] and [BOR]). Multi-dimensional (r-d , r>1) system theory in the engineering sense has been developed in the discrete case of example (7) , primarily for r=2 (image processing) . Survey articles are , for instance , contained in [BOS1] , [BOS2] , [TZ] and [LO] . A new journal is exclusively devoted to multidimensional systems under this title .

2. THE ABSTRACT THEORY OF SYSTEMS

(1) **Assumption:** Let **D** be an arbitrary commutative noetherian ring and **A** a **D**-module. The notations are those from § 1. ||

I am going to develop the theory of **A**-systems according to (1.35) in this general setting, but with strong additional assumptions for the main results. One should always think of the examples (1.7) to (1.24) with the noetherian ring $\mathbf{D} = F[s_1,\cdots,s_r]$ for which the following theory is applicable. As usual I

(2) identify $\mathbf{D}^{k,l} = \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^l,\mathbf{D}^k)$, $R = (w \longmapsto Rw)$

where $\mathrm{Hom}_{\mathbf{D}}(M,N)$ denotes the **D**-module of all **D**-linear maps between **D**-modules M and N.

(3) **Motivation:** Let $R \in \mathbf{D}^{k,l}$ be an arbitrary matrix. Consider the **A**-system

$$S = \{ w = (w_1 \cdots w_l)^T \in \mathbf{A}^l;\ Rw = 0 \} \subset \mathbf{A}^l$$

according to (1.35). The image of the transposed matrix $R^T \in \mathbf{D}^{l,k} = \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^k,\mathbf{D}^l)$ is the column module $R^T\mathbf{D}^k = \mathrm{im}(R^T) \subset \mathbf{D}^l$ of R^T. It gives rise to the quotient module $M := \mathrm{cok}(R^T) := \mathbf{D}^l/R^T\mathbf{D}^k$. An easy, but essential observation (see (13) below for the details) furnishes the **D**-isomorphism

(4) $$\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^l/R^T\mathbf{D}^k,\mathbf{A}) \cong S = \{ w \in \mathbf{A}^l; Rw = 0 \} ,\quad f \longmapsto (f(\bar{e}_1 \cdots, f(\bar{e}_l))^T$$

where $e_1 = (10\cdots0)^T \in \mathbf{D}^l, \cdots, e_l = (0\cdots01)^T$ are the standard basis of $\mathbf{D}^l$ and $\bar{e}_1,\cdots,\bar{e}_l \in M = \mathbf{D}^l/R^T\mathbf{D}^k$ are the corresponding residue classes. In the context of systems of partial differential equations the preceding observation was already used by Malgrange in [MAL]. The isomorphism (3) suggests to consider its left side $\mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A})$ as a system too. ||

Functors of the type $\mathrm{Hom}_{\mathbf{D}}(-,\mathbf{A})$ have been used to give concrete descriptions of the duals of abelian categories. The Pontrjagin duality, for instance, is given by

$$M \longmapsto Hom(M,S^1), M \text{ an abelian group,}$$

where $S^1:=\{z\in\mathbb{C}; |z|=1\}$ is the circle group and Hom(M,S) carries the topology induced from the product topology of $(S^1)^M$, and furnishes a duality between discrete and compact abelian groups. The most general duality of this type was developed in [ROO] ans [OB1] and will be described below. The crucial condition for $Hom_{\mathbf{D}}(-,\mathbf{A})$ to be a complete duality is that **A** is a large injective cogenerator in the sense of the following definitions. I refer to [BOU2], § 1, and the exercises, for the basic notions. The theory from [OB1] is valid in more general situations, but this generality is not necessary for the system theoretic applications of this paper.

Let Mod(**D**) denote the category of all **D**-modules and Modf(**D**) its full subcategory of all finitely generated ones. Let further

(5) $$\mathbf{E}:=End_{\mathbf{D}}(\mathbf{A}):=Hom_{\mathbf{D}}(\mathbf{A},\mathbf{A})=\{e:\mathbf{A}\longrightarrow\mathbf{A}\ \mathbf{D}\text{-linear}\}$$

be the **D**-algebra of all **D**-linear endomorphisms of **A**. Then **A** becomes a left **E**-module via

(6) $$ea:=e(a), e\in\mathbf{E}, \mathbf{a}\in\mathbf{A},$$

and is trivially even a **D**-**E**-bimodule with this structure. Also if M is a **D**-module the **D**-module $Hom_{\mathbf{D}}(M,\mathbf{A})$ becomes a **D**-**E**-bimodule via the **E**-structure

(7) $$ef:=e \text{ composed with } f,\quad e\in\mathbf{E},\ f\in Hom_{\mathbf{D}}(M,\mathbf{A}).$$

Then the canonical **D**-isomorphism

(8) $$Hom_{\mathbf{D}}(\mathbf{D},\mathbf{A})\cong\mathbf{A},\ f\longmapsto f(1),$$

is a **E**-isomorphism too as is, more generally, the **D**-isomorphism

(9) $$Hom_{\mathbf{D}}(\mathbf{D}^l,\mathbf{A})\cong\mathbf{A}^l,\ f\longmapsto(f(e_1),\cdots,f(e_l))^T,$$

where $e_1,\cdots,e_l$ denotes the standard basis. Normally I will identify

(10) $$Hom_{\mathbf{D}}(\mathbf{D}^l,\mathbf{A})=A^l \text{ via (9).}$$

Denoting by Mod(**E**) the category of all left **E**-modules we obtain the contravariant functor

(11) $$Hom_{\mathbf{D}}(-,\mathbf{A}) : Mod(\mathbf{D})^{op}\longrightarrow Mod(\mathbf{E}),\ M\mapsto Hom_{\mathbf{D}}(M,\mathbf{A})$$
$$(f:M\longrightarrow N)\longmapsto(Hom(f,\mathbf{A}):Hom_{\mathbf{D}}(N,\mathbf{A})\longrightarrow Hom_{\mathbf{D}}(M,\mathbf{A}), g\longmapsto gf).$$

Since $\mathbf{D}$ is assumed noetherian every finitely generated $\mathbf{D}$-module M is, up to isomorphism , of the form

(12) $$M:=\mathbf{D}^{l}/R^{T}\mathbf{D}^{k},\ R\in\mathbf{D}^{k,l},$$

and the $\mathbf{D}$-$\mathbf{E}$- isomorphism (10) induces the $\mathbf{D}$-$\mathbf{E}$-isomorphism

(13) $$\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^{l}/R^{T}\mathbf{D}^{k},\mathbf{A})\cong\{w\in\mathbf{A}^{l};Rw=0\}\subset\mathbf{A}^{l},\ f\longmapsto(f(\bar{e}_1,\cdots,f(\bar{e}_l))^{T}$$

Here $\bar{e}_1,\cdots,\bar{e}_l$ are the residue classes in $M=\mathbf{D}^{l}/R^{T}\mathbf{D}^{k}$ of the standard basis vectors $e_1,\cdots,e_l\in\mathbf{D}^{l}$, and $Rw=0$ is short for the linear system

(14) $$\Sigma\{R_{ij}w_j;\ j=1,\cdots,l\}=0\ ,\ i=1,\cdots,k,$$

with $R_{ij}\in\mathbf{D}$ and $w_j\in\mathbf{A}$. The following definition generalizes (1.35).

(15) **Definition and Corollary** ($\mathbf{A}$-systems) : Assumption (1). The $\mathbf{E}$-modules of the form

$$\mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A})\cong\{w\in\mathbf{A}^{l};\ Rw=0\},\ M=\mathbf{D}^{l}/R^{T}\mathbf{D}^{k},$$

and their $\mathbf{E}$-isomorphic copies are called ($\mathbf{A}$-) systems. The full subcategory of Mod($\mathbf{E}$) of all $\mathbf{A}$-systems is denoted by Syst($\mathbf{A}$). The functor $\mathrm{Hom}_{\mathbf{D}}(-,\mathbf{A})$ induces the contravariant functor

$$S:=\mathrm{Hom}_{\mathbf{D}}(-,\mathbf{A}):\mathrm{Modf}(\mathbf{D})^{op}\longrightarrow\mathrm{Syst}(\mathbf{A})$$
$$S(M):=\mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A})\ ,\ S(f):=\mathrm{Hom}(f,\mathbf{A}).$$

For a matrix $R\in\mathbf{D}^{k,l}$ with its transposed matrix $R^{T}:\mathbf{D}^{k}\longrightarrow\mathbf{D}^{l}$ in $\mathbf{D}^{l,k}=\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^{k},\mathbf{D}^{l})$ the diagram

$$\begin{array}{ccc} S(D^{l})=\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^{l},\mathbf{A}) & \xrightarrow{S(R^{T})} & S(\mathbf{D}^{k})=\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^{k},\mathbf{A}) \\ \wr\Vert & & \wr\Vert \\ \mathbf{A}^{l} & \xrightarrow{(w\longmapsto Rw)} & \mathbf{A}^{k} \end{array}$$

commutes where the vertical maps are the canonical identifications (9) or (10). Hence

$$S(R^{T})=\mathrm{Hom}(R^{T},\mathbf{A})=R\ (w\longmapsto Rw)$$

for short. But caution: The map $w\longmapsto Rw$ on $\mathbf{A}^{l}$ may be zero, for instance if $\mathbf{A}$=0 , without R itself being zero. ||

The main goal of this paper is to describe and prove conditions under which S is a (contravariant) equivalence, i.e. a duality, and to point out the system theoretic consequences of this result.

The results of this section can also be expressed by lattice dualities as in projective geometry. For this purpose consider a $n>0$ and the **D**-bilinear map

$$\langle -,- \rangle : \mathbf{D}^n \times \mathbf{A}^n \longrightarrow \mathbf{A}\,,\ (q,w) \longmapsto \langle q,w \rangle := \sum\{q_i w_i\ ;\ i=1,\cdots,n\}$$

where $q=(q_1\cdots q_n)^T \in \mathbf{D}^n, w=(w_1\cdots w_n)^T \in \mathbf{A}^n$ and where $\langle q,w\rangle$ is given like the usual euclidean scalar product. The form $\langle -,- \rangle$ is not only **D**-bilinear, but even **E**-linear in the second variable. The induced map

(17) $$\mathbf{A}^n \cong \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^n, \mathbf{A}),\ w \longmapsto \langle -,w \rangle,$$

is obviously an isomorphism, inverse to (9). If U is a subset of $\mathbf{D}^n$ its orthogonal (polar, conjugate) complement is the **E**–submodule

(18) $$U^\perp := \{w \in \mathbf{A}^n;\ \langle q,w \rangle = 0 \text{ for all } q \in U\}$$

of $\mathbf{A}^n$. Obviously $U^\perp = ({}_{\mathbf{D}}\langle U \rangle)^\perp$ where ${}_{\mathbf{D}}\langle U\rangle$ denotes the (finitely generated) submodule of $\mathbf{D}^n$ generated by U.

(19) **Example:** For a matrix $R \in \mathbf{D}^{k,l}$ the identities

$$R_{i-}w = \sum\{R_{ij}w_j;\ j=1,\cdots,l\} = \langle (R^T)_{-i}, w \rangle,\ i=1,\cdots,k,$$

furnish the equation

(20) $$\{w \in \mathbf{A}^l;\ Rw = 0\} = (R^T\mathbf{D}^k)^\perp \cong \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^l/R^T\mathbf{D}^k, \mathbf{A})\,.$$

In the same fashion one obtains the **D**-submodule

(21) $$S^\perp := \{q \in \mathbf{D}^n;\ \langle q,w \rangle = 0 \text{ for all } w \in S\} \subset \mathbf{D}^n$$

for a subset $S \subset \mathbf{A}^n$ and also $S^\perp = ({}_{\mathbf{E}}\langle S \rangle)^\perp$. If

(22) $$\mathbf{P}(\mathbf{D}^n) \text{ respectively } \mathbf{P}(\mathbf{A}^n)$$

denote the lattices of **D**-submodules of $\mathbf{D}^n$ respectively of **E**-submodules of $\mathbf{A}^n$, ordered by inclusion and usually called the *projective geometries* of $\mathbf{D}^n$ resp. $\mathbf{A}^n$, the maps

(23) $$(-)^\perp : \mathbf{P}(\mathbf{D}^n) \longleftrightarrow \mathbf{P}(\mathbf{A}^n)$$

form a Galois correspondence. This means, by definition, that the maps $(-)^\perp$ reverse the inclusion order and satisfy $U \subset U^{\perp\perp}$ for $U \subset \mathbf{D}^n$, $S \subset S^{\perp\perp}$ for $S \subset \mathbf{A}^n$. In particular $U^\perp = U^{\perp\perp\perp}$, $S^\perp = S^{\perp\perp\perp}$ for $U \subset \mathbf{D}^n, S \subset \mathbf{A}^n$.

The first lemma gives an elementary characterization of certain exact sequences . Recall the following

(24) **Definition** (see [BOU1], Ch .I, § 1.3) : A short sequence

(25) $$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$$

of modules M_i over some ring and of linear maps f_i is called a *complex* if $f_2 f_1 = 0$, or, in other terms, if $\mathrm{im}(f_1) \subset \ker(f_2)$. The sequence is called *exact* if even $\mathrm{im}(f_1) = \ker(f_2)$.

A longer sequence is called a complex respectively exact if all three term subsequences are such. ||

Consider in particular a sequence

$$(26) \qquad \mathbf{D}^k \xrightarrow{Q^T} \mathbf{D}^l \xrightarrow{P^T} \mathbf{D}^m \ , \quad Q \in \mathbf{D}^{k,l}, P \in \mathbf{D}^{l,m} \ .$$

This sequence is a complex if $P^T Q^T = (QP)^T = 0$ or $QP = 0$. It is exact if, additionally, the relation $x^T \in \ker(P^T)$, i.e. $P^T x^T = 0$ or $xP = 0$, implies that $x^T \in \mathrm{im}(Q^T)$, i.e. $x^T = Q^T y^T$ or $x = yQ$, $y \in \mathbf{D}^{1,k}$. We obtain the following easy

(27) **Lemma and Definition:** Let $Q \in \mathbf{D}^{k,l}$ and $P \in \mathbf{D}^{l,m}$ be two matrices. The following assertions are equivalent:

(i) The sequence (26) is exact.

(ii) (a) $QP = 0$. (b) If $XP = 0$ for some other matrix X then there is a matrix Y with $X = YQ$.

If these equivalent conditions are satisfied I call the matrix Q "*universal with respect to* $QP = 0$". For every matrix P there is always a matrix Q universal with respect to $QP = 0$.

Proof: The last statement follows from the noetherianess of **D**. As a submodule of $\mathbf{D}^l$ the kernel $\ker(P^T)$ is finitely generated. A matrix Q such that the columns of Q^T from a finite generator system of $\ker(P^T)$ is universal with $QP = 0$. ||

Generalizing the preceding terminology I make the

(28) **Definition and Corollary:** Consider block matrices $Q = (Q_1, -Q_2)$ and $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ such that $QP = Q_1P_1 - Q_2P_2 = 0$, i.e. $Q_1P_1 = Q_2P_2$. I say that Q_1 and Q_2 are universal with $Q_1P_1 = Q_2P_2$ if Q is universal with $QP = 0$ according to (27). If this is the case and if X_1, X_2 are some other matrices satisfying $X_1P_2 = X_2P_2$ there is a matrix Y such that

$X_1 = Y Q_1, X_2 = Y Q_2$. ||

(29) **Example** ($\mathbf{D} = F[s]$, $r=1$) Let **D** be the polynomial ring in one indeterminate s over a field F and $P \in F[s]^{l,m}$ an arbitrary matrix. By elementary row and column operations P can be transformed into the matrix $D = \begin{pmatrix} D' & 0 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ where $D' = \mathrm{diag}(d_1, \cdots, d_r)$, $r = \mathrm{rank}(P)$ and $d_1, \cdots, d_r$ are elementary divisor polynomials of P . In other terms, this means

$$UPV = \begin{pmatrix} D' & 0 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad, U \in Gl_l(F[s]) \,,\quad V \in Gl_m(F[s])$$

where one obtains U from the identity matrix I with the same elementary row operations as D from P. Write U in block form $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ with

$U_1 \in F[s]^{r,l}, U_2 \in F[s]^{l-r,l}$. Then obviously $U_2 P = 0$. It is easy to see that $Q := U_2 \in F[s]^{k,l}$, $k := l - r$, is universal with $QP = 0$. ||

(30) **Remark:** For $r > 1$ and $\mathbf{D} = F[s_1, \cdots, s_r]$ a matrix Q universal with respect to $QP = 0$ can always be constructively found by the method of Gröbner bases (see [BU] , [PAU] , [WIN]). The details will be explained in §5 . ||

Injective modules

The next lemma gives the standard characterization of injective modules for noetherian rings .

(31) **Lemma and Definition** (Injectivity and fundamental principle):

Assumption (1). The following statements are equivalent:

(i) The module **A** is injective, i.e. by definition. the functor $\mathrm{Hom}_{\mathbf{D}}(-, \mathbf{A})$ is exact. This means that this functor transforms exact sequences into exact sequences or, sufficiently, injections into surjections.

(ii) The module **A** satisfies the fundamental principle in the following sense: If a sequence

$$\mathbf{D}^k \xrightarrow{Q^T} \mathbf{D}^l \xrightarrow{P^T} \mathbf{D}^m, \quad Q \in \mathbf{D}^{k,l}, \; P \in \mathbf{D}^{l,m},$$

is exact or, in other words, if Q is universal with $QP = 0$ then also the transformed sequence

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^m, \mathbf{A}) & \xrightarrow{S(P^T)} & \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^l, \mathbf{A}) & \xrightarrow{S(Q^T)} & \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^k, \mathbf{A}) \\
\| & & \| & & \| \\
\mathbf{A}^m & \xrightarrow{\quad P \quad} & \mathbf{A}^l & \xrightarrow{\quad Q \quad} & \mathbf{A}^k
\end{array}$$

is exact. This means: The equation $Pw = u$, $u \in \mathbf{A}^l$, has a solution $w \in \mathbf{A}^m$ if and only if $Qu = 0$.

Proof: Obviously (ii) is a special case of (i). The implication (ii)$\Rightarrow$(i) is "well known", and follows easily from Baer's criterion (see [BOU2], §1.7, Prop. 10) and the assumed noetherian property of $\mathbf{D}$. ||

In connection with systems of partial differential equations the term '*fundamental principle*' is due to Ehrenpreis (compare [EH]) whereas the use of homological methods is due to Malgrange ([MAL]) as far as I know.

As an application of injectivity consider a submodule U of $\mathbf{D}^n$ and the exact sequence

(32) $$0 \longrightarrow U \xrightarrow{\text{inj}} \mathbf{D}^n \xrightarrow{\text{can}} \mathbf{D}^n/U \longrightarrow 0.$$

Application of $S = \mathrm{Hom}_{\mathbf{D}}(-,\mathbf{A})$ to (32) furnishes the exact sequence

$$0 \longrightarrow S(\mathbf{D}^n/U) \xrightarrow{S(\text{can})} S(\mathbf{D}^n) = \mathbf{A}^n \xrightarrow{S(\text{inj})} S(U), \quad S(\text{inj})(w) = \langle -, w\rangle | U$$

Hence $$\mathrm{im}(S(\text{can})) = \ker(S(\text{inj})) = U^{\perp}.$$

If $\mathbf{A}$ is injective the map $S(\text{inj})$ is surjective and induces the isomorphism $\mathbf{A}^n/U^{\perp} \cong S(U)$ by the homomorphism theorem. We obtain the

(33) **Corollary**: For a $\mathbf{D}$-submodule $U \subset \mathbf{D}^n$ the canonical map $\text{can}: \mathbf{D}^n \longrightarrow \mathbf{D}^n/U$ induces an $\mathbf{E}$-isomorphism

$$S(\text{can}): S(\mathbf{D}^n/U) = \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^n/U, \mathbf{A}) \cong U^{\perp} \subset \mathbf{A}^n.$$

If $\mathbf{A}$ is injective the map

$$\mathbf{A}^n/U^{\perp} \cong S(U) = \mathrm{Hom}_{\mathbf{D}}(U, \mathbf{A}), \quad w \longmapsto \langle -, w\rangle | U,$$

is also a $\mathbf{E}$-isomorphism. ||

The following theorem is of great importance in system theory and depends essentially on the injectivity of $\mathbf{A}$.

(34) **Theorem:** Let $\mathbf{A}$ be an injective $\mathbf{D}$-module,

$$S_1 = \{w_1 \in \mathbf{A}^{k(1)};\ R_1 w_1 = 0\} \subset \mathbf{A}^{l(1)}, R_1 \in \mathbf{D}^{k(1),l(1)},$$

a subsystem of $\mathbf{A}^{l(1)}$ and $P\in\mathbf{D}^{l(2),l(1)}$ a matrix. Then

$$S_2:=P\,S_1:=\{Pw_1;w_1\in S_1\subset \mathbf{A}^{l(1)}\}\subset \mathbf{A}^{l(2)}$$

is not only a $\mathbf{D}$-submodule of $\mathbf{A}^{l(2)}$, but even a subsystem, indeed

$S_2=PS_1=\{w\in\mathbf{A}^{l(2)};R_2w=0\}$ where $R_2\in\mathbf{D}^{k(2),l(2)}$ and some matrix $Y\in\mathbf{D}^{k(2),k(1)}$ are universal matrices with $R_2P=Y\,R_1$ (compare (28)).

Proof: (i) For $w\in S_1$ the relations $R_1w=0$ and $R_2P=Y\,R_1$ imply

$$R_2Pw=YR_1w=Y0=0\ ,\ \text{thus}\ Pw\in\{w_2;R_2w_2=0\}.$$

Hence $PS_1\subset\{w_2\in\mathbf{A}^{l(2)};R_2w_2=0\}$.

The main part of the proof consists in the reverse inclusion. This trivial part (i) of the proof is, however, a good motivation for the statement of the theorem.

(ii) The relation $R_2P=YR_1$ induces a commutative diagram with exact rows

(35)
$$\begin{tikzcd}
0 & \mathbf{D}^{l(1)}/R_1^T\mathbf{D}^{k(1)} \arrow[l] & \mathbf{D}^{l(1)} \arrow[l, "can_1"'] & \mathbf{D}^{k(1)} \arrow[l, "R_1^T"'] \\
0 & \mathbf{D}^{l(2)}/R_2^T\mathbf{D}^{k(2)} \arrow[l] \arrow[u, "(P^T)_{ind}"'] & \mathbf{D}^{l(2)} \arrow[l, "can_2"'] \arrow[u, "P^T"'] & \mathbf{D}^{k(2)} \arrow[l, "R_2^T"'] \arrow[u, "Y^T"']
\end{tikzcd}$$

where $(P^T)_{ind}$ is induced from P^T on the residue classes and *not* given by a matrix since the factor modules are not free in general.

(36) **Assertion:** $(P^T)^{-1}(R_1^T\mathbf{D}^{k(1)})=R_2^T\mathbf{D}^{k(2)}$

Proof: $P^T(R_2^T\mathbf{D}^{k(2)})=P^TR_2^T\mathbf{D}^{k(2)}=R_1^TY^T\mathbf{D}^{k(2)}\subset R_1^T\mathbf{D}^{k(1)}$, hence

$$R_2^T\mathbf{D}^{k(2)}\subset (P^T)^{-1}(R_1^T\mathbf{D}^{k(1)}).$$

Let, on the other side, x^T be a column in $(P^T)^{-1}(R_1^T\mathbf{D}^{k(1)})$, i.e. $P^Tx^T\in R_1^T\mathbf{D}^{k(1)}$, and thus

(37)
$$P^Tx^T=R_1^Ty^T \quad\text{or}\quad xP=yR_1.$$

By assumption the matrices R_2 and Y are universal with $R_2P=YR_1$ (see (27)). This and (37) imply that there is some z such that

$$x=zR_2\ ,\ y=zY\ ,\ \text{hence}\ x^T\in R_2^T\mathbf{D}^{k(2)}.$$

Thus $(P^T)^{-1}(R_1^T\mathbf{D}^{l(1)})=R_2^T\mathbf{D}^{l(2)}$ as desired.

The homomorphism theorem now implies that the map

$(P^T)_{ind}: \mathbf{D}^{l(2)}/R_2^T\mathbf{D}^{k(2)}=\mathbf{D}^{l(2)}/(P^T)^{-1}(R_1^T\mathbf{D}^{k(1)})\longrightarrow\mathbf{D}^{l(1)}/R_1^T\mathbf{D}^{k(1)}$

from (35) is injective. Since $\mathbf{A}$ is an injective module the induced map $S((P^T)_{ind}) = \mathrm{Hom}_{\mathbf{D}}(((P^T))_{ind}, \mathbf{A})$ is surjective.

(iii) Application of the exact functor $S = \mathrm{Hom}_{\mathbf{D}}(-, \mathbf{A})$ to the diagram (35) yields the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & S(\mathbf{D}^{1(1)}/R_1\mathbf{D}^{k(1)}) \arrow[r, "S(can_1)"] \arrow[d, "S((P^T)_{ind}"] & S(\mathbf{D}^{1(1)}) = \mathbf{A}^{1(1)} \arrow[r, "R_1"] \arrow[d, "P"] & \mathbf{A}^{k(1)} \arrow[d, "Y"] \\ 0 \arrow[r] & S(\mathbf{D}^{1(2)}/R_2\mathbf{D}^{k(2)}) \arrow[r, "S(can_2)"] & S(\mathbf{D}^{1(2)}) = \mathbf{A}^{1(2)} \arrow[r, "R_2"] & \mathbf{A}^{k(2)} \end{tikzcd}$$

Then $\qquad \mathrm{im}(S(can_i)) = \{w_i \in \mathbf{A}^{1(i)}; R_i w_i = 0\}$,$i = 1$, 2 .

and, since $S((P^T)_{ind})$ is surjective, this implies

$$P(\{w_1 \in \mathbf{A}^{1(1)}; R_1 w_1 = 0\}) = \{w_2 \in \mathbf{A}^{1(2)}; R_2 w_2 = 0\}$$

as asserted. ||

The preceding theorem yields several important examples. Let, for instance, the subspace $S \subset \mathbf{A}^1$ be given as

$$S = \{w \in \mathbf{A}^1; \exists x \in \mathbf{A}^n : R'w = R''x\}$$

with arbitrary matrices $R' \in \mathbf{D}^{k,1}$ and $R'' \in \mathbf{D}^{k,n}$. Define the system

$$S_1 := \left\{\begin{pmatrix} w \\ x \end{pmatrix} \in \mathbf{A}^{1+n}; R_1\begin{pmatrix} w \\ x \end{pmatrix} = 0\right\}, \quad R_1 = (R', -R''),$$

and the matrix $P := (I \;\; 0) \in \mathbf{D}^{1,1+n}$. Then obviously $S = (I \;\; 0)S_1 \subset \mathbf{A}^1$ and thus S is a subsystem of $\mathbf{A}^1$ by the preceding theorem. Let R_2 and Y be universal with

$$R_2 P = Y R_1, \text{ thus } R_2(I \;\; 0) = Y(R', -R'') \text{ or } R_2 = YR' \text{ and } YR'' = 0.$$

But this means that Y is universal with $YR'' = 0$ and then $R_2 = YR'$ is determined by Y. The preceding theorem yields $S = \{w \in \mathbf{A}^1; R_2 w = YR'w = 0\}$. Remark again that the defining equation $R'w = R''x$ implies $YR'w = YR''x = 0$, hence $S \subset \{w; YR'w = 0\}$. The theorem furnishes a Y such that $S = \{w; YR'w = 0\}$.

(38) **Corollary:** Let $\mathbf{A}$ be an injective $\mathbf{D}$-module and $S \subset \mathbf{A}^1$ a $\mathbf{D}$-submodule defined by

(39) $$S = \{w \in \mathbf{A}^1; \exists x \in \mathbf{A}^n : R'w = R''x\}$$

with matrices $R' \in \mathbf{D}^{k,1}$ and $R'' \in \mathbf{D}^{k,n}$. Then S is a subsystem of $\mathbf{A}^1$, and indeed $S = \{w \in \mathbf{A}^1; YR'w = 0\}$ where Y is a matrix universal with respect to $YR'' = 0$. ||

A special case of the preceding corollary yields the Rosenbrock systems (compare [ROS]). Consider a system of equations

$$(40)\qquad \begin{aligned} Px &= Qu\ ,\ x \in \mathbf{A}^n, u \in \mathbf{A}^m \\ y &= Rx + Wu\ ,\ y \in \mathbf{A}^p \end{aligned}$$

with polynomial matrices of appropriate sizes. Then

$$S := \left\{ \binom{u}{y} \in \mathbf{A}^{m+p} ; \exists x \in \mathbf{A}^n \text{ such that (40) is satisfied} \right\}$$

is of the form (39) by writing (40) as

$$R'\binom{u}{y} = R''x,\ R' := \begin{pmatrix} Q & 0 \\ W & -I \end{pmatrix},\ R'' := \begin{pmatrix} P \\ -R \end{pmatrix}\ .$$

Let (XY) be universal with $(XY)\begin{pmatrix} P \\ -R \end{pmatrix} = 0$, i.e. $XP = YR$. Then

$$(XY)R' = (XY)\begin{pmatrix} Q & 0 \\ W & -I \end{pmatrix} = (XQ+YW, -Y)$$

$$(XY)R'\binom{u}{y} = 0 \text{ if and only if } Yy = (XQ+YW)u\ .$$

The preceding corollary is applicable and yields the

(41) **Corollary and Definition** (Abstract Rosenbrock systems): Let $\mathbf{A}$ be an injective $\mathbf{D}$-module. Consider the system (40) of linear equations. Let X and Y be universal with $XP = YR$. Then

$$S := \left\{ \binom{u}{y} \in \mathbf{A}^{m+p} ; \exists x \in \mathbf{A}^n \text{ such that } Px = Qu, y = Rx + Wu \right\}$$

is a subsystem of $\mathbf{A}^{m+p}$, and

$$S = \left\{ \binom{u}{y} \in \mathbf{A}^{m+p} ; Yy = (XQ+YW)u \right\}.$$

I call such a system S an abstract *Rosenbrock system* . ||

Remark again that the equations

$Px = Qu, y = Rx + Wu$, $XP = YR$ imply $Yy = YRx + YWu = XPx + YWu = (XQ+YW)u$ so that the vector x is eliminated from the equation. The main result is that the latter equation is sufficient for (uy) to be in S.

(42) **Standard example:** Let $\mathbf{A}$ be an injective module over the polynomial ring $\mathbf{D} = F[s] = F[s_1]$ in *one* indeterminate over the field F. Take, for instance, the module $\mathbf{A}$ from examples (1.7) to (1.24) for $r = 1$. Consider the standard state space equations

$$sx=Ax+Bu, y=Cx+Du \text{ with } A\in F^{n,n}, B\in F^{n,m}, C\in F^{p,n}, D\in F^{n,n}.$$

In the form $(sI_n-A)x=Bu, y=Cx+Du$ these are Rosenbrock equations with $P=sI_n-A, Q=B, R=C, W=D$. The matrix $\begin{pmatrix} P \\ -R \end{pmatrix} = \begin{pmatrix} sI_n-A \\ -C \end{pmatrix}$ has rank n.

By (29) construct a matrix $(XY)\in F[s]^{p,n+p}$, $p=(n+p)-n$, universal with respect to

$$(XY)\begin{pmatrix} sI_n-A \\ -C \end{pmatrix} = 0 \text{ or } X(sI_n-A)=YC.$$

The preceding corollary yields

$S:=\{(uy)\in \mathbf{A}^{m+p}; \exists x\in \mathbf{A}^n$ such that $sx=Ax+Bu, y=Cx+Du\} =$

$=\{(uy)\in \mathbf{A}^{m+p}; Yy=(XB+YD)u\}$ with $Y\in F[s]^{p,p}, XB+YD\in F[s]^{p,m}$.

The preceding calculation gives the well-known passage from the state space to the transfer function description of a 1-d system. ||

Injective cogenerators

Next I investigate the notion of a cogenerator.

(43) **Lemma and Definition** (Cogenerator): For a **D**-module **A** over an arbitrary ring **D** the following assertions are equivalent:

(i) Every **D**-module M can be embedded into a direct product of copies of **A**, i.e. there is a monomorphism $f: M \longrightarrow \mathbf{A}^I, m \longmapsto f(m)=(f_i(m); i\in I)$ for some possibly infinite index set I.

(ii) For arbitrary **D**-modules M and N the map

$$S: \mathrm{Hom}_{\mathbf{D}}(M.N) \longrightarrow \mathrm{Hom}_{\mathbf{E}}(S(N), S(M)), f \longmapsto S(f),$$

$$S(N)=\mathrm{Hom}_{\mathbf{D}}(N,A), S(M)=\mathrm{Hom}_{\mathbf{D}}(M,A), S(f)=\mathrm{Hom}(f,\mathbf{A})$$

$$S(f)(g)=gf \text{ for } f\in \mathrm{Hom}_{\mathbf{D}}(M,N), g\in \mathrm{Hom}_{\mathbf{D}}(N,A)$$

is injective.

Under the additional assumption that **A** is an *injective* module (i) and (ii) are also equivalent to

(iii) A **D**-module M is zero if and only if $S(M)=\mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A})$ is zero.

A module **A** satisfying these equivalent conditions is called a *cogenerator* .

The proof of the preceding lemma is simple and known (see [BOU2], §1.8). ||

The circle group $S^1:=\{z\in\mathbb{C};|z|=1\}$, for instance, is an injective cogenerator in the category of abelian groups.

Assume that **A** is an injective cogenerator and $f:M\longrightarrow N$ a **D**-linear map. The latter gives rise to the exact sequence

(44) $$0\longrightarrow \ker(f)\xrightarrow{\text{inj}} M\xrightarrow{f} N\xrightarrow{\text{can}} \operatorname{cok}(f)=N/\operatorname{im}(f)\longrightarrow 0$$

Since **A** is injective the functor $S=\operatorname{Hom}_{\mathbf{D}}(-,\mathbf{A})$ is exact and transforms (44) into the exact sequence

$$0\longrightarrow S(\operatorname{cok}(f)\longrightarrow S(N)\xrightarrow{S(f)} S(M)\longrightarrow S(\ker(f))\longrightarrow 0,$$

in particular

(45) $S(\operatorname{cok}(f))\cong\ker S(f), S(\ker(f))\cong\operatorname{cok}(S(f))=S(M)/\operatorname{im}(S(f))$.

If f is a monomorphism (epimorphism, isomorphism) then $S(f)$ is an epimorphism (monomorphism, isomorphism). If, on the other side, S(f) is injective then $S(\operatorname{cok}(f))=0$ by (45). The cogenerator property from (43), (iii), of **A** implies $\operatorname{cok}(f)=0$ and thus that f is surjective.

(46) **Corollary:** If **A** is an injective cogenerator a **D**-linear map $f:M\longrightarrow N$ is injective resp. surjective resp. bijective if and only if

$$S(f)=\operatorname{Hom}(f,\mathbf{A}):\operatorname{Hom}_{\mathbf{D}}(N,\mathbf{A})\longrightarrow\operatorname{Hom}_{\mathbf{D}}(M,\mathbf{A}),\ g\longmapsto gf\ ,$$

is surjective resp. injective resp. bijective. ||

(47) **Corollary:** If **A** is an injective cogenerator and $U\subset\mathbf{D}^n$ is a **D**-submodule then $U=U^{\perp\perp}$ with respect to the Galois correspondence $(-)^{\perp}$ from (23).

Proof: Consider the commutative diagram

$$\begin{array}{ccc}
\mathbf{A}^n/U^{\perp} & \xrightarrow{\text{can}_1} & S(U)\\
\cong\big\downarrow \text{can}_3 & & \cong\big\downarrow S(\text{inj})\\
\mathbf{A}^n/U^{\perp\perp\perp} & \xrightarrow{\text{can}_2} & S(U^{\perp\perp})
\end{array}$$

where can_1, can_2 are the isomorphisms from (33) and can_3 is isomorphic since $U^{\perp}=U^{\perp\perp\perp}$ for every Galois correspondence. But then S(inj) is isomorphic too. The preceding lemma implies that $\text{inj}:U\longrightarrow U^{\perp\perp}$ is an isomorphism, hence $U=U^{\perp\perp}$. ||

(48) **Corollary:** Let **A** be an injective cogenerator. If $S:=\{w\in\mathbf{A}^1;\ Rw=0\}$,

$R \in \mathbf{D}^{k,l}$, is a subsystem of $\mathbf{A}^l$ then $S^\perp := R^T\mathbf{D}^k$ and $S = S^{\perp\perp}$.

Hence the maps $(-)^\perp$ induce order antiisomorphisms

(49) $$(-)^\perp :\ \mathbf{P}(\mathbf{D}^l) \overset{\cong}{\longleftrightarrow} \mathbf{Pf}(\mathbf{A}^n)$$

where $\mathbf{P}(\mathbf{D}^l)$ as in (22) and $\mathbf{Pf}(\mathbf{A}^n)$ are the ordered sets of all submodules of $\mathbf{D}^l$ resp. all subsystems S of $\mathbf{A}^l$.

Proof: Let $U := R^T\mathbf{D}^k$ be the column module of R^T. Then $S = U^\perp$ by (20) and hence $S^{\perp\perp} = U^{\perp\perp\perp} = U^\perp = S$ since $(-)^\perp$ is a Galois correspondence. Moreover $U = U^{\perp\perp} = S^\perp$ by (47). The equations $U = U^{\perp\perp}$ and $S = S^{\perp\perp}$ imply that (49) are inverse bijections. ||

(50) **Remark:** The module $S^\perp \subset \mathbf{D}^l$ consists of all vectors $q = (q_1 \cdots q_l)^T \in \mathbf{D}^l$ such that $q_1w_1 + \cdots + q_lw_l = 0$ for all $w \in S \subset \mathbf{A}^l$, and is thus the largest submodule U of **D** with $U \perp S$ (with respect to $\langle -,- \rangle$). For the special case $\mathbf{D} = F[s], r = 1$ and l=1 the module $S^\perp$ is an ideal of F[s] and generated by a unique monic polynomial which is usually called the *minimal polynomial* of S. Thus $S^\perp$ generalizes the minimal polynomial of a recursive sequence. ||

(51) **Injective modules over commutative noetherian rings according to Matlis**

Now and later I need the following facts from the original paper [MAT] by Matlis which are also well presented in [BOU2], §1.9, 1.10, Ex. 1.27. The assumption (1) is in force, in particular **D** denotes a commutative noetherian ring.

Every injective **D**-module is a direct sum of indecomposable direct summands which are, of course, also injective. Assume thus that E is an indecomposable injective **D**-module. Then E is the injective envelope (or hull) of its non-zero submodules (see [BOU2], §1.9, Prop. 19). The set of annihilator ideals

$$(0:x) := \{p \in \mathbf{D};\, px = 0\}\ ,\ 0 \neq x \in E,$$

admits a largest ideal $\mathbf{p}(E)$ and this is a prime ideal, the associated prime ideal of E. Then, up to isomorphism, E is the injective envelope of $\mathbf{D}/\mathbf{p}(E)$. The map $E \longmapsto \mathbf{p}(E)$ induces a bijection between the class of indecomposable injective **D**-modules and the set Spec(**D**) of prime ideals of **D**, the inverse map being given by $\mathbf{p} \longmapsto E(\mathbf{D}/\mathbf{p})$ where E(M) denotes the injective envelope of a

D-module M, unique up to isomorphism.

If, for instance, **D** is a noetherian integral domain with quotient field **K** then

$$\mathbf{K}=\mathrm{Quot}(\mathbf{D})=E(\mathbf{D}/0)=E(\mathbf{D})$$

is an indecomposable injective with associated prime ideal $\mathbf{p}(\mathbf{K})=0$ (see [BOU2], §1, Cor. 2 of Prop. 10). ||

(52) **Lemma and Definition** (Large injective cogenerator) : Let **D** be a commutative noetherian ring and **A** an injective **D**-module. The following assertions are equivalent:

(i) For every finitely generated **D**-module M there is a **D**-linear monomorphism

$$f: M \longrightarrow \mathbf{A}^k,\ m \longmapsto (f_i(m); i=1,\cdots,k)\ ,\ k \in \mathbb{N},$$

i.e. every such M can be embedded into a *finite* product $\mathbf{A}^k$ of copies of **A.**

(ii) The statement of (i) is true for all $M=\mathbf{D}/\mathbf{p}$ where **p** is a prime ideal of **D**.

If (i) and (ii) are satisfied then **A** is a cogenerator and called a *large* injective cogenerator.

Proof : (ii) is a special case of (i). Assume that (ii) is satisfied and that M is a finitely generated **D**-module with injective envelope $E(M) \supset M$. Then E(M) decomposes into a direct sum $E(M)=\oplus\{E_i; i \in I\}$ of indecomposable injectives E_i. Since M is an essential submodule of E(M) the intersection modules $E_i \cap M$ are not zero and thus $\oplus\{E_i \cap M; i \in I\}$ is a direct sum of non-zero modules in the noetherian module M. This implies that I is finite.

By (51) $\qquad E_i \cong E(\mathbf{D}/\mathbf{p}_i)\ ,\ \mathbf{p}_i := \mathbf{p}(E_i) \in \mathrm{Spec}(\mathbf{D})\ ,\ i \in I.$

The assumption (ii) yields embeddings $\mathbf{D}/\mathbf{p}_i \longrightarrow \mathbf{A}^{k(i)}$, $i \in I$, which can be extended to **D**-linear maps

$$f_i : E(\mathbf{D}/\mathbf{p}_i) \longrightarrow \mathbf{A}^{k(i)},\ f_i|(\mathbf{D}/\mathbf{p}_i) \text{ injective,}$$

since **A** and thus $\mathbf{A}^{k(i)}$ are injective. But then

$$\ker(f_i) \cap (\mathbf{D}/\mathbf{p}_i) = \ker(f_i|(\mathbf{D}/\mathbf{p}_i)) = 0$$

which implies $\ker(f_i)=0$ since $\mathbf{D}/\mathbf{p}_i$ is essential in $\mathbf{E}(\mathbf{D}/\mathbf{p}_i)$. We derive embeddings $\qquad E_i \cong E(\mathbf{D}/\mathbf{p}_i) \longrightarrow \mathbf{A}^{k(i)}$, $i \in I$, $\qquad$ and finally

$$M \subset E(M) = \oplus E_i \cong \Pi_{i \in I} E_i \longrightarrow \Pi_{i \in I} \mathbf{A}^{k(i)} = \mathbf{A}^k$$

with $k := \sum\{k(i); i \in I\} \in \mathbb{N}$. Thus (i) is proven.

Assume finally that N is an arbitrary non-zero **D**-module. Choose a finitely generated, non-zero submodule M of N and an embedding $f: M \longrightarrow \mathbf{A}^k$ as in (i). Without loss of generality the component f_1 of f is non-zero. Since **A** is injective this can be extended to $g_1: N \longrightarrow \mathbf{A}$ with $g_1|M = f_1$. But then

$$0 \neq g_1 \in S(N) = \mathrm{Hom}_{\mathbf{D}}(N, \mathbf{A}), \text{ thus } S(N) \neq 0.$$

Condition (iii) of (43) yields that N is a cogenerator. ||

(53) **Remark** (A new interpretation of the preceding theorem): An embedding $g: \mathbf{D}/\mathbf{I} \longrightarrow \mathbf{A}^k, k \in \mathbb{N}$, of a cyclic **D**-module **D/I** where **I** is an ideal of **D** can be interpreted in the following fashion: Let

$$f := g\ \mathrm{can}: \mathbf{D} \longrightarrow \mathbf{D}/\mathbf{I} \longrightarrow \mathbf{A}^k,\ p \longmapsto (f_i(p); i = 1, \cdots, k),$$

be the composed map with $\ker(f) = \ker(\mathrm{can}) = \mathbf{I}$ since g is injective. The components f_i of f are given as $f_i: \mathbf{D} \longrightarrow \mathbf{A}$, $p \longmapsto p a_i, a_i := f_i(1)$, hence

$$\ker(f_i) = \{p \in \mathbf{D}; p a_i = 0\} =: (0 : a_i) \text{ and finally}$$

$$\mathbf{I} = \ker(f) = \cap\{\ker(f_i); i = 1, \cdots, k\} = \cap\{(0 : a_i)\ ;\ i = 1, \cdots, k\}.$$

Thus the injective module **A** is a large injective cogenerator if and only if for every ideal **I** of **D** there are finitely many elements $a_1, \cdots, a_k$ of **A** such that $\mathbf{I} := \{p \in \mathbf{D}; p a_1 = \cdots = p a_k = 0\}$ is the exact annihilator of these a_i. By lemma (52) it is sufficient that this condition is satisfied for all prime ideals $\mathbf{I} = \mathbf{p}$ of **D**. ||

The main theorems

The technicalities developped above are justified by the following

(54) **Main Theorem**: The $F[s_1, \cdots, s_r]$-modules **A** from the examples (1.7), (1.13), (1.18), (1.22), their real forms $\mathbf{A}_{\mathbb{R}}$ and (1.24) are large injective cogenerators. The proof of this theorem is the main content of the paragraphs 3 and 4. The injectivity in the continuous case is derived from famous work of Ehrenpreis, Malgrange and Palamodov (see [EH], [MAL], [PAL]). The proof of the large cogenerator property is new. ||

(55) **Assumption**: In the remainder of this paragraph I assume that **A** is a large injective cogenerator over the commutative noetherian ring **D**. Always recall that this assumption is satisfied for examples (1.7) to (1.24). ||

The next results describe the importance of assumption (55) for system theory.

(S6) **Duality Theorem** (Systems and **D**-modules): Assumption (S5), notations as above.

(i) A left $\mathbf{E}:=\mathrm{End}_{\mathbf{D}}(\mathbf{A})$-module S is a system, i.e. **E**-isomorphic to some **E**-module

$$S(M)=\mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A}),M\in\mathrm{Modf}(\mathbf{D}),$$

if and only if S is a finitely generated **E**-submodule of some $\mathbf{A}^l$, $l\in\mathbb{N}$, up to isomorphism. In particular , a **E**-finitely generated submodule of a system S is again a system.

(ii) The functor $\mathrm{Hom}_{\mathbf{D}}(-,\mathbf{A})$ induces the duality (=contravariant equivalence)

$$S=\mathrm{Hom}_{\mathbf{D}}(-,\mathbf{A}):\mathrm{Modf}(\mathbf{D})^{op}\longrightarrow\mathrm{Syst}(\mathbf{A}).$$

This means that for arbitrary finitely generated **D**-modules M and N the map

$$S:\mathrm{Hom}_{\mathbf{D}}(M,N)\longrightarrow\mathrm{Hom}_{\mathbf{E}}(S(N),S(M)),\ f\longmapsto S(f),$$

$$S(N)=\mathrm{Hom}_{\mathbf{D}}(N,\mathbf{A}),S(M)=\mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A}),S(f)=\mathrm{Hom}(f,\mathbf{A}),\ S(f)(g)=gf$$

is bijective and even a **D**-isomorphism. In other words: A map $F:S(N)\longrightarrow S(M)$ is of the form $F=S(f)$ if and only if F is **E**-linear, and this f is then unique.

(iii) The full subcategory Syst(**A**) of Mod(**E**) consisting of all systems and **E**-linear maps between these is closed under taking kernels, images and finite direct sums. In particular, Syst(**A**) is an abelian category and, of course, $S=\mathrm{Hom}_{\mathbf{D}}(-,\mathbf{A})$ is an exact functor from Modf(**D**) to Syst(**A**). Also the set $\mathbf{Pf}(\mathbf{A}^l),l\geq 0$, of all subsystems of $\mathbf{A}^l$ (compare (49)), i.e. of all **E**-finitely generated submodules of $\mathbf{A}^l$, is closed under finite sums and intersections and thus a lattice.

This theorem is a special case of proposition 3.3 on page 488 of [OB1]. The data $\mathfrak{A}$, $\mathfrak{N}$, E from [loc.cit.] correspond to Mod(**D**), Modf(**D**) respectively **A** here. The assertion (i) of the theorem follows from the fact that **A**=S(**D**) is coherent in the sense of [loc. cit] and that a finitely generated subobject of a coherent one is again coherent. ||

(S7) **Corollary:** Assumption (S5). The map

$$S:\mathbf{D}^{l,k}=\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^k,\mathbf{D}^l)\cong\mathrm{Hom}_{\mathbf{E}}(\mathbf{A}^l,\mathbf{A}^k),\ R^T\longmapsto\ S(R^T)=R(\ w\longmapsto Rw)$$

is a **D**-isomorphism. In particular

$$\mathbf{D}=\mathrm{Hom}_{\mathbf{D}}(\mathbf{D},\mathbf{D})\cong \mathrm{End}_{\mathbf{B}}(\mathbf{A})\subset \mathrm{End}_{\mathbf{D}}(\mathbf{A})=\mathbf{E}.$$

In other words this means that **D** is the center of the **D**-algebra **E** or again that a **D**-linear endomorphism $e\in\mathbf{E}$ of **A** is in the center if and only if e has the form $e(a)=pa, a\in\mathbf{A}$, some $p\in\mathbf{D}$. ||

(58) **Corollary and Definition:** Assumption (55). The quasi-inverse functor of $S:\mathrm{Modf}(\mathbf{D})^{op}\cong \mathrm{Syst}(\mathbf{A})\subset \mathrm{Mod}(\mathbf{E})$ is denoted by

$$M:\mathrm{Syst}(\mathbf{A})\cong \mathrm{Modf}(\mathbf{D})^{op},\ S\longmapsto M(S).$$

This means: If S is a system then M(S) is the **D**-module, unique up to **D**-isomorphism, for which there is a **E**-isomorphism

$$\Phi_S: S(M(S))=\mathrm{Hom}_{\mathbf{D}}(M(S),\mathbf{A})\cong S\quad .$$

If $F:S_1\longrightarrow S_2$ is a **E**-linear map between systems S_1 and S_2 and $M_i:=M(S_i)$, $\Phi_i=\Phi_{S_i}, i=1,2$, the map $f:=M(F):M_2\longrightarrow M_1$ is the unique **D**-linear map such that

$$\begin{array}{ccc} S(M_1) & \xrightarrow{\Phi_1} & S_1 \\ \downarrow{\scriptstyle S(f)} & & \downarrow{\scriptstyle F} \\ S(M_2) & \xrightarrow{\Phi_2} & S_2 \end{array}$$

commutes. ||

(59) **Corollary:** Let $w^1,\cdots,w^m$ be m vectors in $\mathbf{A}^l$. Then $S:=\mathbf{E}w^1+\cdots+\mathbf{E}w^m$ is a system and $S=U^{\perp}$ where

$$U=\{q=(q_1\cdots q_l)^T\in\mathbf{D}^l;\ \langle q,w^j\rangle=q_1w_1^{\,j}+\cdots+q_lw_l^{\,j}=0 \text{ for } j=1,\cdots,m\}$$

Proof: By (56), (i), this S is a subsystem. The rest follows from (48). ||

System theoretic applications of the main duality theorem

In the sequel I use the main duality theorem to prove several results of system theoretic significance.

(60) **Corollary:** The system $\mathbf{A}=S(\mathbf{D})$ is injective in Syst(**A**). This means that if $F:S_1\longrightarrow S_2$ is a **E**-monomorphism between systems S_1 and S_2 any **E**-linear map $G_1:S_1\longrightarrow\mathbf{A}$ can be extended to a **E**-linear map G_2 such that $G_2F=G_1$.

Proof: The proof follows directly from the duality $S:\mathrm{Modf}(\mathbf{D})^{op}\cong\mathrm{Syst}(\mathbf{A})$ and the trivial fact that the **D**-module **D** is free and thus projective which is the dual property of injectivity. ||

This corollary is the main ingredient for the following important

(61) **Theorem:** Let

$$S_i := \{w \in \mathbf{A}^l ; R_i w = 0\} \ , \ R_i \in \mathbf{D}^{k(i),l}, \ i=1,2,$$

be two systems in $\mathbf{A}^l$. Then $S_1 \subset S_2$ if and only if here is a matrix $X \in \mathbf{D}^{k(2),k(1)}$ such that $XR_1 = R_2$.

Proof: If $XR_1 = R_2$ then the relation $R_1 w = 0$ implies $R_2 w = X(R_1 w) = 0$ and thus $S_1 \subset S_2$. Assume in return that $S_1 \subset S_2$. The system maps

$$F_i := S(R_i^T) : \mathbf{A}^l \longrightarrow \mathbf{A}^{k(i)}, w \longmapsto R_i w, i=1,2 \ ,$$

with $\ker(F_i) = S_i$ factorize as

$$\mathbf{A}^l \xrightarrow{can_i} \mathbf{A}^l/S_i \xrightarrow{(F_i)_{ind}} \mathbf{A}^{k(i)}, \ F_i = (F_i)_{ind} \cdot can_i,$$

with a **E**-monomorphism $(F_i)_{ind}$. There results the commutative diagram

$$\begin{array}{ccc} \mathbf{A}^l/S_1 & \xrightarrow{(F_1)_{ind}} & \mathbf{A}^{k(1)} \\ \downarrow{can} & & \downarrow{G} \\ \mathbf{A}^l/S_2 & \xrightarrow{(F_2)_{ind}} & \mathbf{A}^{k(2)} \end{array}$$

where the canonical map *can* is defined since $S_1 \subset S_2$ and where G with $G \cdot (F_1)_{ind} = (F_2)_{ind} \cdot can$ exists according to the preceding corollary. We obtain

$$GF_1 = G(F_1)_{ind} \cdot can_1 = (F_2)_{ind} \cdot can \cdot can_1 = (F_2)_{ind} \cdot can = F_2.$$

Let $X \in \mathbf{D}^{k(2),k(1)}$ be the unique matrix with $S(X^T) = G$ (see (57)). Thus

$$S((XR_1)^T)) = S(R_1^T X^T) = S(X^T) S(R_1^T) = GF_1 = F_2 = S(R_2^T).$$

Since S is injective on maps according to the main duality theorem or (57) this implies $XR_1 = R_2$ as asserted . ||

(62) **Remark:** For the preceding theorem it is essential that **A** is a *large* injective cogenerator. The theorem is interesting and new even for the standard cases with $r = 1$. (Compare [BY] , Th. 1,2 on page 91). ||

(63) **Corollary** (Quasi-uniqueness of the representing matrix): If a system $S \subset \mathbf{A}^l$ is defined by two matrices $R_i \in \mathbf{D}^{k(i),l}, i=1,2$, i.e. $S = \{w \in \mathbf{A}^l ; R_1 w = 0\} = \{w \in \mathbf{A}^l ; R_2 w = 0\}$ then there are matrices

$X_1 \in \mathbf{D}^{k(2),k(1)}$ and $X_2 \in \mathbf{D}^{k(1),k(2)}$ such that $R_2 = X_1 R_1$ and $R_1 = X_2 R_2$. ||

Rank, IO-systems and transfer matrix

I sharpen the assumption (55) to

(64) **Assumption:** Let $\mathbf{D}$ be a commutative, noetherian *integral domain* and $\mathbf{A}$ a large injective cogenerator. Let

$\mathbf{K} := \mathrm{Quot}(\mathbf{D}) \supset \mathbf{D}$ be the *quotient field* of $\mathbf{D}$. ||

This assumption is satisfied in the examples (1.7) to (1.24). The quotient field of the polynomial algebra $F[s_1, \cdots, s_r]$ is the field $\mathbf{K} := F(s) = F(s_1, \cdots, s_r)$ of rational functions.

(65) **Theorem and Definition** (Input and output dimension of a system) Assumption (64). Let

$$S := \{w \in \mathbf{A}^l ; Rw = 0\}, R \in \mathbf{D}^{k,l},$$

be a subsystem of $\mathbf{A}^l$. As a matrix with coefficients in $\mathbf{K} \supset \mathbf{D}$ the matrix R has a rank $p := \mathrm{rank}(R)$. This rank depends on S only and not on the special choice of the defining matrix R. The numbers p and $l-p=:m$ are called the

(66) *output dimension* of $S = \text{o-dim}(S) := p$

(67) *input dimension* of $S = \text{i-dim}(S) = m := l - p$.

Proof : Assume that S is defined by two matrices $R_i \in \mathbf{D}^{k(i),l}, i=1,2,$ thus $S = \{w \in \mathbf{A}^l ; R_1 w\} = \{w \in \mathbf{A}^l ; R_2 w = 0\}$. By (63) there are matrices X_1, X_2 with coefficients in $\mathbf{D}$, hence in $\mathbf{K}$, satisfying $R_2 = X_1 R_1$ and $R_1 = X_2 R_2$. But then

$$\mathrm{rank}(R_2) = \mathrm{rank}(X_1 R_1) \leq \mathrm{rank}(R_1)$$

and similarly $\mathrm{rank}(R_1) \leq \mathrm{rank}(R_2)$, hence $\mathrm{rank}(R_1) = \mathrm{rank}(R_2)$ as asserted. ||

That the terminology in (66) and (67) is justified will be demonstrated in the next theorem (69),(iii). As a preparation for this consider a subset $O \in \{1, \cdots, l\}$ and define $p := |O|$, $I := \{1, \cdots, l\} \setminus O$, $m := l - p = |I|$. Then

(68) $$\{1, \cdots, l\} = I \cup O$$

is a decomposition of $\{1, \cdots, l\}$ into two disjoint subsets with m respectively p elements. Using (68) identify

$$\mathbf{A}^l = \mathbf{A}^{m+p} = \mathbf{A}^I \times \mathbf{A}^O = \mathbf{A}^m \times \mathbf{A}^p, \quad w = (w_i ; i = 1, \cdots, l) = (u, y)$$

where $u := (w_i ; i \in I) \in \mathbf{A}^I = \mathbf{A}^m$, $y := (w_i ; i \in O) \in \mathbf{A}^O = \mathbf{A}^p$.

I often write (uy) instead of $\binom{u}{y}$ for typographical simplicity. In the same manner I write

$R=(-Q,P)\in \mathbf{D}^{k,l}=\mathbf{D}^{k,I}\times\mathbf{D}^{k,O}=\mathbf{D}^{k,m}\times\mathbf{D}^{k,p}$ with $Q=-(R_{-j};j\in I),P=(R_{-j};j\in O)$

where the minus sign in Q is just a convention. Then

$$Rw=(-Q,P)\binom{u}{y}=-Qu+Py.$$

(69) **Theorem and Definition** (Input-output structure and transfer matrix) Assumption (64). Let $S=\{w\in\mathbf{A}^{l};Rw=0\}$ be a system and

(70) $$\{1,\cdots,l\}=I\,\dot{\cup}\,O\ ,m:=|I|,p:=|O|,$$

a decomposition of the index set $\{1,\cdots,l\}$ with the corresponding decompositions $w=(uy)\in\mathbf{A}^{m+p}=\mathbf{A}^{m}\times\mathbf{A}^{p}$ and $R=(-Q,P)$ such that

(71) $$S=\{w=(uy)\in\mathbf{A}^{m+p};\ Py=Qu\}$$

Assume that

(72) $$\text{o-dim}(S)=\text{rank}(R)\geq p$$

but not necessarily o-dim(S)= p. The following assertions are equivalent:

(i) $\text{o-dim}(S)=\text{rank}(P)=p$, i.e. $\text{o-dim}(S)=p$ and the p columns R_{-j}, $j\in O$, of R are **K**-linearly independent.

(ii) There is a matrix $H\in\mathbf{K}^{p,m}$ such that $PH=Q$.

(iii) For every $u\in\mathbf{A}^{m}$ there is a $y\in\mathbf{A}^{p}$ such that $(uy)\in S$ or, in other terms, the projection $\text{proj}:S\longrightarrow\mathbf{A}^{m},(u,y)\longmapsto u$, is surjective.

If these equivalent conditions are satisfied the decompositions (70) or w=(uy) are called an *input-output structure* (IO-structure) and u resp. y are called an input-resp. output vector of S. By (iii) this notion depends on S, but not on the special choice of R. In general, a system admits several distinct IO-structures, indeed every family of $p:=\text{rank}(R)$ **K**-linearly independent column vectors of R gives rise to one.

If (70) is a IO-structure of S the matrix H from (ii) is uniquely determined by the equation PH=Q and depends on S and the chosen IO-structure (70), but not on the special choice of R=(-Q,P). The matrix is called the *transfer* matrix of S with respect to the IO-structure w=(u,y) and S is called a *realization* of H .

Proof : (i) $\Rightarrow$(ii) From $\text{o-dim}(S)=\text{rank}(R)=\text{rank}(-Q,P)=\text{rank}(P)=p$ we conclude that the p columns $R_{-j}, j\in O$, of R are a basis of the column space of R in $\mathbf{K}^k$ and hence that the columns of Q are $\mathbf{K}$-linearly dependent on those of P. But this means $PH=Q$ with a matrix $H\in\mathbf{K}^{p,m}$ (not $\mathbf{D}^{p,m}$!). Since rank(P) = p the map $P:\mathbf{K}^p\longrightarrow\mathbf{K}^k$ in $\mathbf{K}^{k,p}=\text{Hom}_{\mathbf{K}}(\mathbf{K}^p,\mathbf{K}^k)$ is injective and this implies that H is uniquely determined by the equation $PH=Q$.

(ii) $\Rightarrow$ (i) From $PH=Q$ we conclude

(73) $$R=(-Q,P)=(-H,I_p)P, \text{ hence } \text{rank}(R)\le\text{rank}(P).$$

By assumption (72)

(74) $$\text{rank}(R)=\text{o-dim}(S)\ge p\ge\text{rank}(P).$$

The latter inequality is trivial since P has only p columns. The combination of (73) and (74) yields (i).

(i), (ii) $\Rightarrow$(iii) Let X be universal with $XP=0$. Then, by (31), the equation $Py=v$ has a solution y if and only if $Xv=0$. Let now $u\in\mathbf{A}^m$ be arbitrary and $v:=Qu$, hence $Xv=XQu$. But $XQ=XPH=0$ since $XP=0$. There follows $Xv=0$ and the existence of $y\in\mathbf{A}^p$ with $Py=v=Qu$, i.e. $(uy)\in S$, as asserted.

(iii) $\Rightarrow$ (i), (ii) By assumption (iii) the system map

$$S=S(\mathbf{D}^{m+p}/R^T\mathbf{D}^k)\subset\mathbf{A}^{m+p}\xrightarrow{(I_m 0)=S\binom{I_m}{0}}\mathbf{A}^m=S(\mathbf{D}^m)$$

is surjective. By the main duality theorem this implies that the $\mathbf{D}$-linear map

$$\begin{array}{ccccc}\mathbf{D}^m & \xrightarrow{\binom{I_m}{0}} & \mathbf{D}^{m+p} & \xrightarrow{\text{can}} & \mathbf{D}^{m+p}/R^T\mathbf{D}^k \\ x & \longmapsto & \binom{x}{0} & \longmapsto & \text{can}\binom{x}{0}\end{array}$$

is injective or, in other terms, that $\left(\binom{x}{0}=R^Ty,\ y\in\mathbf{D}^k\right)$ implies $x=0$. But $R=(-Q,P)$ hence

$$\left(x=-Q^Ty,\ P^Ty=0\right) \text{ implies } x=0 \text{ or } \{y\in\mathbf{D}^k;P^Ty=0\}\subset\{y\in\mathbf{D}^k;Q^Ty=0\}.$$

By considering common denominators we conclude that also

(75) $$\{y\in\mathbf{K}^k;P^Ty=0\}\subset\{y\in\mathbf{K}^k;Q^Ty=0\}$$

But if $\perp$ denotes the orthogonal complement with respect to the standard non-degenerate, symmetric $\mathbf{K}$-bilinear form

$$\langle -,- \rangle : \mathbf{K}^{k} \times \mathbf{K}^{k} \longrightarrow \mathbf{K}\ ,\ \langle x,y \rangle := x^{T}y = x_{1}y_{1} + \cdots + x_{k}y_{k}$$

there is the well-known result from linear algebra

$$(P\mathbf{K}^{p})^{\perp} = \{y \in \mathbf{K}^{k} ; P^{T}y = 0\},\ \text{hence}\ P\mathbf{K}^{p} = \{y \in \mathbf{K}^{k} ; P^{T}y = 0\}^{\perp}\ \text{in}\ \mathbf{K}^{k}.$$

This, by the way, is a special case of (48). From (75) we thus conclude $P\mathbf{K}^{p} \supset Q\mathbf{K}^{m}$ for the column spaces since $(-)^{\perp}$ reverses the order. Hence the columns of Q are contained in the columns space of P or, in other terms, are **K**-linear combinations of the columns of P. But this means $PH=Q$ for some $H \in \mathbf{K}^{p,m}$, and thus (ii) is satisfied.

I show finally that H depends on S and the IO-structure (70), but not on the special choice of $R=(-Q,P)$. Assume that S is given in two representations

$$S = \{w = (uy) \in \mathbf{A}^{m+p} ; R_{i}w = 0\}\ ,\ R_{i} = (-Q_{i}, P_{i}), i = 1,2,\ \text{hence}$$

$$S = \{(uy) \in \mathbf{A}^{m+p} ; P_{i}y = Q_{i}u\}, i = 1,2.$$

Let $H_{i} \in \mathbf{K}^{p,m}$ be defined by $P_{i}H_{i} = Q_{i}$, $i=1,2$, as in (ii) . By (63) there are matrices X_1 and X_2 such that

$$R_{2} = (-Q_{2}, P_{2}) = X_{1}R_{1} = X_{1}(-Q_{1}, P_{1})\ \text{and}\ R_{1} = X_{2}R_{2},\ \text{hence}$$

$$Q_{2} = X_{1}Q_{1}, P_{2} = X_{1}P_{1}\ \text{and}\ Q_{1} = X_{2}Q_{2}, P_{1} = X_{2}P_{2}.$$

These equations yield

$$P_{2}H_{2} = Q_{2} = X_{1}Q_{1} = X_{1}P_{1}H_{1} = P_{2}H_{1}.$$

Since (70) is an IO-structure P_2 has rank p and can thus be cancelled on the left hence $H_1=H_2$ as desired. ||

(76) **Remark:** For the special case of example (1.18) and r=1 consult the book [BY] where differential input-output systems and relations are considered. Willems [WIL] , for the case of example (1.7) and r=1 , talks of IO-systems only if the transfer matrix H is proper. At this stage of my paper the transfer matrix H always exists uniquely as was shown above, but there is no causality structure. In particular H *cannot* be considered as an operator from $\mathbf{A}^{m}$ to $\mathbf{A}^{p}$ which assigns an output y to every input u for appropriate initial conditions. The theory of transfer functions or operators and their connection with the transfer matrix will be treated later and presupposes a suitable operational calculus which I will develop along lines given in the literature. Also the theory of minimal realizations will be treated later . ||

(77) **Corollary:** Assume that the system S admits a IO-form

(78) $$S=\{(uy)\in\mathbf{A}^{m+p};\ Py=Qu\}$$

with a square matrix P, i.e. $P\in\mathbf{D}^{p,p}, \det(P)\neq\mathbf{0}$, $Q\in\mathbf{D}^{p,m}$. Then the transfer matrix of S with respect to this IO-structure is the traditional

(79) $$H:=P^{-1}Q\in\mathbf{K}^{p,m}$$

where $P,P^{-1}\in Gl_p(\mathbf{K})$ since $\det(P)\neq 0$. Moreover P is unique up to **D**-left equivalence. This means that if also

$$S=\{(uy)\in\mathbf{A}^{m+p};P_1y=Q_1u\},\ P_1\in\mathbf{D}^{p,p},$$

the matrix $X:=P_1P^{-1}$ is not only in $Gl_p(\mathbf{K})$, but even in $Gl_p(\mathbf{D})$, i.e. both X and X^{-1} lie in $\mathbf{D}^{p,p}$.

Proof: The equations $PH=Q, P\in\mathbf{D}^{p,p}$ and $\mathrm{rank}\,(P)=p$ imply directly $P\in Gl_p(\mathbf{K}), \det(P)\neq 0$, and $H=P^{-1}Q$. Assume that also $S=\{(uy)\in\mathbf{A}^{m+p};P_1y=Q_1u\}, P_1\in\mathbf{D}^{p,p}$. From (63) we derive

$$(-Q_1,P_1)=X_1(-Q,P),(-Q,P)=X_2(-Q_1,P_1)$$

with matrices $X_i,X_2\in\mathbf{D}^{p,p}$. Hence in particular $P_1=X_1P, P=X_2P_1$ in $\mathbf{D}^{p,p}$ or $X=P_1P^{-1}=X_1$, $X^{-1}=PP_1{}^{-1}=X_2$ in $\mathbf{D}^{p,p}$ as asserted. ||

In 1-d system theory (79) is called a matrix fraction description of H (Compare [KAI], 6.23). For arbitrary S a representation (78) with square P and thus a matrix fraction representation (79) of H, however, cannot generally be achieved. This depends on the global dimension of **D** and will be explained in a later paragraph. ||

(80) **Remark:** Assume that

$$S=\{(uy)\in\mathbf{A}^{m+p};Py=Qu\},\quad P\in\mathbf{D}^{k,p},\ \mathrm{rank}(P)=p,\ Q=PH\in\mathbf{D}^{k,m},$$

is given in IO-form and that P can be transformed by elementary row operations into a block matrix $\begin{pmatrix}P_1\\0\end{pmatrix}, P_1\in\mathbf{D}^{p,p}$. This can always be achieved if **D** is a principal ideal domain, for instance the polynomial ring $F[s_1]$ in *one* indeterminate over a field. Then

$$UP=\begin{pmatrix}P_1\\0\end{pmatrix}\ \text{with}\ U\in Gl_k(\mathbf{D}),\det(P_1)\neq 0,\ \text{hence also}$$

$$UQ=UPH=\begin{pmatrix}P_1\\0\end{pmatrix}H=\begin{pmatrix}Q_1\\0\end{pmatrix}, Q_1\in\mathbf{D}^{p,m}.$$

Since $U \in Gl_k(\mathbf{D})$ this implies $S=\{(uy);P_1y=Q_1u\}$ and $H=\left(P_1\right)^{-1}Q_1$ as in the preceding corollary. ||

I am going to calculate the transfer matrix of Rosenbrock systems. But first I give another useful

(81) **Theorem:** Let $F:\mathbf{A}^{l(1)} \longrightarrow \mathbf{A}^{l(2)}$ be a system morphism, i.e. a $\mathbf{E}$-linear map, and let $S_1 \subset \mathbf{A}^{l(1)}$ be a subsystem with image $S_2:=F(S_1) \subset \mathbf{A}^{l(2)}$. Then

$$\text{i-dim}(S_2) \leq \text{i-dim}(S_1).$$

Proof : By the main duality theorem of (57) F has uniquely the form $F=S(P^T), P \in \mathbf{D}^{l(2),l(1)}$. The theorem (34) and its proof are valid, thus

(36) $$\left(P^T\right)^{-1}\left(R_1{}^T\,\mathbf{D}^{k(1)}\right)=R_2^T\mathbf{D}^{k(2)}.$$

Taking common denominators we derive

$\left(P^T\right)^{-1}\left(R_1{}^T\mathbf{K}^{k(1)}\right)=R_2{}^T\mathbf{K}^{k(2)}$ and as in (34) the injection

$$\left(P^T\right)_{ind}:\mathbf{K}^{l(2)}/\,R_2{}^T\,\mathbf{K}^{k(2)} \longrightarrow \mathbf{K}^{l(1)}/\,R_1{}^T\,\mathbf{K}^{k(1)}.$$

We conclude

$\text{i-dim}(S_2)=l(2)-\text{rank}(R_2)=l(2)-\text{rank}\left(R_2{}^T\right)=\dim_{\mathbf{K}}\mathbf{K}^{l(2)}/\,R_2{}^T\,\mathbf{K}^{k(2)} \leq$
$\leq \dim_{\mathbf{K}}\mathbf{K}^{l(1)}/\,R_1{}^T\mathbf{K}^{k(1)}=\text{i-dim}(S_1)$ as asserted. ||

(82) **Theorem** (Transfer matrix of Rosenbrock systems) Consider the Rosenbrock equations

(40) $$Px=Qu,\ y=Rx+Wu$$

with the additional property that the system

(83) $$S'=\{(u,x) \in \mathbf{A}^{m+n};Px=Qu\}$$

is in IO-form, i.e. $\text{rank}(P)=n$ and $Q=PH'$, $H' \in \mathbf{K}^{n,m}$, where H' is the transfer matrix of S'. Then the Rosenbrock system

$$S:=\{(uy) \in \mathbf{A}^{m+p};\exists x \in \mathbf{A}^n \text{ such that } Px=Qu,\ y=Rx+Wu\}$$

has the IO-form

(84) $$S=\{(uy) \in \mathbf{A}^{m+p};Yy=(XQ+YW)u\}$$

where X and Y are universal with $XP=YR$. The transfer matrix of S is $H=W+RH'$.

Proof: The representation (84) has been proven in (41). Since S is an image of S' the preceding theorem implies

$\text{i-dim}(S) \le \text{i-dim}(S') = m$, hence $\text{o-dim}(S) = (m+p) - \text{i-dim}(S) \ge p$. This is the general condition (72) in (69). Moreover

$$XQ+YW=XPH'+YW=YRH'+YW=Y(W+RH')=YH.$$

Thus $H:=W+RH'$ satisfies condition (ii) from (69) and theorem (69) shows that (84) is a IO-form of S with transfer matrix $H=W+RH'$. ||

(85) **Example:** In the situation of example (42) the state space equations $(sI_n-A)x=Bu, y=Cx+Du$ satisfy the property that

$$S':=\{(u,x)\in \mathbf{A}^{m+n};(sI_n-A)x=Bu\}$$

is in IO-form with transfer matrix $H'=(sI_n-A)^{-1}B$. Thus

$$S:=\{(uy)\in \mathbf{A}^{m+p};\exists x\in \mathbf{A}^n \text{ with } sx=Ax+Bu, y=Cx+Du\}=$$

$$=\{(uy)\in \mathbf{A}^{m+p};Yy=(XB+YD)u\} \text{ with } Y\in F[s]^{p,p}, XB+YD\in F[s]^{p,m}$$

is in IO-form with transfer matrix $H=D+CH'=D+C(sI_n-A)^{-1}B$ as usual. In particular $\text{rank}(Y)=p$, hence $\det(Y)\neq 0$ and

$$H=D+C(sI_n-A)^{-1}B=Y^{-1}(XB+YD). \quad ||$$

The assocciated signal flow space

Assumption (64) is in force. In 1-d system theory the assignment of a signal flow graph to a block diagram plays an important role, at least in the engineering literature, and whole books are devoted to this subject. (see, for instance, [CH]). The following considerations generalize those ideas substantially and give the algebraic, but not the graph theoretical background. Even for the 1-d case the following results give something new. The graph theoretical part will be treated later. The main idea is to compare a system

(86) $$S=\{w\in \mathbf{A}^l;Rw=0\}\subset \mathbf{A}^l,\ R\in \mathbf{D}^{k,l},$$

with the solution space

(87) $\widehat{S}:=\{\widehat{w}\in \mathbf{K}^l;R\widehat{w}=0\}\subset \mathbf{K}^l$ of the system $R\widehat{w}=0, R\in \mathbf{D}^{k,l}\subset \mathbf{K}^{k,l}, \widehat{w}\in \mathbf{K}^l$ of linear equations with coefficients in $\mathbf{K}$. The main observation is that the assignment $S\longmapsto \widehat{S}$ is well-defined and an exact functor. For this purpose consider the exact functor

(88) $$\mathbf{K}\otimes_{\mathbf{D}}(-):\text{Modf}(\mathbf{D})\longrightarrow \text{Modf}(\mathbf{K}),\ M\longmapsto \mathbf{K}\otimes_{\mathbf{D}}M,$$

given by extension of scalars. In more constructive terms $\mathbf{K}\otimes_{\mathbf{D}}M$ is the module of fractions $\mathbf{K}\otimes_{\mathbf{D}}M=\{m/t; t\in\mathbf{D}, t\neq 0\}$ (Compare [5], ch. II, § 2.2). Consider also the standard duality of finite dimensional vector spaces

$$(-)^*: \mathrm{Modf}(\mathbf{K})^{op} \cong \mathrm{Modf}(\mathbf{K}), V\longmapsto V^*,$$

with $V^*=\mathrm{Hom}_{\mathbf{K}}(V,\mathbf{K})$. Finally recall the duality (see (58))

$$M: \mathrm{Syst}(\mathbf{A})^{op}\cong \mathrm{Modf}(\mathbf{D}), S\longmapsto M(S),$$

with $\mathrm{Hom}_{\mathbf{D}}(M(S),\mathbf{A})\cong S$. Composition of these three functors yields

$$\mathrm{Syst}(\mathbf{A})^{op} \overset{M}{\cong} \mathrm{Modf}(\mathbf{D}) \xrightarrow{\mathbf{K}\otimes_{\mathbf{D}}(-)} \mathrm{Modf}(\mathbf{K}) \overset{(-)^*}{\cong} \mathrm{Modf}(\mathbf{K})^{op}$$

and hence the covariant exact functor

(90) $$\mathrm{Syst}(\mathbf{A})\longrightarrow \mathrm{Modf}(\mathbf{K}), S\longmapsto \tilde{S},$$

$$\text{where } \tilde{S}:=\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}\otimes_{\mathbf{D}}M(S),\mathbf{K})\cong \mathrm{Hom}_{\mathbf{D}}(M(S),\mathbf{K}), f\longmapsto g,$$

$$\text{with } f(k\otimes m)=kg(m), \ k\in\mathbf{K}, \ m\in M(S).$$

If S is given by (86) the module M(S) is $M(S)=\mathbf{D}^{1}/R^{T}\mathbf{D}^{k}$ due to (13) and

$$\text{thus } \tilde{S}=\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^{1}/R^{T}\mathbf{D}^{k},\mathbf{K}) \cong \hat{S}, \ g\longrightarrow (g(\bar{e}_1),\cdots,g(\bar{e}_1))^{T}$$

where the latter isomorphism is deduced like (13). I identify $\tilde{S}=\hat{S}$ and obtain

(91) **Theorem and Definition** (signal flow system) The exact functor (90) furnishes, by identification, the exact functor

(92) $$(-)^{\wedge}: \mathrm{Syst}(\mathbf{A})\longrightarrow \mathrm{Modf}(\mathbf{K}), \ S\longmapsto\hat{S},$$

$$\text{where } \hat{S}:=\{\hat{w}\in\mathbf{K}^{1}; R\hat{w}=0\} \text{ if } S=\{w\in\mathbf{A}^{1}; Rw=0\}, \ R\in\mathbf{D}^{k,1}.$$

This means:

(i) $\hat{S}$ is well-defined, i.e. $\hat{S}$ depends on S only and not on the special choice of the defining matrix R.

(ii) If $S_i=\{w_i\in\mathbf{A}^{1(i)}; R_i w_i=0\}$, $R_i\in\mathbf{D}^{k(i),1(i)}$, $i=1,2$, are two systems and

$$P = S(P^{T}) : S_1 \longrightarrow S_2, \ P\in\mathbf{D}^{1(2),1(1)},$$

is a system morphism , i.e. $PS_1\subset S_2$, then also $P\hat{S}_1\subset\hat{S}_2$.

(ii) The assignment $S\longmapsto\hat{S}$ is exact.

Moreover $\dim_{\mathbf{K}}\hat{S}=\text{i-dim}(S)$.

The **K**-space $\hat{S}$ is called *the associated signal flow space* of S.

The last equation comes from the standard equality

$$\dim_{\mathbf{K}}\{\hat{w}\in\mathbf{K}^{1}; R\hat{w}=0\}=1-\mathrm{rank}(R)=\text{i-dim}(S). \ ||$$

I will show instantly that $\hat{S}$ contains all the information on the IO-structures and the transfer matrices of S. Consider first the system

$$S=\{(uy)\in\mathbf{A}^{m+p};\ Py=Qu\},\ PH=Q,$$

in IO-form with transfer matrix H. Then $\widehat{S}=\{(\widehat{u}\widehat{y})\in\mathbf{K}^{m+p};P\widehat{y}=Q\widehat{u}\}$. Since rank(P)=p the map $P:\mathbf{K}^p\longrightarrow\mathbf{K}^k$ is injective ($P:\mathbf{A}^p\longrightarrow\mathbf{A}^k$ is not!) and thus $P\widehat{y}=Q\widehat{u}=PH\widehat{u}$ and $\widehat{y}=H\widehat{u}$ are equivalent. This yields

$$\widehat{S}=\{(\widehat{u}\widehat{y})\in\mathbf{K}^{m+p};\widehat{y}=H\widehat{u}\}=:\mathrm{Graph}(H)$$

where Graph(H) is the graph of the **K**-linear map $H:\mathbf{K}^m\longrightarrow\mathbf{K}^p$. Assume on the other side that for a system $S=\{w\in\mathbf{A}^l;Rw=0\},R\in\mathbf{D}^{k,l}$, the associated signal flow space $\widehat{S}=\{\widehat{w}\in\mathbf{K}^l;R\widehat{w}=0\}$ is written in the form

(93) $$\widehat{S}=\{\widehat{w}=(\widehat{u}\widehat{y})\in\mathbf{K}^{m+p};P\widehat{y}=Q\widehat{u}\}=\mathrm{Graph}(\widehat{H})$$

with $(-Q,P):=R$, $\widehat{H}\in\mathbf{K}^{p,m}$. This can always be achieved for a suitable decomposition $\widehat{w}=(\widehat{u}\ \widehat{y})$ and $\widehat{H}\in\mathbf{K}^{m+p}$ by standard linear algebra results. In particular then

$$m=\dim\widehat{S}=\text{i-dim}(S),\ \text{hence}\ p=\text{o-dim}(S).$$

Moreover (93) implies $P(\widehat{H}\widehat{u})=Q\widehat{u}$ for all $\widehat{u}\in\mathbf{K}^m$ and finally $P\widehat{H}=Q$. From (69), (ii), we conclude that $w=(uy)\in\mathbf{A}^{m+p}$ is a IO-structure of S and that $\widehat{H}$ is its transfer matrix. Hence the

(94) **Theorem** (IO-structures and transfer matrix via the signal flow space). A system

$$S=\{w\in\mathbf{A}^l;Rw=0\},R\in\mathbf{D}^{k,l},$$

admits the IO-form

$$S=\{w=(uy)\in\mathbf{A}^{m+p};Py=Qu\}$$

with transfer matrix $H\in\mathbf{K}^{p,m}$ if and only if $\widehat{S}=\{\widehat{w}\in\mathbf{K}^l;R\widehat{w}=0\}$ is the graph of H with respect to the corresponding decomposition $\widehat{w}=(\widehat{u}\ \widehat{y})$ or, in other terms, the map

(95) $$\begin{pmatrix}I_m\\H\end{pmatrix}:\mathbf{K}^m\cong\widehat{S}=\{\widehat{w}=\begin{pmatrix}\widehat{u}\\\widehat{y}\end{pmatrix}\in\mathbf{K}^{m+p};\ \widehat{y}=H\widehat{u}\}.$$

$$\widehat{u}\longmapsto\begin{pmatrix}I_m\\H\end{pmatrix}\widehat{u}=\begin{pmatrix}\widehat{u}\\H\widehat{u}\end{pmatrix}=(\widehat{u},H\widehat{u})$$

is an isomorphism. ||

(96) **Example** (Transfer matrix of Rosenbrock systems) Consider the situation of theorem (82). The system S' has the transfer matrix H' which implies the **K**-isomorphism

(97) $$\begin{pmatrix} I_m \\ H' \end{pmatrix} : \mathbf{K}^m \cong (S')^\wedge \; , \; \hat{u} \longmapsto \begin{pmatrix} \hat{u} \\ H'\hat{u} \end{pmatrix}$$

Furthermore the map

(98) $$\begin{pmatrix} I_m & 0 \\ W & R \end{pmatrix} : S' \longrightarrow S, \begin{pmatrix} u \\ x \end{pmatrix} \longmapsto \begin{pmatrix} u \\ Wu+Rx \end{pmatrix}$$

is a system epimorphism by definition of S. Since the functor $(-)^\wedge$ is exact

(98) $$\begin{pmatrix} I_m & 0 \\ W & R \end{pmatrix} : (S')^\wedge \longrightarrow \hat{S}$$ is surjective.

Composition of (97) and (98) yields the surjection

$$\mathbf{K}^m \longrightarrow \hat{S} \; , \; \hat{u} \longmapsto (\hat{u}, (W+RH')\hat{u}).$$

But this map is obviously injective, thus an isomorphism and hence $\hat{S} = \mathrm{Graph}(W+RH')$. This says that $H := W+RH'$ is the transfer matrix of the Rosenbrock system . Remark that the IO-representation (84) of (82) cannot be derived from $\hat{S}$. ||

Induced dualities

Here I assume that $\mathbf{D}$ is a commutative noetherian ring, not necessarily integral, and that $\mathbf{A}$ is a large injective cogenerator. Thus theorem (56) is applicable. Furthermore let $\mathbf{I} \subset \mathbf{D}$ be an ideal and $\mathbf{D}/\mathbf{I}$ the factor ring. A $\mathbf{D}/\mathbf{I}$-module M is the same as a $\mathbf{D}$-module annihilated by $\mathbf{I}$, i.e. with $\mathbf{I}M=0$. The scalar multiplications are conncected by

$$\bar{p}m = pm \; , \; p \in \mathbf{D} \; , \; m \in M, \; \bar{p} \in \mathbf{D}/\mathbf{I}.$$

In this fashion I consider the category Mod($\mathbf{D}/\mathbf{I}$) as a full subcategory of Mod($\mathbf{D}$) and similarly for the finitely generated modules. thus

$$\mathrm{Modf}(\mathbf{D}/\mathbf{I}) \subset \mathrm{Modf}(\mathbf{D}).$$

In both categories the morphisms are just the $\mathbf{D}$-linear maps. The finitely generated $\mathbf{D}/\mathbf{I}$ -modules are exactly the $\mathbf{D}$-epimorphic images of the modules $(\mathbf{D}/\mathbf{I})^k = \mathbf{D}^k/\mathbf{I}\mathbf{D}^k$, $k \in \mathbb{N}$. Thus the duality

$$S = \mathrm{Hom}_{\mathbf{D}}(-, \mathbf{A}) : \mathrm{Modf}(\mathbf{D})^{op} \cong \mathrm{Syst}(\mathbf{A})$$

maps the full subcategory $\mathrm{Modf}(\mathbf{D}/\mathbf{I})$ onto the category of those systems S which can be $\mathbf{E}$-linearly embedded into some $S(\mathbf{D}/\mathbf{I})^k$. But the $\mathbf{E} = \mathrm{End}_{\mathbf{D}}(\mathbf{A})$-isomorphism $S(\mathbf{D}) = \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}, \mathbf{A}) \cong \mathbf{A}, f \longmapsto f(1)$, induces the isomorphism

$$S(\mathbf{D}/\mathbf{I})=\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}/\mathbf{I},\mathbf{A})\cong\{a\in\mathbf{A};\mathbf{I}a=0\}=:(0:\mathbf{I})_{\mathbf{A}}=:(0:\mathbf{I})$$

Hence, by identification,

$$\begin{array}{ccccc} S(\mathbf{D}) & = & \mathrm{Hom}_{\mathbf{D}}(\mathbf{D},\mathbf{A}) & = & \mathbf{A} \\ \cup & & \cup & & \cup \\ S(\mathbf{D}/\mathbf{I}) & = & \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}/\mathbf{I},\mathbf{A}) & = & (0:\mathbf{I}):=\{a\in\mathbf{A};\mathbf{I}a=0\}. \end{array}$$

(99) **Theorem and Definition:** Assumption (55). Let $\mathbf{I}$ be an ideal of $\mathbf{D}$. The duality $S=\mathrm{Hom}_{\mathbf{D}}(-,\mathbf{A}) : \mathrm{Modf}(\mathbf{D})^{op}\cong\mathrm{Syst}(\mathbf{A})$ induces the duality $S:\mathrm{Modf}(\mathbf{D}/\mathbf{I})^{op}\cong\mathrm{Syst}((0:\mathbf{I})_{\mathbf{A}})$ where $(0:\mathbf{I})_{\mathbf{A}}:=\{a\in A;\mathbf{I}a=o\}\subset\mathbf{A}$ and where $\mathrm{Syst}((0:\mathbf{I})_{\mathbf{A}})$ consists of those $\mathbf{A}$-systems S annihilated by $\mathbf{I}$, i.e. $\mathbf{I}S=0$, or, in other terms, which can be embedded into some system $(0:\mathbf{I})^k, k\in\mathbb{N}$. ||

(100) **Remark:** The preceding theorem can also be proven by the remark that $(0:\mathbf{I})_{\mathbf{A}}=\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}/\mathbf{I},\mathbf{A})$ is a large injective cogenerator over $\mathbf{D}/\mathbf{I}$. For the injectivity this is an easy lemma (see [BOU2], §1.8, Prop.11). If further M is a finitely generated $\mathbf{D}/\mathbf{I}$ -module and $f:M\longrightarrow\mathbf{A}^k, k\in\mathbb{N}$, a $\mathbf{D}$-linear embedding then $\mathbf{I}M=0$ implies $\mathbf{I}\cdot\mathrm{im}(f)=0$, i.e. $\mathrm{im}(f)\subset(0:\mathbf{I})^k$, such that f induces the embedding $f:M\longrightarrow(0:\mathbf{I})^k$ as desired. The theorem follows from the identification $S(M)=\mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A})=\mathrm{Hom}_{\mathbf{D}/\mathbf{I}}(M,(0:\mathbf{I})_A), M\in\mathrm{Modf}(\mathbf{D}/\mathbf{I})$, and the main duality theorem (56) applied to $(0:\mathbf{I})_{\mathbf{A}}$ over $\mathbf{D}/\mathbf{I}$. ||

(101) **Corollary:** If $R\in\mathbf{D}^{k,l}$ the module $M:=(\mathbf{D}/\mathbf{I})^l/\bar{R}^T(\mathbf{D}/\mathbf{I})^k$, $\bar{R}\in(\mathbf{D}/\mathbf{I})^{k,l}$ gives rise to the system

$$S:=S(M)=\{w\in(0:\mathbf{I})^l;Rw=0\}=\{w\in\mathbf{A}^l;\ Rw=0\ ,\ \mathbf{I}w=0\ \}$$

Hence, to consider systems in $\mathrm{Syst}((0:\mathbf{I})_{\mathbf{A}})$ means to solve systems $Rw=0, R\in\mathbf{D}^{k,l}$, with "*functions*" $w_j\in\mathbf{A}$ which are themselves solutions of the equations $\mathbf{I}w_j=0$, i.e. $pw_j=0$ for all $p\in\mathbf{I}$. ||

(101) **Example:** Consider example (1.18) with $\mathbf{D}=\mathbb{C}[s_1,\cdots,s_r]$, $\mathbf{A}=C^{\infty}(\mathbb{R}^r)$ and $pa=p(\partial_1,\cdots,\partial_r)(a)$. For the principal ideal $\mathbf{I}:=\mathbb{C}[s]\Delta, \Delta=(s_1)^2+\cdots+(s_r)^2$, one obtains $\mathbf{H}:=(0:\mathbf{I})_{\mathbf{A}}=\{a\in C^{\infty}(\mathbb{R}^r);\Delta a=0\}$, the set of all harmonic functions. Theorem (99) induces the duality

(102) $$\mathrm{Mod}(\mathbb{C}[s]/\langle\Delta\rangle)^{op}\cong\mathrm{Syst}(\mathbf{H})$$

The systems in $\mathrm{Syst}(\mathbf{H})$ of the form $S=\{w\in\mathbf{H}^l;Rw=0\}, R\in\mathbb{C}[s]^{k,l}$, consist of solutions $w=(w_1 \cdots w_l)^T$ of systems of partial differential equations $R(\partial)(w)=0$ whose components w_j are harmonic. The modules on the left side of (102) are $\mathbb{C}[s]$-modules M satisfying $\Delta M=((s_1)^2+\cdots+(s_r)^2)M=0$.||

(103) **Example**: In the case of example (1.18) and r=1, i.e. $\mathbf{D}=\mathbb{C}[s]$, $\mathbf{A}=C^\infty(\mathbb{R})$ with $sa=\dot{a}$ consider the ideal $\mathbf{I}=\mathbb{C}[s](s^2+\omega^2)$, $\omega>0$ real. Then

$$\mathbf{P}:=(0:(s^2+\omega^2))_\mathbf{A}=\{a\in C^\infty(\mathbb{R});\ddot{a}+\omega^2a=0\}$$

consists of all complex oszillations $a(t)=a_o\sin(\omega t+\varphi), a_o\in\mathbb{C}, \varphi\in\mathbb{R}$, with complex *amplitude* a_o, *period* $T:=2\pi/\omega$ and *phase* φ. The factorization $s^2+\omega^2=(s-i\omega)(s+i\omega)$ induces the algebra isomorphism

$$\mathbb{C}[s]/\langle s^2+\omega^2\rangle \cong \mathbb{C}^2,\ p\longmapsto(p(+i\omega),p(-i\omega))$$

by the *Chinese remainder* theorem. This decomposition again gives the identification

$$\mathrm{Modf}\,(\mathbb{C}[s]/\langle s^2+\omega^2\rangle) = \mathrm{Modf}(\mathbb{C})^2 \subset \mathrm{Modf}(\mathbb{C}[s])$$

where a pair $\left(\mathbb{C}^{n(+)},\mathbb{C}^{n(-)}\right)=\mathbb{C}^{n(+)}\times\mathbb{C}^{n(-)}$ of complex vector spaces is considered as a $\mathbb{C}[s]$-module via $p\cdot(v_+,v_-)=(p(i\omega)v_+,p(-i\omega)v_-)$ for $p\in\mathbb{C}[s], v_+\in\mathbb{C}^{n(+)}, v_-\in\mathbb{C}^{n(-)}$. The corresponding decomposition of $\mathbf{P}$ is

$$\mathbf{P}=\mathbb{C}\,e^{i\omega t}\oplus\mathbb{C}e^{-i\omega t}\ \text{with}\ \mathbb{C}e^{i\omega t}=\{a;\dot{a}=i\omega a\},\ \mathbb{C}e^{-i\omega t}=\{a;\dot{a}=-i\omega a\}.$$

System theory in Syst($\mathbf{P}$) plays an important role in electrical engineering and is thus reduced to the easy theory of pairs (V_+,V_-) of complex vector spaces with no interaction between V_+ and V_-. Replacing $s^2+\omega^2$ by an arbitrary complex polynomial $p=s^d+p_{d-1}s^{d-1}+\cdots+p_o\in\mathbb{C}[s]$ gives the similar theory of systems in

$$\mathrm{Syst}((0:p)),\ (0:p)=\{a\in C^\infty(\mathbb{R});a^{(d)}+\cdots+p_1\dot{a}+p_o\,a=0\}.$$

The main ingredient is again the Chinese remainder isomorphism

$$\mathbb{C}[s]/\langle p\rangle\cong\Pi\{\mathbb{C}[s]/\langle(s-\lambda_i)^{d(i)}\rangle\ ;\ i=1,.,k\}$$

where $p=\Pi\{(s-\lambda_i)^{d(i)};i=1,.,k\}$ is the decomposition of p into linear factors. For r>1 , on the contrary , the structure of finitely generated modules over $\mathbb{C}[s_1,.,s_r]$ or $\mathbb{C}[s]/\mathbf{I}$ is very difficult in general (a *wild* problem in the sense of representation theory).||

(104) **Example** (General multiperiodic functions) This example generalizes

the preceding one, but this time I formulate it for the discrete case with $\mathbf{D}=\mathbb{C}[s_1,..,s_r]$ and $\mathbf{A}=\mathbb{C}^{\mathbb{N}^r}$. Choose r "periods" $d(j)>1$, $j=1,..,r$, and define the ideal $\mathbf{I}:=<s_1{}^{d(1)}-1,\cdots,s_r{}^{d(r)}-1> \subset \mathbb{C}[s]$. Then

$$\mathbf{P}:=(0:\mathbf{I})=\{a\in\mathbb{C}^{\mathbb{N}^r};\ s_j{}^{d(j)}\ a=a\ ,\ j=1,..,r\ \}.$$

But $\qquad (s_j{}^{d(j)}a)(n)=a(n(1),\cdots,n(j-1),n(j)+d(j),n(j+1),\cdots,n(r))$

Hence $\mathbf{P}=(0:\mathbf{I})$ consists of all multiindexed, multiperiodic sequences $a=(a(n);n\in\mathbb{N}^r)$ which are periodic with period d(j) in the j.th direction. Let $\zeta_j:=\exp(2\pi i/d(j)),j=1,..,r$, be the primitive d(j).th root of unity, and define the index set $K:=\{k\in\mathbb{N}^r;o\le k(j)<d(j),j=1,.,r\}$.

The Fourier transform for the group $\mathbb{Z}/\mathbb{Z}(1)\times\cdots\times\mathbb{Z}/\mathbb{Z}d(r)$ or the Chinese remainder theorem furnish the algebra isomorphism

$$\mathbb{C}[s]/\mathbf{I}\cong\mathbb{C}^K\ ,\ \bar{p}\longmapsto(p(\zeta_1{}^{k(1)},..,\zeta_r{}^{k(r)});\ k\in K\ ).$$

This algebra isomorphism induces the category equivalence

(105) $$\mathrm{Mod}(\mathbb{C}[s]/\mathbf{I})\cong\mathrm{Mod}(\mathbb{C})^K\ ,\ V\longmapsto(V_k;k\in K)$$

in the following fashion: If V is a $\mathbb{C}[s]$-module with $\mathbf{I}V=0$, i.e. $(s_j{}^{d(j)}-1)v=0$ for $v\in V$, $j=1,\cdots,r$, then

$$V=\oplus\{\ V_k;k\in K\},V_k:=\{v\in V;\ pv=p(\zeta_1{}^{k(1)},\cdots,\ \zeta_r{}^{k(r)})v\ \text{for all}\ p\in\mathbb{C}[s]\}.$$

If, on the other side, the family $(V_k;k\in K)$ of $\mathbb{C}$-vector spaces is given the $\mathbb{C}[s]$-module V is defined by $V=\oplus\{V_k;k\in K\}=\prod\{V_k;k\in K\}$ with the $\mathbb{C}[s]$-multiplication

$$p(\Sigma\{v_k;k\in K\}):=\Sigma\{p(\zeta_1{}^{k(1)},\cdots,\zeta_r{}^{k(r)})v_k;k\in K\}.$$

The equivalence (105) induces the equivalence $\mathrm{Modf}(\mathbb{C}[s]/\mathbf{I})\cong\mathrm{Modf}(\mathbb{C})^K$.

Also $\mathbf{P}=\oplus\{\mathbf{P}_k;k\in K\}$. It is not difficult to prove that

$$\mathbf{P}_k=\mathbb{C}a_k,a_k\in\mathbb{C}^{\mathbb{N}^r}=\mathbb{C}\{t_1,\cdots,t_r\}\ \text{where}\ a_k=(1-\zeta_1{}^{k(1)}t_1)^{-1}\cdots(1-\zeta_r{}^{k(r)}t_r)^{-1}$$

$$\text{or}\ a_k(n)=\zeta_1{}^{k(1)n(1)}\cdots\zeta_r{}^{k(r)n(r)},\ k\in K\subset\mathbb{N}^r,\ n\in\mathbb{N}^r.$$

These conditions yield the following consequences for systems. The endomorphism ring $\mathbf{E}$ is given by

$$\mathbf{E}=\mathrm{End}_{\mathbb{C}[s]/\mathbf{I}}(\mathbf{P},\mathbf{P})\cong\mathrm{End}_{\mathbb{C}^K}(\mathbb{C}^K,\mathbb{C}^K)\cong\mathbb{C}^K.$$

Every system is, up to isomorphism, of the form

$$S=\oplus\{S_k;k\in K\}\ ,\ S_k=(\mathbb{C}a_k)^{l(k)},l(k)\ge 0,$$

and the l(k) are a complete set of invariants of S. The system morphisms

(=**E**-linear maps) between S and $T=\oplus\{T_k;k\in K\}, T_k=(\mathbb{C}a_k)^{m(k)}, k\in K$, are given by $\mathrm{Hom}_{\mathbf{E}}(S,T)=\mathrm{Hom}_{\mathbb{C}}K(\oplus S_k,\oplus T_k)\cong\Pi\{\mathbb{C}^{m(k),l(k)};k\in K\}$. In this fashion system theory in Syst(**P**) is completely reduced to linear algebra over $\mathbb{C}$. In the continuous case with $\mathbf{A}:=\mathbb{C}^{\infty}(\mathbb{R}\ )$ the same results hold with the modification

$$\mathbf{P}_k=\mathbb{C}a_k\ ,\ a_k(t_1,\cdots,t_r)=\exp[2\pi i(\zeta_1{}^{k(1)}t_1+..+\zeta_r{}^{k(r)}t_r)], k\in K.$$

(106) **Example:** Consider the example (1.18) in an even dimension $2r$ with the following notation. Identify $\mathbb{C}^r=\mathbb{R}^{2r}$ with the variables $z_\rho=x_\rho+iy_\rho=(x_\rho,y_\rho)\in\mathbb{C}=\mathbb{R}^2, \rho=1,..,r$. The functions *a* in $\mathbf{A}:=\mathbb{C}^{\infty}(\mathbb{C}^r)$ are written as $a=a(z_1,\cdots,z_r)=a(x_1,y_1,\cdots,x_r,y_r)$. The polynomial algebra $\mathbb{C}[s,t]:=\mathbb{C}[s_1,t_1,\cdots,s_r,t_r]$ in $2r$ indeterminates operates on **A** via

(107) $$s_\rho a=\partial a/\partial x_\rho, t_\rho a=\partial a/\partial y_\rho, \rho=1,..,r.$$

Consider the ideal $\mathbf{I}:=\langle s_1+it_1,\cdots,s_r+it_r\rangle\subset\mathbb{C}[s,t]$. Remark that $(s_\rho+it_\rho)a=\partial a/\partial x_\rho+i\partial a/\partial y_\rho=\partial a/\partial\bar{z}_\rho$ in the customary notation. Hence

$$(0:\mathbf{I})_{\mathbf{A}}=\{a\in C^{\infty}(\mathbb{R}^{2r});\partial a/\partial\bar{z}_\rho=0,\rho=1,..,r\}=\mathcal{O}(\mathbb{C}^r)$$

is the space of holomorphic functions on $\mathbb{C}^r$. Further, by identification

$$\mathbb{C}[s]=\mathbb{C}[s,t]/\mathbf{I},\ \bar{s}_\rho=s_\rho\ ,\ \bar{t}_\rho=i\bar{s}_\rho=is_\rho.$$

The operation of $\mathbb{C}[s,t]$ on $C^{\infty}(\mathbb{R}^{2r})$ given by (107) induces the operation

(108) $$s_\rho h=\partial h/\partial x_\rho=\partial h/\partial z_\rho\ , h\in\mathcal{O}(\mathbb{C}^r),$$

of $\mathbb{C}[s]$ on $\mathcal{O}(\mathbb{C}^r)$. This is exactly the structure from example (1.24). Theorem (99) or Remark (100) imply that $\mathcal{O}(\mathbb{C}^r)$ is a large injective cogenerator over $\mathbb{C}[s]$ if $\mathbf{C}^{\infty}(\mathbb{R}^{2r})$ is one over $\mathbb{C}[s,t]$. This observation reduces the work in the proof of theorem (54). Ehrenpreis in [EH] proves the injectivity of $\mathbb{C}^{\infty}(\mathbb{R}^{2r})$ and $\mathcal{O}(\mathbb{C}^r)$ separately. ||

3. THE MAIN THEOREM IN THE DISCRETE CASE

In this paragraph I prove for the discrete case of example (1.7) that $F^{\mathbb{N}^r}$ is a large injective cogenerator over $\mathbf{D}=F[s_1,\cdots,s_r]$. Both for applications and the proof I generalize the theorem to arbitrary affine algebras.

Vector space duality

The following remarks of a technical nature are the simplest case of the main duality theorem (2.56). Let F be a field and

(1) $$\langle -,- \rangle : V \times \widehat{V} \longrightarrow F$$

a F-bilinear form. As usual, (1) induces F-linear maps

(2) $$\widehat{V} \longrightarrow \mathrm{Hom}_F(V,F),\ \widehat{v} \longmapsto \langle -,\widehat{v} \rangle, \text{ and}$$

(3) $$V \longrightarrow \mathrm{Hom}_F(\widehat{V},F),\ v \longmapsto \langle v,- \rangle.$$

I call $\langle -,- \rangle$ non-degenerate if the induced map (2) is an isomorphism, i.e. if every linear function on V has uniquely the form $\langle -,\widehat{v} \rangle, \widehat{v} \in \widehat{V}$. Remark that for infinite dimensional spaces this definition is not symmetric in V and $\widehat{V}$. If $\langle -,- \rangle$ is non- degenerate the map (3) is injective, but bijective only if V is finite-dimensional (see also [JA], Ch.4, §4, Th1).

(4) **Examples:** (i) The evaluation map

$$\langle -,- \rangle : V \times \mathrm{Hom}_F(V,F) \longrightarrow F,\ \langle v,v^* \rangle := v^*(v),$$

is trivially non-degenerate since the corresponding map (2) is the identity. Up to isomorphism this is the only example.

(ii) Let I be an index set and

$$F^{(I)} := \{x \in F^I;\ x(i)=0 \text{ for almost all } i \in I\} \subset F^I.$$

Then the standard "scalar product"

$$\langle -,- \rangle : F^{(I)} \times F^I \longrightarrow F,\ \langle x,y \rangle := \Sigma\{x(i)y(i);\ i \in I\},$$

is non-degenerate . ||

If $\langle -,- \rangle_V : V \times \widehat{V} \longrightarrow F$ and $\langle -,- \rangle_W : W \times \widehat{W} \longrightarrow F$ are non- degenerate any F-linear map $f : V \longrightarrow W$ induces the adjoint map $f^* : \widehat{W} \longrightarrow \widehat{V}$ via

(5) $$\langle f(v),\widehat{w} \rangle_W = \langle v, f^*(\widehat{w}) \rangle_V,\ v \in V, \widehat{w} \in \widehat{W}.$$

The correspondence $f \longmapsto f^*$ has the usual properties. ||

I return to system theory and make the

(6) **Assumption:** Let F be a field, $\mathbf{D}$ a commutative F-algebra and $\mathbf{A} := \mathrm{Hom}_F(\mathbf{D},F)$ its dual space. ||

It is a standard fact that $\mathbf{A}$ itself becomes a $\mathbf{D}$-module via the operation

(7) $$(pa)(q) = \langle q,pa \rangle := \langle qp,a \rangle = a(pq) \text{ for } p,q \in \mathbf{D} \text{ and } a \in \mathbf{A}$$

or, in other terms, $(a \longmapsto pa) = (q \longmapsto pq)^*$.

(8) **Examples:** (i) The polynomial algebra $\mathbf{D} := F[s_1,\cdots,s_r]$ has the basis

$s^n = (s_1)^{n(1)} \cdots (s_r)^{n(r)}, n \in \mathbb{N}^r$. The F-isomorphism

$$\mathbf{A} := \mathrm{Hom}_F(F[s],F) \longrightarrow F^{\mathbb{N}^r} = F\{t_1,\cdots,t_r\}$$

$$a \longmapsto (a(s^n);\ n \in \mathbb{N}^r) = \sum\{a(s^n)t^n;\ n \in \mathbb{N}^r\}$$

suggests the identification

$$\mathbf{A} = \mathrm{Hom}_F(F[s],F) = F^{\mathbb{N}^r} = F\{t\}\ ,\ a = (a(s^n); n \in \mathbb{N}^r) = \sum a(s^n)t^n\ .$$

The $F[s]$-structure of $\mathbf{A}$ from (7) is given by

$$(s^k a)(n) = (s^k a)(s^n) = a(s^{k+n}) = a(k+n) \text{ for } k,n \in \mathbb{N}^r,\ a \in \mathbf{A} = F^{\mathbb{N}^r},$$

and coincides with the left shift structure from example (1.7).

(ii) Consider the quotient ring

$$F[s,s^{-1}] := F[s_1,\cdots,s_r,s_1^{-1},\cdots,s_r^{-1}] = \{\ ps^{-m};\ p \in F[s],\ m \in \mathbb{N}^r\ \}$$

of $F[s]$ of all *Laurent* polynomials. As a vector space this has the basis $s^n = s_1^{n(1)} \cdots s_r^{n(r)}, n \in \mathbb{Z}^r$, with $n(i) \in \mathbb{Z}$ instead of $n(i) \in \mathbb{N}$ as in (i). Again I identify

$$\mathrm{Hom}_F(F[s],F) = F^{\mathbb{Z}^r},\ a = \left(a(s^n);\ n \in \mathbb{N}^r\right)\ .$$

The $F[s,s^{-1}]$-module structure of $F^{\mathbb{Z}^r}$ is given by left shifts. For $r = 1$ Willems [WIL] considers this case too. The distinctive feature of this example compared to (i) is that the left shifts are given by the operation of the *group* $\mathbb{Z}^r$ and are thus bijective. ||

If M is a $\mathbf{D}$-module and V a F-vector space the F-space $\mathrm{Hom}_F(M,V)$ becomes a $\mathbf{D}$-module too via the structure

(9) $$\mathbf{D} \times \mathrm{Hom}_F(M,V) \longrightarrow \mathrm{Hom}_F(M,V), (p,f) \longmapsto pf\ ,$$

$$(pf)(m) = f(pm)\ ,\ f \in \mathrm{Hom}_F(M,V)\ , p \in \mathbf{D}\ , m \in M.$$

This is a generalization of (7). In particular, for $M = \mathbf{D}$, one obtains the functor

(10) $$\mathrm{Hom}_F(\mathbf{D},-) : \mathrm{Mod}(F) \longrightarrow \mathrm{Mod}(\mathbf{D}),\ V \longmapsto \mathrm{Hom}_F(\mathbf{D}, V)\ ,$$

which is right adjoint to the "forgetful" functor

$$\mathrm{Mod}(\mathbf{D}) \longrightarrow \mathrm{Mod}(F)\ ,\ M \longmapsto M$$

where a $\mathbf{D}$-module is just considered as a F-vector space only. The adjointness isomorphism is given for $M \in \mathrm{Mod}(\mathbf{D})$ and $V \in \mathrm{Mod}(F)$ by

(11) $$\mathrm{Hom}_F(M,V) \cong \mathrm{Hom}_{\mathbf{D}}(M,\mathrm{Hom}_F(\mathbf{D},V))\ ,\ f \longleftrightarrow g,$$

$$f(m)=g(m)(1),\ g(m)(p)=f(pm).$$

Since every F-vector space V is injective as a F-module the functors $\mathrm{Hom}_F(-,V)$ and hence also, by (11), $\mathrm{Hom}_{\mathbf{D}}(-,\mathrm{Hom}_F(\mathbf{D},V))$ are exact. This means that $\mathrm{Hom}_F(\mathbf{D},V)$, in particular $\mathrm{Hom}_F(\mathbf{D},F)=\mathbf{A}$, are injective $\mathbf{D}$-modules. This easy and well-known argument (see [BOU2], §1.8, Prop. 11) furnishes the important

(12) **Corollary:** If $\mathbf{D}$ is a F-algebra its dual space $\mathbf{A}:=\mathrm{Hom}_F(\mathbf{D},F)$ is an injective $\mathbf{D}$-module. ||

(13) **Remark:** By a similar argument ([BOU2], §1.8, Prop. 13) it can be easily shown that $\mathbf{A}$ in (12) is a cogenerator . It is much more work to show below that $\mathbf{A}$ is a *large* injective cogenerator and consequently satisfies the hypotheses of (2.56) . ||

(14) **Assumption:** In the remainder of this paragraph I assume that $\mathbf{D}$ is an affine F-algebra , i.e. a commutative algebra which is finitely generated (=of finite type) over the field F . As above the dual space is denoted by $\mathbf{A}:=\mathrm{Hom}_F(\mathbf{D},F)$. ||

(15) **Theorem :** Under assumption (14) $\mathbf{A}=\mathrm{Hom}_F(\mathbf{D},F)$ is a large injective cogenerator . ||

The proof of this theorem proceeds by a series of reduction steps . Consider for a an arbitrary affine algebra $\mathbf{D}$ the following

(16) **Property :** $\mathbf{D}$ can be $\mathbf{D}$-linearly embedded into some $\mathbf{A}^k=\mathrm{Hom}_F(\mathbf{D},F)^k$, i.e. there is a $\mathbf{D}$-monomorphism $\mathbf{D}\rightarrowtail\mathbf{A}^k$, $k\in\mathbb{N}$. Obviously this is a special case of the large cogenerator property (see (2.52)) . ||

(17) **Lemma** (Reduction to property (16)): For the proof of theorem (15) it is sufficient to show that each affine integral domain has property (16) .

Proof: Assume that each affine integral domain has property (16) . Let now $\mathbf{D}$ be an arbitrary affine algebra with $\mathbf{A}:=\mathrm{Hom}_F(\mathbf{D},F)$ and $\mathbf{p}\subset\mathbf{D}$ a prime ideal. By (2.52) it is sufficient to show that $\mathbf{D}/\mathbf{p}$ can be $\mathbf{D}$-linearly embedded into some $\mathbf{A}^k$, $k\in\mathbb{N}$. But the $\mathbf{D}$-linear surjection $can:\mathbf{D}\longrightarrow\mathbf{D}/\mathbf{p}$ gives rise to the

D-linear injection

$$(18)\qquad \mathrm{Hom}(\mathrm{can},F):\mathrm{Hom}_F(\mathbf{D}/\mathbf{p},F)\longrightarrow \mathrm{Hom}_F(\mathbf{D},F)=\mathbf{A}$$

By assumption there is a $\mathbf{D}/\mathbf{p}$-, hence **D**-linear injection

$$(19)\qquad \mathbf{D}/\mathbf{p}\longrightarrow \mathrm{Hom}_F(\mathbf{D}/\mathbf{p},F)^k\ ,\ k\in\mathbb{N}\ .$$

Composing (18) and (19) yields a **D**-linear embedding of $\mathbf{D}/\mathbf{p}$ into $\mathbf{A}^k$. ||

(20) **Lemma** (Reduction to polynomial algebras) For the proof of theorem (15) it is sufficient to prove property (16) for all polynomial algebras .

Proof: I assume that all polynomial algebras have property (16) and prove the same for all affine integral domains . By (17) this is sufficient . By the Noetherian normalization lemma (see [BOU1], Ch. 5, § 3.1) **D** contains a polynomial algebra F[s] , $F\subset \mathbf{D}':=F[s_1,\cdots,s_r]\subset \mathbf{D}$, such that $\mathbf{D}'\subset\mathbf{D}$ is an integral extension . The $\mathbf{D}'$-linear injection $inj:\mathbf{D}'\longrightarrow\mathbf{D}$ induces the $\mathbf{D}'$-linear surjection

$$(21)\qquad \mathrm{Hom}(\mathrm{inj},F):\mathbf{A}=\mathrm{Hom}_F(\mathbf{D},F)\longrightarrow \mathbf{A}':=\mathrm{Hom}_F(\mathbf{D}',F),\ a\longmapsto a|\mathbf{D}'$$

where **A** is considered as $\mathbf{D}'$-module by restriction of scalars from **D** to $\mathbf{D}'$. Using property (16) for the polynomial algebra $\mathbf{D}'=F[s]$ yields a $\mathbf{D}'$-linear

$$(22)\qquad \text{injection}\quad \mathbf{D}'\longrightarrow \mathbf{A}'^k\ ,\ p'\longmapsto(p'a_i'\,;\ i=1,\cdots,k)$$

with elements $a_1',\cdots,a_k'$ in $\mathbf{A}'$. Since (21) is surjective the functions a_i' can be extended to linear functions $a_i:\mathbf{D}\longrightarrow F$, $i=1,\cdots,k$, with $a_i|\mathbf{D}'=a_i'$. These a_i give rise to the **D**-linear map

$$(23)\qquad \mathbf{D}\longrightarrow\mathbf{A}^k\ ,\ p\longmapsto(pa_i\,;\ i=1,\cdots,k)\ .$$

Its kernel **I** is an ideal of **D** . Since $(p'a_i)|\mathbf{D}'=p'a_i'$, $i=1,\cdots,k$, and since (22) is injective there results $\mathbf{I}\cap\mathbf{D}'=\ker(\mathbf{D}'\longrightarrow\mathbf{A}'^k)=0$. Since $\mathbf{D}'\subset\mathbf{D}$ is an integral extension of integral domains the relation $\mathbf{I}\cap\mathbf{D}'=0$ implies $\mathbf{I}=0$ (compare [BOU1], Ch. 5, § 2.1 , corollary of lemma 2) . Thus (23) is the desired **D**-linear embedding . ||

Finally I prove property (16) for polynomial algebras . Consider first a polynomial algebra F[s] , $r=1$, $s=s_1$, in *one* indeterminate s . It is well known in this case that there is even a **D**-linear injection $F[s]\longrightarrow\mathbf{A}=F^{\mathbb{N}}$, $p\longmapsto pa$, $1\longmapsto a$, or , in other words , a **D**-linear independent element $a\in F^{\mathbb{N}}$. To recall the construction of such an element *a* remark that the equation $pa=0$ for

(24) $p = s^d + p(d-1)s^{d-1} + \cdots + p(0) \in F[s]$, $\deg(p) = d$, means

(25) $a(n) + p(d-1)a(n-1) + \cdots + p(0)a(n-d) = 0$ for $n \geq d$.

Choose an increasing sequence

(26) $\mu(1) < \mu(2) < \mu(3) < \cdots\cdots$ in $\mathbb{N}$ such that

(27) $\lim_{j \longrightarrow \infty} (\mu(j+1) - \mu(j)) = \infty$

Define $S := \{\mu(1), \mu(2), \cdots\}$ and $a \in F^{\mathbb{N}}$ as the characteristic function of S , i.e.

(28) $a := \delta_S$, $a(n) = 1$ if $n = \mu(l) \in S$, $a(n) = 0$ if $n \notin S$.

If $p \in F[s]$ is given as in (24) and if $\mu(k) - \mu(k-1) > d$ for $k \geq k_0$ then for any $k \geq k_0$ and $n := \mu(k)$

$$a(n) = a(\mu(k)) = 1, \text{ but } a(n-1) = \cdots = a(n-d) = 0 \text{ , hence}$$

$$(pa)(n) = a(n) + p(d-1)a(n-1) + \cdots + p(0)a(n-d) = a(n) = 1 \neq 0.$$

and $pa \neq 0$. Thus $a = \delta_S$ from (28) is **D**-linearly independent.

In order to strengthen the preceding result consider, for an integral domain **D** , the following

(29) **Property** : The **D**-module $\mathbf{A} := \mathrm{Hom}_F(\mathbf{D}, F)$ admits a countable family of **D**-linearly independent elements. ||

(30) **Lemma**: The polynomial ring $\mathbf{D} = F[s]$ in *one* indeterminate satisfies property (29).

Proof: (i) Generalizing the construction of (28) I construct countably many characteristic functions

(31) $$a_k = \delta_{S(k)} , \quad a_k(n) = \begin{cases} 1 & \text{if } n \in S(k) \\ 0 & \text{if } n \notin S(k) \end{cases}$$

where the $S(k) = \{ \alpha(k,1) < \alpha(k,2) < \alpha(k,3) \cdots \}$, $k = 1, 2, \cdots$ are countable subsets of $\mathbb{N}$ which are selected by induction on k below.

(ii) Consider first an arbitrary sequence $\mu(1), \mu(2) \cdots$ in $\mathbb{N}$ with (26) and (27). Using (27) select an increasing sequence

$$i(1) < i(2) < \cdots < i(k) < i(k+1) < \cdots$$

such that

$$i(k+1) - i(k) \geq 2 \text{ and } \mu(j+1) - \mu(j) \geq 2k \quad \text{for } j \geq i(k).$$

Define

(32) $\nu(k) := \mu(i(k)) + k$, $k = 1, 2, \cdots$.

Arranging the infinite sequences μ and ν in *one* strictly increasing sequence one obtains

(33) $\cdots < \mu(i(k)) < \nu(k) < \mu(i(k)+1) < \cdots < \mu(i(k+1)) < \nu(k+1) < \cdots$

Since

$$\nu(k)-\mu(i(k)) = k \leq \mu(i(k)+1)-\nu(k),\ k \in \mathbb{N},$$

the sequence (33) satisfies condition (27) too. In particular the sequence ν satisfies (27) and

(34) $$|\nu(k)-\mu(j)| \geq k \text{ for all } k,j.$$

The construction of ν from μ can now also be applied to the composite sequence (33).

(iii) The construction of (ii) is now applied to the construction of a double sequence $\alpha(k,l)$, $k,l=1,2\cdots$. Start with an arbitrary sequence $\alpha(1,1) < \alpha(1,2) < \alpha(1,3) < \cdots$ such that (27) is satisfied, i.e.

$$\lim_{j\to\infty}(\alpha(1,j+1)-\alpha(1,j)) = \infty.$$

Pose $\beta(1,-) := \alpha(1,-)$ and construct a sequence $\alpha(2,1) < \alpha(2,2)\cdots$ from $\beta(1,-)=\alpha(1,-)$ as ν from μ in (ii). Also call $\beta(2,-)$ the respective composite sequence (33). Then both $\alpha(2,-)$ and $\beta(2,-)$ satisfy (27) and moreover $|\alpha(2,k)-\alpha(1,j)| \geq k$ for all j,k. Construct $\alpha(3,-)$ from $\beta(2,-)$ as ν from μ and so on. Inductively one obtains a double sequence $\alpha(k,l)$, $k,l \geq 1$, of pairwise different elements with the following properties:

(35) $\alpha(k,1) < \alpha(k,2) < \cdots$ and $\lim_{l\to\infty}(\alpha(k,l+1)-\alpha(k,l)) = \infty$, $k=1,2\cdots$,

(36) $$|\alpha(k,l)-\alpha(i,j)| \geq i \text{ for } k > i, \text{ all } j.$$

Finally define the pairwise disjoint sets

$$S(k) = \{\alpha(k,1), \alpha(k,2),\cdots\},\ k = 1,2,\cdots.$$

(iv) Now define $a_k := \delta_{S(k)}$, $k=1,2,\ldots$, as in (31). I am going to show that these $a_1,a_2,\cdots$ are $F[s]$-linearly independent. Assume this is not the case and

(37) $$p_m a_m + \cdots + p_1 a_1 = 0,\ p_i \in F[s],\ p_m \neq 0,$$

is a non-trivial linear relation. Let $p_i = \sum\{p_i(j)s^j;\ j \leq d(i)\} \in F[s]$, $p_m(d(m)) \neq 0$. The relation (37) means that for all n

(38) $$\sum_{i,j} p_i(j)a_i(n+j) = I + II = 0 \text{ where}$$

$$I := p_m(d(m))a_m(n+d(m)) + \sum\{p_m(j)a_m(n+j);\ j \le d(m)-1\}$$

$$II := \sum\{p_i(j)a_i(n+j);\ i \le m-1,\ j \le d(i)\}\ .$$

Let then l_0 be such that $\alpha(m,l) - \alpha(m,l-1) > d(m)$ for $l \ge l_0$ and define, for a suitable such l (see below),

$$n := \alpha(m,l) - d(m), \text{ hence } n+d(m) = \alpha(m,l) \in S(m).$$

By definition of a_m this gives $a_m(n+d(m)) = 1$. On the other side

$$\alpha(m,l-1) < n+j < \alpha(m,l) \text{ for } j=0,\cdots,d(m)-1\ ,$$

thus $n+j \notin S(m)$ and $a_m(n+j)=0$. The preceding calculations yield $I = p_m(d(m)) \neq 0$ in (38). Consider now the arguments

$$n+j = \alpha(m,l)-d(m)+j\ ,\ j \le d(i)\ ,\ i \le m-1\ ,$$

appearing in II of (38). These satisfy

(39) $$|\alpha(m,l)-(n+j)| = |d(m)-j| \le M := \mathrm{Max}\ (d(i);\ i=1,\cdots,m).$$

Now I choose l above such that $l \ge l_0$ and $l \ge M+1$ and use

(40) $$|\alpha(m,l) - \alpha(i,j')| \ge l > M \text{ for } i<m, \text{ all } j',$$

from (36). The inequalities (39) and (40) imply that

$$n+j \neq \alpha(i,j') \text{ for } i=1,\cdots,m-1\ ,\ j \le d(i)\ ,\ \text{all } j'\ ,\ \text{hence}$$

$$n+j \notin S(i) \text{ and } a_i(n+j)=0 \text{ for } i=1,\cdots,m-1\ ,\ j \le d(i).$$

This yields $II=0$ in (38) and finally the contradiction

$$0 = I + II = p_m(d(m)) \neq 0.$$

Hence a linear relation (37) cannot hold. ||

The last step in the proof of theorem (15) consists in showing property (29) for the polynomial algebra $F[s_1,\cdots,s_r]$ in more than one indeterminate.

(41) **Technical remark:** If

$$\langle -,- \rangle_i : V(i) \times \widehat{V}(i) \longrightarrow F\ ,\ i \in I\ ,$$

is a family of non-degenerate bilinear forms like (1) then also the bilinear form

$$\langle -,- \rangle : \oplus_{i\in I} V(i) \times \prod_{i\in I} \widehat{V}(i) \longrightarrow F$$

$$\langle (v(i);i\in I),(\widehat{v}(i);i\in I) \rangle := \sum \langle v(i),\ \widehat{v}(i) \rangle_i$$

is non-degenerate. Consider, in particular, an affine integral domain **D** and its dual space $\mathbf{A} := \mathrm{Hom}_F(\mathbf{D},F)$ with the evaluation form

$$\mathbf{D} \times \mathbf{A} \longrightarrow F,\ (p,a) \longmapsto \langle p,a\rangle := a(p).$$

The construction above yields the non-degenerate form

$$\langle -,- \rangle : \mathbf{D}^{(\mathbb{N})} \times \mathbf{A}^{\mathbb{N}} \longrightarrow F$$

$$\langle p,a\rangle := \langle (p(n);\ n\in \mathbb{N}),\ (a(n);\ n\in \mathbb{N})\rangle = \sum \langle p(n),a(n)\rangle .$$

Identify

$\mathbf{D}^{(\mathbb{N})} = \mathbf{D}[s]$ $\ p=\sum\{p(n)s^n;\ n\in \mathbb{N}\}$ and $\mathbf{A}^{\mathbb{N}} = \mathrm{Hom}_F(\mathbf{D}[s],\ F)$ where

$$a = (a(n);n\in \mathbb{N}) = \langle -,a\rangle = \left(\sum p(n)s^n \longmapsto \sum \langle p(n),a(n)\rangle\right).$$

The $\mathbf{D}[s]$- structure on $\mathbf{A}^{\mathbb{N}}$ defined according to (7) is then given by the componentwise $\mathbf{D}$- multiplication on $\mathbf{A}^{\mathbb{N}}$ and $(s^m a)(n) = a(m+n), m,n\in\mathbb{N}$. ||

(42) **Lemma:** Assume that the affine integral domain $\mathbf{D}$ satisfies property (29) , i.e. that the $\mathbf{D}$-module $\mathbf{A} := \mathrm{Hom}_F(\mathbf{D},F)$ admits a countable family of $\mathbf{D}$-linearly independent elements. Then the polynomial algebra $\mathbf{D}[s]$ in one indeterminate s satisfies (29) too .

Proof : I use the notations of the preceding remark, in particular the identification $\mathbf{A}^{\mathbb{N}} = \mathrm{Hom}_F(\mathbf{D}[s],F)$. By assumption there is a countable family of $\mathbf{D}$-linearly independent elements in $\mathbf{A}$. Using the bijection $\mathbb{N} \cong \mathbb{N}^2$ choose a double indexed family $a_k(l)\in \mathbf{A}, k,l = 0,1,\cdots,$ of $\mathbf{D}$-linearly independent elements in $\mathbf{A}$ and define

$$a_k := (a_k(0),a_k(1),\cdots)\in \mathbf{A}^{\mathbb{N}},\ k=0,1,\cdots .$$

I am going to show that these a_k are $\mathbf{D}[s]$-linearly independent. Assume that $\sum p_k a_k = 0$ where $p_k = \sum\{p_k(l)s^l;\ l=0,1,\cdots\}\in \mathbf{D}[s]$ and almost all $p_k(l)$ are zero in $\mathbf{D}$. Then

$$0 = \left(\sum p_k a_k\right)(0) = \sum_{k,l} p_k(l)\,a_k(l)$$

implies that all $p_k = 0$ since the $a_k(l)$, k , $l\in \mathbb{N}$, are $\mathbf{D}$-linearly independent. ||

(43) **Proof of theorem** (15): Using (30) and (42) furnishes property (29), in particular (16), for every polynomial algebra $F[s_1,\cdots,s_r]$. The lemma 20 then implies the theorem. ||

(44) **Remark:** The proofs of (20), (30) and (42) show that an affine integral domain admits even a $\mathbf{D}$-linear embedding

$$\mathbf{D} \longrightarrow \mathbf{A} = \mathrm{Hom}_F(\mathbf{D},F),\ p \longmapsto p\,a .$$

If $\mathbf{D} = F[s_1,\cdots,s_r]$ is a polynomial algebra and $\mathbf{A} = F^{\mathbb{N}^r}$ then a can be chosen as a characteristic function, i.e. as a sequence of 0's and 1's only. If $\mathbf{p} \subset F[s_1,\cdots,s_r]$ is a prime ideal there is a $F[s]$- linear embedding

$$F[s]/\mathbf{p} \longrightarrow F^{\mathbb{N}^r} \quad , \bar{p} \longmapsto p a \ ,$$

i.e. a multiindexed sequence $a \in F^{\mathbb{N}^r}$ such that

$$\mathbf{p} = \{p \in F[s] \ ; \ p\,a = p(L)(a) = 0\}$$

is the exact annihilator of a . ||

In [WIL] Willems characterizes discrete systems as closed subspaces of $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{Z}}$ for the cases

$$r = 1 \ , \ \mathbf{D} = \mathbb{R}[s], \ \mathbf{A} = \mathbb{R}^{\mathbb{N}} \text{ and } r = 1 \ , \ \mathbf{D} = \mathbb{R}[s,s^{-1}] \ , \ \mathbf{A} = \mathbb{R}^{\mathbb{Z}} \ .$$

This characterization can be generalized to arbitrary affine F-algebras considered above. For this purpose I need some technical preparations concerning linearly compact vector spaces.

Linearly compact vector spaces

The notion is due to S. Lefschetz. The basic material is, for instance, exposed in [KÖ], Kap. II, §10.9. The categorical formulations below are a standard technique of the theory of formal groups and schemes à la francaise (see, for instance, [GA]) and are easily derived from [KÖ], but I cannot give a reference where all the *easy* results and their proofs are written down in detail. Consider the field F as a topological one with the discrete topology and a topological vector spaces X over F. Such a space is called linearly compact (l.c.) if it satisfies one of the following equivalent conditions:

(45) X is (Hausdorff and) complete and admits a basis of neighborhoods of zero (short: 0-basis) of finite codimensional subspaces.

(46) X is topologically isomorphic to a space F^I. Here I is some index set and F^I carries the product topology with the 0-basis of all $F^J \subset F^I$ where J runs over the cofinite (i.e. with a finite complement) subsets of I.

(47) X is topologically isomorphic to a dual space $V^* = \mathrm{Hom}_F(V,F)$ of a (discrete) space V. The topology of V^*, the so-called finite topology, has

the 0-basis of all subspaces

$$\mathrm{Hom}_F(V/U,F)=\{v^*\in V\ ;\ v^*(U)=0\}=\{v^*\in V\ ;v^*(u_1)=\cdots=v^*(u_k)=0\}$$

where $U=Fu_1+\cdots+Fu_k$ runs over the finite dimensional subspaces of V. If, in (47), $(v_i;i\in I)$ is a F-basis of V then

$$V^*=\mathrm{Hom}_F(V,F)\longrightarrow F^I,\ v^*\longmapsto(v^*(v_i);\ i\in I\ ),$$

is a topological isomorphism. The l.c. F-spaces with the continuous F-linear maps as morphisms form the category $\widehat{\mathrm{Mod}}(F)$. Denote by $\widehat{\mathrm{Hom}}_F(X,Y)$, $X,Y\in\widehat{\mathrm{Mod}}(F)$, the F-space of continuous F-linear maps. The category $\widehat{\mathrm{Mod}}(F)$ is abelian. The kernel, image etc. are the algebraic ones with the induced topologies, the cokernel is the algebraic one with the coinduced topology. The image of a map $f:X\longrightarrow Y$ in $\widehat{\mathrm{Mod}}(F)$ is a *closed* subspace of Y. Every continuous F-linear bijection is bicontinuous, i.e. a topological isomorphism. For a discrete space V the non-degenerate bilinear map

(48) $$\langle -,-\rangle : V\times V^*\longrightarrow F\ ,\ \langle v,v^*\rangle:=v^*(v),$$

from (4), (i), is continuous in the second variable and induces the Gelfand isomorphism

(49) $$V\cong\widehat{\mathrm{Hom}}_F(V^*,F)\ ,\ v\longmapsto\langle v,-\rangle\ .$$

This is most easily seen for $V=F^{(I)}$ where (48) specializes to (4),(ii). Thus the introduction of the topology on V^* reduces $\mathrm{Hom}_F(V^*,F)$ to $\widehat{\mathrm{Hom}}_F(V^*,F)$ in such a way that the injection (3) becomes the bijection (49). These considerations imply easily that the functor

(50) $$(-)^*=\mathrm{Hom}_F(-,F):\mathrm{Mod}(F)^{op}\cong\widehat{\mathrm{Mod}}(F)$$
$$V\longmapsto V^*=\mathrm{Hom}_F(V,F),\ f\longmapsto f^*:=\mathrm{Hom}(f,F)$$

is a duality, in particular that for $V,W\in\mathrm{Mod}(F)$ the map

(51) $$\mathrm{Hom}_F(V,W)\cong\widehat{\mathrm{Hom}}_F(W^*,V^*),\ f\longmapsto f^*,$$

is an isomorphism. The duality (50) is the oldest and simplest of those which I derived in [OB1] and used in (2.56). ||

I return to system theory and assumption (14). If M is a **D**-module and hence a F-space in particular, the dual space $M^*=\mathrm{Hom}_F(M,F)$ is l.c.

and a $\mathbf{D}$-module according to (9). The scalar multiplication satisfies

$$(dm^*)(m) = \langle m, dm^*\rangle = \langle dm, m^*\rangle = m^*(dm)$$

$$\text{hence } (m^* \longmapsto dm^*) = (m \longmapsto dm)^*.$$

As an adjoint map the multiplication $m^* \longmapsto dm^*$ is continuous. Let then $\widehat{\mathrm{Mod}}(\mathbf{D})$ denote the category of all l.c. $\mathbf{D}$-modules X, i.e. of l.c. F-vector spaces with a $\mathbf{D}$-module structure such that the homotheties $x \longmapsto dx, d \in \mathbf{D}$, are continuous. The morphisms of $\widehat{\mathrm{Mod}}(\mathbf{D})$ are the continuous $\mathbf{D}$-linear maps. The duality (50) induces the duality

$$(52) \qquad (-)^* = \mathrm{Hom}_F(-,F) : \mathrm{Mod}(\mathbf{D})^{op} \cong \widehat{\mathrm{Mod}}(\mathbf{D})$$

$$M \longmapsto M^* = \mathrm{Hom}_F(M,F),\ f \longmapsto f^* = \mathrm{Hom}(f,F).$$

For $M \in \mathrm{Modf}(\mathbf{D})$ identify

$$(53) \quad M^* = \mathrm{Hom}_F(M,F) = \mathrm{Hom}_{\mathbf{D}}(M, \mathrm{Hom}_F(\mathbf{D},F)) = \mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A}) = S(M)$$

$$m^*(m)(d) = m^*(dm) = \langle dm, m^*\rangle = \langle m, dm^*\rangle$$

via (11) and conclude that the system functor $S = \mathrm{Hom}_{\mathbf{D}}(-,\mathbf{A})$ from (2.56) and the duality $(-)^*$ coincide on $\mathrm{Modf}(\mathbf{D})$. In particular

$$(\mathbf{D}^k)^* = S(\mathbf{D}^k) = \mathbf{A}^k,\ k \in \mathbb{N},$$

can be considered as a $\mathbf{E} = \mathrm{End}_{\mathbf{D}}(\mathbf{A})$- module or as a l.c. $\mathbf{D}$- module. For finitely generated $\mathbf{D}$- modules $M_1, M_2 \in \mathrm{Modf}(\mathbf{D})$ and the systems $S_i := M_i^* = S(M_i)$, $i = 1,2$, the dualities (2.56) and (52) furnish the isomorphisms

$$(54) \qquad \mathrm{Hom}_{\mathbf{D}}(M_1,M_2) \cong \mathrm{Hom}_{\mathbf{E}}(S(M_2), S(M_1)),\ f \longmapsto S(f)$$

$$(55) \qquad \mathrm{Hom}_{\mathbf{D}}(M_1,M_2) \cong \widehat{\mathrm{Hom}}_{\mathbf{D}}(M_2^*, M_1^*),\ f \longmapsto f^* = \mathrm{Hom}(f,F).$$

In analogy to (53) I identify

$$(56) \quad f^* = \mathrm{Hom}(f,F) = \mathrm{Hom}(f,\mathbf{A}) = S(f) : M_2^* = S(M_1) \longrightarrow M_1^* = S(M_1).$$

With this identification the maps (54) and (55) coincide and give

$$\mathrm{Hom}_{\mathbf{E}}(S(M_2), S(M_1)) = \widehat{\mathrm{Hom}}_{\mathbf{D}}(M_2^*, M_1^*).$$

(57) **Theorem** (Topological characterization of discrete systems) Assumptions and notations as above, in particular $\mathbf{D}$ is an affine F-algebra and $\mathbf{A} = \mathrm{Hom}_F(\mathbf{D}, F)$ with its linearly compact topology.

(i) For a F-subspace S of $\mathbf{A}^l$, $l \in \mathbb{N}$, the following properties are equivalent:

a) S is a system, i.e. by definition of the form

$$S = \{w \in \mathbf{A}^l;\ Rw = 0\}\ ,\ R \in \mathbf{D}^{k,l}.$$

b) S is a finitely generated $\mathbf{E} = \mathrm{End}_{\mathbf{D}}(\mathbf{A})$-submodule of $\mathbf{A}^l$.

c) S is a closed $\mathbf{D}$-submodule of $\mathbf{A}^l$.

(ii) For systems $S_i = M_i^* = S(M_i)$, $i=1,2$, $M_i \in \mathrm{Modf}(\mathbf{D})$,

the isomorphism

$$\mathrm{Hom}_{\mathbf{D}}(M_1,M_2) \cong \mathrm{Hom}_{\mathbf{E}}(S_2,S_1) = \widehat{\mathrm{Hom}}_{\mathbf{D}}(M_2^*,M_1^*) \subset \mathrm{Hom}_F(S_2,S_1)$$

$$g \longmapsto S(g) = \mathrm{Hom}(g,\mathbf{A}) = \mathrm{Hom}(g,F) = g^*$$

holds. This means that a F-linear map $G: S_2 \longrightarrow S_1$ is $\mathbf{E}$-linear and thus a system morphism if and only if G is of the form $G = S(g) = g^*$, $g \in \mathrm{Hom}_{\mathbf{D}}(M_1,M_2)$, or if and only if G is $\mathbf{D}$-linear and continuous.

Proof: Part (ii) follows directly from (54), (55) and (56). The equivalence of (a) and (b) in (i) is a consequence of theorem (2.56). If

$$S = \{w \in \mathbf{A}^l;\ Rw = 0\} = \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^l/R^T\mathbf{D}^k, \mathbf{A}) =$$
$$= \left(\mathbf{D}^l/R^T\mathbf{D}^k\right)^* \subset (\mathbf{D}^l)^* = \mathbf{A}^l$$

is a subsystem it is closed in $\mathbf{A}^l$ as the image of

$$\mathrm{can}^* : \left(\mathbf{D}^l/R^T\mathbf{D}^k\right)^* \longrightarrow \left(\mathbf{D}^l\right)^* .$$

If, on the other hand, $S \subset \mathbf{A}^l = \left(\mathbf{D}^l\right)^*$ is a closed $\mathbf{D}$-submodule of $\mathbf{A}^l$ it is a subobject of $\mathbf{A}^l$ in $\widehat{\mathrm{Mod}}(\mathbf{D})$. The duality (52) implies that then S comes from a factor module $M = \mathbf{D}^l/R^T\mathbf{D}^k$ of $\mathbf{D}^l$ in the form

$$S = M^* = S(M) = S(\mathbf{D}^l/R^T\mathbf{D}^k) = \{w \in \mathbf{A}^l;\ Rw = 0\}$$

and is thus a subsystem of $\mathbf{A}^l$. ||

(58) **Remark:** The preceding theorem significantly sharpens and generalizes the proposition 4 of [WIL]. The topological characterizations of systems and their maps is much easier to obtain than theorem (2.56) since only the duality (52), derived from the standard vector space duality, is needed whereas the difficult theorem (15) is not used. For the continuous case, however, there is no analogue to theorem (57). ||

(59) **Historical remark:** A completely different proof of theorem (15) for a field F of characteristic zero was given by G. Hauger [HA], Satz on page 196.

The main work of the proof of (15), namely the construction of a $F[s]$-linear embedding

$$F[s] \longrightarrow F[s]^* = F^{\mathbb{N}^r} = F\{t_1,\cdots,t_r\},$$

is easier in characteristic zero. Hauger also hints at the connection of the large injective cogenerator property of $F^{\mathbb{N}^r} = F\{t\}$ with the work of Macaulay in [MAC1]. Indeed, identify as above

$$\mathrm{Hom}_F(F[s], F) = F^{\mathbb{N}^r} = F\{t_1,\cdots,t_r\}$$

and consider the non-degenerate bilinear form

(60) $\quad \langle -,- \rangle : F[s] \times F\{t\} \longrightarrow F, \ \langle \sum p_n s^n, \sum a_n t^n \rangle = \sum p_n a_n .$

Of course, Macaulay considers the case $F = \mathbb{C}$ only. The *inverse system* of a *modular system* , i.e., by definition , an ideal M of $F[s]$ is nothing else than

$$M^\perp = (F[s]/M)^* = \mathrm{Hom}_F(F[s]/M, F) \subset F[s]^* = F\{t\}$$

where $\perp$ denotes the orthogonal complement with respect to $\langle -,- \rangle$ from (60) (See the definition on p. 68 of [loc. cit.]). Thus the inverse system $M^\perp$ of Macaulay coincides with the discrete system $(F[s]/M)^*$ of this paper (compare (57)). By the proposition on page 91 of [loc. cit.] every ideal $M \subset \mathbb{C}[s]$ has the form $M = (\sum\{F[s]E_i; i=1,\cdots,k\}^\perp$ with finitely many $E_1,\cdots,E_k \in \mathbb{C}\{t\}$. But this means exactly that the $\mathbb{C}[s]$-linear map

$$\mathbb{C}[s]/M \longrightarrow \mathbb{C}\{t\}^k, \ \bar{p} \longmapsto (pE_1,\cdots, pE_k)^T$$

is injective. This is the main part of the proof of (15) for $F = \mathbb{C}$. Macaulay proves this result for zero-dimensional ideals M, i.e. for which $\mathbb{C}[s]/M$ is finite dimensional over $\mathbb{C}$, and proceeds then by reduction to the zero-dimensional case. As far as I understand it this reduction procedure is not valid, at least not for $M = 0$. In [loc. cit.], p. 71, it is also asserted that in the above situation

(61) $$M^\perp = \sum\{F[s]E_i; i = 1,\cdots,k\} .$$

This is false. By theorem (57) it is true that

$$M^\perp = \text{closure}(\sum\{F[s]E_i; i = 1,\cdots, k\}).$$

On the other side it is easily seen that a discrete system S in the sense of this paper, i.e. a closed $F[s]$-submodule of some $F\{t\}^n$, is finitely generated as an $F[s]$-module if and only if S is a finite dimensional F-vector space. Thus the relation (61) holds if and only if M is a zero-dimensional ideal.

In remark (44) I indicated that every affine integral domain $\mathbf{D} = F[s]/M$, M a prime ideal of $F[s]$, admits a $\mathbf{D}$-, hence $F[s]$-linear injection $\mathbf{D} = F[s]/M \longrightarrow \mathbf{D}^* = (F[s]/M)^* = M^\perp \subset F[s]^* = F\{t\}$, $\overline{p} \longmapsto pE$, $E \in F\{t\}$, which implies that $M = (0 : E) = \{p \in F[s];\ pE = 0\}$.

In the language of [loc. cit.], definition on page 70, such an ideal is called a *principal* system. In [loc. cit.], p. 72, it is proved that a zero-dimensional ideal M which is a complete intersection, i.e. which can be generated by $r = \dim \mathbb{C}[s_1, \cdots, s_r]$ generators, is a principal system. But this means that the finite dimensional algebra $\mathbf{D} := \mathbb{C}[s]/M$ admits a $\mathbf{D}$-linear injection and thus bijection $\mathbf{D} \cong \mathbf{D}^*$. In the terminology of ring theory such a $\mathbf{D}$ is called a *Frobenius* algebra. In commutative algebra such a $\mathbf{D}$ is called an (Artinian) *Gorenstein* ring. ||

4. THE MAIN THEOREM IN THE CONTINUOUS CASE

In this paragraph I show that the $\mathbb{C}[s]$-submodules of the space of distributions appearing in connection with partial differential equations are large injective cogenerators.

Topological algebras of convergent power series and analytic functionals

The material in this section is essentially known, but I do not know a suitable reference for all details and my specific exposition. My main sources were the books [GR1], [GR2] and [GR3] by Grauert and Remmert and [HÖ2], [LE]. Define $\mathbb{R}_+ := \{\alpha \in \mathbb{R}; \alpha > 0\}$. For a vector $T = (T(1), \cdots, T(r)) \in (\mathbb{R}_+)^r$ and a formal power series

$$a = \sum a_n t^n = \sum\{ a_n t_1^{n(1)} \cdots t_r^{n(r)};\ n \in \mathbb{N}^r\} \in \mathbb{C}\{t\} = \mathbb{C}\{t_1, \cdots, t_r\}$$

define, according to [GR1], p. 15,

(1) $$|a|_T := \sum |a_n|\, T^n = \sum |a_n| T^{n(1)} \cdots T^{n(r)} \leq \infty$$ and

(2) $$B_T := B_T\{t\} = \{a \in \mathbb{C}\{t\} ;\ |a|_T < \infty\}.$$

Then B_T is a subalgebra of $\mathbb{C}\{t\}$ and indeed a Banach algebra with the norm

$|-|_T$ (see [GR1], p. 16, Satz 1) . The union

(3) $$\mathbb{C}\langle t\rangle := \bigcup\{ B_T ;\ T \in (\mathbb{R}_+)^r \}$$

is the $\mathbb{C}$-algebra of *convergent power series* ([loc.cit.], p. 27) . On $\mathbb{C}\langle t\rangle$ consider the final topology coinduced by the injections $B_T \subset \mathbb{C}\langle t\rangle$, $T \in (\mathbb{R}_+)^r$. This topology is called *Folgentopologie* , the *topology of analytic convergence* or *Silva topology* in [loc.cit.] , p.31 , 32 and Kap. I , §6,7 . With this topology $\mathbb{C}\langle t\rangle$ is a topological algebra ([loc.cit.], Satz 7 on p.66) . By

(4) $$\mathbb{C}\langle t\rangle' := \widehat{\mathrm{Hom}}_{\mathbb{C}}(\mathbb{C}\langle t\rangle, \mathbb{C})$$

I denote the dual space of continuous , $\mathbb{C}$-linear functions on $\mathbb{C}\langle t\rangle$ .By definition of the topology of $\mathbb{C}\langle t\rangle$ a $\mathbb{C}$-linear map $\nu : \mathbb{C}\langle t\rangle \longrightarrow \mathbb{C}$ is continuous if and only if its restrictions $\nu | B_T$, $T \in (\mathbb{R}_+)^r$, are continuous . For $T \in (\mathbb{R}_+)^r$ let

(5) $$Z(T) := \{ t \in \mathbb{C}^r ;\ |t_i| < T(i), i = 1, \cdots, r\} \ , \ \bar{Z}(T) := \{ t \in \mathbb{C}^r ;\ |t_i| \le T(i), i = 1, \cdots, r\}$$

be the open resp. closed polydisk. Denote by $\mathcal{O}(Z(T))$ resp. $C^0(\bar{Z}(T))$ the $\mathbb{C}$-algebra of holomorphic resp. continuous complex-valued functions on $Z(T)$ resp. $\bar{Z}(T)$. The algebra $C^0(\bar{Z}(T))$ is a Banach algebra with the norm

(6) $$\| f \|_T := \mathrm{Max}\{ |f(t)| ;\ |t_i| \le T(i) \ , \ i = 1, \cdots, r\}$$

for $f \in C^0(\bar{Z}(T))$. For $a \in B_T \subset \mathbb{C}\langle t\rangle$ the series $a(t) = \sum a_n t^n$ converges uniformly on $\bar{Z}(T)$ and is thus a holomorphic function of t on $Z(T)$ and continuous on $\bar{Z}(T)$. There results the inclusion

$$B_T \subset \mathcal{O}(Z(T)) \cap C^0(\bar{Z}(T)) \ .$$

Obviously $|a(t)| \le \sum |a_n| |t^n| \le \sum |a_n| T^n = |a|_T$ and hence

(7) $$\| a \|_T \le |a|_T \quad \text{for } a \in B_T \ .$$

On the other hand Cauchy's inequality (see any book on complex variables, for instance [HÖ2], Th. 2.27) furnishes

(8) $$|a_n| T^n \le \| a \|_T \quad \text{for all } n \in \mathbb{N}^r \ .$$

(9) **Corollary** : $\mathbb{C}\langle t\rangle = \{ a = (a_n ;\ n \in \mathbb{N}^r) \in \mathbb{C}\{t\}$; There are $C > 0$ and $S \in (\mathbb{R}_+)^r$ such that $|a_n| \le C S^n$ for all $n \in \mathbb{N}$ $\}$.

In words: $\mathbb{C}\langle t\rangle$ contains exactly the sequences of at most exponential growth.

Proof : If $|a_n| \le C S^n$ for all n, $S \in (\mathbb{R}_+)^r$, then $a = \sum a_n t^n$ has the geometric series $C(\sum S^n t^n)$ as a majorant with convergence for $|t_i| < 1/S(i)$, $i = 1, \cdots, r$. If, on the other hand, $a \in B_T \subset \mathbb{C}\langle t\rangle$ then , by (8),

$$|a_n| \le \|a\|_T S^n \ , \ S(i) := 1/T(i) \ , \ i=1,\cdots,r \ . \ \|$$

It is now obvious that the spaces l^p, $0 < p \le \infty$, and the space of asymptotically stable sequences are contained in $\mathbb{C}\langle t\rangle$. This was already stated in (1.13).

The algebra $C^o(\mathbb{C}^r)$ of all continuous functions on $\mathbb{C}^r$ is a *Frechet* algebra with respect to the topology of compact convergence, i.e. uniform convergence on compact subsets of $\mathbb{C}^r$ (compare [GR3], Kap. V, § 6, or [LE], Ch. I., § 3), its topology being given by the semi-norms $\| - \|_S$ where S runs over a countable, increasing family of a vectors in $(\mathbb{R}_+)^r$.

The subalgebra $\mathcal{O}(\mathbb{C}^r)$ of $C^O(\mathbb{C}^r)$ of all entire functions, i.e. of all holomorphic functions on $\mathbb{C}^r$, is a closed subalgebra of $C^O(\mathbb{C}^r)$ and also Frechet. Since every entire function has a everywhere convergent Taylor series one obtains the natural inclusion

(10) $$\mathcal{O}(\mathbb{C}^r) = \bigcap\{B_S(s);\ S \in (\mathbb{R}_+)^r\} \subset \mathbb{C}\langle s\rangle$$

where the canonical coordinates on $\mathbb{C}^r$ are denoted by $s_1,\cdots,s_r$ instead of $t_1,\cdots,t_r$ as above. For $f = \sum f_n s^n \in \mathcal{O}(\mathbb{C}^r)$ the inequalities (7) and (8) imply

(11) $$|f_n| S^n \le \|f\|_S \le |f|_S \text{ for all } S \in (\mathbb{R}_+)^r .$$

The dual space

(12) $$\mathcal{O}(\mathbb{C}^r)' := \widehat{\mathrm{Hom}}_{\mathbb{C}}(\mathcal{O}(\mathbb{C}^r), \mathbb{C})$$

of $\mathcal{O}(\mathbb{C}^r)$ of all continuous, $\mathbb{C}$-linear functions on $\mathcal{O}(\mathbb{C}^r)$ is called the space of *analyic functionals on* $\mathbb{C}^r$ (compare [LE], p. 37 ff , or [HÖ2], p. 100 ff).

I am now going to define a bilinear form on $\mathcal{O}(\mathbb{C}^r) \times \mathbb{C}\langle t\rangle$ which is analogous to $\langle -,-\rangle$ on $\mathbb{C}[s]\times\mathbb{C}\{t\}' = \mathbb{C}^{(\mathbb{N}^r)} \times \mathbb{C}^{\mathbb{N}^r}$ from (3.4), (ii). Let $f = \sum f_n s^n \in \mathcal{O}(\mathbb{C}^r)$ and $a = \sum a_n t^n \in B_T\langle t\rangle \subset \mathbb{C}\langle t\rangle$ be power series. By (11) $|f_n| \le \|f\|_{1/T} T^n$ for all $n \in \mathbb{N}^r$, hence

$$\sum |f_n|\ |a_n| \le \sum \|f\|_{1/T} T^n |a_n| = \|f\|_{1/T} |a|_T < \infty .$$

In particular

(13) $$\langle f,a\rangle := \sum\{f_n a_n;\ n \in \mathbb{N}^r\}$$

converges absolutely and

(14) $$|\langle f,a\rangle| \le \|f\|_{1/T} |a|_T \quad \text{for } a \in B_T,\ f \in \mathcal{O}(\mathbb{C}^r).$$

This means that

(15) $$\langle -,- \rangle : \mathcal{O}(\mathbb{C}^r) \times \mathbb{C}\langle t\rangle \longrightarrow \mathbb{C}$$

is a $\mathbb{C}$-bilinear form which, by (14), is partially continuous in both variables $f \in \mathcal{O}(\mathbb{C}^r)$ and $a \in \mathbb{C}\langle t\rangle$ where the topologies on $\mathcal{O}(\mathbb{C}^r)$ resp. $\mathbb{C}\langle t\rangle$ are those of compact resp. analytic convergence. In particular, $\langle -,- \rangle$ induces $\mathbb{C}$-linear maps

(16) $$\mathbb{C}\langle t\rangle \longrightarrow \mathcal{O}(\mathbb{C}^r)', \ a \longmapsto \langle -,a\rangle$$ and

(17) $$\mathcal{O}(\mathbb{C}^r) \longrightarrow \mathbb{C}\langle t\rangle', \ f \longmapsto \langle f,-\rangle .$$

By definition

(18) $$\langle s^n, a\rangle = a_n \ , \quad \langle f, t^n\rangle = f_n \ , \ n \in \mathbb{N}^r ,$$

which implies that the maps (16) and (17) are injective. In particular,

(19) $$\langle s^m, t^n\rangle = \delta_{m,n}$$

so that the s^m resp. t^n are dual bases with respect to $\langle -,- \rangle$.

(20) **Theorem:** Situation as above. The bilinear form

$$\langle -,- \rangle : \mathcal{O}(\mathbb{C}^r) \times \mathbb{C}\langle t\rangle \longrightarrow \mathbb{C}$$

from (15) is non-degenerate in the strong sense that the induced maps (16) and (17) are $\mathbb{C}$-isomorphisms.

Proof: It remains to show that the maps (16) and (17) are surjective.

(i) Let first $\mu \in \mathcal{O}(\mathbb{C}^r)'$ be an analytic functional. Any $f = \sum f_n s^n \in \mathcal{O}(\mathbb{C}^r)$ is the compact limit of its finite partial sums

$$f = \sum f_n s^n = \lim_I \sum\{ f_n s^n ; \ n \in I\} \ , \ I \subset \mathbb{N}^r \text{ finite} .$$

Since μ is $\mathbb{C}$-linear and continuous this implies

(21) $$\mu(f) = \lim_I \sum\{f_n \mu(s^n) ; \ n \in I\} = \sum f_n \mu(s^n).$$

Further, since μ is continuous, there is a (semi-) norm $\| \ \|_S$ on $\mathcal{O}(\mathbb{C}^r)$ and a neighborhood

$$U = \{ f \in \mathcal{O}(\mathbb{C}^r); \ \|f\|_S \le \varepsilon \} \text{ of } 0 \text{ in } \mathcal{O}(\mathbb{C}^r), \ \varepsilon > 0 ,$$

such that $|\mu(U)| \le 1$, hence

$$|\mu(f)| \le 1 \text{ if } \|f\|_S \le \varepsilon \quad \text{or} \quad |\mu(f)| \le (1/\varepsilon)\|f\|_S \text{ for all } f \in \mathcal{O}(\mathbb{C}^r).$$

In particular $|\mu(s^n)| \le (1/\varepsilon) S^n$ for all $n \in \mathbb{N}^r$. This again implies that

(22) $$\sum \mu(s^n) t^n \in B_T\langle t\rangle \subset \mathbb{C}\langle t\rangle$$

for all $T \in (\mathbb{R}_+)^r$ with $T(i) < 1/S(i)$, $i = 1,\cdots,r$. The results (21) and (22) yield

$$\mu = \langle -,a\rangle \quad \text{with} \quad a := \sum \mu(s^n) t^n \in \mathbb{C}\langle t\rangle .$$

and therefore the surjectivity of (16).

(ii) For a continuous, $\mathbb{C}$-linear $\nu \in \mathbb{C}\langle t\rangle' = \widehat{\mathrm{Hom}}_{\mathbb{C}}(\mathbb{C}\langle t\rangle, \mathbb{C})$ I show in the same fashion that $\sum \nu(t^n)s^n \in \mathcal{O}(\mathbb{C}^r)$ and $\nu = \langle \sum \nu(t^n)s^n, -\rangle$. Indeed, the restrictions of ν to all $B_T\langle t\rangle$ are $\mathbb{C}$-linear and continuous. The same proof as that of (21) gives

(23) $\qquad \nu(\sum a_n t^n) = \sum a_n \nu(t^n)$ for $a := \sum a_n t^n \in B_T\langle t\rangle$.

In particular, for $S \in (\mathbb{R}_+)^r$, the geometric series $\sum S^n t^n$ is convergent. The relation (23) yields $\nu(\sum S^n t^n) = \sum S^n \nu(t^n)$, $S \in (\mathbb{R}_+)^r$. But this means that the series $f := \sum\{\nu(t^n)s^n;\ n \in \mathbb{N}^r\}$ converges everywhere and represents an entire function $f \in \mathcal{O}(\mathbb{C}^r)$ with $\nu(a) = \langle f, a\rangle$ by (23). This says that (17) is surjective too. ||

The following theorem shows that the examples (1.13) and (1.25) are essentially the same. Trivially the map

(24) $\qquad \mathbb{C}\{t\} \longrightarrow \mathbb{C}\{t\},\ \sum a_n t^n \longmapsto \tilde{a} := \sum (a_n/n!)t^n$

is a $\mathbb{C}$-linear isomorphism. This map is even $\mathbb{C}[s_1, \cdots, s_r]$-linear if $\mathbb{C}[s]$ operates on the left side by left shifts and on the right by partial differentiation. I am going to show that the image of $\mathbb{C}\langle t\rangle$ under (24) is the space of entire functions of exponential type. I derived the fact, the idea of the proof and the relevant notions from [HÖ2], Ch. IV, § 5, p. 100 ff. The details are different, however. Take

$a = \sum a_n t^n \in B_T \subset \mathbb{C}\langle t\rangle$, $T \in (\mathbb{R}_+)^r$. Then $|a_n|T^n \le \|a\|_T$ by (8), hence $\sum |a_n/n!|\,|t|^n \le \|a\|_T (\sum (1/n!)|t|^n T^{-n}) = \|a\|_T \exp(|t_1|/T_1 + \cdots + |t_r|/T_r)$ with $|t| := (|t_1|, \cdots, |t_r|)$. This implies that $\tilde{a}(t) = \sum (a_n/n!)t^n$ is entire and satisfies

(25) $\qquad |\tilde{a}(t) \le \|a\|_T \exp(\|t\|)$

where $\|t\| := |t_1|/T_1 + \cdots + |t_r|/T_r$ is a norm on $\mathbb{C}^r$. An entire function $f \in \mathcal{O}(\mathbb{C}^r)$ is called of *exponential type* if $|f(t)| \le C \exp(\|t\|)$, $t \in \mathbb{C}^r$, for some constant $C > 0$ and some norm $\|-\|$ on $\mathbb{C}^r$. Call

(26) $\qquad \mathcal{O}(\mathbb{C}^r; \exp) \subset \mathcal{O}(\mathbb{C}^r)$

the algebra of alle entire functions of exponential type.

(27) **Theorem:** The isomorphism

$$\mathbb{C}\{t\} \longrightarrow \mathbb{C}\{t\},\ a=\sum a_n t^n \longmapsto \tilde{a} := \sum (a_n/n!)t^n$$

induces the $\mathbb{C}$-isomorphism

(28) $$\mathbb{C}\langle t\rangle \cong \mathcal{O}(\mathbb{C}^r;\ \exp),\ a \longmapsto \tilde{a}.$$

The map is even $\mathbb{C}[s]$-linear if the $\mathbb{C}[s]$-structures on $\mathbb{C}\langle t\rangle$ resp. $\mathcal{O}(\mathbb{C}^r;\ \exp)$ are given by left shifts resp. differentiation as in examples (1.13) resp. (1.25).

Proof: By (25) the image of $\mathbb{C}\langle t\rangle$ under the bijection (24) lies in $\mathcal{O}(\mathbb{C}^r;\ \exp)$. Let, on the other side, $f=\sum f_n t^n \in \mathcal{O}(\mathbb{C}^r)$ be holomorphic and of exponential type with $|f(t)| \le C \exp(\|t\|)$, $C>0$, for some norm $\|-\|$ on $\mathbb{C}^r$. I am going to show that f comes from $\mathbb{C}\langle t\rangle$ by an argument from [15], bottom of page 101. Since all norms on $\mathbb{C}^r$ are equivalent there are constants $S(i)>0$, $i=1,\cdots,r$, such that

$$\|t\| \le S(1)|t_1|+\cdots+S(r)|t_r| \text{ for all } t\in\mathbb{C}^r, \qquad \text{hence}$$

(29) $$|f(t)| \le C\exp(S(1)|t_1|)\cdots\exp(S(r)|t_r|).$$

The inequalities (8) and (29) imply for every $T\in(\mathbb{R}_+)^r$ and $n\in\mathbb{N}^r$ the inequality

$$|f_n|T^n \le \|f\|_T \le C\exp(S(1)T(1))\cdots\exp(S(r)T(r)).$$

in particular, choosing $T(i) := n(i)/S(i)$, $i=1,..,r$, we obtain

$$|f_n| \le C\prod\{\exp(n(i))\,S(i)^{n(i)}/\,n(i)^{n(i)}\ ;\ i=1,..,r\} \text{ and finally}$$

$$|f_n|n! \le C\prod\{n(i)!\exp(n(i))S(i)^{n(i)}/n(i)^{n(i)};\ i=1,\cdots,r\}$$

By Stirling's formula the limit of $k!\exp(k)/k^{k+1/2}$, $k\in\mathbb{N}$, exists and the sequence is bounded in particular. This yields

$$|f_n|n! \le C_1\ (n(1)\cdot\cdots\cdot n(r))^{1/2}\ S^n \text{ for all } n\in\mathbb{N}^r$$

and some constant $C_1>0$. This again shows $\sum f_n n!t^n \in \mathbb{C}\langle t\rangle$ as in (9). Under (28) this function is mapped onto f. Hence (28) is surjective. ||

(30) **Corollary:** The composition of the isomorphisms (28) and (16) yields the isomorphism

(31) $$\mathcal{O}(\mathbb{C}^r)' \cong \mathbb{C}\langle t\rangle \cong \mathcal{O}(\mathbb{C}^r;\ \exp)$$

$$\mu \longmapsto \sum\mu(s^n)t^n \longmapsto \tilde{\mu}(t) := \sum(\mu(s^n)/n!)t^n$$

$$\tilde{\mu}(t) = \sum(\mu(s^n)/n!)t^n = \mu(\sum s^n t^n/n!) = \mu_s(\exp(s\cdot t))$$

with $s\cdot t = s_1t_1+\cdots+s_rt_r$. This shows that $\tilde{\mu}$ is exactly the Fourier -

Laplace transform of μ, and that the isomorphism (31) coincides with the one in [HÖ2], Th. 4.5.3 on p. 101, so that the theorems (20) and (27) are just a reformulation of [loc. cit.], Th. 4.5.3. I regard the detailed exposition above justified because the formulations are better adapted to system theory and the proofs are elementary which is not so obvious when reading [HÖ2] or [LE].

Injectivity

The basic results in this context were obtained by Ehrenpreis as the so-called "fundamental principle" for systems of constant linear partial differential equations. Other principal contributors were Malgrange [MAL] and Palamodov. Both Ehrenpreis and Palamodov wrote books on the subject ([EH], [PAL]) which, however, are not easy to read. Refer to [PAL], p. 427 ff, for a historical account.

(32) **Theorem:** The $\mathbb{C}[s_1,\cdots,s_r]$- modules $C^\infty(\mathbb{R}^r)$ and $\mathcal{D}'(\mathbb{R}^r)$ from (1.18) and (1.22) are injective $\mathbb{C}[s]$-modules. More generally the same is true for the spaces $C^\infty(\Omega)$, $\mathcal{D}'(\Omega)$ and $\mathcal{D}'^{F}(\Omega)$ where $\Omega \subset \mathbb{R}^r$ is an open convex subset.

Proof: These are, for arbitrary Ω, exactly the theorems th.1 on page 300, th. 2 on page 304 and th. 3 on page 305 in [PAL]. For $\Omega = \mathbb{R}^r$ these results are contained in [EH] as th. 5.11 on page 145 for $\mathcal{D}'^{F}(\mathbb{R}^r)$, th. 5.14 on page 150 for $\mathcal{D}'(\mathbb{R}^r)$ and th. 5.20 on page 156 for $C^\infty(\mathbb{R}^r)$. ||

(33) **Corollary:** The space

$$\mathcal{O}(\mathbb{C}^r) \subset C^\infty(\mathbb{R}^{2r}),\ \mathbb{C}^r = \mathbb{R}^{2r},$$

of all entire functions is an injective $\mathbb{C}[s_1,\cdots,s_r]$-module.

Proof: This follows from the preceding theorem via (2.99) or coincides with th. 5.4 on page 126 of [EH]. ||

(34) **Theorem:** The space $\mathcal{O}(\mathbb{C}^r;\exp) \subset \mathcal{O}(\mathbb{C}^r)$ of all entire holomorphic functions of exponential type (see (26)) is $\mathbb{C}[s]$-injective. Hence, by (28), also the $\mathbb{C}[s]$- module $\mathbb{C}\langle t\rangle$ is injective.

Proof: This is example 2 on page 138 of [EH]. ||

(35) **Corollary**: The $\mathbb{C}[s]$-modules **A** from the examples (1.13), (1.18), (1.22), (1.24) are injective or, in other words, satisfy the fundamental principle (compare (2.31)). ||

(36) **Corollary**: The injective $\mathbb{C}[s]$-modules **A** from the preceding corollary are related by the $\mathbb{C}[s]$-monomorphisms

(37)
$$\begin{array}{ccccc} \mathbb{C}\langle t\rangle \cong \mathcal{O}(\mathbb{C}^r;\exp) & \subset & \mathcal{O}(\mathbb{C}^r) \longrightarrow C^\infty(\mathbb{R}^r) & \subset & \mathcal{D}'(\mathbb{R}^r) \\ \sum a_n t^n \longmapsto \sum (a_n/n!)t^n\,, & & f \longmapsto f\mid \mathbb{R}^r\,. & & \end{array}$$

In particular, if $\mathbb{C}\langle t\rangle$ is a large injective cogenerator then so are all other $\mathbb{C}[s]$-modules appearing in (35). ||

The remainder of this paragraph is devoted to the proof that the injective $\mathbb{C}[s]$- module

(38)
$$\mathbb{C}\langle t\rangle \cong \mathcal{O}(\mathbb{C}^r;\exp) \cong \mathcal{O}(\mathbb{C}^r)'$$

(Compare (31)) is a large injective cogenerator. The proof copies that of theorem (3.15) by replacing formal power series by convergent ones, and requires several results on complex spaces wich I recall or cite first.

Reduced complex spaces and Stein spaces

For the known, but non-trivial technical remarks which follow I refer to the books [GR1], [GR2], [GR3], [HÖ2] and [LE].

(39) **Assumption**: In this paragraph X denotes a reduced complex space with its sheaf $\mathcal{O}_X$ of holomorphic functions on X. The $\mathbb{C}$-algebra of global sections of $\mathcal{O}_X$, i.e. of holomorphic functions on X, is denoted by $\mathcal{O}_X(X)$. Since X is reduced $\mathcal{O}_X$ is a subsheaf of the sheaf C_X of continuous functions on X (Compare [GR2], Kap. 1, § 1.6). ||

(40) **Standard example**: This is the complex manifold $\mathbb{C}^r$ with the usual holomorphic (=analytic) functions on open subsets of $\mathbb{C}^r$. In this particular case I omit the index by writing

$$\mathcal{O} := \mathcal{O}_{\mathbb{C}^r} \text{ and } \mathcal{O}(U) := \mathcal{O}_{\mathbb{C}^r}(U),\ U \subset \mathbb{C}^r \text{ open. } \|$$

(41) **Example** (see [GR2], p. 88): Let X be a reduced complex space, for instance $\mathbb{C}^r$, and let $Y \subset X$ be an analytic subset. Let $J := i(Y) \subset \mathcal{O}_X$ be the ideal sheaf of holomorphic functions vanishing on Y. By [loc.cit.], p.77,

$$J(U) = \{f \in \mathcal{O}_X(U) \; ; \; f(Y \cap U) = 0\}.$$

Then Y itself is a reduced complex subspace of X with the function sheaf

$$\mathcal{O}_Y := (\mathcal{O}_X/J)|Y \subset C_Y .$$

A continuous function $g \in C^0(Y) = C_Y(Y)$ on Y is holomorphic and contained in $\mathcal{O}_Y(Y)$ if and only if for every $y \in Y$ there is an open neighborhood U of y in X and $f \in \mathcal{O}_X(U)$ such that $f|Y \cap U = g|Y \cap U$. I need this construction for $X = \mathbb{C}^r$ only. ||

(42) **Definition and Remark** (Stein space, see [GR3], p. 128 and Ch. V): For a complex space $(X, \mathcal{O}_X)$ the property of being a *Stein space* is defined. The manifold $\mathbb{C}^r$ and its closed subspaces are Stein according to [loc.cit.], p. 130, Bemerkung, and p. 129, Satz 1. If $(Y, \mathcal{O}_Y) := (\mathcal{O}_X/J)|Y)$ is a closed reduced subspace of the reduced Stein space $(X, \mathcal{O}_X)$ the canonical map

(43) $$\mathcal{O}_X(X) \longrightarrow \mathcal{O}_Y(Y) = (\mathcal{O}_X/J)(X), \; f \longmapsto f|Y ,$$

is surjective or, in other words, every holomorphic function on Y can be extended to a holomorphic function on X (see [GR3], Satz 4 on p. 156). Again I need this only for $X = \mathbb{C}^r$. ||

The topology on $\mathcal{O}_X(X)$ (see [GR3], Kap. V, §6, p. 166 ff)

Let $(X, \mathcal{O}_X)$ be a reduced complex space and **S** a *coherent* $\mathcal{O}_X$ - module or *analytic sheaf*. The $\mathcal{O}_X(X)$ - module **S**(X) of global sections becomes a Frechet space with respect to the canonical topology ([loc. cit.], Satz 4 on page 168 and Satz 5 on p. 170). In particular, for $\mathbf{S} = \mathcal{O}_X$, the canonical topology on $\mathcal{O}_X(X)$ coincides with the topology of compact convergence (= uniform convergence on compact subsets), see [loc. cit.], Satz 8 on p. 174. Moreover $\mathcal{O}_X(X)$ is a closed subalgebra of the algebra $C^0(X)$ of continuous functions on X ([loc. cit.], lemma on p. 174). Since $C^0(X)$ is a topological algebra with continuous multiplication the same is true for $\mathcal{O}_X(X)$.

(44) **Corollary**: If $(X, \mathcal{O}_X)$ is a reduced complex space the $\mathbb{C}$ - algebra $\mathcal{O}_X(X)$ of holomorphic functions on X is a topological Frechet algebra with respect to the topology of compact convergence. ||

Analytic functionals

The following considerations generalize (12).

(45) **Definition:** Let $(X, \mathcal{O}_X)$ be a reduced complex space. By (44) $\mathcal{O}_X(X)$ is a Frechet algebra with the topology of compact convergence. Its dual space

$$\mathcal{O}_X(X)' := \widehat{\mathrm{Hom}}_{\mathbb{C}}(\mathcal{O}_X(X), \mathbb{C})$$

of all continuous, $\mathbb{C}$-linear functions on $\mathcal{O}_X(X)$ is called the space of analytic functionals on X. As in (3.4), (i), define the $\mathbb{C}$-bilinear form

(46) $$\langle -,- \rangle : \mathcal{O}_X(X) \times \mathcal{O}_X(X)' \longrightarrow \mathbb{C}, \quad \langle f, \mu \rangle := \mu(f)$$

for $f \in \mathcal{O}_X(X)$, $\mu \in \mathcal{O}_X(X)'$. ||

In analogy to (3.7) the scalar multiplication

(47) $$\mathcal{O}_X(X) \times \mathcal{O}_X(X)' \longrightarrow \mathcal{O}_X(X)', \quad (f,\mu) \longmapsto f\mu$$

$$(f\mu)(g) = \langle g, f\mu \rangle := \langle fg, \mu \rangle = \mu(gf), \quad g \in \mathcal{O}_X(X),$$

is well-defined and turns $\mathcal{O}_X(X)'$ into a $\mathcal{O}_X(X)$-module. The linear function $f\mu$ is again continuous since, by (44), the multiplication on $\mathcal{O}_X(X)$ is continuous. A holomorphic map $F: Y \longrightarrow X$ between reduced complex spaces induces a $\mathbb{C}$-algebra homomorphism

(48) $$\begin{array}{ccccl} F^*: \mathcal{O}_X(X) & \longrightarrow & \mathcal{O}_Y(Y) & & \\ \cap & & \cap & & \\ F^*: C^0(X) & \longrightarrow & C^0(Y), & f \longmapsto & F^*(f) := f \circ F \end{array}$$

(Compare [GR2], 4.3.3 on page 89). This map is continuous on $C^0(X)$ and thus on $\mathcal{O}_X(X)$ and induces the adjoint $\mathbb{C}$-linear map

(49) $$F_* := \mathrm{Hom}(F^*, \mathbb{C}) : \mathcal{O}_Y(Y)' \longrightarrow \mathcal{O}_X(X)'$$

$$\langle f, F_*(\nu) \rangle = \langle F^*(f), \nu \rangle = \nu(f F)$$

for $\nu \in \mathcal{O}_Y(Y)'$ and $f \in \mathcal{O}_X(X)$. This map F_* satisfies the relation

(50) $$F_*[F^*(f)\nu] = f\, F_*(\nu), \quad f \in \mathcal{O}_X(X), \ \nu \in \mathcal{O}_Y(Y)'.$$

This means that F_* is $\mathcal{O}_X(X)$-linear if $\mathcal{O}_Y(Y)'$ is considered as a $\mathcal{O}_X(X)$-module via (47) and the scalar restriction F^*. ||

After these technical preparations I return to the main task, namely to prove the large cogenerator property of the $\mathbb{C}[s]$-module $\mathbb{C}\langle t \rangle \cong \mathcal{O}(\mathbb{C}^r; \exp) \cong \mathcal{O}(\mathbb{C}^r)'$ in analogy to the proof of theorem (3.15). Remark that $\mathbb{C}[s] = \mathbb{C}[s_1, \cdots, s_r] \subset \mathcal{O}(\mathbb{C}^r)$.

(51) **Lemma:** There is a $\mathcal{O}(\mathbb{C}^r)$-monomorphism

(52) $$\mathcal{O}(\mathbb{C}^r) \longrightarrow \mathcal{O}(\mathbb{C}^r)', \ f \longmapsto f\mu,$$

i.e. a $\mathcal{O}(\mathbb{C}^r)$-linearly independent analytic functional μ on $\mathbb{C}^r$. The $\mathcal{O}(\mathbb{C}^r)$-structure of $\mathcal{O}(\mathbb{C}^r)'$ is that of (47).

Proof: Choose a test function $\varphi \neq 0$, $\varphi \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^r)$, i.e. a real-valued C^∞-function with compact support. Now define μ by

$$\mu(g) := \int_{\mathbb{R}^r} \varphi(s)g(s)ds, \ g \in \mathcal{O}(\mathbb{C}^r).$$

Remark that the integral is taken only over the real domain $\mathbb{R}^r$. This definition was suggested to me by my colleague Peter Wagner. There is no convergence problem since φ has compact support. Choose a vector $S = (S(1),\cdots,S(r)) \in (\mathbb{R}_+)^r$ such that $\{s;\ \varphi(s) \neq 0\} \subset \bar{Z}(S)$ (see (5)). Then

(53) $$|\mu(g)| \leq \left(\int|\varphi(s)|ds\right)\|g\|_S, \ g \in \mathcal{O}(\mathbb{C}^r)$$

where $\|-\|_S$ is the maximum norm from (6). The inequality (53) implies the continuity of μ with respect to compact convergence and hence $\mu \in \mathcal{O}(\mathbb{C}^r)'$. Assume now that $f\mu = 0$ for some $f \in \mathcal{O}(\mathbb{C}^r)$ and thus $(f\mu)(g) = \mu(fg) = 0$ for all $g \in \mathcal{O}(\mathbb{C}^r)$. The assertion of the lemma is that f is zero. In particular,

$0=(f\mu)(s^k)=\mu(fs^k)=\int_{\mathbb{R}^r}\varphi(s)f(s)s^k\,ds$ for all $k \in \mathbb{N}^r$. Since $s=\mathrm{Re}(s)$ on $\mathbb{R}^r$ and $\varphi(s) \in \mathbb{R}^r$ the preceding equality implies

$$0=\int_{\mathbb{R}^r}\varphi(s)\mathrm{Re}(f(s))s^k\,ds = \int_{\mathbb{R}^r}\varphi(s)\mathrm{Im}(f(s))s^k\,ds \quad \text{for all } k \in \mathbb{N}^r.$$

Hence the real-valued C^∞-functions $\varphi(s)\mathrm{Re}(f(s))$ and $\varphi(s)\mathrm{Im}(f(s))$ with compact support are orthogonal to all polynomials and thus zero by the Stone-Weierstraß theorem. Choose a $s_0 \in \mathbb{R}^r$ such that $\varphi(s_0) \neq 0$. Then $\mathrm{Re}(f(s))$, $\mathrm{Im}(f(s))$ and finally $f(s)$ itself are zero near for real s_0 and hence f is identically zero by the identity theorem. ||

The following theorem is the main new result of this paragraph.

(54) **Theorem:** For every prime ideal $\mathbf{p}$ of $\mathbb{C}[s]=\mathbb{C}[s_1,\cdots,s_r]$ there is a $\mathbb{C}[s]$-linear embedding

(55) $$\mathbb{C}[s]/\mathbf{p} \longrightarrow \mathcal{O}(\mathbb{C}^r)', \ \bar{1} \longmapsto \lambda, \ \bar{f} \longmapsto f\lambda.$$

This means in other terms that $\mathbf{p} = \{f \in \mathbb{C}[s];\ f\lambda = 0\}$ for this particular analytic functional λ. Since $\mathbb{C}\langle t\rangle \cong \mathcal{O}(\mathbb{C}^r)'$ by (20) this result implies that all the injective $\mathbb{C}[s]$-modules appearing in (35) are large injective cogenerators to which the main duality theorem (2.56) is applicable.

Proof : (i) Let

$$X := V(\mathbf{p}) = \{z \in \mathbb{C}^r;\ p(z) = 0 \text{ for all } p \in \mathbf{p}\}$$

be the vanishing set of this prime ideal $\mathbf{p}$. By Hilbert's Nullstellensatz

(56) $$\mathbf{p} = \{f \in \mathbb{C}[s] \subset \mathcal{O}(\mathbb{C}^r)\ ;\ f(X) = 0\}.$$

This X is an irreducible algebraic subset of $\mathbb{C}^r$ and thus an affine variety with $\mathbb{C}[s]/\mathbf{p}$ as $\mathbb{C}$-algebra of regular functions (compare [MU], Ch. I, § 4, Prop 3).

Since X is algebraic it is also an analytic subset of $\mathbb{C}^r$. We obtain the closed, reduced complex subspace $(X, \mathcal{O}_X := (\mathcal{O}/J)|X)$ of $(\mathbb{C}^r, \mathcal{O})$ and the canonical injection

$$F := \mathrm{inj}: (X, \mathcal{O}_X) \longrightarrow (\mathbb{C}^r, \mathcal{O})$$

as in example (41). Here $J \subset \mathcal{O}$ is the coherent ideal of all holomorphic functions vanishing on X and given by

(57) $$J(U) := \{f \in \mathcal{O}(U) := \{f \in \mathcal{O}(U);\ f(U \cap X) = 0\}$$

for an open subset U of $\mathbb{C}^r$. Since $\mathbb{C}^r$ is a Stein manifold the map

(58) $$F^* : \mathcal{O}(\mathbb{C}^r) \longrightarrow \mathcal{O}_X(X) = \mathcal{O}(\mathbb{C}^r)/J(\mathbb{C}^r),\ f \longmapsto f|X,$$

is a surjective and continuous $\mathbb{C}$-algebra homomorphism by (43) and (48) and induces the injective $\mathbb{C}$-linear map

(59) $$F_* = \mathrm{Hom}(F^*, \mathbb{C}) : \mathcal{O}_X(X)' \longrightarrow \mathcal{O}(\mathbb{C}^r)'$$

according to (49). Moreover F_* is $\mathcal{O}(\mathbb{C}^r)$-linear by (50). The equalities (56) and (57) imply $\mathbf{p} = \mathbb{C}[s] \cap J(\mathbb{C}^r)$ from which I derive the commutative diagram

(60) $$\begin{array}{ccccc} \mathbb{C}[s] & \xrightarrow{\text{can}} & \mathbb{C}[s]/\mathbf{p} & \ni & \bar{f} \\ \cap & & \cap & & \downarrow \\ \mathcal{O}(\mathbb{C}^r) & \xrightarrow{F^*} & \mathcal{O}_X(X) & \ni & f|X = \bar{f} \text{ (by identification)} \end{array}$$

of canonical horizontal surjections and vertical injections. By this identification the algebra $\mathbb{C}[s]/\mathbf{p}$ of regular or polynomially defined functions on the affine variety X lies inside the algebra $\mathcal{O}_X(X)$ of all holomorphic functions.

(ii) In analogy to the proof of (3.20) I apply Noether's normalization lemma to $\mathbb{C}[s]/\mathbf{p}$ and obtain an injective and integral $\mathbb{C}$-algebra homomorphism

(61) $$G^* : \mathbb{C}[z] = \mathbb{C}[z_1,\cdots,z_d] \longrightarrow \mathbb{C}[s]/\mathbf{p} \subset \mathcal{O}_X(X)$$
$$z_i \longmapsto G^*(z_i) = \bar{Q}_i = Q_i|X$$

where $\mathbb{C}[z]$ is a polynomial algebra and the $\bar{s}_i \in \mathbb{C}[s]/\mathbf{p}, i=1,\cdots,r$, are integral over $\mathrm{im}(G^*)$. The number d is the Krull dimension of $\mathbb{C}[s]/\mathbf{p}$ or X. The Q_i induce the polynomial map

$$Q = (Q_1,\cdots,Q_d) : \mathbb{C}^r \longrightarrow \mathbb{C}^d =: Z$$

which is a morphism both with respect to the algebraic and the analytic structures (structure sheaves) of $\mathbb{C}^r$ and $\mathbb{C}^d$. The same results for the restriction

(62) $$G := Q|X = (Q_1|X,\cdots,Q_d|X) : X \longrightarrow \mathbb{C}^d = Z.$$

This map induces the injective, integral map G^* from (61) on the affine algebras and is thus a finite, surjective morphism of affine algebraic varieties according to [MU], § 7, Prop. 3 on page 77. Since G is surjective the induced map

(63) $$G^* : \mathcal{O}(\mathbb{C}^d) \longrightarrow \mathcal{O}_X(X),\ h \longmapsto G^*(h) = hG,$$

is also an injective $\mathbb{C}$-algebra homomorphism. I will show below in (iv) that the induced map

(64) $$G^* : \mathcal{O}(\mathbb{C}^d) \cong \mathrm{im}(G^*) \subset \mathcal{O}_X(X)$$

is a *topological* isomorphism where the topologies on $\mathcal{O}(\mathbb{C}^d)$ and $\mathcal{O}_X(X)$ are those of compact convergence and $\mathrm{im}(G^*)$ carries the induced topology from $\mathcal{O}_X(X)$. Since $\mathcal{O}_X(X)$ is a Frechet space and locally convex in particular every continuous $\mathbb{C}$ - linear function on $\mathrm{im}(G^*)$ can be extended to $\mathcal{O}_X(X)$ due to the Hahn-Banach theorem. With the topological isomorphism (64) this implies that the induced map

$$G_* = \mathrm{Hom}(G^*, \mathbb{C}) : \mathcal{O}_X(X)' \longrightarrow \mathcal{O}(\mathbb{C}^d)'$$

is surjective. This means that every analytic functional ν on $\mathbb{C}^d$ has the form

(65) $$\nu = G_*(\mu) = \mu G^*,\ \nu(h) = \mu(hG),\ \mu \in \mathcal{O}_X(X)',\ h \in \mathcal{O}(\mathbb{C}^d).$$

(iii) Postponing to (iv) the proof that G^* from (64) is a topological isomorphism I apply (51) to $\mathbb{C}^d$ and choose an analytic functional $\nu \in \mathcal{O}(\mathbb{C}^d)'$ such that the map

$$\tilde{N} : \mathcal{O}(\mathbb{C}^d) \longrightarrow \mathcal{O}(\mathbb{C}^d)' , \; h \longmapsto h\,\nu ,$$

is injective where $h\nu$ is defined according to (47). Then write $\nu = G_*(\mu)$, $\mu \in \mathcal{O}_X(X)'$, as in (65). Finally define the analytic functional

$$\lambda := F_*(\mu) \in \mathcal{O}(\mathbb{C}^r)' \quad \text{with } F_* \text{ from (60).}$$

Altogether there results a commutative diagram

$$\text{(66)} \qquad \begin{array}{ccccc}
\mathbb{C}[z] & \xrightarrow{G^*} & \mathbb{C}[s]/\mathbf{p} & \xleftarrow{\text{can}} & \mathbb{C}[s] \\
\cap & & \cap & & \cap \\
\mathcal{O}(\mathbb{C}^d) & \xrightarrow{G^*} & \mathcal{O}_X(X) & \xleftarrow{F^*} & \mathcal{O}(\mathbb{C}^r) \\
\downarrow{\scriptstyle \tilde{N}} & & \downarrow{\scriptstyle \tilde{M}} & & \downarrow{\scriptstyle \tilde{L}} \\
\mathcal{O}(\mathbb{C}^d)' & \xleftarrow{G_*} & \mathcal{O}_X(X)' & \xrightarrow{F_*} & \mathcal{O}(\mathbb{C}^r)'
\end{array}$$

where the maps $\tilde{N}$, $\tilde{M}$ and $\tilde{L}$ are defined by

$$\tilde{N}(h) = h\nu , \quad \tilde{M}(g) = g\mu, \quad \tilde{L}(f) = f\,\lambda .$$

The commutativity of (66) follows from (50). Denote the restrictions of these maps to the affine algebras of regular functions by

$$N := \tilde{N}|\mathbb{C}[z] , \; M := \tilde{M}|(\mathbb{C}[s]/\mathbf{p}) , \; L := \tilde{L}|\mathbb{C}[s].$$

By construction the maps $\tilde{N}$ and N are injective. Consider the kernel $\mathbf{m} := \ker(M) \subset \mathbb{C}[s]/\mathbf{p}$ and let $h \in (G^*)^{-1}(\mathbf{m}) \subset \mathbb{C}[z]$. Then

$$\tilde{N}(h) = h\nu = h\,G_*(\mu) = G_*(G^*(h)\,\mu) = G_*(0) = 0$$

since $G^*(h) \in \ker(M) \subset \ker(\tilde{M})$. This gives

$$h \in \ker(\tilde{N}) = 0 \text{ and } (G^*)^{-1}(\mathbf{m}) = 0.$$

Since G^* is injective and integral this yields $\mathbf{m} = 0$ (Compare the same argument in the proof of (3.20)). Hence M is injective too. Finally, since F^* is surjective and thus F_* injective

$$\ker(L) = \ker(F_* \cdot M \cdot \text{can}) = \ker(\text{can}) = \mathbf{p}$$

and L induces the desired $\mathbb{C}[s]$-linear embedding .

(iv) I prove here that the bijection

$$\text{(64)} \qquad G^* : \mathcal{O}(\mathbb{C}^d) \cong \operatorname{im}(G^*) \subset \mathcal{O}_X(X)$$

is a topological isomorphism. As a finite algebraic morphism the map G is closed with respect to the Zariski topology. By lemma (65) below G is also closed with respect to the strong topologies (those induced from the metric topologies on $\mathbb{C}^d$ and $\mathbb{C}^r$) (Compare [MU], § 10, p. 109 ff). This implies that G is also finite as a morphism of complex spaces according to [GR2], p. 47. By the finite mapping theorem ([GR2], th. 3 on p. 64) the image sheaf $G_*(\mathcal{O}_X)$ is coherent, i.e. a coherent $\mathcal{O}_{\mathbb{C}^d}$-module. Since G is surjective the sheaf map $G^*: \mathcal{O}_{\mathbb{C}^d} \longrightarrow G_*(\mathcal{O}_X)$ given by

$$G^*: \mathcal{O}_{\mathbb{C}^d}(V) \longrightarrow G_*(\mathcal{O}_X)(V) = \mathcal{O}_X(G^{-1}(V)),\ h \longmapsto h(G|G^{-1}(V)),$$

is injective and thus a monomorphism between coherent $\mathcal{O}_{\mathbb{C}^d}$-modules. By the "closure theorem" of [GR3], p. 172, G^* induces an injection

$$G^*: \mathcal{O}(\mathbb{C}^d) \longrightarrow G_*(\mathcal{O}_X)(\mathbb{C}^d)$$

onto a *closed* subspace where $G_*(\mathcal{O}_X)(\mathbb{C}^d)$ is endowed with the canonical topology (see [GR3], Kap. V, § 6). The canonical topology on $G_*(\mathcal{O}_X)(\mathbb{C}^d) = \mathcal{O}_X(X)$ derived from the coherent $\mathcal{O}_{\mathbb{C}^d}$-module $G_*(\mathcal{O}_X)$ coincides with that induced from the coherent $\mathcal{O}_X$-module $\mathcal{O}_X$ by [GR3], Kap. V, § 6, Satz 6c), and is the topology of compact convergence on $\mathcal{O}_X(X)$ according to [loc.cit.], Satz 8 on p. 174. Altogether the preceding considerations imply that the map $G^*: \mathcal{O}(\mathbb{C}^d) \longrightarrow \mathcal{O}_X(X)$ is $\mathbb{C}$-linear, injective, continuous with closed image between Frechet spaces, the topologies being those of compact convergence. Since a closed subspace of a Frechet space is again such the map

(64)
$$G^*: \mathcal{O}(\mathbb{C}^d) \cong \operatorname{im}(G^*)$$

is a continuous $\mathbb{C}$-linear bijection between Frechet spaces and then a topological isomorphism by the "open mapping" theorem. But this was to be shown. ||

(65) **Lemma:** The map G from (62) is closed with respect to the strong (standard) topologies on $\mathbb{C}^d$ and $X \subset \mathbb{C}^r$.

Proof: The lemma is known, but I cannot find an explicit reference. I learned its proof from Richard Swan. Both the lemma and its proof are a relative version of the result that complete and compact complex varieties coincide (see [MU], § 10, Th. 2). The argument appears at the end of the proof of [loc.cit.], th. 2. The notations are those from theorem (54). Since G^* is integral there are equations

$$s_i^{d(i)} + \sum\{ G^*(p_{ij})s_i^j ;\ 0 \le j \le d(i)-1\} = 0,\ i = 1,\cdots,r$$

of integral dependence where the $p_{ij} \in \mathbb{C}[z]$ are polynomials. In other terms this means

$$(66) \qquad x_i^{d(i)} + \sum\{p_{ij}(G(x))x_i^j ,\ 0 \le j \le d(i) - 1\} = 0$$

for all $i = 1,\cdots,r$ and $x \in X$ with $G(x) = (Q_1(x),\cdots,Q_d(x))$. I am going to prove that for every compact set $K \subset \mathbb{C}^d$ the inverse image $G^{-1}(K)$ is compact, i.e. bounded, in $X \subset \mathbb{C}^r$. By [GR2], Ch. 9, § 2.4, this implies that G is closed and the assertion of the lemma. But define

$$M := \mathrm{Max}\{|p_{ij}(z)|;\ z \in K,\ i = 1,\cdots,r, 0 \le j \le d(i)-1\}.$$

This maximum exists since K is compact and the p_{ij} are continuous. From (66) we derive

$$1 = -\sum\left\{p_{ij}(G(x))\,x_i^{j-d(i)} ;\ j = 0,\cdots,d(i)-1\right\}$$

if $x_i \neq 0$ and conclude

$$1 \le M(|x_i|^{-1} + \cdots + |x_i|^{-d(i)}) \text{ for } x \in G^{-1}(K).$$

Hence the $|x_i|^{-1}$, $i = 1,\cdots,r$, are bounded from below and thus $G^{-1}(K)$ is bounded from above as asserted. ||

The proof of the duality theorem in the real case

The proof of the theorem 2.54 for the examples (1.13 real), (1.18 real) and (1.22 real) can be easily deduced from the complex case by means of a descent argument with respect to the field extension $\mathbb{R} \subset \mathbb{C}$. Assume more generally that $F \subset K$ is a field extension, that **D** is a finitely generated commutative. F-algebra and **A** a **D**-module. By scalar extension $K\otimes\mathbf{D} := K\otimes_F\mathbf{D}$ is a finitely generated commutative K-algebra and $\mathbf{B} := K\otimes_F\mathbf{A}$ is a $K\otimes\mathbf{D}$-module. I am going to show by a well-known argument that if $\mathbf{B} = K\otimes_F\mathbf{A}$ is a large injective cogenerator over $K\otimes_F\mathbf{D}$ then so is **A** over **D** . Indeed, consider a finitely generated **D**-module M and the canonical map

(67) $$\mathrm{can}: K\otimes_F \mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A}) \longrightarrow \mathrm{Hom}_{K\otimes\mathbf{D}}(K\otimes M, K\otimes\mathbf{A}),$$
$$\alpha\otimes f \longmapsto \alpha\otimes f,\ (\alpha\otimes f)(\beta\otimes m):=\alpha\beta\otimes f(m).$$

This map is obviously an isomorphism for $M=\mathbf{D}$ and then for $M=\mathbf{D}^k$, $k\in\mathbb{N}$. Since $K\otimes_F(-)$ is exact and Hom is left exact both functors of M in (67) are left exact and the standard argument implies that (67) is an isomorphism for all finitely generated modules M over the noetherian ring $\mathbf{D}$. The injectivity of $\mathbf{B}=K\otimes_F\mathbf{A}$ over $K\otimes\mathbf{D}$ implies the exactness of $\mathrm{Hom}_{K\otimes\mathbf{D}}(-,K\otimes\mathbf{A})$, hence of $K\otimes_F\mathrm{Hom}_{\mathbf{D}}(-,\mathbf{A})\cong\mathrm{Hom}_{K\otimes\mathbf{D}}(K\otimes(-),K\otimes\mathbf{A})$ and finally of $\mathrm{Hom}_{\mathbf{D}}(-,\mathbf{A})$ on $\mathrm{Modf}(\mathbf{D})$ since $F\subset K$ is faithfully flat. Since $\mathbf{D}$ is noetherian this implies by (2.31) that $\mathbf{A}$ is an injective $\mathbf{D}$ - module. Also, for any $M\in\mathrm{Modf}(\mathbf{D})$, there is a $K\otimes\mathbf{D}$-linear monomorphism

$$\Phi=(\Phi_i;\ i=1,\cdots,m): K\otimes M \longrightarrow (K\otimes\mathbf{A})^m = K\otimes\mathbf{A}^m\ .$$

Because of the isomorphism (67) the Φ_i have the form

$$\Phi_i=\sum\{\alpha_{ij}\otimes\varphi_{ij};\ j=1,\cdots,n\},\ \varphi_{ij}\in\mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A})$$

and give rise to the $\mathbf{D}$-linear map

$$\varphi:=(\varphi_{ij};\ i=1,\cdot,m,\ j=1,\cdots,n): M \longrightarrow \mathbf{A}^{[m]\times[n]}.$$

If $\varphi(x)=(\varphi_{ij}(x);\,i,j)$ is zero then so is $\Phi(1\otimes x)=(\sum_j\alpha_{ij}\otimes\varphi_{ij}(x);\,i=1,\cdots,m)$. The injectivity of Φ implies $1\otimes x=0$ and finally $x=0$ since $M\subset K\otimes M, x\mapsto 1\otimes x$, is injective too. We have proven the

(68) **Lemma:** Assume that $F\subset K$ is a field extension and that $\mathbf{A}$ is a $\mathbf{D}$-module over the finitely generated commutative F-Algebra $\mathbf{D}$ such that $K\otimes_F\mathbf{A}$ is a large injective cogenerator over $K\otimes_F\mathbf{D}$. Then $\mathbf{A}$ is a large injective $\mathbf{D}$-cogenerator too. ||

(69) **Theorem:** The $\mathbb{R}[s]=\mathbb{R}[s_1,\cdots,s_r]$ modules $\mathbf{A}_{\mathbb{R}}$ from the examples (1.13 real), (1.18 real) and (1.22 real) are large injective cogenerators, hence the duality theorem is valid in the *real continuous case* .

Proof: The theorem follows from the canonical $\mathbb{C}[s]=\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}[s]$- isomorphism

(70) $$\mathbf{A}_{\mathbb{R}}{}^2 = \mathbb{C}\otimes_{\mathbb{R}}\mathbf{A}_{\mathbb{R}}\cong\mathbf{A}_{\mathbb{R}}\oplus i\mathbf{A}_{\mathbb{R}}=\mathbf{A}\ ,\ (b,c)\longmapsto b+ic$$

where $\mathbf{A}$ is the $\mathbb{C}[s]$ - cogenerator from (1.13) resp. (1.18) resp. (1.22). The map inverse to (69) is given as

(71) $$\mathbf{A} \longrightarrow \mathbf{A}_{\mathbb{R}}{}^2 , \; a \longmapsto (\mathrm{Re}(a), \mathrm{Im}(a)).$$

For instance, if $\mathbf{A}_{\mathbb{R}} = \mathbb{R}<t>$ and $\mathbf{A} = \mathbb{C}<t>$, then

$$\mathbb{C}<t> = \mathbb{R}<t> \oplus i \, \mathbb{R}<t>, \; a = \mathrm{Re}(a) + i \, \mathrm{Im}(a)$$

with $a = \sum a(n) t^n$, $\mathrm{Re}(a) := \sum \mathrm{Re}(a(n)) t^n$, $\mathrm{Im}(a) := \sum \mathrm{Im}(a(n)) t^n$. ||

Appendix : Partial difference equations of infinite order

In § 3 I gave an easy proof for the fact that $\mathrm{Hom}_F(\mathbf{D}, F)$ is an injective $\mathbf{D}$ - module, i.e. satisfies the fundamental principle , for an affine F-algebra $\mathbf{D}$. This result has a topological analogue which I mention without proof. The needed technical prerequisites are contained in [GR3].

Consider a Stein space X and its ring $\mathcal{O}(X) := \mathcal{O}_X(X)$ of global sections which is a Stein algebra (see [GR3], Kap. V, § 7). Stein algebras have many interesting algebraic properties, but are not noetherian in general. Consider the dual space $\mathcal{O}(X)' := \widehat{\mathrm{Hom}}_{\mathbb{C}}(\mathcal{O}(X), \mathbb{C})$ of analytic functionals as a $\mathcal{O}(X)$-module via (47). As in the abstract situation of § 2 the linear system

(72) $$Py = u, \; P \in \mathcal{O}(X)^{1,m}, \; y \in \mathcal{O}(X)'^{m}, \; u \in \mathcal{O}(X)'^{1} ,$$

makes sense. It is reasonable to ask under which circumstances (72) has a solution y for given P and u.

(73) **Theorem:** Let X be a Stein space, $P \in \mathcal{O}_X(X)^{1,m}$ a matrix of holomorphic functions and $u = (u_1 \cdots u_l)^T \in \mathcal{O}_X(X)'^{l}$ a vector of analytic functionals. The system

$$Py = u , \; y \in \mathcal{O}_X(X)'^{m} ,$$

has a solution y of analytic functionals if and only if for all solutions $q = (q_1, \cdots, q_l) \in \mathcal{O}_X(X)^{1,l}$ of $qP = 0$ the relation

(74) $$qu = q_1 u_1 + \cdots + q_l u_l = 0 \quad \text{in} \quad \mathcal{O}_X(X)'$$ holds . ||

The $\mathcal{O}(X)$ - submodule $\{q \in \mathcal{O}(X)^{1,l} ; \; qP = 0\} \subset \mathcal{O}(X)^{1,l}$ is not finitely generated in general or, in other words, there is no matrix $Q \in \mathcal{O}(X)^{k,l}$ which is universal with respect to $QP = 0$ (see (2.27)).

This means that the condition (74) of the preceding theorem has to be checked for infinitely many q . For practical purposes, therefore, the preceding theorem is of limited value only.

The theorem (73) is applicable to $X = \mathbb{C}^r$ and $\mathbb{C}\langle t\rangle \cong \mathcal{O}(\mathbb{C}^r)'$ by (20). The $\mathcal{O}(\mathbb{C}^r)$ - structure of $\mathbb{C}\langle t\rangle$ extends the $\mathbb{C}[s]$ - structure of $\mathbb{C}\langle t\rangle$ from example (1.13) and is given by

$$(75) \qquad (fa)(n) = \sum\{f_m a(m+n);\ m \in \mathbb{N}^r\}$$

$$\text{for } f = \sum f_m s^m \in \mathcal{O}(\mathbb{C}^r)\ ,\ a = (a(n);\ n \in \mathbb{N}^r) \in \mathbb{C}\langle t\rangle .$$

Since infinitely many $a(m+n)$ appear in (74) a corresponding system $Py = u$ as in (71) is called a system of partial difference equations of *infinite order* . ||

5. THE CONSTRUCTIVE SOLUTION OF THE CANONICAL CAUCHY PROBLEM FOR DISCRETE SYSTEMS

The paragraph heading explains what I am going to do in this paragraph .

The abstract Cauchy problem

The considerations in this section were influenced by ideas , but not the details from J. Gregor [GRE] .

(1) **Assumption** : I consider discrete IO-systems

$S = \{(u,y) \in \mathbf{A}^{m+p};\ P(L)(y) = Q(L)(u)\}$, $P \in F[s]^{k,p}$, $Q \in F[s]^{k,m}$, rank$(P) = p$, $Q = PH$,

as in (2.69) with the signal space $\mathbf{A} = F^{\mathbb{N}^r} = F\{t_1,\cdots,t_r\} = F\{t\}$ from (1.7) and the transfer matrix $H \in F(s)^{p,m}$, $F[s] = F[s_1,\cdots,s_r]$, $F(s) = F(s_1,\cdots,s_r)$. The vector $L = (L_1,\cdots,L_r)$ consists of the r left shifts $L_i : F\{t\} \longrightarrow F\{t\}$ from (1.10). ||

In (2.69) I have shown that the projection $S \longrightarrow \mathbf{A}^m, (u,y) \longmapsto u$, is surjective or, in other words, that the equation $P(L)(y) = Q(L)(u)$ has a solution y for any given $u \in \mathbf{A}^m$. Because of this I have called u resp. y an input resp. an output vector of S. The Cauchy problem consists in solving $Py = Qu$ under a suitable initial condition which forces the solution y to be unique.

(2) **Identification:** Let $[p] := \{1,\cdots,p\}$ denote the set of numbers from 1 to

p. I identify

(3) $$\mathbf{A}^p = (F^{\mathbb{N}^r})^p = (F^p)^{\mathbb{N}^r} = F^{[p]\times\mathbb{N}^r} \text{ via}$$

$$y = (y_1 \dots y_p)^T = \left((y_1(n) \dots y_p(n))^T \; ; \; n \in \mathbb{N}^r\right) = \left(y_j(n) \; ; \; j=1,\cdots,p, n \in \mathbb{N}^r\right)$$

with $y_j = \left(y_j(n); n \in \mathbb{N}^r\right) = \sum\left\{y_j(n)t^n; n \in \mathbb{N}^r\right\}$ and $[p]\times\mathbb{N}^r = \{(j,n); j=1,\cdots,p, n \in \mathbb{N}^r\}$. Remember that $\mathbf{A}^p$, F^p etc. always mean the spaces of *column* vectors. Analogous identifications are valid for $\mathbf{A}^m$, $\mathbf{A}^k$ etc. If $G \subset [p]\times\mathbb{N}^r$ is an arbitrary subset I identify $F^G \subset F^{[p]\times\mathbb{N}^r} = \mathbf{A}^p$ as a subspace by extension by zero, i.e. for $x \in F^G$ I define $x_j(n) := 0$ for $(j,n) \notin G$ and identify $x = (x_j(n); (j,n) \in G) = (x_j(n); (j,n) \in [p]\times\mathbb{N}^r)$. The complement of G is denoted by $G' := \left([p]\times\mathbb{N}^r\right)\setminus G$, hence $[p]\times\mathbb{N}^r = G \mathbin{\dot\cup} G'$ is a disjoint union. This disjoint union induces the direct sum decomposition

(3) $$\mathbf{A}^p = F^G \oplus F^{G'} \text{ with } y = (y_j(n); (j,n) \in [p]\times\mathbb{N}^r) = y|G + y|G'$$

where $y|G = (y_j(n); (j,n) \in G), y|G' = (y_j(n); (j,n) \in G')$. The projection from $\mathbf{A}^p$ to F^G is just the map $\mathrm{proj}: \mathbf{A}^p \longrightarrow F^G$, $y \longmapsto y|G$. ||

Since $\ker(P(L)) \subset \mathbf{A}^p = F^{[p]\times\mathbb{N}^r}$ is a closed subspace with respect to the linearly compact product topology on $F^{[p]\times\mathbb{N}^r}$ there is a subset $G' \subset [p]\times\mathbb{N}^r$ such that

(4) $$\ker(P(L)) \oplus F^{G'} = F^{[p]\times\mathbb{N}^r} = \mathbf{A}^p$$

The existence of such a G' follows from the Steinitz exchange theorem for infinite dimensional spaces via the duality (2.50). I omit the detailed proof since it is unconstructive and has no practical significance. The main constructive task below will be to find a set $G' \subset [p]\times\mathbb{N}^r$ and its complement $G = G'' = [p]\times\mathbb{N}^r \setminus G'$ which are canonically associated with S or P only.

(5) **Theorem and Definition** (the abstract Cauchy problem). Assumption (1). In addition let $[p]\times\mathbb{N}^r = G \mathbin{\dot\cup} G'$ be a disjoint decomposition. The following assertions are equivalent:

(i) The *inhomogeneous Cauchy* (*or initial value*) *problem*

(6) $$P(L)(y) = Q(L)(u), \; y|G = x, \; u \in \mathbf{A}^m, \; x \in F^G,$$

has a unique solution $y \in \mathbf{A}^p$ for any given input u and *initial data* $x \in F^G$.

(ii) The homogeneous Cauchy problem

(7) $$P(L)(y) = 0\ ,\ y|G = x\ ,\ x \in F^G ,$$

has a unique solution $y \in \mathbf{A}^p$ for any given initial data x.

(iii) The decomposition (4) holds, i.e.

(4) $$\mathbf{A}^p = \ker(P(L)) \oplus F^{G'} .$$

Proof: Obviously (ii) is a special case of (i).

(ii)$\Rightarrow$(i) Consider the equations $P(L)(y) = Q(L)(u), y|G = x$, $u \in \mathbf{A}^m$, $x \in F^G$. By (2.69) there is a $z \in \mathbf{A}^p$ such that $P(L)(z) = Q(L)(u)$ and then, by (ii), a unique $w \in \mathbf{A}^p$ satisfying $P(L)(w) = 0$ and $w|G = x - z|G$. Then $y := z + w$ satisfies

$$P(L)(y) = P(L)(z) + P(L)(w) = Q(L)(u) + 0 = Q(L)(u)$$

$$\text{and } y|G = z|G + w|G = z|G + x - z|G = x.$$

Thus y is the desired solution. Assume that y_1 is another solution of (6). Then $P(L)(y-y_1)=0$ and $(y-y_1)|G=x-x=0$, hence $y-y_1=0$ by (ii). Thus y is the unique solution of (6).

(i), (ii)$\Rightarrow$(iii) Any $y \in \ker(P(L)) \cap F^{G'}$ satisfies $P(L)(y)=0$ and $y|G=0$ since $F^{G'}=\ker(\mathrm{proj}: \mathbf{A}^p \longrightarrow F^G,\ y \longmapsto y|G)$. Condition (ii) implies $y=0$ and thus $\ker(P(L) \cap F^{G'} = 0$.

For $x \in F^G$ the solution y of (7) according to (ii) implies the decomposition $x=y+(x-y)$ with $y \in \ker(P(L))$ and $(x-y)|G=x|G-y|G=x-y|G=0$, thus $x-y \in F^{G'}$ and $x \in \ker(P(L))+F^{G'}$. We deduce $F^G \subset \ker(P(L))+F^{G'}$, thus $\mathbf{A}^p=F^G+F^{G'}=\ker(P(L))+F^{G'}$ and finally $\mathbf{A}^p=\ker(P(L)) \oplus F^{G'}$.

(iii)$\Rightarrow$(i) The decomposition (4) and the homomorphism theorem imply the isomorphism $P(L)|F^{G'}: F^{G'} \xrightarrow{\cong} \mathrm{im}(P(L))$ and its inverse $\tilde{K} := (P(L)|F^{G'})^{-1}: \mathrm{im}(P(L)) \longrightarrow F^{G'} \subset \mathbf{A}^p$. By definition $\tilde{K}$ satisfies the relation $P(L)\tilde{K}=\mathrm{id}_{\mathrm{im}(P(L))}$ and is thus a linear section of $P(L)$. According to the following standard lemma 9 this section induces the direct sum decomposition

(8) $$\mathbf{A}^p=\ker(P(L)) \oplus \mathrm{im}(\tilde{K}) = \ker(P(L)) \oplus F^{G'} \ni\ y = \pi(y)+(\mathrm{id}_{\mathbf{A}^p}-\pi)(y)$$

where $\pi := \mathrm{id}_{\mathbf{A}^p} - \tilde{K}P(L)$ resp. $\mathrm{id}-\pi = \tilde{K}P(L)$ are the projections onto $\ker(P(L))$ resp. $\mathrm{im}(\tilde{K}) = F^{G'}$ with respect to this decomposition. The direct decompositions (4) in (iii) and (8) obviously coincide.

The surjectivity of $S=\{(u,y);Py=Qu\}\longrightarrow \mathbf{A}^m,(u,y)\longmapsto u$, implies or is actually equivalent to the inclusion $\mathrm{im}(Q(L))\subset \mathrm{im}(P(L))$. Since $\tilde{K}$ is defined on $\mathrm{im}(P(L))$ the map

$$\tilde{H}:=\tilde{K}Q(L):\mathbf{A}^m\longrightarrow \mathbf{A}^p,\ u\longmapsto \tilde{H}(u):=\tilde{K}(Q(L)(u)),$$

is well-defined.

Now let $u\in A^m$ and $x\in F^G$ be given as in (6) and define $y:=\pi(x)+\tilde{H}(u)$. This vector satisfies

$$P(L)(y)=P(L)(\pi(x))+P(L)\tilde{K}Q(L)(u)=0+Q(L)(u)=Q(L)(u)$$

since π is the projection onto $\ker(P(L))$ and $P(L)\tilde{K}=\mathrm{id}_{\mathrm{im}(P(L))}$. Moreover $y|G=\pi(x)|G+\tilde{H}(u)|G$. But $\pi(x)|G=x|G=x$ according to the following lemma 10 and $\tilde{H}(u)|G=0$ since $\tilde{H}(u)\in \mathrm{im}(\tilde{K})=F^{G'}=\ker(\mathbf{A}^p\longrightarrow F^G)$. Hence $y|G=x$ and y is a solution of (6).

Assume finally that z is another solution of (6). Then by (8)

$$z=\pi(z)+\tilde{K}P(L)(z)=\pi(z)+\tilde{K}Q(L)(u)=\pi(z)+\tilde{H}(u).$$

Also $z=z|G+z|G'=x+z|G'$ and $z|G'\in F^{G'}=\ker(\pi)$, thus $\pi(z|G')=0$. We derive $z=\pi(z)+\tilde{H}(u)=\pi(x)+\tilde{H}(u)=y$ and hence the uniqueness of the solution y of (6). ||

(9) **Lemma:** Let $f:M\longrightarrow N$ be an epimorphism between modules over some ring and g a linear section of N satisfying $fg=\mathrm{id}_N$. Then M decomposes as

$$M=\ker(f)\oplus \mathrm{im}(g)\ni y=(y-(gf)(y))+(gf)(y)=\pi(y)+(\mathrm{id}_M-\pi)(y)$$

where $\pi:=\mathrm{id}_M-gf$ resp. $\mathrm{id}_M-\pi=gf$ are the projections onto $\ker(f)$ resp. $\mathrm{im}(g)$ with respect to this decomposition. In particular $\ker(f)=\mathrm{im}(\pi)$ and $\mathrm{im}(g)=\ker(\pi)$. ||

(10) **Lemma:** In the situation of theorem 5 one has $\pi(y)|G=y|G$ and $\pi(y|G)=\pi(y)$ for all $y\in \mathbf{A}^p$. In particular $\pi(x)|G=x$ for all $x\in F^G$.

Proof: The assertion follows from
$y-\pi(y)=\tilde{K}P(L)(y)\in F^{G'}=\ker(\mathbf{A}^p\longrightarrow F^G,\ y\longmapsto y|G)$, hence $(y-\pi(y))|G=0$. Similarly $y-y|G\in F^{G'}=\ker(\pi)$ and $\pi(y)=\pi(y|G)$. ||

The proof of (iii) $\rightarrow$ (i) in theorem 5 implies and suggests the following

(11) **Corollary and Definition** (State space and transfer function): Assume that the equivalent assertions of theorem 5 are satisfied. Define $\tilde{K} := (P(L)|F^{G'})^{-1} : \mathrm{im}(P(L)) \longrightarrow \mathbf{A}^p$, $\tilde{H} := \tilde{K}Q(L) : \mathbf{A}^m \longrightarrow \mathbf{A}^p$ and $\pi := \mathrm{id}_{\mathbf{A}^p} - \tilde{K}P(L)$ as in theorem 5, proof of (iii)$\Rightarrow$(i). Then:

(i) π resp. $\tilde{K}P(L)$ are the projections onto $\ker(P(L))$ resp. $F^{G'}$ in (4) and $P(L)\tilde{H} = Q(L)$.

(ii) The vector $y := \pi(x) + \tilde{H}(u)$ is the unique solution of $P(L)(y) = Q(L)(u)$, $y|G = x$.

(iii) The maps

(12) $$\ker(P(L)) \longleftrightarrow F^G,\ y = \pi(x) \longleftrightarrow x = y|G, \text{ and}$$

(13) $$S \longleftrightarrow \mathbf{A}^m \times F^G,\ (u,y) \longleftrightarrow (u,x),\ y = \pi(x) + \tilde{H}(u),\ y|G = x,$$

are inverse isomorphisms. In particular

(14) $$\mathbf{A}^m \longrightarrow S,\ u \longmapsto (u, \tilde{H}(u)),$$

is a linear section of the projection $S \longrightarrow \mathbf{A}^m$.

In analogy to the 1-dimensional situation I call F^G the *statespace*, $\tilde{H} : \mathbf{A}^m \longrightarrow \mathbf{A}^p$ the (0-state) *transfer function* (or operator) and $\pi : F^G \longrightarrow \mathbf{A}^p$ the *0-input transfer function* of *the IO-System* S *with respect to the initial set* G. It is also customary to consider the *mixed transfer operator*

(15) $$\mathbf{A}^m \times F^G \cong S \xrightarrow{\mathrm{proj}} \mathbf{A}^p,\ (u,x) \longmapsto y = \pi(x) + \tilde{H}(u)$$

where $\pi(x) + \tilde{H}(u)$ is the unique solution of (6) and $\pi(x)$ resp. $\tilde{H}(u)$ are the components of the output depending on the initial state x resp. the input u.

Proof: (i) and (ii) were proven in (iii)$\Rightarrow$(i) of theorem 5. The bijections (12) resp. (13) follow directly from (5), (ii), resp. (5), (i). ||

(16) **Remark**: In the situation of (5) and (10) we have the transfer matrix $H \in F(s)^{p,m}$ with $PH = Q$ and the transfer function $\tilde{H} \in \mathrm{Hom}_F(\mathbf{A}^m, \mathbf{A}^p)$ with $P(L)\tilde{H} = Q(L)$. In general there is no direct connection between H and $\tilde{H}$.||

The following sections give an introduction to the theory of *Gröbner* or *standard bases* adapted to the needs of this paper. The ideas go back to

Macaulay [MAC2] (compare [LJ]) , the algorithms are due to Buchberger [BU] and many others. I learned the theory from my colleague Franz Pauer [PAU] . I omit the proofs or give indications only.

The orders on $\mathbb{N}^r$

(a) **The cw-(componentwise) order** : The cw-order on $\mathbb{Z}^r$ and then $\mathbb{N}^r$ is given by

$m \le_{cw} n \Leftrightarrow \forall\ i = 1,\cdots,r : m(i) \le n(i) \Leftrightarrow 0 \le_{cw} n-m \Leftrightarrow \exists k \in \mathbb{N}^r$ with $n = m+k$.

The monoid $\mathbb{N}^r$ is the positive cone with respect to this order. The cw-order is the standard (partial) ordering of $\mathbb{Z}^r$ and makes it a lattice with $\sup_{cw}(m,n) = (\max(m(i),n(i));\ i=1,\cdots,r)$ and $\inf_{cw}(m,n) = (\min(m(i),n(i));\ i=1,\cdots,r)$. The monoid $\mathbb{N}^r$ is a sublattice of $\mathbb{Z}^r$ and artinian, i.e. every non-empty subset of $\mathbb{N}^r$ (trivially) admits a minimal element with respect to the cw-order.

(17) **Lemma**: Let G be a subset of $\mathbb{N}^r$ and $D := \mathrm{Min}_{cw}(G)$ the set of its minimal elements with respect to the cw-order. Then D is a finite, discrete subset of $(\mathbb{N}^r, \le_{cw})$ and $G \subset D + \mathbb{N}^r$. If $G = G + \mathbb{N}^r$ or, in other words, if $m \in G$ and $m \le_{cw} n$ implies $n \in G$ then $G = D + \mathbb{N}^r$.

The proof is easy and known. (Compare [HI], [BU] , [LJ] , [PAU]) . ||

(b) **Well-orders on $\mathbb{N}^r$**: Consider orders $\le$ on $\mathbb{N}^r$ with the following

(18) **Property** : (i) the order is strict, i.e. $m \le n$ or $n \le m$ for all $m,n \in \mathbb{N}^r$. (ii) The order is compatible with the algebraic structure, i.e. $m < n$ and $k \in \mathbb{N}^r$ implies $m+k \le n+k$. (iii) 0 is the smallest element of $\mathbb{N}^r$, i.e. $0 \le m$ for all $m \in \mathbb{N}^r$. ||

The standard example is the lexicographic order on $\mathbb{N}^r$. All orders on $\mathbb{N}^r$ with the property (18) are classified by Hahn's theorem (see Fuchs [FU], p. 91).

(19) **Corollary**: Any order $\le$ on $\mathbb{N}^r$ satisfying (18) is a well-order or , equivalently, artinian. This means that every non-empty subset G of $\mathbb{N}^r$ admits a smallest element with respect to $\le$. With the data from (17) the smallest element of G is the smallest element of the finite set D with respect to $\le$. ||

(20) **Assumption:** For the remainder of this paper choose a fixed order $\leq$ on $\mathbb{N}^r$ satisfying (18). Several later definitions depend on this order. On the other side, the order can often be suitably chosen for a given system S. ||

The degree of functions with finite support: Assume that F is a field and $(I, \leq)$ a strictly ordered index set. The support of a function $x=(x(i);\, i\in I)\in F^I$ is given by $\mathrm{supp}(x):=\{\, i\in I;\ x(i)\neq 0\,\}$. Then $F^{(I)}:=\{x\in F^I;\ \mathrm{supp}(x) \text{ is finite }\}$ is the subspace of F^I of all functions of finite support (compare (3.4,(ii))) and has the standard basis $\delta_i, i\in I$, $\delta_i(j)=\delta_{ij}$.

(21) **Definition** (degree) : For $x\neq 0$ in $F^{(I)}$ the non-empty, finite set $\mathrm{supp}(x)$ admits a largest element d with respect to the strict order $\leq$. This d is called the *degree* $\deg(x)$ of x and $x(d)=x(\deg(x))$ is its *leading* coefficient. Define $\deg(0)=-\infty$ with $-\infty < i$ for all $i\in I$. ||

Obviously the vectors $\delta_i, i\in I$, of the standard basis have the degree $\deg(\delta_i)=i$. The basis representation of $x\in F^{(I)}, x\neq 0$, can be written as

(22) $$x=\sum\{x(i)\delta_i;\ i\in I\,\}=x(d)\,\delta_d+\sum\{x(i)\delta_i;\ i<d\,\}\ ,\ d:=\deg(x).$$

The representation (22) implies that for vectors x and y of the same degree $d\in I$ the vector $x-(x(d)/y(d))y$ has a degree smaller than d. By transfinite induction with respect to the degree one derives the easy

(23) **Lemma:** Let I be a well-ordered index set, e.g. $\mathbb{N}^r$ from (20), and F a field. Any family $x_i, i\in I$, of vectors in $F^{(I)}$ with $\deg(x_i)=i$ for all $i\in I$ is a basis of $F^{(I)}$. ||

(24) **Standard example** (Macaulay [MAC2]) : Consider the well-order $\leq$ on $\mathbb{N}^r$ from (20) and identify

$$F[s]=F[s_1,\cdots,s_r]=F^{(\mathbb{N}^r)},\ p=\sum\{p(n)s^n;\ n\in\mathbb{N}^r\}=(p(n);\ n\in\mathbb{N}^r).$$

If $d:=\deg(p)\in\mathbb{N}^r$ denotes the degree of $p\in F[s]$ this polynomial can be written as

$$p=p(d)s^d+\sum\{p(n)s^n;\ n<d\},\ p(d)\neq 0\ ,$$

a special case of (22). The preceding lemma shows that any family $(p_d;\ d\in\mathbb{N}^r)$ of polynomials $p_d\in F[s]$ of degree $\deg(p_d)=d$ is a F-basis of $F[s]$. For polynomials in one variable this fact is proven by induction instead of the transfinite induction in the proof of (23). ||

The degree of polynomial vectors : I consider the module of polynomial row vectors $q=(q_1,\cdots,q_p)\in F[s]^{1,p}$ ($F[s]=F[s_1,\cdots,s_r]$) with the notations of (1.28). In analogy to (2) I identify

(25) $$F[s]^p = (F^{(\mathbb{N}^r)})^{1,p} = (F^{1,p})^{(\mathbb{N}^r)} = F^{([p]\times\mathbb{N}^r)}$$

$$q=(q_1,\cdots,q_p)=((q_1(m),\cdots,q_p(m));m\in\mathbb{N}^r)=(q_i(m);i=1,\cdots,p,m\in\mathbb{N}^r)$$

$$\text{where } q_i=(q_i(m);\ m\in\mathbb{N}^r)=\sum\{q_i(m)s^m;m\in\mathbb{N}^r\}.$$

Using the fixed well-order $\le$ on $\mathbb{N}^r$ from (20) I order $[p]\times\mathbb{N}^r$ lexicographically via

(26) $$(i,m)<(j,n) \text{ if } i<j \text{ or } (i=j \text{ and } m<n).$$

This order on $[p]\times\mathbb{N}^r$ is again a well-order on $[p]\times\mathbb{N}^r$, and hence the considerations in (21) and (23) are applicable to $F^{([p]\times\mathbb{N}^r)}=F[s]^{1,p}$. The standard F-basis of $F[s]$ are the monomials $s^m, m\in\mathbb{N}^r$, the standard $F[s]$- basis of the $F[s]$-module $F[s]^{1,p}$ is given by the vectors $e_1=(10\cdots0),\cdots,e_p=(0\cdots01)$ in $F^{1,p}\subset F[s]^{1,p}$. Finally the standard F-basis $\delta_{i,m},(i,m)\in[p]\times\mathbb{N}^r$, of $F[s]^{1,p}=F^{([p]\times\mathbb{N}^r)}$ is formed by the vectors $\delta_{i,m}=s^m e_i=(0\cdots0s^m0\cdots0)$, s^m at the i.th place, $(i,m)\in[p]\times\mathbb{N}^r$. Since the degree of a non-zero vector $q=(q_1,\cdots,q_p)\in F[s]^{1,p}$ is defined as $\deg(q)=\mathrm{Max}\{(i,m)\in[p]\times\mathbb{N}^r;\ q_i(m)\neq0\}\in[p]\times\mathbb{N}^r$ one obtains the

(27) **Corollary:** A non-zero vector $q=(q_1,\cdots,q_p)\in F[s]^{1,p}$ has the degree $(i,d)\in[p]\times\mathbb{N}^r$ with respect to the order (26) if and only if q has the form

$$q=(q_1,\cdots,q_i,0,\cdots,0),\ q_i=q_i(d)s^d+\sum\{q_i(m)s^m;m<d\},\ q_i(d)\neq0$$

In other terms this means that q can be written as

$$q=q_i(d)s^d e_i+\sum\{q_i(m)s^m e_i;\ m<d\}+\sum\{q_j(m)s^m e_j;\ j<i,\ m\in\mathbb{N}^r\}.$$

In particular $\deg(q)\in\{i\}\times\mathbb{N}^r$ if and only if $q_i\neq0$, but $q_{i+1}=\cdots=q_p=0$. ||

For a non-zero F-subspace $U\subset F[s]^p$ define

(28) $$\deg(U):=\{\deg(q)\in[p]\times\mathbb{N}^r\ ;\ q\in U,\ q\neq0\}\subset[p]\times\mathbb{N}^r$$

as the set of all degrees of non-zero vectors in U.

(29) **Lemma:** If U is a non-zero $F[s]$-submodule of $F[s]^{1,p}$ and $(i,d)\in\deg(U)$ then $\{i\}\times(d+\mathbb{N}^r)\subset\deg(U)$.

Proof: The proof is simple. The relation $deg(q)=(i,d), q=(q_1,\cdots,q_p)\in U$, means $q_{i+1}=\cdots=q_p=0$ and $deg(q_i)=d$. But then, for any $m\in\mathbb{N}^r, s^m q=(\cdots,s^m q_i,0,\cdots,0)\in U$ and $deg(s^m q_i)=deg(s^m)+deg(q_i)=m+d$, hence $deg(s^m q)=(i,d+m)\in deg(U)$. ||

(30) **Corollary and Definition:** Let $U\subset F[s]^{1,p}$ be a non-zero $F[s]$-submodule. For $i=1,\cdots,p$ define $E(i):=\{m\in\mathbb{N}^r;\ (i,m)\in deg(U)\}$ and $D(i):=Min_{cw}(E(i))$ as in (17). Then $deg(U)=\bigcup\{\{i\}\times(D(i)+\mathbb{N}^r);\ i=1,\cdots,p\}$. ||

The notations in the corollary come from Hironaka [HI], p. 244 .

(31) **Lemma:** Let $U=F[s]^{1,k}P\subset F[s]^{1,p}$ be the row module of a polynomial matrix $P\in F[s]^{k,p}$, i.e. the $F[s]$-submodule generated by the rows $P_{i-}, i=1,\cdots,k$, of P. Then the following assertions are equivalent:

(i) $rank(P)=p$.

(ii) $M:=F[s]^{1,p}/U=F[s]^{1,p}/F[s]^{1,k}P$ is a torsion module.

(iii) All the sets $E(i):=\{m\in\mathbb{N}^r;(i,m)\in deg(U)\}, i=1,\cdots,p$, are non-empty.

Proof: The equivalence of (i) and (ii) is an easy fact for torsion modules over integral domains. The module $M=F[s]^{1,p}/U$ is torsion if and only if there is a polynomial $a\neq 0$ in $F[s]$ such that $aF[s]^{1,p}=(aF[s])^{1,p}\subset U$. But $deg(aF[s]^{1,p})=[p]\times(deg(a)+\mathbb{N}^r)$.

(ii) $\Rightarrow$ (iii) If $F[s]^{1,p}/U$ is torsion then the sets $E(i)$ contain $deg(a)+\mathbb{N}^r$ by the preceding considerations and are consequently not empty.

(iii) $\Rightarrow$ (i) Since all $E(i), i=1,\cdots,p$, are assumed non-empty it is possible to choose a $p\times p$-matrix Q such that the i.th row Q_{i-} of Q is in U and has the degree $(i,d(i)), d(i)\in\mathbb{N}^r$. This means that Q_{i-} has the form $Q_{i-}=(Q_{i1},\cdots,Q_{ii},0,\cdots,0)$ with $deg(Q_{ii})=d(i)$, in particular $Q_{ii}\neq 0$. Hence Q is lower triangular with the non-zero elements Q_{ii} in the main diagonal and has therefore the rank p. Moreover the row module $F[s]^{1,p}Q$ is contained in $U=F[s]^{1,k}P$ which implies a representation $Q=XP, X\in F[s]^{p,k}$, and finally $rank(P)\geq rank(Q)=p$, thus rank $(P)=p$ as desired. ||

The canonical Cauchy problem for a IO-system and its Gröbner or standard basis.

I return to assumption (1) and the IO-system $S=\{(u,y)\in\mathbf{A}^{m+p};\ P(L)(y)=Q(L)(u)\}$ where in particular $\operatorname{rank}(P)=p$ such that (31) is applicable.

(32) **Lemma:** Assumption (1). The row module $U:=F[s]^{1,k}P$ of the matrix P depends on S and its given IO-structure $(u,y)\in\mathbf{A}^{m+p}$, but not on the special choice of the matrix P.

Proof: Assume that S is also given as $S=\{(u,y)\in\mathbf{A}^{m+p}; P_1(L)(y)=Q_1(L)(u)\}$ where $P_1\in F[s]^{k(1),p}, Q_1\in F[s]^{k(1),m}$. By (2.63) there are matrices X and X_1 such that $P_1=XP$ and $P=X_1P_1$. In particular $F[s]^{1,k(1)}P_1=F[s]^{1,k(1)}XP\le F[s]^{1,k}P=U$ and similarly $U\subset F[s]^{1k(1)}P_1$, hence $U=F[s]^{1,k}P=F[s]^{1,k(1)}P_1$. ||

(33) **Corollary and Definition** (the canonical Cauchy problem of a system) Let S be a IO-system as in (1) and choose orders $\le$ on $\mathbb{N}^r$ and $[p]\times\mathbb{N}^r$ as in (20) and (26). The module $U:=F[s]^{1,k}P$ depends on S and its given IO-structure, but not on the special choice of P. The same is true for the torsion factor module $F[s]^{1,p}/F[s]^{1,k}P$. The degree set $G':=\deg(U)=\{\deg(q); 0\neq q\in U\}$ depends on S, its IO-structure and the given order $\le$, but not on the special choice of P or Q, and has the form

$$G'=\deg(U)=\bigcup\Big\{\{i\}\times E(i); i=1,\cdots,p\Big\}=\bigcup\Big\{\{i\}\times(D(i)+\mathbb{N}^r);\ i=1,\cdots,p\Big\}\quad\text{where}$$

$E(i):=\{m\in\mathbb{N}^r;(i,m)\in\deg(U)\}=D(i)+\mathbb{N}^r\neq\emptyset$, $D(i)=\operatorname{Min}_{cw}(E(i)), i=1,\cdots,p$.

The complementary set $G:=([p]\times\mathbb{N}^r)\setminus\deg(F[s]^{1,k}P)$ of initial conditions (compare (5), (11)), the state space F^G and the Cauchy problem $P(L)(y)=Q(L)(u),\ y|G=x, u\in\mathbf{A}^m, x\in F^G$, are termed *canonical*. They depend on S, its IO-structure and the chosen order $\le$. ||

(34) **Theorem:** Under the assumptions of definition (33) the canonical decomposition

(35) $$F[s]^{1,p}=F^{(G)}\oplus F[s]^{1,k}P$$

holds.

Proof : Pose $U:=F[s]^{1,k}P$. For each of the finitely many elements (i,d), $i=1,\cdots,p$, $d\in D(i)$, choose a vector $q^{(i,d)}\in U$. For every $(i,m)\in G'=\bigcup\left\{\{i\}\times(D(i)+\mathbb{N}^r);\ i=1,\cdots,p\right\}$ choose a representation $m=d+k$, $d\in D(i), k\in\mathbb{N}^r$, or $d\leq_{cw} m$ and $k:=m-d$, and define $q^{(i,m)}=s^k q^{(i,d)}$. Then $q^{(i,m)}\in U$ and $\deg(q^{(i,m)})=(i,m)\in G'$. For $(i,m)\in G=([p]\times\mathbb{N}^r)\setminus G'$ the standard basis vector $\delta_{(i,m)}=s^m e_i$ has the degree (i,m). By lemma 23 the family

$$s^m e_i,(i,m)\in G \quad , \quad q^{(i,m)},(i,m)\in G'=\deg(U),$$

is a F-basis of $F[s]^{1,P}$, hence

(36) $$F[s]^{1,P}=\oplus\{Fs^m e_i\ ;\ (i,m)\in G\}\oplus\oplus\{\ Fq^{(i,m)};\ (i,m)\in G'\}.$$

Since the $\delta_{(i,m)}=s^m e_i$ come from the standard basis the first summand on the right is $F^{(G)}=\oplus\{Fs^m e_i\,;(i,m)\in G\}$. The vectors $q^{(i,m)}$ of the second summand are in U, thus (36) implies $F[s]^{1,P}=F^{(G)}+U$. But the intersection $F^{(G)}\cap U$ is zero. For assume that $q\neq 0$ is contained in $F^{(G)}$ and U. Then $\mathrm{supp}(q)\subset G$ and $\deg(q)\in G$ and $\deg(q)\in\deg(U)=G'$, a contradiction. We conclude $F[s]^{1,P}=F^{(G)}\oplus U$. ||

Moreover (35) and (36) together furnish $U=\oplus\{Fq^{(i,m)};(i,m)\in G'\}$. This implies $U=\sum\{F[s]q^{(i,d)};\ i=1,\cdots,p,\ d\in D(i)\}$ since $q^{(i,m)}=s^k q^{(i,d)}$ by definition. Thus the $q^{(i,d)}$, $i=1,\cdot\cdot,p$, $d\in D(i)$, from a finite $F[s]$-generating set of $U=F[s]^{1,k}P$. These generating systems can be constructed from P and are called standard of Gröbner bases of U [loc.cit] .

(37) **Definition and Corollary** (*standard or Gröbner basis*) In the situation of the preceding theorem a finite family of vectors from $U=F[s]^{1,k}P$ is called a standard or Gröbner basis of U or S if all the degrees (i,d), $i=,\cdots,p$, $d\in D(i)$, appear among the degrees of vectors from the family. A Gröbner basis of U generates U as an $F[s]$-module. This notion depends on S, its IO-structure and the order on $\mathbb{N}^r$, but not on the special choice of P. ||

(38) **Division algorithm** (see [PAU]) : In the situation of theorem 34 the division algorithm furnishes a decomposition $q=q'+q''$, $q'\in F^{(G)}$, $q''\in F[s]^{1,k}P$,

for any $q \in F[s]^{1,p}$ in finitely many steps . ||

The decomposition (35) implies that the canonical Cauchy problem of (33) is uniquely solvable as I am going to show now. For this purpose I consider the non-degenerate F-bilinear form

(39) $$\langle -,- \rangle : F[s]^{1,p} \times F\{t\}^p = F^{([p]\times\mathbb{N}^r)} \times F^{[p]\times\mathbb{N}^r} \longrightarrow F$$
$$\langle (q_i(m);(i,m)\in[p]\times\mathbb{N}^r),(y_j(n);(j,n)\in[p]\times\mathbb{N}^r)\rangle := $$
$$=\sum\{q_i(m)y_i(m);\ (i,m)\in[p]\times\mathbb{N}^r\}$$

In particular $\langle s^m e_i,\ t^n e_j\rangle = \delta_{m,n}\delta_{i,j}$ for the standard F-basis $s^m e_i$ of $F^{([p]\times\mathbb{N}^r)}$ and the topological F-basis $t^m e_j$ of $\mathbf{A}^p = F\{t\}^p$. The form $\langle -,- \rangle$ is $F[s]$-bilinear where the $F[s]$-structure of $F\{t\} = \mathbf{A} = F^{\mathbb{N}^r}$ is that given by left shifts (see (1.10), (3.53)). For $q=(q_1,\cdots,q_p)\in F[s]^{1,p}$ and $y=(y_1\cdots y_p)^T$ the matrix product furnishes $q\,y = q_1y_1+\cdots+q_py_p$. Using $q_i=\sum\{q_i(m)s^m; m\in\mathbb{N}^r\}$ we obtain $q_iy_i=\sum\{q_i(m)s^m y_i;\ m\in\mathbb{N}^r\}$. But $(s^m y_i)(l)=y_i(l+m)$, hence the

(40) **Corollary:** For $q=(q_1\cdots q_p)\in F[s]^{1,p}$ and $y=(y_1\cdots y_p)^T\in\mathbf{A}^p$

(a) $\langle s^m e_i, y\rangle = y_i(m)$, $(i,m)\in[p]\times\mathbb{N}^r$

(b) $(q\,y)(0)=\langle q,y\rangle = \sum\{q_i(m)y_i(m);(i,m)\in[p]\times\mathbb{N}^r\}$

(c) $(q\,y)(l)=(s^l q y)(0)=\langle s^l q,y\rangle=\langle q,s^l y\rangle=\sum\{q_i(m)y_i(m+l);(i,m)\in[p]\times\mathbb{N}^r\}$. ||

I use the linearly compact product topology on $F^{[p]\times\mathbb{N}^r}=\mathbf{A}^p$ and the duality $V\longmapsto V^*$ from (3.50) and identify $(F[s]^{1,p})^*=\mathbf{A}^p=F^{[p]\times\mathbb{N}^r}$ as in (3.8, i)), i.e. $y=\langle -,y\rangle$. In particular the orthogonal complement $(-)^\perp$ with respect to $\langle -,- \rangle$ defines a order reversing involution between the $F[s]$-submodules of $F[s]^{1,p}$ and the closed $F[s]$-submodules of $\mathbf{A}^p$. The duality (3.50) also implies that for a F-linear map $\psi: F[s]^{1,k}\longrightarrow F[s]^{1,p}$ the adjoint map $\psi^*: \mathbf{A}^p\longrightarrow\mathbf{A}^k$, defined by

$$\langle\psi(q),y\rangle=\langle q,\psi^*(y)\rangle, q\in F[s]^{1,k},\ y\in\mathbf{A}^p,$$

satisfies $\ker(\psi^*)=\mathrm{im}(\psi)^\perp$ and $\mathrm{im}(\psi^*)=\ker(\psi)^\perp$. A direct decomposition $F[s]^{1,p}=U\oplus V$ with projection φ onto V, i.e. $\varphi=\varphi^2$, $\mathrm{im}(\varphi)=V, \ker(\varphi)=U$, implies in this fashion a direct decomposition $\mathbf{A}^p=U^\perp\oplus V^\perp$ with projection $\varphi^*=(\varphi^*)^2=(\varphi^2)^*$ onto $U^\perp$, i.e. $\mathrm{im}(\varphi^*)=U^\perp$ and $\ker(\varphi^*)=V^\perp$. These considerations are now applied to the direct decomposition

$F[s]^{1,p} = F^{(G)} \oplus U$, $U = F[s]^{1,k}P$ from (35) . Denote by φ its projection onto $F^{(G)}$. In this case (40) implies $(F^{(G)})^{\perp} = F^{G'}$. Moreover $U = F[s]^{1,k}P = \mathrm{im}(F[s]^{1,k} \xrightarrow{P} F[s]^{1,p}, z \longmapsto zP)$ and $(F[s]^{1,k} \xrightarrow{P} F[s]^{1,p})^{*} = (P(L): \mathbf{A}^{p} \longrightarrow \mathbf{A}^{k})$ as in (2.15), thus $U^{\perp} = \ker(P(L))$. Hence $\mathbf{A}^{p} = \ker(P(L)) \oplus F^{G'}$ with the projection $\pi := \varphi^{*}$ onto $\ker(P(L))$. But this is exactly proposition (iii) of theorem 5. We have thus proven the following

(41) **Theorem** (Unique solution of the canonical Cauchy problem) Let

$$S = \{(u,y) \in \mathbf{A}^{m+p};\ P(L)(y) = Q(L)(u)\}, P \in F[s]^{k,p}\ ,\ Q \in F[s]^{k,m},$$

$\mathrm{rank}(P) = p$, $PH = Q$, be a IO-system as in (1). Choose an order $\leq$ on $\mathbb{N}^{r}$ as in (20) and consider the canonical data (33). Then the canonical Cauchy problem

$$P(L)(y) = Q(L)(u),\ y|G = x,\ u \in \mathbf{A}^{m}, x \in F^{G}\ ,$$

is uniquely solvable or, equivalently, $\mathbf{A}^{p} = \ker(P(L)) \oplus F^{G'}$. The projection π onto $\ker(P(L))$ with respect to this decomposition is $\pi = \varphi^{*}$ where φ is the projection onto $F^{(G)}$ belonging to $F[s]^{1,p} = F^{(G)} \oplus F[s]^{1,k}P$ and constructively given by the division algorithm.

The inverse $\tilde{K} := (P(L)|F^{G'})^{-1}$ of the induced isomorphism $P(L)|F^{G'}: F^{G'} \cong \mathrm{im}(P(L))$ is related to π via $\tilde{K}P(L) = \mathrm{id} - \pi$, i.e. $\tilde{K}(P(L)(z)) = z - \pi(z)$ (see (11)). In particular if $u \in \mathbf{A}^{m}$ and $z \in \mathbf{A}^{p}$ is a solution of $P(L)(z) = Q(L)(u)$ then $\tilde{H}(u) = \tilde{K}(Q(L)(u)) = z - \pi(z)$ is the unique solution of $P(L)(\tilde{H}(u)) = Q(L)(u)$ with $\tilde{H}(u)|G = 0$ (see (11) for *the* transfer operator $\tilde{H}$). ||

The projection π in the preceding theorem can also be given by the matrix coefficients of φ. Consider for this purpose the decompositions

(42) $s^{n}e_{j} = (0 \cdots s^{n} 0 \cdots) = \varphi(s^{n}e_{j}) + (s^{n}e_{j} - \varphi(s^{n}e_{j})) \in F[s]^{1,p} = F^{(G)} \oplus F[s]^{1,k}P$

of the standard basis vectors $s^{n}e_{j}$ of $F[s]^{1,p} = F^{([p] \times \mathbb{N}^{r})}$ which can be constructively found by the division algorithm (38). Write

(43) $\varphi(s^{n}e_{j}) = \sum\{e_{i}s^{m}M((i,m),(j,m)); (i,m) \in G\} \in F^{(G)} = \oplus\{Fs^{m}e_{i}; (i,m) \in G\}$

Writing the $G\times([p]\times\mathbb{N}^r)$-matrix M in block form with respect to $[p]\times\mathbb{N}^r = G\dot{\cup} G'$ furnishes $M=(M(G,G)\ M(G,G'))$ and $M(G,G)=I_G$ = the $G\times G$-identity matrix since φ is the projection onto $F^{(G)}$ and hence $\varphi(s^m e_i)=s^m e_i$ for $(i,m)\in G$. For $y=(y_j(n);(j,n)\in[p]\times\mathbb{N}^r)\in F^{[p]\times\mathbb{N}^r}=\mathbf{A}^p$ the equation $y_i(m)=\langle s^m e_i,y\rangle$ from (40) yields

$$\pi(y)_j(n)=\langle s^n e_j,\pi(y)\rangle=\langle s^n e_j,\varphi^*(y)\rangle=$$
$$=\langle\varphi(s^n e_j),y\rangle=\langle\sum\{s^m e_i M\big((i,m),(j,n)\big);(i,m)\in G\},y\rangle=$$
$$=\sum\{\langle s^m e_i,y\rangle M((i,m),(j,n));(i,m)\in G\}=\sum\{y_i(m)M((i,m),(j,n));(i,m)\in G\}.$$

In particular we see that $\pi(y)$ depends on $y|G$ only which, however , is clear since $y-y|G\in F^{G'}=\ker(\pi)$.

(44) **Corollary** : Situation of theorem 41 and (43). Then the projection π is given by the formula

$$\pi(y)_j(n)=\sum\{y_i(m)\ M((i,m),\ (j,n));\ (i,m)\in G\}.$$

In particular, for all $y\in\mathbf{A}^p$, $y|G=\pi(y)|G$, and $\pi(y)=\pi(y|G)$. ||

The constructive calculation of functions defined by transfinite recursion

By (41) the canonical Cauchy problem can be uniquely solved. The next goal is to show that the solution can be constructively found. In this section I develop a general technical preparation for this purpose.

Consider a well-ordered index set I and a disjoint decomposition $I=G\dot{\cup} G'$. The later application will be to the case $I=[p]\times\mathbb{N}^r=G\dot{\cup}G'$ as in (33) and (41). Moreover let F be any field; for the following abstract argument an arbitrary set F or an even more general situation would suffice.

(45) **Lemma** (Definition by transfinite recursion) Assume that for all $i\in G'$ a function $\varphi_i:F^{(-\infty,i)}\longrightarrow F$ is given where $(-\infty,i):=\{j\in I;j<i\}$ is the open interval ending in i. Then, for every $x\in F^G$, there is a unique $y\in F^I$ satisfying the equations

(46) $\quad y(i)=x(i)$ for $i\in G$ or $y|G=x$ and $y(i)=\varphi_i(y|(-\infty,i))$ for $i\in G'$.

One says that y is constructed from x by transfinite recursion via the φ_i.

This construction gives rise to a section

(47) $\quad \Phi:F^G\longrightarrow F^I,\ x\longmapsto y$, of the projection $\ \mathrm{proj}: F^G\longrightarrow F^I, y\longmapsto y|G$.

The proof is a special case of [BOU3], Ch. III, § 2, C 60, p. 42. ||

(48) **Corollary** (The constructive case of (45)): In the situation of the preceding lemma assume that $\varphi_i(z), z \in F^{(-\infty,i)}$, depends only on finitely many $z(j)$ with $j<i$. More precisely, assume that for every $i \in G'$ a finite subset $S(i) \subset (-\infty, i) = \{j; j<i\}$ and a function $\varphi_i : F^{S(i)} \longrightarrow F$ are given. Then, given $x \in F^G$, there is a unique solution $y \in F^I$ of the equations

(49) $\qquad y(i) = x(i)$ for all $i \in G$ and $y(i) = \varphi_i(y(j); j \in S(i))$ for $i \in G'$.

This construction gives rise to the section

(50) $\qquad \Phi : F^G \longrightarrow F^I,\ x \longmapsto y$, of $\mathrm{proj} : F^G \longrightarrow F^I, y \longmapsto y|G$.

The proof is a special case of that of (45). ||

The following considerations are intuitively clear and show that in the situation of (48) every $y(i)$ can be calculated from x in finitely many steps. The data of (48) are assumed.

Consider on I the relation $<<.$ defined by

(51) $$j <<. i :\Leftrightarrow i \in G' \text{ and } j \in S(i).$$

In particular $j<<.i$ implies $j<i$ since $S(i) \subset (-\infty, i)$ by assumption. The relation $<<.$ induces the new order relation $<<=$ on I defined via

(52) $$j <<= i : \Leftrightarrow \text{There is a chain } j = j(0) <<. j(1) <<. \cdots <<. j(r) = i$$

Again $j<<i$, i.e. $j<<=i$ and $j \neq i$, implies $j<i$, in particular the ordered set $(I, <<=)$ is artinian since the well-ordered set $(I, <)$ is. Also $j<<i$ implies $i \in G'$ since by (51) the relation $j(r-1) <<. j(r) = i$ implies $i \in G'$. Hence the set G is a discrete subset of $(I, <<=)$. Let

(53) $$I(i) := \{j \in I; j <<= i\}$$

be the closed interval of $(I, <<=)$ ending in i. Then

(54) $\qquad I(i) = \{i\}$ if $i \in G$, $\qquad$ and $\qquad I(i) = \{i\} \cup \bigcup \{I(j);\ j \in S(i)\}$ for $i \in G'$.

The following argument is trivial but nevertheless crucial for the constructiveness argument.

(55) **Lemma:** The sets $I(i), i \in I$, are finite.

Proof: This follows directly from (54) by transfinite induction. If $i \in G$ then $I(i) = \{i\}$ consists only of the element i. If $i \in G'$ then $S(i)$ is finite by assumption and the $I(j), j\in S(i)$, thus $j<i$, are finite by the inductive hypothesis. But then, by (54), $I(i)$ is a finite union of finite sets and hence finite itself. ||

In analogy to the argument in (48) we now construct a section

$\Phi_i : F^{I(i)\cap G} \longrightarrow F^{I(i)}, x \longmapsto y$, of $\mathrm{proj} : F^{I(i)} \longrightarrow F^{I(i)\cap G}, y \longmapsto y|I(i)\cap G$.

by finite induction. Indeed, the finite subset $I(i)$ of the well-ordered set I is strictly ordered with respect to $<$ and has i as its largest element. Assume that $x\in F^{I(i)\cap G}$ is given and that for any $k\in I(i)$ the values $y(j)$ for all $j<k$ in $I(i)$ have already been constructed. If k is contained in G and thus in $I(i)\cap G$ pose $y(k):=x(k)$. If k is in G' then all elements $j\in S(k)$ are contained in $I(i)$ and satisfy $j<k$ such that the $y(j), j\in S(k)$, are known by inductive hypotheses. We define $y(k):=\varphi_k(y(j);\, j\in S(k))$ as in (49). We have thus proven the

(56) **Lemma:** Assumptions as in (48). For every $i\in I$ and $x\in F^{I(i)\cap G}$ there is a unique $y\in F^{I(i)}$ which solves the equations

(57) $y(k)=x(k)$ for all $k\in I(i)\cap G$ and $y(k)=\varphi_k(y(j);j\in S(k)$ for $k\in I(i)\cap G'$.

The preceding construction gives rise to a section

$$\Phi_i : F^{I(i)\cap G} \longrightarrow F^{I(i)}, x \longmapsto y, \text{ of proj: } F^{I(i)} \longrightarrow F^{I(i)\cap G}. \quad ||$$

A simple induction in the finite set $I(i)$ using the equations (49) and (57) proves the

(58) **Corollary** (Constructive calculation in finitely many steps) Assumptions and notations as in (48) and (56). The diagram

$$\begin{array}{ccc} F^G & \xrightarrow{\quad\Phi\quad} & F^I \\ \downarrow \mathrm{proj} & & \downarrow \mathrm{proj} \\ F^{I(i)\cap G} & \xrightarrow{\quad\Phi_i\quad} & F^{I(i)} \end{array}$$

commutes. In particular, the value $y(i)$ of $y:=\Phi(x), x\in F^G$, can be calculated from $x|I(i)\cap G$ in finitely many steps via φ_i as $y(i)=\Phi_i(x|I(i)\cap G)(i)$.

Suggestive language: The solution y of (49) can be constructively calculated from $x\in F^G$ in finitely many steps. ||

Remark that I do not make any statement on the number of necessary steps to calculate $y(i)$, i.e. on the size of $I(i)$. This, however, is not necessary when, for instance, one writes a recursive solution program.

The constructive solution of the canonical Cauchy problem

In this section I explain what the heading indicates. The situation is that of theorem 41. I will apply the preceding section to the well-ordered set $[p]\times\mathbb{N}^r=\{(i,m);i=1,\cdots,p,m\in\mathbb{N}^r\}$ and its canonical decomposition

$$[p]\times\mathbb{N}^r=G\dot{\cup}G',\quad G':=\bigcup\{\{i\}\times(D(i)+\mathbb{N}^r);\ i=1,\cdots,p\}$$

(59) **Definition** (Gröbner or standard matrix) A matrix P^g whose rows are a Gröbner basis of the row module $U:=F[s]^{1,k}P\subset F[s]^{1,p}$ and are normalized with highest coefficient one for simplicity is called a Gröbner or standard matrix of U respectively $S=\{(uy);Py=Qu\}$. ||

(60) **Result** (Existence of Gröbner matrices) Given P a Gröbner matrix P^g of $F[s]^{1,k}P$ can be constructed by Buchberger's algorithm and its variants (see [PAU] and [WIN]). Also matrices X and Y such that $P^g=XP$ and $P=YP^g$ can be found in finitely many steps according to [loc.cit.] . ||

Remark that a Gröbner matrix P^g for P is automatically constructed when the canonical decomposition $[p]\times\mathbb{N}^r=G\dot{\cup}G'$ from (33) is determined.

Consider now the Cauchy problem

$$P(L)(y)=Q(L)(u),y|G=x,u\in\mathbf{A}^m,x\in F^G$$

from theorem 41 where the input $u\in\mathbf{A}^m$ and the initial data $x\in F^G$ are arbitrarily given and the unique solution $y\in\mathbf{A}^p=F^{[p]\times\mathbb{N}^r}$ shall be calculated. Choose a Gröbner matrix P^g for P as in (60) and a matrix X such that $P^g=XP$. Pose $v:=Q(L)(u)$ such that y satisfies the equation $P(L)(y)=Py=v$. For every index $(i,m)\in G'=\bigcup\{\{i\}\times(D(i)+\mathbb{N}^r);\ i=1,\cdots,p\}$ choose an index $d=d(i,m)\in D(i)$ such that $m\in d+\mathbb{N}^r$ or $m-d\in\mathbb{N}^r$. Since the rows of P^g are a Gröbner basis of $U=F[s]^{1,k}P$ I can and do choose a row

(61) $$q:=(P^g)_{\mu-}=s^d e_i+\sum\{q_j(l)s^l e_j;\ (j,l)<(i,d)\},$$

$\mu=\mu(i,d)$, $e_i=(0\cdots1(\text{i.th place})\cdots0)$, of P^g of degree $\deg(q)=(i,d)$. The equations $Py=Qu=v$ and $P^g=XP$ imply

$$s^{m-d}qy=s^{m-d}(P^g)_{\mu-}y=s^{m-d}X_{\mu-}Py=s^{m-d}X_{\mu-}v \quad \text{and , using (61),}$$

$$s^{m-d}qy=s^m e_i y+\sum\{q_j(l)s^{l+m-d}e_j y\ ;(j,l)<(i,d)\}=s^{m-d}X_{\mu-}v.$$

For $(s^{m-d}qy)(0)$ the preceding equation and (40)(c) imply

$$y_i(m)+\sum\{q_j(l)y_j(l+m-d);(j,l)<(i,d)\}=\sum\{X_{\mu j}(l)v_j(l+m-d);(j,l)\in[p]\times\mathbb{N}^r\}$$

or

(62) $$y_i(m)=-\sum\{q_j(l)y_j(l+m-d);(j,l)<(i,d)\}+$$
$$+\sum\{X_{\mu j}(l)v_j(l+m-d);\ j=1,\cdots,p,l\in\mathbb{N}^r\}\quad .$$

Of course, since $q=(P^g)_{\mu-}$ is a polynomial row vector, almost all $q_j(l)$ up to finitely many are zero, and the same holds for the $X_{\mu j}(l)$. Moreover the relations $(j,l)<(i,d)$ for the indices of the first sum in (62) imply $(j,l+m-d)<(i,d+m-d)=(i,m)$. Hence the indices $(j,l+m-d)$ in the first sum of (62) are smaller than (i,m) with respect to the lexicographic order of $[p]\times\mathbb{N}^r$. With the abbreviation

$$S(i,m):=\{(j,l+m-d)\in[p]\times\mathbb{N}^r;\ (j,l)<(i,d),\ q_j(l)\neq 0\}$$

the equations (62) are of the form

$$y_i(m)=\varphi_{(i,m)}(y_j(n);(j,n)\in S(i,m))\ ,\ (i,m)\in G',$$

as in (48). Thus (58) is applicable and furnishes the

(63) **Theorem** (Constructive solution of the Cauchy problem) Situation of theorem 41. Additional data as chosen above. The unique solution y of the canonical Cauchy problem

$$P(L)(y)=Q(L)(u)=:v,\ y|G=x,\ u\in\mathbf{A}^m,\ x\in F^G,$$

can be calculated in finitely many steps from the recursive equations

(62) $$y_i(m)=x_i(m)\ \text{for}\ (i,m)\in G$$
$$y_i(m)=-\sum\{q_j(l)y_j(l+m-d);(j,l)<(i,d)\}+\sum\{X_{\mu j}(l)v_j(l+m-d);$$
$$j=1,\cdots,p,l\in\mathbb{N}^r\}\ \text{for}\ (i,m)\in G'\ .\quad ||$$

The following choices are involved: $d\in D(i)$ is chosen such that $m-d\in\mathbb{N}^r$; μ is a row index of the Gröbner matrix P^g with $\deg(P^g)_{\mu-}=(i,d)$; $q:=(P^g)_{\mu-}$.

Normal forms for IO-systems

In this section I assume more generally than in (1) that $\mathbf{A}$ is an arbitrary large injective cogenerator over the polynomial algebra $F[s]=F[s_1,\cdots,s_r]$. The main examples have been derived in theorem 2.54. The module $\mathbf{A}=F\{t\}$ of formal power series is one of these. But also the "function" modules of the continuous case can be used. Assumption (20) is in force.

I consider IO-systems S with a fixed IO-structure $(u\,y)\in\mathbf{A}^{m+p}$ and a transfer matrix $H \in F(s)^{p,m}$. These have the form

$$S=\{(uy)\in \mathbf{A}^{m+p};\ Py=Qu\},\ PH=Q,\ \mathrm{rank}(P)=p.$$

where P and Q are polynomial matrices with k rows, $k \geq p$.

Let now P^g be a Gröbner matrix of S whose rows are a Gröbner basis of the row module $U=F[s]^{1,k}P$, uniquely associated with S according to (32). Let also X and Y denote polynomial matrices satisfying $P^g=XP$ and $P=YP^g$ as in (60). Then

(64) $$Q^g:=XQ=XPH=P^gH$$

is polynomial too and satisfies

(65) $$YQ^g=YP^gH=PH=Q.$$

We obtain the new IO-representation

(66) $$S=\{(uy)\in \mathbf{A}^{m+p};P^gy=Q^gu\}.$$

Indeed, if (uy) is in S and satisfies $Py=Qu$ then $P^gy=XPy=XQu=Q^gu$ by (64). In the same fashion $P^gy=Q^gu$ and (65) imply $Py=Qu$ and finally (66). The representation (66) is called a *standard or Gröbner representation* of S. Remember that, by definition, all degrees $(i,d), d\in D(i)$, in

$$\deg(S):=\deg(F[s]^{1,k}P)=\bigcup\left\{\{i\}\times(D(i)+\mathbb{N}^r);i=1,\cdots,p\right\}$$

appear among the degrees of the rows of P^g. The requirements for the rows of P^g can be sharpened and lead to special types of Gröbner matrices.

A Gröbner basis of a module $U=F[s]^{1,k}P\subset F[s]^{1,p}$ is called *simplified* according to [PAU] if it does not contain any superfluous vectors. A matrix $P^{sg}\in F[s]^{k,p}$ is called a *simplified Gröbner matrix* of $U=F[s]^{1,k}P$ if its rows are a simplified Gröbner basis of U. This means that for all $(i,d), d\in D(i), i=1,\cdots,p$ there is a *unique* row index μ such that $\deg\left((P^{sg})_{\mu-}\right)=(i,d)$. For simplicity I include in this definition of P^{sg} again that the rows of P^{sg} are normalized, i.e. that $(P^{sg})_{\mu-}=s^d e_i+\cdots$. One derives a simplified Gröbner matrix P^{sg} from an arbitrary Gröbner matrix P^g of $U=F[s]^{1,k}P$ by omitting as many rows as possible without destroying the Gröbner property.

(67) **Definition** (Reduced Gröbner matrix, see [PAU],[WIN]): A Gröbner matrix P^{rg} of the module $U=F[s]^{1,k}P$ or of the system $S=\{uy);Py=Qu\}$ is

called *reduced* if it is simplified and if the row $(P^{rg})_{\mu-}$ of P^{rg} of degree (i,d) has the form

$$(P^{rg})_{\mu-}=s^d e_i+\sum\left\{\left(P^{rg}\right)_{\mu j}(m)s^m e_j;\ (j,m)\in G\right\}$$

with $G:=[p]\times\mathbb{N}^r\setminus\deg(F[s]^{1,k}P)$ as in (33) or, in other terms, if the support of $(P^{rg})_{\mu-}\in F[s]^{1,p}=F^{([p]\times\mathbb{N}^r)}$ satisfies

$$\mathrm{Supp}((P^{rg})_{\mu-})\cap\deg(U)=\{(i,d)\}$$

A representation

(68) $$S=\{(uy)\in A^{m+p};P^{rg}y=Q^{rg}y=Q^{rg}u\},\quad Q^{rg}=P^{rg}H,$$

is called a *reduced standard or Gröbner representation* of S. ||

(69) **Result** ([PAU], [WIN]): Every submodule $U=F[s]^{1,k}P\subset F[s]^{1,p}$ admits a, up to a permutation of the rows, unique reduced Gröbner matrix P^{rg} such that U=row module of P = row module of P^{rg}. This P^{rg} can be constructed from P in finitely many steps. ||

The preceding result and the earlier considerations imply that the matrices P^{rg} and H form a *complete system of invariants* for the system S. In more detail: Denote by **Syst** the set of all IO-systems in $\mathbf{A}^{m+p}$ with the given IO-structure $(uy)\in\mathbf{A}^{m+p}$, i.e. of all systems

$$S=\{(uy)\mathbf{A}^{m+p};Py=Qu\},\ \mathrm{rank}(P)=p,PH=Q.$$

According to (32) the row module $U=F[s]^{1,k}P$ is uniquely associated with S and, by (69), gives rise to its reduced Gröbner matrix P^{rg} which is polynomial and unique up to a permutation of the rows, and to the reduced Gröbner representation

(68) $$S=\{(uy)\in\mathbf{A}^{m+p};\ P^{rg}y=Q^{rg}u\},\ Q^{rg}=P^{rg}H.$$

The matrix Q^{rg} is also polynomial by (64). The matrix P^{rg} has rank p like P itself . Thus let **Inv** be the set of all pairs (P,H) where P is a polynomial reduced Gröbner matrix with p columns and of rank p and where $H\in F(s)^{p,m}$ is rational such that PH is again polynomial. For simplicity I identify pairs (P,H) and (P',H) when P' results from P by a permutation of the rows. The above considerations furnish the map

(70) $$\mathbf{Syst}\longrightarrow\mathbf{Inv},\ S\longmapsto(P^{rg},H).$$

(71) **Theorem** (Normal form of IO-systems) Let $\mathbf{A}$ be an arbitrary large injective cogenerator over $F[s]=F[s_1,\cdots,s_r]$ (see theorem 2.54) and $\le$ an order on $\mathbb{N}^r$ and $[p]\times\mathbb{N}^r$ as in (20). Any IO-system $S=\{(uy)\in\mathbf{A}^{m+p};\ Py=Qu\}$ with IO-structure (uy) and transfer matrix H admits a reduced Gröbner representation

$$(68)\qquad S=\{(uy)\in\mathbf{A}^{m+p};P^{rg}y=Q^{rg}u\},Q^{rg}=P^{rg}H,$$

where P^{rg} is the reduced Gröbner matrix of S or $U:=F[s]^{1,k}P$. The pair (P^{rg},H) is a complete system of invariants for S, i.e. the map

$$(70)\qquad \mathbf{Syst}\longrightarrow\mathbf{Inv}\ ,\ S\longmapsto(P^{rg},H)\ ,$$

is bijective. Here I identify the matrix P^{rg} with all matrices derived from P^{rg} by permutation of the rows.

Proof: By the preceding considerations and essentially due to (32) and (69) the map (70) is well-defined. It is injective since S can be reconstructed from (P^{rg},H) according to (68). If finally (P,H) is any pair in **Inv** the system

$$S:=\{(uy)\in\mathbf{A}^{m+p};\ Py=Qu\},\ Q:=PH\ ,$$

is a system in **Syst** which has (P,H) as its invariants. Thus (70) is surjective. ||

The one-dimensional "classical" case. Connection with the Hermite form of matrices

In the situation of the preceding section assume in addition that $F[s]=F[s_1,\cdots,s_r]$ has just one indeterminate s_1, i.e. $r=1$ and $s=s_1$. In this situation the reduced Gröbner matrix P^{rg} is in Hermite form (compare [KAI], 6.7.1, p. 476) as I am going to show below. The notations are those of the preceding section. The order on $\mathbb{N}=\mathbb{N}^r$ is of course the natural one and and on $[p]\times\mathbb{N}$ the lexicographic one. Consider the IO-system S with transfer matrix H

$$S=\{(uy)\in\mathbf{A}^{m+p};\ Py=Qu\},P\in F[s]^{k,p},\mathrm{rank}(P)=p,\ PH=Q.$$

As is well-known and recalled in (2.29) P can be transformed into a matrix $\binom{P'}{0}$, $P'\in F[s]^{p,p},\det(P')\neq 0$, by elementary row operations, i.e. by left multiplication with an invertible (or unimodular) polynomial matrix. The rows of P' are then a F[s]-basis of $F[s]^{1,k}P$. Without loss of generality I assume $P'=P$ or, in other terms, that P is already a square matrix with $\det(P)\neq 0$. This matrix P is now subjected to further elementary row operations and transformed into the reduced Gröbner matrix P^{rg}. By a sequence of elementary row operations as in [KAI], p. 375, 476 , or in [BY] , p. 327 , 328 , P can be transformed into a lower triangular matrix

$$R=(R_{ij};i,j=1,\cdots,p)\ ,\ R_{ii}\neq 0,\ R_{ij}=0 \text{ for } i<j\ .$$

Let $d(j):=\deg(R_{jj})$, $j=1,\cdots,p$, and hence $\deg(R_{j-})=(j,d(j))$. Without loss of generality R_{jj} has the highest coefficient 1 , so $R_{jj}=s^{d(j)}+\cdots$ and $R_{j-}=s^{d(j)}e_j+\cdots$. I am going to show now that $P^{sg}:=R$ is a simplified Gröbner matrix of U=row module of P=row module of R. This means that

$$\deg(S)=\deg(U)=G'=\dot{\bigcup}\left\{\{j\}\times(d(j)+\mathbb{N});j=1,\cdots,p\right\}$$

Let for this purpose $x=qR\neq 0$ be an arbitrary nonzero vector in U of degree (k,m). It has to be shown that $m\geq d(k)$. But $\deg(x)=(k,m)$ means that $x=(x_1,\cdots,x_k,0,\cdots,0)$ with $\deg(x_k)=m$. Write

$$x=(x_I\ 0)\ ,\ q=(q_I\,q_{II}) \text{ and } R=\begin{pmatrix} R_{I,I} & 0 \\ R_{II,I} & R_{II,II}\end{pmatrix}$$

in block form where x_I and q_I have k components and $R_{I,I}$ is a $k\times k$ - matrix . Then $x=qR$ means

$$q_IR_{I,I}+q_{II}\,R_{II,I}=x_I\ ,\ q_{II}R_{II,II}=0\ .$$

But the lower triangular matrices R and $R_{II,II}$ with nonzero coefficients in the main diagonal are invertible as rational matrices which implies $q_{II}=0$ and $x_I=q_IR_{I,I}=(\cdots q_kR_{kk})$, hence $x_k=q_kR_{kk}$ and $\deg(x_k)=m\geq\deg(R_{kk})=d(k)$. Thus $R=P^{sg}$ is indeed a simplified Gröbner matrix of U.

Again as in [KAI] or [BY] further elementary row operations can be applied to $P^{sg}=R$ in order to obtain a lower triangular matrix P^{rg} like R with the additional property that

$$\deg((P^{rg})_{ij})<\deg((P^{rg})_{jj})=d(j) \text{ for } i>j\ .$$

(72) **Theorem** (Normal form of one-dimensional IO-systems) In addition to the data of (71) let r be equal one, i.e. let F[s] be the polynomial ring in one indeterminate $s=s_1$. The above algorithm $(P\longmapsto P^{rg})$ yields the unique

reduced Gröbner matrix of the IO-system

$$S=\{(uy)\in\mathbf{A}^{m+p};\ Py=Qu\},\ P\in F[s]^{k,p},\ \mathrm{rank}(P)=p,\ PH=Q$$

with transfer matrix $H\in F(s)^{p,m}$. This P^{rg} is also the unique square matrix in row Hermite form such that P and $\binom{P^{rg}}{0}$ are row equivalent. The pair (P^{rg},H) is a complete system of invariants for S in the sense of (71).

Proof : As shown above the matrix P^{rg} is a simplified Gröbner matrix of S and $\deg(S)=\deg(U)=\dot{\bigcup}\{\{j\}\times(d(j)+\mathbb{N});j=1,\cdots,p\}$ where

$$\deg((P^{rg})_{jj})=d(j)\ ,\ (P^{rg})_{jj}=s^{d(j)}+\cdots \text{ and } \deg((P^{rg})_{j-})=(j,d(j))\ .$$

Additionally the inequalities $\deg((P^{rg})_{ij})<d(j)$ for $i>j$ hold . For $i=1,\cdots,p$ the i.th row of P^{rg} has thus the form

$$(P^{rg})_{i-}=((P^{rg})_{i1},\cdots,(P^{rg})_{ii},0,\cdots,0)=s^{d(i)}e_i+\sum\{(P^{rg})_{ij}(m)s^m e_j;j\le i,m<d(j)\}\ .$$

The indices $(j,m),j\le i,m<d(j)$, of the right sum lie in

$$G:=([p]\times\mathbb{N})\setminus G'=[p]\times\mathbb{N}\setminus\bigcup\{\{j\}\times(d(j)+\mathbb{N})\ ;j=1,\cdots,p\}=\{(j,m);j=1,\cdots,p,m<d(j)\}\ ,$$

hence $\mathrm{Supp}((P^{rg})_{i-})\cap G'=\{(i,d(i))\}$. By (67) this means that P^{rg} is the unique reduced Gröbner matrix . The remainder follows from (67) . ||

The preceding theorem is an extension and sharpening of [KAI] , § 6.7 , and [BY], p. 93 .

The alternative order on $[p]\times\mathbb{N}^r$

The preceding considerations are also valid for the following well-order on $[p]\times\mathbb{N}^r$. Consider an arbitrary order on $\mathbb{N}^r$ satisfying (18) and define

(73) $\quad (i,m)<(j,n)$ if and only if $m<n$ or ($m=n$ and $i<j$) .

This is the lexicographic order on $[p]\times\mathbb{N}^r$ where , contrary to (26) , the second component $m\in\mathbb{N}^r$ of $(i,m)\in[p]\times\mathbb{N}^r$ is the dominant one . The monoid $\mathbb{N}^r$ operates on $[p]\times\mathbb{N}^r$ via $k+(i,m):=(i,k+m)$, $i\in[p]$, $k,m\in\mathbb{N}^r$, as before , and the orders (26) and (73) satisfy the following conditions (74) in generalization of (18) :

(74) (i) The order is a well-ordering .

(ii) The order is compatible with the operation of $\mathbb{N}^r$ on $[p]\times\mathbb{N}^r$.

(iii) $(i,0)<(i,m)$ for all $i=1,\cdots,p$ and $0\ne m\in\mathbb{N}^r$.

Consider the order < from (73) in the remainder of this section . Let

$$0\ne q=(q_1,\cdots,q_p)=\sum\{q_i(l)s^l e_i;i=1,\cdots,p \text{ and } l\in\mathbb{N}^r\}\in F[s]^{1,p}=F^{([p]\times\mathbb{N}^r)}$$

be a non-zero vector of degree

$$\deg(q)=(i,d):=\mathrm{Max}\{(j,m)\in[p]\times\mathbb{N}\ ;\ q_j(m)\neq 0\}\ .$$

Then q can be written as

$q=q(d)s^d+\tilde{q}$, $\tilde{q}:=\sum\{q(l)s^l;\ l<d\}$, $q(d):=(q_1(d),\cdots,q_i(d),0,\cdots,0)$, $q_i(d)\neq 0$

where $q(l):=(q_1(l),\cdots,q_p(l))\in F^{1,p}$. Obviously d has the property of a degree of a vector as often in the one-dimensional case . I therefore define

(75) $\quad\quad\mathbb{N}^r$-degree of q $:=\deg_{\mathbb{N}^r}(q):=d$

such that $\mathbb{N}^r$-$\deg(q)$ is the second component of $\deg(q)$. Consider now a matrix $P\in F[s]^{k,p}$ with non-zero rows for simplicity . Let $d(i):=\mathbb{N}^r$-$\deg(P_{i-})$ be the $\mathbb{N}^r$-degree of the i.th row of P . Each row can thus be written as

$$P_{i-}=s^{d(i)}P_{i-}(d(i))+\tilde{P}_{i-}\ \text{ with }\ \mathbb{N}^r\text{-}\deg(\tilde{P}_{i-})<d(i)=\mathbb{N}^r\text{-}\deg(P_{i-})\ .$$

Altogether P admits the unique representation

(76) $$P=\Delta(s)P_{hc}+\tilde{P}$$

with the following specifications : $\Delta(s):=\mathrm{diag}(s^{d(1)},\cdots,s^{d(k)})$ where $d(i):=\mathbb{N}^r$-$\deg(P_{i-})$, $P_{hc}:=(P_{ij}(d(i));\ i=1,\cdots,k\ ,\ j=1,\cdots,p)\in F^{k,p}$,

$\tilde{P}:=(\ \sum\{P_{ij}(l)s^l;\ l<d(i)\}\ ;\ i=1,\cdots,k$ and $j=1,\cdots,p)$ with

$\mathbb{N}^r$-$\deg(\tilde{P}_{i-})<d(i)=\mathbb{N}^r$-$\deg(P_{i-})$. If , in particular , P is a $p\times p$-square-matrix and $\deg(P_{i-})=(i,d(i))$ for all $i=1,\cdots,p$ then

$P_{i-}(d(i))=(\text{-----},P_{ii}(d(i))\neq 0,0,\cdots,0)$ and hence

(77) $$P=\Delta(s)P_{hc}+\tilde{P}\ ,\ P_{hc}\in Gl_p(F)\ \text{ lower triangular }\ .$$

Compare [KAI], (3), p.404 , in the one-dimensional situation . The following lemma says that (31) is valid for the alternative order used in this section too .

(78) **Lemma :** Let $P\in F[s]^{k,p}$ be a matrix with non-zero rows and $P=\Delta(s)P_{hc}+\tilde{P}$ a representation as in (77) . The following assertions are equivalent : (i) $\mathrm{rank}(P)=p$. (ii) For every $j=1,\cdots,p$ there is a row index i such that $\deg(P_{i-})=(j,d(j))$ for some $d(j)\in\mathbb{N}^r$, i.e.

$$\deg(F[s]^{1,k}P)=\bigcup\Big\{\{j\}\times(\ D(j)+\mathbb{N}^r\ );\ j=1,\cdots,p\Big\}\ \text{and}\ D(j)\neq\Phi\ !$$

Proof : The implication (i)$\Rightarrow$(ii) is the same as (i),(ii)$\Rightarrow$(iii) from (31) .

(ii) $\Rightarrow$ (i) After a permutation of the rows and after omitting the last $k-p$ rows of P I can and do assume without loss of generality that P is

$p \times p$-square with the representation (77). It has to be shown that $\mathrm{rank}(P) = p$ or, in other terms, that the linear system

$$Px = 0 \ , \ x = (x_1, \cdots, x_p)^T \in F[s]^p \ ,$$

has only the trivial solution $x = 0$. But

(79) $$0 = Px = \Delta(s)y + Qy \ , \ y := P_{hc} x \text{ and } Q := \tilde{P} P_{hc}^{-1} \ .$$

Since P_{hc} is invertible it is sufficient to show that $y = 0$. Since $Q_{ij} = \sum \tilde{P}_{ik} (P_{hc}^{-1})_{kj}$ and $\deg(\tilde{P}_{ik}) < d(i)$, $(P_{hc}^{-1})_{kj} \in F$ also $\deg(Q_{ij}) < d(i)$. Assume that y is not zero and that y_i is a component of highest degree. From (79) we conclude $s^{d(i)} y_i + \sum_j Q_{ij} y_j = 0$. But $\deg(s^{d(i)} y_i) = d(i) + \deg(y_i)$ and $\deg(\sum_j Q_{ij} y_j) \le \mathrm{Max}(\deg(Q_{ij}) + \deg(y_j); j = 1, \cdots, p) < d(i) + \deg(y_i)$, a contradiction, hence $y = 0$ and $x = 0$. ||

Since the Gröbner basis theory is valid for the new alternative order too (see [FSK]) and indeed for any order satisfying (74) (oral communication by Franz Pauer) one easily sees that the results (31) pp. of this paragraph are true for the alternative order mutatis mutandis. I elaborate only on the constructive solution of the Cauchy problem in analogy to (63). I consider the Cauchy problem

(80) $$P(L)(y) = v := Q(L)(u) \ , \ y|G = x \text{ for given } u \in \mathbf{A}^m \text{ and } x \in F^G$$

where $G' := \deg(F[s]^{1,k} P)$. I indicate the recursive equations for the values $y_j(n)$ of the unique solution y. Without loss of generality I assume that P is already a Gröbner matrix, i.e. that for every (j,d), $d \in D(j)$, there is a row index $\mu = 1, \cdots, k$ such that $\deg(P_{\mu -}) = (j,d)$. Let now $(j,n) \in [p] \times \mathbb{N}^r$ be arbitrary such that $y_{j'}(n')$ is already known for all $(j',n') < (j,n)$. If (j,n) is in G then the initial condition $y_j(n) = x_j(n)$ holds. If $(j,n) \in G'$ choose a $d \in D(j)$ and a row index μ such that $n = d + k \in d + \mathbb{N}^r$ and $\deg(P_{\mu -}) = (j,d)$. The μ.th row of the equation $y = (\Delta(s) P_{hc} + P) y = 0$ evaluated at k gives

(81) $$P_{\mu j}(d) y_j(n) + \sum\{P_{\mu i}(d) y_i(n); i \le j-1\} + \sum\{P_{\mu i}(l) y_i(l+k); i \in [p], l < d\} = v_\mu(k) .$$

The indices (i,n), $i \le j-1$, and $(i, l+k)$, $l < d$ and $l + k < d + k = n$, are smaller than (j,n) in the alternative order and $P_{\mu j}(d)$ is not zero as the leading coefficient of $P_{\mu -}$. Thus $y_j(n)$ can be determined from (81) recursively in finitely many steps as in (63).

§ 6. CONVOLUTIONARY TRANSFER OPERATORS

Motivation: The canonical Cauchy problem for a IO-system

$S=\{(uy)\in \mathbf{A}^{m+p};Py=Qu\}, P\in F[s]^{k,p}, \operatorname{rank}(P)=p, Q\in F[s]^{k,m}, PH=Q$

gives rise to the operator $\tilde{K}:=(P(L)\mid F^{G'})^{-1}$, the transfer operator $\tilde{H}=\tilde{K}Q(L)$ and the projection π as in theorem 5.41. Remark that $\tilde{K}$ depends on P and not on S only. According to (5.11) the unique solution y of the canonical Cauchy problem

$$P(L)(y)=Q(L)(u), u\in \mathbf{A}^{m}, x\in F^{G}, \text{ is } y=\pi(x)+\tilde{H}(u) .$$

Theorem 5.63 contains a constructive algorithm for the calculation of y, hence of $\tilde{H}$ and π. However, in general it is rather difficult to derive the properties of $\tilde{H}$, for instance BIBO-boundedness, from the algorithm. In the one-dimensional case it is known that such conclusions can be drawn if the transfer function is given by convolution with the transfer matrix, and that every system admits at least one IO-structure where this is the case. This paragraph contains generalizations of these results to the multidimensional case. It was again influenced by ideas, but not by any details from the paper [GRE] of J. Gregor.

It is possible and useful here to consider multidimensional systems over rings K instead of fields F. I develop the theory in this more general frame-work. There is a connection with the usual theory of one-dimensional linear systems over rings (see for instance [BM]), but only a slight one.

Discrete systems over rings

(1) **Assumption:** Let K be a noetherian integral domain and **B** a K-module. ||

(2) **Standard example:** This is given by the polynomial algebra $K=F^{(\mathbb{N}^{\rho})}=F[\sigma]=F[\sigma_1,\cdots,\sigma_\rho]$ in indeterminates $\sigma_1,\cdots,\sigma_\rho$ over a field F and the F[σ]-module $\mathbf{B}:=F^{\mathbb{N}^{\rho}}$. The F[σ]-module structure of **B** is given by left shifts as in (1.11), i.e.

$$(\sigma^{\mu}b)(v)=b(\mu+v), \sigma^{\mu}=\sigma_1^{\mu(1)}\cdots\sigma_\rho^{\mu(\rho)}, \mu,v\in\mathbb{N}^{\rho} . \quad ||$$

Now define

(3) $\quad \mathbf{D} := K[s] = K[s_1,\cdots,s_r]$ and $\mathbf{A} := \mathbf{B}^{\mathbb{N}^r} = \mathbf{B}\{t\} = \mathbf{B}\{t_1,\cdots,t_r\}$

The elements of $\mathbf{A}$ are power series in $t_1,\cdots,t_r$ with coefficients in the module $\mathbf{B}$ and can be written as

$$a = (a(n); n \in \mathbb{N}^r) = \sum a(n) t^n \ , \ a(n) \in \mathbf{B} \ .$$

With the componentwise addition and scalar multiplication $\mathbf{A}$ is a K-module. It becomes a $\mathbf{D} = K[s]$-module given by left shifts via

(4) $\quad (s^m a)(n) = a(m+n), \ a = (a(n); n \in \mathbb{N}^r) \in \mathbf{A} = \mathbf{B}^{\mathbb{N}^r}, \ m, n \in \mathbb{N}^r$.

For the case $K = \mathbf{B} = F$ the preceding data are those of (1.7). The algebra $\mathbf{D} = K[s]$ is again a noetherian integral domain like K itself ([MATS], Th. 3.3)

(5) **Standard example:** In the situation of (2), i.e. $K = F[\sigma]$ and $\mathbf{B} = F^{\mathbb{N}^\rho}$, the preceding construction gives rise to

$$\mathbf{D} = K[s] = F[\sigma][s] = F[\sigma, s] = F[\sigma_1,\cdots,\sigma_\rho, s_1,\cdots,s_r] \quad \text{and}$$
$$\mathbf{A} = \mathbf{B}\{t\} = F^{\mathbb{N}^\rho}\{t\} = \mathbf{B}^{\mathbb{N}^r} = (F^{\mathbb{N}^\rho})^{\mathbb{N}^r} = F^{\mathbb{N}^\rho \times \mathbb{N}^r} = F^{\mathbb{N}^{\rho+r}}$$

The elements of $\mathbf{A}$ can be written as

$$a = (a(\nu, n); (\nu, n) \in \mathbb{N}^{\rho+r}) = (a(-,n)); \ n \in \mathbb{N}^r) = \sum\{a(-,n) t^n; \ n \in \mathbb{N}^r\}$$

where $a(-, n) := (a(\nu, n); \ \nu \in \mathbb{N}^\rho) \in F^{\mathbb{N}^\rho} = \mathbf{B}$.

Altogether the polynomial algebra $\mathbf{D} = F[\sigma, s]$ in the $\rho+r$ indeterminates $\sigma_1,\cdots,s_r$ operates on $\mathbf{A} = F^{\mathbb{N}^{\rho+r}}$ by left shifts as in example (1.7) and gives nothing really new. However, in the following theory the two families σ and s of indeterminates will not be treated symmetrically which will create new solution possibilities. ||

Under the general assumptions (1), (3) and (4) above I will consider IO-systems of the form

(6) $$S = \{(u\,y) \in \mathbf{A}^{m+p}; P(L)(y) = Q(L)(u)\} \subset \mathbf{A}^{m+p}$$

where $P \in K[s]^{p,p}$ is a square matrix with nonzero determinant, $\det(P) \neq 0$, and $Q \in F[s]^{p,m}$. Of course, S is again a $F[s]$-submodule of $\mathbf{A}^{m+p}$. I will show later in (84) and (86) that in connection with the question whether the transfer operator $\tilde{K}$ is a convolution essentially only systems of the form (6) with square P and $\det(P) \neq 0$ appear.

Convolution and right shifts

The assumption (1), (3) and (4) are in force. In addition consider the power series algebra

(7) $K\{t\}=K\{t_1,\cdots,t_r\}=K^{\mathbb{N}^r}$ with elements $a=(a(n);n\in\mathbb{N}^r)=\sum a(n)t^n, a(n)\in K$.

The multiplication of this algebra is the convolution. Since K is a noetherian integral domain by (1) so is $K\{t\}$ according to [MATS], Th. 3.3.

The algebra $K\{t\}$ contains the polynomial algebra

$$K[t]=K[t_1,\cdots,t_r]=K^{(\mathbb{N}^r)}\subset K\{t\}=K^{\mathbb{N}^r}$$

as a subalgebra. Again with the convolution as scalar multiplication the K-module $\mathbf{A}=\mathbf{B}\{t\}=\mathbf{B}^{\mathbb{N}^r}$ becomes a $K\{t\}$-module via

(8) $$a*b=(\sum\{a(k)b(l);\ k+l=n\};\ n\in\mathbb{N}^r), a\in K\{t\}, b\in\mathbf{B}\{t\}$$

where $a=(a(n);n\in\mathbb{N}^r)=\sum a(n)t^n\in K\{t\}$ and $b=(b(n);n\in\mathbb{N}^r)=\sum b(n)t^n\in\mathbf{B}\{t\}$, $b(n)\in\mathbf{B}$. Since $\mathbf{B}$ is a K-module by assumption the finite sums $\sum\{a(k)b(l);k+l=n\}$ make sense. In particular

(9) $$t^m*(\sum\{b(n)t^n;n\in\mathbb{N}^r\})=\sum\{b(n)t^{m+n};n\in\mathbb{N}^r\}.$$

The *right shift* R_i in the i.th direction is defined via

(10) $$R_i:\mathbf{A}=\mathbf{B}\{t\}\to\mathbf{A}\ ,b\longrightarrow R_i(b)=t_i*b=\sum b(n)t^{n+e_i}$$

with $e_i=(0\cdots 1\ 0\cdots 0)$, one at the i.th component, or, in other terms

(11) $$R_i(b)(n)=\begin{cases} b(n-e_i) & \text{if } n(i)>0\\ 0 & \text{if } n(i)=0\end{cases}$$

For $r=1$ the map $R=R_1$ is given by $R(b(0),\cdots)=(0,b(0),b(1),\cdots)$ and this explains the terminology "right shift" as usual. The $K\{t\}$-linear maps $R_1,\cdots,R_r$ in $\mathrm{End}_K(\mathbf{A})$ commute pairwise which implies that for any polynomial $a=\sum a(n)t^n\in K[t]\subset K\{t\}$ the operator $a(R):=\sum a(n)R^n=$ $=\sum a(n)R_1^{n(1)}\cdot\cdots\cdot R_r^{n(r)}$ in $\mathrm{End}_K(\mathbf{A})$ is well-defined. Since R_i is the convolution with t_i the map $a(R)$ is the convolution with $a(t)$, i.e.

(12) $$a(R)(b)=a*b\ ,\ a\in K[t],\ b\in\mathbf{A}=\mathbf{B}\{t\}\ .$$

Finally, as for any commutative ring, there is the $K\{t\}$-algebra homomorphism

(13) $$K\{t\}\longrightarrow \mathrm{End}_{K\{t\}}(\mathbf{A}), a\longmapsto(b\longmapsto a*b).$$

The preceding formula (12) suggests the definition

(14) $$a(R)(b):=a*b \text{ also for } a\in K\{t\}, b\in\mathbf{B}\{t\}.$$

Remark that at this stage I do not define $a(R)=\sum a(n)R_1^{n(1)}\cdot\cdots\cdot R_r^{n(r)}$ as a power series of operators in $\mathrm{End}_K(\mathbf{A})$ since this would require a notion of convergence and since (14) gives an easier alternative definition. The "operator calculus" in more general situations, for instance in the continuous cases, will be treated later. By (14) the map (13) gets the form

(15) $$K\{t\} \longrightarrow \mathrm{End}_{K\{t\}}(\mathbf{B}\{t\}),\ a \longmapsto a(R).$$

More generally we obtain the $K\{t\}$-linear map

(16) $$K\{t\}^{m,n} \longrightarrow \mathrm{Hom}_{K\{t\}}(\mathbf{B}\{t\}^n, \mathbf{B}\{t\}^m), P \longmapsto P(R),$$

where $P(R)(x) = P*x = (\sum\{P_{ij}*x_j;\ j=1,\cdots,n\},\ i=1,\cdots,m)$.

The map $P(R)$ is called the convolution with the matrix P. For $m=n$ the map

(17) $$K\{t\}^{n,n} \to \mathrm{End}_{K\{t\}}(\mathbf{B}\{t\}^n), P \longmapsto P(R),$$

is even a $K\{t\}$-algebra homomorphism.

(18) **Remark and Definition** (The two module structures on $\mathbf{A}$) There are two structures on $\mathbf{A}$ of a module over the polynomial algebra $K[s]$ or $K[t]$ defined by left respectively right shifts. These have to be carefully distinguished. For the scalar multiplication I use the distinguishing symbols

(19) $$p\cdot_L b = p(L_1,\cdots,L_r)(b),\ p \in K[s], b \in \mathbf{A}\ , \text{ and}$$

(20) $$p\cdot_R b = p(R_1,\cdots,R_r)(b) = p(t)*b(t), p \in K[t], b \in \mathbf{A}\ .$$

I use the ambiguous symbol pb only if its meaning is clear from the context. ||

The algebra homomorphisms (15) and (17) induce the group homomorphisms

(21) $$U(K\{t\}) \longrightarrow Gl_{K\{t\}}(\mathbf{B}\{t\}) \text{ and } Gl_n(K\{t\}) \longrightarrow Gl_{K\{t\}}(\mathbf{B}\{t\}^n)$$

where $U(K\{t\}) := \{a = \sum a(n)t^n \in K\{t\};\ a(0) \in U(K)\}$ and $U(K)$ denote the groups of multiplicatively invertible elements or units of the commutative rings $K\{t\}$ resp. K and where

$Gl_n(K\{t\}) = \{P \in K\{t\}^{n,n}; \det(P) \in U(K\{t\}),\ \text{i.e. } \det(P) = a(0) + \cdots\ , a(0) \in U(K)\}$

is the general linear group of $K\{t\}$ and consists exactly of those matrices whose determinant in $K\{t\}$ has an invertible constant term .

(22) **Corollary** : For the special case $\mathbf{B}=K$ the maps (15), (16), (17) and (21) are isomorphisms. ||

The commutator rules

The data of the preceding section are given. The operators $R_1,\cdots,R_r$ resp. $L_1,\cdots,L_r$ in $\mathrm{End}_K(\mathbf{A})$ commutative pairwise. It is also easy to see that the relations

(23) $$L_iR_i=id_{\mathbf{A}},\ i=1,\cdots,r \text{ and } L_iR_j=R_jL_i \text{ for } i\neq j$$

hold. The equation $L_iR_i=id_{\mathbf{A}}$ means that the right shift R_i is a right inverse or, in other words, a section of the left shift L_i, and induces by (5.9) the direct decomposition

(24) $$\mathbf{A}=\mathbf{B}^{\mathbb{N}^r}=\ker(L_i)\oplus \mathrm{im}(R_i),\quad a=(a-R_iL_i(a))+R_iL_i(a)$$

with $\ker(L_i)=\mathbf{B}^{\{n\in\mathbb{N}^r;\ n(i)=0\}}$ and $\mathrm{im}(R_i)=\mathbf{B}\{t\}t_i=\mathbf{B}^{\{n\in\mathbb{N}^r;\ n(i)>0\}}$

which is of the type (5.3). The projection onto the kernel of L_i is given by

(25) $$\varepsilon_i:=Id-R_iL_i,\ \varepsilon_i(a)(n)=\begin{cases} 0 & \text{if } n(i)>0 \\ a(n) & \text{if } n(i)=0\end{cases}$$

and coincides of course with the canonical projection from $\mathbf{A}=\mathbf{B}^{\mathbb{N}^r}$ onto $\ker(L_i)=\mathbf{B}^{\{n\in\mathbb{N}^r;n(i)=0\}}$. The main observation is that L_i and R_i satisfy the easy rules

(26) $$L_iR_i=id_{\mathbf{A}} \text{ and } R_iL_i=id_{\mathbf{A}}-\varepsilon_i,\ i=1,\cdots,r,$$

but they do not commute. The above derived commutator rules for the R_i and L_j are the basis of an algebraic "operator calculus" and are fully developped, mainly for $r=1$, in the standard book [BE] and the papers of L. Berg and the comprehensive book [PR].

As seen above, the maps R_i and L_i are not bijective, but are one-sided inverses of each other. Therefore I define

(27) $$R_i^{(k)}:=\begin{cases} R_i^{\ k} & \text{if } k\geq 0 \\ (L_i)^{-k} & \text{if } k<0\end{cases},\ k\in\mathbb{Z},\ \text{and } R^{(k)}:=R_1^{(k(1))}\cdot\ldots\cdot R_r^{(k(r))},\ k\in\mathbb{Z}^r.$$

The commutator rules for the R_i and L_j imply the

(28) **Corollary :** Data as above. Then

$$L^mR^n=R^{(n-m)} \text{ for } n,m\in\mathbb{N}^r.$$ In particular (see (5.17) for $\leq_{cw}$)

$$L^mR^n=R^{n-m} \quad \text{if } n-m\in\mathbb{N}^r,\ \text{i.e. } m\leq_{cw}n, \text{ and}$$

$$L^mR^n=L^{m-n} \quad \text{if } m-n\in\mathbb{N}^r,\ \text{i.e. } n\leq_{cw}m. \quad ||$$

Derived Algebras

The data are those of (1), (3) and (4). The rings $K, K[t], \mathbf{D}:=K[s]$ and $K\{t\}$ are integral by assumption (1) and admit quotient fields $Q:=Q(K)\subset Q(K[t])\subset Q(K\{t\})$. All rings I consider in this context are contained in the large field $Q(K\{t\})$. In particular

(29) $$Q(t):=Q(t_1,\cdots,t_r):=Q(K[t]\subset Q(K\{t\})$$

is the field of rational functions in $t_1,\cdots,t_r$ over $Q:=Q(K)$. This field contains $t_1^{-1}.\cdots,t_r^{-1}$ and the polynomial algebra $K[t_1^{-1},\cdots,t_r^{-1}]$. I identify

(30) $$s_1=t_1^{-1},\cdots,s_r=t_r^{-1}, \text{ hence } \mathbf{D}=K[s]=K[t^{-1}]\subset Q(K\{t\})$$

which implies that

(31) $$Q(s):=Q(K[s])=Q(\mathbf{D})=Q(K[t])=Q(t)\subset Q(K\{t\}).$$

Next I introduce the algebra $K\{\{t\}\}$ of *formal Laurent series* as the quotient algebra of $K\{t\}$ with respect to the monomials $t^m, m\in\mathbb{N}^r$, via

(32) $$K\{\{t\}\}:=K\{t\}[t_1^{-1},\cdots,t_r^{-1}\}=:K\{t\}[t^{-1}]=K\{t\}[s]:=\{t^{-m}a=s^m a;\ m\in\mathbb{N}^r, a\in K\{t\}\}.$$

A Laurent series $t^{-m}a$, $a=\sum\{a(n)t^n;n\in\mathbb{N}^r\}$, is formally written as

(33) $$t^{-m}a=\sum\{a(n)t^{-m+n};\ n\in\mathbb{N}^r\} =$$
$$=\sum\{a(m+n)t^n;\ n\in-m+\mathbb{N}^r\subset \mathbb{Z}^r, \text{ i.e.} -m\le_{cw}n\}= =(a(m+n);\ n\in\mathbb{Z}^r,-m\le_{cw}n).$$

The latter sequence can be considered as a sequence in $K^{\mathbb{Z}^r}$ whose support is bounded from below. Thus

(34) $$K\{\{t\}\}=\{b\in K^{\mathbb{Z}^r};\ \mathrm{Supp}(b) \text{ is bounded from below}\}$$

where $\mathrm{Supp}(b):=\{n\in\mathbb{Z}^r;b(n)\neq o\}$. The boundedness condition means that there is a $m\in\mathbb{N}^r$ such that $\mathrm{Supp}(b)\subset -m+\mathbb{N}^r=\{n\in\mathbb{Z}^r;-m\le_{cw}n\}\subset\mathbb{Z}^r$.
In analogy to (33) the sequence $b=(b(n);n\in\mathbb{Z}^r)$ with $\mathrm{Supp}(b)\subset -m+\mathbb{N}^r$ is then identified with the Laurent series $t^{-m}(\sum\{b(-m+n)t^n;n\in\mathbb{N}^r\})$.
The algebra $K\{\{t\}\}$ contains both $\mathbf{D}=K[s]$ and $K[t]$ as subrings. This is one of its main advantages for system theory.

(35) **Theorem** (The extended operator calculus) Let $\lambda=s^m a=t^{-m}a\in K\{\{t\}\}$, $m\in\mathbb{N}^r$, $a\in K\{t\}$, be a Laurent series. Then the operator $\lambda(R):=L^m a(R)$ in $\mathrm{End}_K(\mathbf{A}),\mathbf{A}=\mathbf{B}\{t\}$, is well-defined, i.e. $\lambda(R)$ does not depend on the special representation $\lambda=s^m a$ of λ. The map

(36) $$K\{\{t\}\}\longrightarrow \mathrm{End}_K(\mathbf{A}),\ \lambda\longmapsto\lambda(R),$$

is K-linear, however not an algebra homomorphism, but satisfies the restricted product rule

(37) $$(p\lambda)(R)=p(L)\lambda(R) \text{ for } p\in K[s],\lambda\in K\{t\}.$$

Proof : (i) Assume that $\lambda=s^m a=s^n b$ where $m,n\in\mathbb{N}^r,a,b\in K\{t\}$, $s_i=t_i^{-1}$,

$i=1,\cdots,r$. Multiplication with t^{m+n} yields $t^{m+n}\lambda=t^n a=t^m b$. Substituting R_i for $t_i=s_i^{-1}, i=1,\cdots,r$, implies $R^n a(R)=R^m b(R)$ since (15) is a algebra homomorphism. Multiplication of this equality with L^{m+n} on the left finally gives

$$L^{m+n}R^n a(R)=L^{m+n}R^m b(R) \text{ or } L^m a(R)=L^n b(R)$$

by (28). This shows that $\lambda(R):=L^m a(R)$ is well-defined.

(ii) The elements 1, s_i resp. t_i are mapped onto id_A, L_i resp. R_i by (36). Thus the product $1=s_i t_i=t_i s_i$ is mapped onto $id_A=L_iR_i \neq R_iL_i=id_A-\varepsilon_i$. This shows that (36) is not multiplicative in general.

(iii) For a Laurent Series $\lambda=s^m a, m\in\mathbb{N}^r$, $a\in K\{t\}$, and a monomial $s^k\in K[s]\subset K\{\{t\}\}, k\in\mathbb{N}^r$, one obtains

$$(s^k\lambda)(R)=(s^k s^m a)(R)=(s^{k+m}a)(R)=L^{k+m}a(R) = L^k L^m a(R)=L^k\lambda(R)$$

which means that (37) holds for monomials $p=s^k\in K[s]$ and then for arbitrary polynomial $p\in K[s]$ by linear extension. ||

(38) **Cautionary remark**: There is a *notational* problem in the preceding theorem. A monomial $\lambda:=s^m=s^m 1\in K[s]$ is mapped onto $\lambda(R)=L^m$ by (36). On the other side $s^m(R)=R_1^{m(1)}\cdot\ldots\cdot R_r^{m(r)}=R^m$ makes good sense. The reason for this ambiguity is that λ is a rational function in t or s and that in an expression like $\lambda(R)$ it does not follow from the notation whether s_i or rather t_i should be replaced by R_i. I will use the notation

$$\lambda(t=R):=\lambda(t_1=R_1,\cdots,t_r=R_r):=\lambda(R)$$

if necessary. The idea behind (36) is that R_i is substituted for t_i and the left inverse $L_i=R_i^{(-1)}$ (see (27)) for $s_i=t_i^{-1}$. ||

The map (36) induces (15) , i.e. the diagram

$$\text{(39)}\qquad \begin{array}{ccccl} K\{\{t\}\} & \longrightarrow & End_K(\mathbf{B}\{t\}) & , & \lambda\longmapsto\lambda(R) \\ \cup & & \cup & & \\ K\{t\} & \longrightarrow & End_{K\{t\}}(\mathbf{B}\{t\}) & , & a\longmapsto a(R), \end{array}$$

commutes.

(40) **Theorem** Data as above. Assume in addition that $\mathbf{B}$ is a faithful K-module, i.e. that the map

$$K\longrightarrow End_K(\mathbf{B}),\ \alpha\longmapsto\alpha\, id_{\mathbf{B}}=(b\longmapsto\alpha b),$$

is injective or, in other terms, that $\alpha\cdot b=0$ for all $b\in\mathbf{B}$ implies $\alpha=0$. Then the map

(41) $$K\{\{t\}\}\longrightarrow \mathrm{End}_K(\mathbf{A}),\ \lambda=s^m a(t)\longmapsto \lambda(R)=L^m a(R),$$

is injective. The operator $\lambda(R)=L^m a(R)$ is $K\{t\}$-linear if and only if $\lambda=s^m a(t)\in K\{t\}$.

Proof: (i) Assume that $\lambda(R)=L^m a(R)=0$. This implies for every $b\in\mathbf{B}$

$$0=L^m a(R)(t^m b)=L^m(a(t)t^m b)=L^m(t^m ab)=L^m R^m(ab)=ab=$$
$$=\sum\{a(n)bt^n;\ n\in\mathbb{N}^r\},\ a(n)\in K,\ b\in B.$$

The equality $L^m R^m=\mathrm{id}$ follows from (28). Hence for all n and $b\in\mathbf{B}$ $a(n)b=0$ which implies $a(n)=0$ for all n since $\mathbf{B}$ is faithful. But this means $a=0$ and $\lambda=0$ and finally that (41) is injective.

(ii) If $\lambda\in K\{t\}$ then $\lambda(R)\in\mathrm{End}_{K\{t\}}(\mathbf{A})$ by (39). Assume on the other side that $\lambda(R)=L^m a(R),\ \lambda=s^m a(t)\in K\{\{t\}\},\ a\in K\{t\}$, is $K\{t\}$-linear. Then in particular $\lambda(R)(t^m b)=t^m\lambda(R)(b)$ for all $b\in\mathbf{B}\subset\mathbf{A}$. But

(42) $$\lambda(R)(t^m b)=L^m(a(t)t^m b)=L^m R^m(a(t)b)=$$
$$=a(t)b=\sum\{a(n)bt^n;\ n\in\mathbb{N}^r\}$$ and

(43) $$t^m\lambda(R)(b)=t^m L^m(a(t)b)=t^m L^m(\sum\{a(n)b\,t^n;\ n\in\mathbb{N}^r\})=$$
$$=t^m\sum a(m+n)bt^n=\sum\{a(m+n)bt^{m+n};\ n\in\mathbb{N}^r\}=\sum\{a(n)bt^n;n\in m+\mathbb{N}^r\}.$$

Comparing (42) and (43) gives $a(n)b=0$ for all $n\notin m+\mathbb{N}^r$ and $b\in\mathbf{B}$ and again, since ${}_K\mathbf{B}$ is faithful, $a(n)=0$ for $n\notin m+\mathbb{N}^r$ or $a=t^m a'$, $a'\in K\{t\}$, and finally $\lambda=s^m a=s^m t^m a'=a'\in K\{t\}$. ||

The theorems (35) and (40) can be generalized to matrices. If $H\in K\{\{t\}\}^{p,m}$ is a matrix with components H_{ij} then the K-linear maps $H_{ij}(R):\mathbf{A}\longrightarrow\mathbf{A}$ induce the K-linear operator

$$H(R):\mathbf{A}^m\longrightarrow\mathbf{A}^p,\ u=(u_1\cdots u_n)^T\longmapsto H(R)(u):=$$
$$=(\sum\{H_{ij}(R)(u_j);\ j=1,\cdots,m\};\ i=1,\cdots,p)$$

and the K-linear map

(44) $$K\{\{t\}\}^{p,m}=\mathrm{Hom}_{K\{\{t\}\}}(K\{\{t\}\}^m,K\{\{t\}\}^p)\longrightarrow\mathrm{Hom}_K(\mathbf{A}^m,\mathbf{A}^p),$$
$$H\longmapsto H(R),$$

(45) **Corollary:** Assumptions as above. (i) For matrices $P\in K[s]^{k,p}$ and

$H \in K\{\{t\}\}^{p,m}$ the product $PH \in K\{\{t\}\}^{k,m}$ and the operators $P(L)$, $H(R)=H(t=R)$ and $(PH)(R)=(PH)(t=R)$ are defined and satisfy

$$P(s=L)\,H(t=R)=(PH)(t=R)$$

(ii) If **B** is a faithful K-module the K - linear map

(44) $$K\{\{t\}\}^{p,m} \longrightarrow \mathrm{Hom}_K(\mathbf{A}^m,\mathbf{A}^p)\ ,\ H \longmapsto H(R),$$

is injective, and the operator $H(R)$ is $K\{t\}$ - linear if and only if $H \in K\{t\}^{p,m}$, and then $H(R)$ is the convolution with H.

Proof : (i) follows at once from (37).

(ii) The K-linear map $H(R):\mathbf{A}^m \longrightarrow \mathbf{A}^p$ is zero resp. $K\{t\}$-linear if and only if all its components $H_{ij}(R), i=1,\cdots,p, j=1,\cdots,m$ have this property. By (40) this is equivalent to $H_{ij}=0$ resp. $H_{ij} \in K\{t\}$ for all i and j , or to $H=0$ resp. $H \in K\{t\}^{p,m}$, $H(R)(u)=H*u$. ||

Invertible Laurent series

It is clear that in $K\{\{t\}\}=K\{t\}[t^{-1}]$ the monomials t^m, $m \in \mathbb{Z}^r$, and more generally the elements $t^m u$, $u \in U(K\{t\})$, are units. Recall that $u \in K\{t\}$ is invertible in $K\{t\}$ if and only if $u(0)$ is invertible in K. The next theorem shows that under an additional assumption there are no other units in $K\{\{t\}\}$.

(46) **Assumption:** In addition to (1) I assume from now on that the ring is factorial and regular (see [MATS] , § 19, 20 , for these notions) .

(47) **Result:** If K is noetherian, integral, factorial and regular then so are the polynomial algebra $K[t]=K[t_1,\cdots,t_r]$ and the power series algebra $K\{t\}=K\{t_1,\cdots,t_r\}$. This is an important result of commutative algebra and proven, for instance, in [MATS] , theorems 19.5, 20.8 . ||

(48) **Corollary :** If F is a field the polynomial algebra $K:=F[\sigma_1,\cdots,\sigma_\rho]$ and the power series algebra $K\{t\}=F[\sigma]\{t\}$ are factorial and regular. In particular assumption (46) is satisfied in the situation of standard example (5). ||

(49) **Theorem** (Units in $K\{\{t\}\}$) Assumption (46). Then

$$U(\ K\{\{t\}\}\ )=\{t^m u;\ m \in \mathbb{Z}^r,\ u \in U(K\{t\})\}.$$

Proof : As explained above the elements $t^m u$ are invertible in $K\{\{t\}\}$.

Assume on the other side that $\lambda = t^{-k}a$, $k \in \mathbb{N}^r$, $a \in K\{t\}$, is invertible in $K\{\{t\}\}$. Then

$$t^{-k}at^{-1}b = 1 \text{ or } ab = t^{k+1} \in K\{t\}$$

for $t^{-1}b := (t^{-k}a)^{-1}$. By (46) and (47) $K\{t\}$ is factorial and obviously $t_1, \cdots, t_r$ are irreducible elements. The unique factorization in $K\{t\}$ and the equation $ab = t^{k+1}$ imply that a has the form $a = ut^m$, $m \in \mathbb{N}^r$, $u \in U(K\{t\})$, and hence $\lambda = t^{-k}a = t^{-k+m}u$, $-k+m \in \mathbb{Z}^r$, $u \in U(K\{t\})$, as asserted. ||

The following theorem is the key result for the construction of column reduced matrices in the multidimensional situation.

(50) **Theorem and Definition:** Assumption (46). Let $p = \sum\{p(k)s^k; k \in \mathbb{N}^r\}$ be a polynomial in $K[s] \subset K\{\{t\}\}$. Then p is invertible in $K\{\{t\}\}$ if and only if p has the form

(51) $$p = p(d)s^d + \sum\{p(k)s^k;\ k <_{cw} d\},\ d \in \mathbb{N}^r,\ p(d) \in U(K).$$

I call such a polynomial cw-unital.

(52) **Remark:** Recall that $k <_{cw} d$ means $k(i) \le d(i)$ for $i = 1, \cdots, r$ and $k \neq d$ (see § 5). For a polynomial p in one indeterminate over a field the form (51) is always valid where d is the degree of p. On the contrary, the polynomial $p := s_1s_2 + s_1 - s_2 + 1$ in two indeterminates is cw-unital, but $s_1 + s_2$ is not.

Proof of (50): The form (51) is sufficient since

$$p = p(d)s^d + \sum\{p(k)s^k;\ k <_{cw} d\} = s^d[p(d) + \sum\{p(k)s^{k-d};\ k <_{cw} d\}] =$$
$$= s^d[p(d) + \sum\{p(d-k)t^k;\ 0 <_{cw} k \le_{cw} d\}]$$

The second factor $u := p(d) + \cdots$ is a polynomial, hence a power series in t with invertible constant term $p(d)$ and thus invertible in $K\{t\}$. This implies that $p = s^d u$ is invertible in $K\{\{t\}\}$.

Assume on the other side that $p(s)$ is invertible in $K\{\{t\}\}$. By (49) p has then the form $p(s) = t^m u$, $m \in \mathbb{Z}^r$, $u = \sum\{u(k)t^k; k \in \mathbb{N}^r\}$, $u(0) \in U(K)$. Choose $N \in \mathbb{N}^r$ large enough such that $p = \sum\{p(k)s^k;\ k \le_{cw} N\}$ and such that $-m \le_{cw} N$, i.e. $m + N \in \mathbb{N}^r$. The equation $p = t^m u$ implies $p(s)t^N = t^{m+N}u$ or

$$\sum\{p(k)t^{N-k}; k \le_{cw} N\} = \sum\{u(k)t^{m+N+k}; k \in \mathbb{N}^r\}$$ and

$$\sum\{p(N-k)t^k; k\le_{cw}N\}=\sum\{u(k-m-N)t^k;\ m+N\le_{cw}k\}$$

Comparing coefficients furnishes

$$(53)\ \begin{cases} p(N-k)=u(k-m-N) & \text{if} \quad m+N\le_{cw}k\le_{cw}N \\ p(N-k)=u(k-m-N)=0 & \text{for all other } k. \end{cases}$$

For $k=m+N$ the element $u(k-m-N)=u(0)\in U(K)$ is not zero. The first row of (53) yields $k\le_{cw}N$, i.e. $m+N\le_{cw}N$ or $-m\in\mathbb{N}^r$, and $p(-m)=$ $=p(N-(m+N))=u(0)\in U(K)$. The polynomial pt^N can thus be written as

$$p(s)t^N=\sum\{p(N-k)t^k;\ m+N\le_{cw}k\le_{cw}N\}\ ,\ \text{hence}$$

$$p(s)=\sum\{p(N-k)s^{N-k};\ m+N\le_{cw}k\le_{cw}N\}=$$

$$=\sum\{p(l)s^l\ ;\ -m\ge l:=N-k\ge_{cw}0\}=p(-m)s^{-m}+\sum\{p(k)s^k;\ k<_{cw}-m\}.$$

But this is exactly the form (51) with $d:=-m$. ||

(54) Definition (cw-degree) A polynomial $p(s)\in K[s]$ has a cw-degree smaller or equal to $d\in\mathbb{N}^r$, written $\text{cw-deg}(p)\le_{cw}d$, if p can be represented as $p=\sum\{p(k)s^k;\ k\le_{cw}d\}$. If additionally $p(d)$ is not zero then I call *d the* cw-degree of p and write $\text{cw-deg}(p)=d$. If the "highest coefficient" $p(d)$ is invertible in K then p is cw-unital. ||

For instance $\text{cw-deg}(s_1+s_2)\le_{cw}(1,1)$, but $\text{cw-deg}(s_1+s_2)$ does not exist, whereas $\text{cw-deg}(s_1s_2+s_1-s_2+1)=(1,1)$.

(55) **Corollary:** A square matrix $P\in K[s]^{p,p}$ is invertible in $K\{\{t\}\}^{p,p}$, i.e. contained in $Gl_p(K\{\{t\}\})$ if and only if its determinant $p:=\det(P)\in K[s]$ is invertible in $K\{\{t\}\}$, i.e. cw-unital by the preceding theorem. Let this be the case and write

$p:=s^d u(t)$, $p(s)^{-1}=t^d v(t)$, $v(t):=u(t)^{-1}=(p(d)^{-1}+\cdots)\in U(K\{t\})$. Then

$$(56)\qquad K(t):=P(s)^{-1}=P_{ad}(s)p(s)^{-1}=P_{ad}(s)t^d v(t)$$

where $P_{ad}(s)$ is the adjoint matrix of cofactors of $P(s)$. It should not create a problem that the letter K denotes both the ground ring and the inverse matrix $K(t):=P(s)^{-1}$ of $P(s)$. Define the operator $\tilde{K}$ by

$$(57)\quad \tilde{K}:=K(t=R)=(P_{ad}(s)t^d v(t))(t=R)=P_{ad}(L)R^d v(R):\mathbf{A}^p\longrightarrow\mathbf{A}^p$$

where the last equality follows from (45), (i). The equation $P(s)K(t)=I_p$ and (45) imply

$$(58)\qquad P(L)\tilde{K}=\mathrm{id}_{\mathbf{A}^p}\ .$$

In particular $P(L)$ is surjective and $\tilde{K}$ is injective, but these maps are not bijective in general since the map $K \longmapsto \tilde{K} := K(t=R)$ is not a ring homomorphism. ||

Theorem 50 induces the notion and properties of *proper* rational functions. Recall that $Q := Q(K)$ is the quotient field of K and that

$$Q(K[s]) = Q(K[t]) = Q(t) = Q(s) \subset Q(K\{t\}).$$

The polynomial $p(s)$ in (50) is invertible in $K\{\{t\}\}$, hence $p(s)$ and $p(s)^{-1}$ are contained in the subalgebra $Q(s) \cap K\{\{t\}\}$ of $Q(s)$. The algebras $Q(s) \cap K\{t\}$ and $Q(s) \cap K\{\{t\}\}$ can be completely characterized. For this purpose consider first the multiplicatively closed subset

(59) $$S := K[t] \cap U(K\{t\}) = \{b \in K[t];\ b(0) \in U(K)\}$$

of $K[t]$. This gives rise to the quotient algebra

$$K[t]_S := \{b^{-1}a;\ a, b \in K[t], b(0) \in U(K)\} \subset Q(t) \cap K\{t\}$$

(60) **Theorem and Definition** (Proper rational functions, compare [BA], proposition 1, for the case of a ground field) Assumptions (46). Then

(61) $$K[t]_S = Q(t) \cap K\{t\},\ S := K[t] \cap U(K\{t\}) = \{b \in K[t]\ ;\ b(0) \in U(K)\}$$

The elements of this algebra are called *proper* (rational functions). A rational function $h = p(s)^{-1}q(s) \in Q(s) = Q(t), p(s), q(s) \in K[s]$, $p \neq 0$, is proper if p is cw-unital and $\text{cw-deg}(q) \leq \text{cw-deg}(p)$ (see (54)). These conditions are also necessary if p and q are relatively prime in the factorial ring $K[s]$.

Proof : (i) Let $h = p(s)^{-1}q(s)$ be a rational function with cw-unital $p = p(d)s^d + \cdots$, $p(d) \in U(K)$, and $q = \sum\{q(k)s^k;\ k \leq_{cw} d\}$. Then

$$p(s) = s^d\, b(t)\ ,\ b(t) := \sum\{p(k)t^{d-k};\ k \leq_{cw} d\}$$

$$q(s) = s^d\, a(t)\ ,\ a(t) := \sum\{q(k)t^{d-k}\ ;\ k \leq_{cw} d\}$$ and finally

$$h = p(s)^{-1}q(s) = b(t)^{-1}a(t) \in K[t]_S \text{ since } b(0) = p(d) \in U(K)$$

(ii) Assume that

$$h = p(s)^{-1}q(s) = b(t)^{-1}a(t) \in K[t]_S\ ,\ b(0) \in U(K),$$

where $p(s)$ and $q(s)$ are relatively prime in $K[s]$. This is a factorial ring by Gauß' Lemma. Choose N large enough such that

$$b(t) = \sum\{b(k)t^k; k \leq_{cw} N\}, a(t) = \sum\{a(k)t^k; k \leq_{cw} N\}\ .$$

Then $s^N b(t) = \sum\{b(N-k)s^k;\ k \leq_{cw} N\}$ is cw-unital of degree N since its highest coefficient $b(N-N)=b(0)$ is invertible in K, and similarly $s^N a(t)=\sum\{s(N-k)s^k; k\le_{cw}N\}$ is of cw-degree $(s^N a(t))\le N$. Furthermore

$$h=b(t)^{-1}a(t)=(s^N b(t))^{-1}(s^N a(t))^{-1}=p(s)^{-1}q(s).$$

Since p and q are relatively prime in $K[s]$ this implies that there is a polynomial $r\in K[s]$ such that

(62) $$pr=s^N b(t)\ ,\ qr=s^N a(t).$$

But $s^N b(t)$ is cw- unital and thus invertible in $K\{\{t\}\}$ by (50). The first equation of (62) implies that p and r are both units in $K\{\{t\}\}$ and thus, again by (50), cw-unital in $K[s]$. In particular (62) implies

$$\text{cw-deg}(p)+\text{cw-deg}(r)=\text{cw-deg}(s^N b(t))=N \text{ and}$$

$$\text{cw-deg}(q)\le_{cw}N - \text{cw-deg}(r)=\text{cw-deg}(p),$$

i.e. in the reduced representation $h=p(s)^{-1}q(s)\in K[t]_S$ of h the polynomial p is cw - unital and $\text{cw-deg}(q)\le_{cw}\text{cw-deg}(p)$ holds as asserted.

(iii) Let finally h be any element in $Q(t)\cap K\{t\}$ and write $h=b^{-1}a$, $a,b\in K[t]$, $b\ne 0$. Since K is factorial so is $K[t]$ again by Gauß' Lemma. Therefore I can and do assume that a and b have no common divisor in K[t] or, equivalently, $K[t]a\cap K[t]b=K[t]ab$. (see [MATS], p. 164, Remark 2). Since the extension $K[t]\subset K\{t\}$ is flat by [MATS], Th. 8.8, the latter equality implies $K\{t\}a\cap K\{t\}b=K\{t\}ab$ or, equivalently, that a and b are relatively prime in $K\{t\}$ too. But $a=hb$ is a factorization in $K\{t\}$ by assumption so that b divides a in $K\{t\}$. With $\gcd(a,b)=1$ in $K\{t\}$ this implies $b\in U(K\{t\})$, hence

$$h=b^{-1}a\in K[t]_S\ ;\ S:=Q(t)\cap U(K\{t\}).\quad ||$$

(63) **Corollary:** A rational function $h\in Q(t)=Q(s)$ is a unit in the ring $K[t]_S$ of proper rational functions if and only if h admits representations

$$h=b(t)^{-1}a(t), a,b\in K[t], a(0),b(0)\in U(K), \quad \text{or}$$

$$h=p(s)^{-1}q(s), p,q\in K[s], \text{ p and q cw-unital of the same degree.} \quad ||$$

The preceding theorem implies an analogous result for rational Laurent series

(64) **Corollary and Definition** (weakly proper rational functions) Assumption (46), Notations as in theorem 60. Then

$$K[t]_T=Q(t)\cap K\{\{t\}\}=K[t]_S[t^{-1}] \text{ with}$$

$$T:=K[t]\cap U(K\{\{t\}\}=\{t^m b;\, m\in\mathbb{N}^r,\ b\in K[t],\ b(0)\in U(K)\}.$$

The rational functions in $K[t]_T$ are called weakly proper (rational functions). A rational function $h=p(s)^{-1}q(s)$, $p,q\in K[s], p\neq 0$, is weakly proper if $p(s)$ is cw- unital. This condition is also necessary if p and q are relatively prime in $K[s]$.

The proof follows easily from (50) and (60). ||

(65) **Example:** The following simple example shows why it is interesting to consider multidimensional systems over rings K instead of fields F. Consider the polynomial $p:=s_1-s_2\in F[s_1,s_2]$ with inverse $p^{-1}=1/(s_1-s_2)$. This p is not cw-unital as a polynomial in s_1 and s_2, thus p^{-1} is not weakly proper according to (50). But

$$p=1.s_1 - s_2\in F[s_2][s_1]=K[s_1]\ ,\ K:=F[s_2]\ .$$

As a polynomial in s_1 with coefficients in $K=F[s_2]$ p is cw-unital and indeed

$$p^{-1}=1/(s_1-s_2)=t_1/(1-s_2t_1)=\sum\{{s_2}^k {t_1}^{k+1};\ k\in\mathbb{N}\}\in F[s_2]\{t_1\}$$

is even proper. In particular the operator calculus from (35) is applicable. This example will be further executed in (118) . ||

The Cauchy problem for systems with proper transfer matrix

(66) **Assumption:** Let K be a regular, factorial, noetherian integral domain, $\mathbf{B}$ a faithful K-module and $\mathbf{A}:=\mathbf{B}\{t\}=\mathbf{B}^{\mathbb{N}^r}$. ||

These conditions are satisfied in the standard example (5) with $K=F[\sigma]$ and $\mathbf{B}=F^{\mathbb{N}^\rho}$. The theorems of the preceding section are applicable. I consider IO - systems

(67) $S:=\{(u\,y)\in\mathbf{A}^{m+p}\ ;\ P(L)(y)=Q(L)(u)\}\subset\mathbf{A}^{m+p}$, rank $(P)=p, PH=Q$, with polynomial matrices $P\in K[s]^{k,p}$ and $Q\in K[s]^{k,m}$ and the rational transfer matrix $H\in Q(s)^{p,m}$, $Q=Q(K)$. In general, H is not weakly proper, i.e. not contained in $K\{\{t\}\}^{p,m}$.

However, assume now that it is, i.e. $H\in(Q(s)\cap K\{\{t\}\})^{p,m}$. By (44) the operator $H(R)=H(t=R):\mathbf{A}^m\longrightarrow\mathbf{A}^p$ is defined, and (45) and $PH=Q$ imply $P(L)H(R)=Q(L)$. For $u\in\mathbf{A}^m$ and $y:=H(R)(u)$ this means $P(L)(y)=Q(L)(u)$ or $(u,y)\in S$. In other words , the map $\mathbf{A}^m\longrightarrow S, u\longmapsto(u,H(R)(u))$, is a section of the projection

$\mathrm{proj}: S \longrightarrow \mathbf{A}^m, (u,y) \mapsto u$. The image of this section is the graph of $H(R)$. Lemma 5.9 yields

$$S = \ker(\mathrm{proj}) \oplus \mathrm{graph}(H(R)) \qquad \text{with}$$

$$\ker(\mathrm{proj}) = (\{0\} \times \mathbf{A}^p) \cap S = \{(0,y); P(L)(y) = 0\} \cong \ker(P(L))$$

and graph $(H(R)) = \{(u, H(R)(u)); u \in \mathbf{A}^m\}$. Thus we have proved

(68) **Theorem and Definition**: Given a system (67) under the assumption (66). Assume in addition that the transfer matrix H satisfying $PH = Q$ is weakly proper, i.e. contained in $K\{\{t\}\}^{p,m}$. Then the K-linear operator $H(R) := H(t=R): \mathbf{A}^m \longrightarrow \mathbf{A}^p$ is defined and induces the decomposition

(70) $$S = S \cap (0 \times \mathbf{A}^p) \oplus \mathrm{graph}(H(R)) \text{ where}$$

$$S \cap (0 \times \mathbf{A}^p) \cong \ker(P(L)), (0,y) \leftrightarrow y \text{ , and}$$

$$\mathbf{A}^m \cong \mathrm{graph}\ (H(R)),\ u \mapsto (u, H(R)(u)).$$

The decomposition in (70) is given by $(u,y) = (0, y - H(R)(u)) + (u, H(R)(u))$. I call $H(R)$ the transfer operator and $S \cap (0 \times \mathbf{A}^p) \cong \ker(P(L))$ the state space of S given as in (67). ||

This theorem should be compared to (5.11). The decomposition (70) respectively the operator $H(R)$ correspond to (5.13) respectively $\tilde{H}$ from (5.11). There is, however, no analogue to the isomorphism $\ker(P(L) \cong F^G$ from (5.12) and the decomposition $\mathbf{A}^p = \ker(P(L)) \oplus F^{G'}$ from (5.4) or the initial condition $y|G = x$. Actually nothing is proven about $\ker(P(L))$. The corollary 45 implies

(71) **Corollary**: Data as in (68). The operator $H(R)$ is $K\{t\}$ - linear, i.e. a convolution with the matrix H or $H(R)(u) = H * u$, if and only if H is proper, i.e. contained in $K\{t\}^{p,m}$. ||

(72) **Corollary**: In the situation of (68) H is weakly proper and can thus be written as $H = s^n H'$ with some $n \in \mathbb{N}^r$ and $H' \in K\{t\}^{p,m}$. Then $H(R) = L^n H'(R)$ by (45) or $H(R)(u) = L^n(H' * u), u \in \mathbf{A}^m$. So $H(R)$ is the convolution with H' followed by the left shift L^n, $(L^n z)(k) = z(n+k)$ for all $k \in \mathbb{N}^r$. ||

In order to have an analogue of the decomposition (5.4) too I specialize the considerations to systems

(73) $\quad S:=\{(uy)\in\mathbf{A}^{m+p};\ P(L)(y) = Q(L)(u)\}$, $P\in K[s]^{p,p}$, $\det(P)\neq 0$,

where I make the additional assumption that $P\in K[s]^{p,p}$ is a square matrix with non-zero determinant $\det(P)$ and weakly proper $K(t):= =P(s)^{-1}$ in $K\{\{t\}\}^{p,p}$. Notice that K denotes both this matrix and the ground ring. By (55) this means that $\det(P)$ is cw-unital. The data and results of corollary 55 are applicable, and I use them here. The following theorem is a discrete variant of the Cauchy-Kowalewskaja theorem (see [HÖ1], §9.4).

(74) **Theorem** (Cauchy problem): Assumption (42). Consider the system (73) with $P(s)\in Gl_p(K\{\{t\}\})$, i.e. with cw-unital $\det(P(s))\neq 0$. Pose $K(t):=P(s)^{-1}$ and $H(t)=P(s)^{-1}Q(s)=K(t)Q(s)$. Data as in (55). Then

(75) $$\mathbf{A}^p=\ker(P(L))\oplus\mathrm{im}(K(R))\ni y=\pi(y)+K(R)P(L)(y)$$

with the projection $\pi:=\mathrm{id}_{\mathbf{A}^p}-K(R)P(L)$ onto $\ker(P(L))$. The "Cauchy problem"

(76) $$P(L)(y)=Q(L)(u) \text{ with the initial condition } \pi(y-x)=0$$

for arbitrarily given $u\in\mathbf{A}^m$ and $x\in\mathbf{A}^p$ has the unique solution $y=\pi(x)+K(R)Q(L)(u)$. The operators $H(R)$ and $K(R)Q(L)$ do not coincide in general, but differ by $\pi H(R)=H(R)-K(R)Q(L)$ which is a K-linear map into $\ker(P(L))$, hence

$$H(R)(u)\equiv K(R)Q(L)(u)(\ker(P(L)).$$

Proof: The equation $P(L)K(R)=\mathrm{id}_{\mathbf{A}^p}$ from (58) and lemma 5.9 induce the decomposition (75). If y is a solution of (76) then

$$y=\pi(y)+K(R)P(L)(y)=\pi(x)+K(R)Q(L)(u).$$

On the other side $y:=\pi(x)+K(R)Q(L)(u)$ satisfies

$$\pi(y)=\pi^2(x)+\pi K(R)Q(L)(u)=\pi(x)$$

since $\pi^2=\pi$ and $\mathrm{im}(K(R))=\ker(\pi)$, and

$$P(L)(y)=P(L)(\pi(x))+P(L)K(R)Q(L)(u)=Q(L)(u)$$

since $\ker(P(L))=\mathrm{im}(\pi)$ and $P(L)K(R)=\mathrm{id}$. This calculation also implies $P(L)(H(R)-K(R)Q(L))=0$ as asserted. ||

(77) **Remark**: By (68) and (74) both $H(R)$ and $K(R)Q(L)$ are operators from $\mathbf{A}^m$ to $\mathbf{A}^p$ and can be considered as transfer operators. Even in the one-dimensional case $r=1$, $s=s_1$, however, they do not coincide. Consider

for instance for $r=m=p=1$ the equations

$$s^2y=(1+s)u\ ,\ P=s^2\ ,\ Q=1+s.$$

Then $K(t)=t^2$, $H(t)=t^2(1+s)=t^2+t$, $K(R)Q(L)=R^2(Id+L)=R^2+RRL=R^2+R(Id-\varepsilon)$ with ε from (25) and $H(R)=R^2+R$, hence

$$H(R)-K(R)Q(L)=R\varepsilon\ ,\ (R\varepsilon)(u)=(0u(0)0\cdots)$$

Whereas H and thus $H(R)$ depend only on S and its IO- structure according to (2.69), at least for a field $K=F$, $\mathbf{B}=F$ and $\mathbf{A}=F\{t\}$, the operators $K(R)$ and $K(R)Q(L)$ and the decomposition (75) depend on the special choice of P. Consider, for instance, the system

$S=\{(uy)\in\mathbf{A}^{2+2}\ ;\ P(L)(y)=Q(L)(u)\}=\{(uy)\in\mathbf{A}^{2+2};P_1(L)(y)=Q_1(L)(u)\}$ with

$$P=\begin{pmatrix}s&0\\0&s^2\end{pmatrix},\ P_1=XP=\begin{pmatrix}s&s^3\\0&s^2\end{pmatrix},\ X:=\begin{pmatrix}1&s\\0&1\end{pmatrix}\in Gl_2(F[s])$$

Then

$$K(t)=\begin{pmatrix}t&0\\0&t^2\end{pmatrix},\ K_1=KX^{-1}=\begin{pmatrix}t&0\\0&t^2\end{pmatrix}\begin{pmatrix}1&-s\\0&1\end{pmatrix}=\begin{pmatrix}t&-1\\0&t^2\end{pmatrix}$$

$im(K(R))=K\{t\}\begin{pmatrix}t\\0\end{pmatrix}\oplus K\{t\}\begin{pmatrix}0\\t^2\end{pmatrix}$, $im\big(K_1(R)\big)=K\{t\}\begin{pmatrix}t\\0\end{pmatrix}\oplus K\{t\}\begin{pmatrix}-1\\t^2\end{pmatrix}$ and

$$\begin{pmatrix}0\\t^2\end{pmatrix}\notin im(K_1(R)),\ \begin{pmatrix}-1\\t^2\end{pmatrix}\notin im(K(R)).$$

The dependence on P is thus demonstrated. So $H(R)$ is the invariant transfer operator whereas $K(R)Q(L)$ is the unique one with "zero initial condition". ||

(78) **Corollary:** Data as in (74). If the matrix Q is constant, i.e. if $Q\in K^{p,m}$, then

$$H(t)=K(t)Q \text{ and } H(R)=K(R)Q=K(R)Q(L).$$

In particular, $K(R)$ can be interpreted as the transfer operator of the system $\{(uy)\in\mathbf{A}^{p+p};\ P(L)(y)=u\}$. ||

Convolutionary transfer operators and column reduced matrices

In the preceding section I have shown for IO-system (73) with weakly proper matrices $K(t)=P(s)^{-1}$ and H over a ring K that a modified Cauchy problem can be formulated and solved and that the transfer operators $K(R)$ and $H(R)$ are given by convolution plus a left shift.

In this section I return to the canonical Cauchy problem of (5.41), in particular to a ground field $F=K$ and the transfer operators

$$\tilde{K}:=(P(L)|F^{G'})^{-1} \text{ and } \tilde{H}=\tilde{K}Q(L).$$

Conversely to (68) and (74) I am going to show that if $\tilde{K}$ is given by convolution then K is proper, $\tilde{K} = K(R)$, and moreover $P(s)$ is column reduced. The notations are those from (5.41), in particular $\mathbf{A} = F\{t\} = F^{\mathbb{N}^r}$. The map (16) specializes to the identification

$$F\{t\}^{k,p} = \mathrm{Hom}_{F\{t\}}(F\{t\}^p, F\{t\}^k),\ P = P(R),$$

where a matrix $P \in F\{t\}^{k,p}$ is identified with the multiplication (here: the convolution) with P, $P(u) = P(R)(u) = P * u$. That a map is given by convolution signifies that both its domain and its codomains are finitely generated free $F\{t\}$-modules and that it is $F\{t\}$-linear. In particular, the operator

$$\tilde{K} = (P(L)|F^{G'})^{-1} : \mathrm{im}(P(L)) \xrightarrow{\cong} F^{G'} \subset F\{t\}^p$$

is given by convolution if $\mathrm{im}(P(L))$ is a free submodule of $F\{t\}^p$ and $\tilde{K}$ is $F\{t\}$-linear. Since $\tilde{K}$ is defined as an inverse this is equivalent to the requirement that $F^{G'}$ is free and that $P(L)|F^{G'}$ is $F\{t\}$-linear.

I first investigate the $F\{t\}$-module structure of $F^{G'}$. Recall that

$$G' = \dot{\bigcup}\{\{j\} \times (D(j) + \mathbb{N}^r);\ j = 1, \cdots, p\} \subset [p] \times \mathbb{N}^r$$

and $F^{G'} \subset F^{[p] \times \mathbb{N}^r} = F\{t\}^p$. Define

$$\mathbf{a}(j) := F^{D(j)+\mathbb{N}^r} = \{\sum\{a(n)t^n;\ \exists d \in D(j) \text{ with } n \in d + \mathbb{N}^r\}\} =$$
$$= \sum\{F\{t\}t^d;\ d \in D(j)\} \text{ for } j = 1, \cdots, p.$$

Hence $\mathbf{a}(j)$ is the ideal of $F\{t\}$ generated by the monomials $t^d, d \in D(j)$. Furthermore

$$F^{G'} = F^{\dot{\bigcup}\{\{j\} \times (D(j)+\mathbb{N}^r);\ j=1,\cdots,p\}} =$$
$$= \prod\{F^{D(j)+\mathbb{N}^r};\ j = 1, \cdots, p\} = \prod\{\mathbf{a}(j);\ j = 1, \cdots, p\}.$$

Hence $F^{G'}$ is the cartesian product of ideals of $F\{t\}$, therefore an ideal of $F\{t\}^p$ with the componentwise ring structure and in particular a $F\{t\}$-submodule of $F\{t\}^p$, generated over $F\{t\}$ by the elements $(0 \cdots 0, t^d, 0 \cdots 0)^T$, $d \in D(j)$, t^d as j.th component. With the abbreviation $D := \{(j,d);\ j = 1, \cdots, p, d \in D(j)\}$ there results the surjective $F\{t\}$-linear map

(80) $F\{t\}^D \longrightarrow F^{G'}$, $a = (a_{(j,d)};\ (j,d) \in D) \longmapsto \left(\sum\{a_{(j,d)}t^d;\ d \in D(j)\}, j \in [p]\right)$

The power series algebra is a local commutative ring. In particular, a $F\{t\}$-module is free if and only if it is projective, i.e. a direct summand of a free module (see [Mats], th. 2.5). Hence the $F\{t\}$-module $F^{G'}=\prod\{\mathbf{a}(j)\,;\ j=1,\cdots,p\}$ is free if and only if all ideals $\mathbf{a}(j)$ are free. But an ideal in a commutative integral domain is free if and only if it is principal. Since the ideal $\mathbf{a}(j)$ is $F\{t\}$-generated by the monomials t^{d}, $d\in D(j)$, and since $D(j)\subset \mathbb{N}^{r}$ is discrete with respect to the cw-order, the ideal is principal if and only if $D(j)$ consist of just one element.

(81) **Corollary**: Assumption and data of theorem (5.41). The $F\{t\}$-module $F^{G'}$ is free if and only if the sets $D(j)=\mathrm{Min}_{cw}\big(\{n\in\mathbb{N}^{r};(j,n)\in\deg F[s]^{1,k}P\}\big)$ consist each of exactly one element, say $D(j)=\{d(j)\}$, $j=1,\cdots,p$. In this case the surjection (80) is the $F\{t\}$-isomorphism

(82) $\quad \Delta(R):F\{t\}^{p}\longrightarrow F^{G'},\ a=(a_1\cdots a_p)^{T}\longmapsto \Delta(t)a=(a_j t^{d(j)}\ ;\ j=1,\cdots,p)^{T}$

where $\Delta(t):=\mathrm{diag}(t^{d(1)},\cdots,t^{d(p)})\in F[t]^{p,p}$.

Moreover the reduced Gröbner matrix P^{rg} is a $p\times p$-matrix satisfying $\deg((P^{rg})_{i-})=(i,d(i))$, $i=1,\cdots,p$ (Compare (5.67)) and the $F[s]$-module $U:=F[s]^{1,k}P$ is free with the rows of P^{rg} as basis . ||

The structure of $F^{G'}$ being clarified I return to the question under which conditions $\tilde{K}$ is given by convolution, i.e. when $F^{G'}$ is $F\{t\}$-free and $P(L)|F^{G'}$ is $F\{t\}$-linear according to the considerations above. Assume that $F^{G'}$ is free and that hence the isomorphism (82) is valid. The restriction $P(L)|F^{G'}$ is then $F\{t\}$-linear if and only if the composed map

(83) $\quad P(L)\Delta(R)=(P(s)\Delta(t))(R):F\{t\}^{p}\cong F^{G'}\longrightarrow F\{t\}^{k}$

is $F\{t\}$-linear or, by (45) (ii), if and only if the components $P_{ij}(s)t^{d(j)}$ of $P(s)\Delta(t)$ are proper, i.e. contained in $F\{t\}$. But this means that $\text{cw-deg}(P_{ij}(s))\le_{cw} d(j)$. We have proven the

(84) **Theorem** (Characterization of convolutionary transfer operators) Assumptions and data from (5.41). The transfer operator $\tilde{K}:=(P(L)|F^{G'})^{-1}$ is given by convolution if and only if

(i) the discrete subsets $D(j):=\mathrm{Min}_{cw}(\{n\in\mathbb{N}^{r};(j,n)\in\deg(F[s]^{1,k}P)\})$ have just one element , $D(j)=\{d(j)\}$ for $j=1,\cdots,p$, and

(ii) $P(s)\Delta(t)$, $\Delta(t)=\mathrm{diag}(t^{d(1)},\cdots,t^{d(p)})$, is proper, i.e. $\text{cw-deg}(P_{ij}(s))\le_{cw} d(j)$ for $i=1,\cdots,k$, $j=1,\cdots,p$.

The condition (ii) is equivalent to $P(s)$ admitting a representation

(85) $$P=P_{hc}\Delta(s)+\tilde{P}\ ,\ P_{hc}\in F^{k,p},\tilde{P}\in F[s]^{k,p},\ \text{with}$$
$$\tilde{P}_{ij}(s)=\sum\{P_{ij}(m)s^m;\ m<_{cw}d(j)\}\ .$$

The matrices P_{hc} and $\tilde{P}$ are uniquely associated with P. ||

The representation (85) is analogous to [KAI], p. 384, (22).

Assume now that the equivalent properties of the preceding theorem are true, in particular, that the system has a reduced Gröbner matrix P^{rg} which is $p\times p$-square and that the module $F[s]^{1,k}P = F[s]^{1,p}P^{rg}$ is free of dimension p . It is thus reasonable to assume from the start that P is a $p\times p$-square matrix itself , and I now make this additional assumption . i.e. $P\in F[s]^{p,p}$, $\det(P)\neq 0$. In this case $P^T:F[s]^p\longrightarrow F[s]^p$ is injective and thus $P(L):\mathbf{A}^p\longrightarrow\mathbf{A}^p$ surjective by (2.56). Then $P(L)$ induces the isomorphism $P(L)|\ F^{G'}:F^{G'}\overset{\cong}{\longrightarrow}\mathbf{A}^p=F\{t\}^p$. If $F^{G'}$ is free the $F\{t\}$-isomorphism $\Delta(R)$ of (82) gives rise to the composed F-linear bijection

$$P(L)\Delta(R)=(P(s)\Delta(t))(R):F\{t\}^p\xrightarrow{\Delta(R)}F^{G'}\xrightarrow{P(L)}F\{t\}^p$$

in analogy to (83). As in (84) it is $F\{t\}$-linear if and only if $U(t):=$ $=P(s)\Delta(t)$ is proper. But the bijectivity of $U(R)$ implies $U\in Gl_p(F\{t\})$ or $U(0)\in Gl_p(F)$. On the other side

$$P=P_{hc}\Delta(s)+\tilde{P},\ U(t)=P(s)\Delta(t)=P_{hc}+\tilde{P}(s)\Delta(t)$$

with $\tilde{P}(s)\Delta(t)\in(F\{t\}_+)^{p,p}$ where $F\{t\}_+=\{a\in F\{t\};\ a(0)=0\}$. Thus $U(0)=P_{hc}\in Gl_p(F)$ and

(86) Theorem and Definition (Column reduced matrices): In addition to the assumptions and data of (84) assume that $P\in F[s]^{p,p}$ is a $p\times p$-square matrix with non-zero determinant. The following assertions are equivalent:

(i) The transfer operator

$$\tilde{K}:=(P(L)|\ F^{G'})^{-1}\ :\ F\{t\}^p\overset{\cong}{\longrightarrow}F^{G'}\subset F\{t\}^p$$

is given by convolution .

(ii) The sets $D(j)$ as in (84) consist each of just one element , say $D(j)=\{d(j)\}$, $j=1,\cdots,p$, in particular $F[s]^{1,p}P$ is free , and the matrix $P\in F[s]^{p,p}$ admits the representation

$$P = P_{hc}\Delta(s)+\tilde{P}(s)\ ,\ P_{hc}\in Gl_p(F),\ \Delta(s)=\mathrm{diag}(s^{d(1)},\cdots,s^{d(p)}) \quad \text{and}$$
(87)
$$\tilde{P}_{ij}(s)=\sum\{P_{ij}(m)s^m\,;m<_{cw}d(j)\} \text{ for all } i,j=1,\cdots,p.$$

A matrix $P(s)$ of the form (87) is called *column reduced* in analogy to the one-dimensional situation. ||

It is an obvious question whether in the situation of the preceding theorem the Gröbner matrix P^{rg} is column reduced too . The answer is contained in

(88) **Corollary** : Let the equivalent assertions of the preceding theorem be satisfied . The matrix P and its reduced Gröbner matrix P^{rg} are $p\times p$-square with the same row module , hence there is an invertible matrix $U\in Gl_p(F[s])$ such that $P^{rg}=UP$. Then P^{rg} is column reduced too if and only if $U\in Gl_p(F)$, i.e. if P can be transformed into P^{rg} by elementary row operations *with coefficients in F* .

Proof : If $U\in Gl_p(F)$ then $P^{rg}=UP=UP_{hc}\Delta(s)+U\tilde{P}$ is a representation as in (87) . Assume on the other side that P^{rg} is column reduced too which , according to (86) , implies that $P^{rg}(L)|F^{G'}:F^{G'}\cong \mathbf{A}^p$ and

$$P^{rg}(L)\Delta(R)=(P^{rg}(s)\Delta(t))(R):\mathbf{A}^p \xrightarrow{\cong} \mathbf{A}^p$$

is an $F\{t\}$-isomorphism . But then $P^{rg}(L)\Delta(R)=(P^{rg}(s)\Delta(t))(R)=$ $=(U(s)P(s)\Delta(t))(R)=U(L)(P(s)\Delta(t))(R)=U(L)(P(L)\Delta(R))$ with $F\{t\}$-isomorphisms $P^{rg}(L)\Delta(R)$ and $P(L)\Delta(R)$, hence $U(L)=U(t=R)$ is a $F\{t\}$-isomorphism and finally $U(s)\in Gl_p(F\{t\})\cap F[s]^{p,p}\subset(F\{t\}\cap F[s])^{p,p}=F^{p,p}$ by (40) as asserted . ||

The Cauchy problem for systems with column reduced matrixes over rings

In the preceding section it was shown that discrete IO-systems over fields admit convolutionary transfer operators only if the defining matrix $P\in F[s]^{p,p}$ is column reduced. In this section I am going to study such systems over rings.

I assume the data of (66) with $\mathbf{A}=\mathbf{B}\{t\}=\mathbf{B}^{\mathbb{N}^r}$ and

$$\mathbf{A}^p=\mathbf{B}\{t\}^p=\mathbf{B}^{[p]\times\mathbb{N}^r}\ni y=(y_1\cdots y_p)^T = (y_j(n);j=1,\cdots,p,n\in\mathbb{N}^r)$$
$$\text{with } y_j=\left(y_j(n);\,n\in\mathbb{N}^r\right)=\sum\{y_j(n)t^n;n\in\mathbb{N}^r\}\ ,\ y_j(n)\in\mathbf{B}\ .$$

I consider systems

(89) $S=\{(uy)\in\mathbf{A}^{m+p};P(L)(y)=Q(L)(u)\}$, $P\in K[s]^{p,p}$, $\det(P)\neq0, Q=PH$, with the additional property that the matrix P is column reduced. This means that P admits a representation

(90) $$P=P_{hc}\Delta(s)+\tilde{P} \text{ with } P_{hc}\in Gl_p(K)\ ,\ \tilde{P}\in K[s]^{p,p}$$
$$\Delta(s)=\operatorname{diag}(s^{d(1)},\cdots,s^{d(p)}),\ d(j)\in\mathbb{N}^r \text{ , and}$$
$$\tilde{P}_{ij}(s)=\sum\{P_{ij}(m)s^m;m<_{cw}d(j)\}$$

The $d(j)$ are just arbitrarily given vectors in $\mathbb{N}^r$ and not derived from $\deg(F[s]^{1,p}P)$ as in (5.33) where a ground *field* F is considered. The representation (90) furnishes $P=P_{hc}[I_p+P_{hc}{}^{-1}\tilde{P}(s)\Delta(t)]\Delta(s)$.
The essential point is that

(91) $$(\tilde{P}(s)\Delta(t))_{ij}=\tilde{P}(s)_{ij}t^{d(j)}=\sum\{P_{ij}(m)s^m;m<_{cw}d(j)\}t^{d(j)}=$$
$$=\sum\{P_{ij}(d(j)-m)t^m;\ 0<_{cw}m\leq_{cw}d(j)\}\in K\{t\}_+\ ,$$

$K\{t\}_+=\{a\in K\{t\};a(0)=0\}$ which implies

(92) $U(t):=I_p+P_{hc}^{-1}\tilde{P}(s)\Delta(t)\in I_p+K\{t\}_+{}^{p,p}\subset Gl_p(K\{t\})$, $U(0)=I_p$, and $P(s)=P_{hc}U(t)\Delta(s)$, $K(t):=P(s)^{-1}=\Delta(t)U(t)^{-1}P_{hc}{}^{-1}$, $U(t)^{-1}(0)=U(0)^{-1}$.
In particular $K(t)=P(s)^{-1}$ is proper and theorem 74 is applicable. The column reducedness of P makes it possible again to replace the modified initial condition $\pi(y-x)=0$ of (74) by a natural initial condition $y|G=x$.
Consider first the special case where $P(s)$ is replaced by $\Delta(s)=I_p(I_p+0)\Delta(s)$. The relation $\Delta(s)\Delta(t)=I_p$ and (45) imply $\Delta(L)\Delta(R)=\mathrm{id}_{\mathbf{A}^p}$, and due to (5.9), the decomposition

(93) $\mathbf{A}^p=\ker(\Delta(L))\oplus\operatorname{im}(\Delta(R))\ni y=(\mathrm{id}-\Delta(R)\Delta(L)(y))+\Delta(R)\Delta(L)(y)$.

But $\Delta(L)(y_1\cdots y_p)^T=(L^{d(1)}y_1,\cdots,L^{d(p)}y_p)^T$ and $(L^{d(j)}y_j)(n)=y_j(d(j)+n)$ which furnishes

$$\ker(\Delta(L))=\{y\in\mathbf{A}^p;\ y_j(n)=0 \text{ for } n\in d(j)+\mathbb{N}^r\}.$$

With the definition

(94) $$G':=\bigcup\{\{j\}\times(d(j)+\mathbb{N}^r);j=1,\cdots,p\}=\{(j,n)\in[p]\times\mathbb{N}^r\ ;d(j)\leq_{cw}n\} \text{ and}$$
$$G:=[p]\times\mathbb{N}^r\setminus G'=\{(j,n)\in[p]\times\mathbb{N}^r;\ n\notin d(j)+\mathbb{N}^r\}$$

we obtain

(95) $$\ker(\Delta(L))=\mathbf{B}^G=$$
$$=\{y=(y_j(n);(j,n)\in[p]\times\mathbb{N}^r)\in\mathbf{A}^p=\mathbf{B}^{[p]\times\mathbb{N}^r};\ y_j(n)=0 \text{ for } (j,n)\in G'\}$$

where I have used the identification from (5.2). In generalization of (82) the matrix $\Delta(t)$ induces the $K\{t\}$-isomorphism

(96) $$\Delta(R):\mathbf{B}\{t\}^{p} \xrightarrow{\cong} \mathbf{B}^{G'} \subset \mathbf{B}\{t\}^{p}\ ,\ \Delta(R)(y)=\Delta(t)*y\ ,$$ where

$$\mathbf{B}^{G'}=\{y=(y_j(n);(j,n)\in[p]\times\mathbb{N}^r);\ y_j(n)=0 \text{ for } (j,n)\in G\}\ .$$

Using (95) and (96) we see that the decomposition (93) coincides with the natural decomposition (5.3)

(97) $$\mathbf{A}^p=\mathbf{B}^G\oplus\mathbf{B}^{G'}\ ,\ \pi_G:=\mathrm{id}-\Delta(R)\Delta(L)$$

$$\text{where } \pi_G:\mathbf{A}^p=\mathbf{B}^{[p]\times\mathbb{N}^r}\longrightarrow\mathbf{B}^G\ ,\ y\longmapsto y|G\ ,$$

denotes the natural (restriction-) projection.

We now return to general systems (89) satisfying (90). The representation $P(s)=P_{hc}U(t)\Delta(s)$ implies $P(s)\Delta(t)=P_{hc}U(t)$ and finally $P(L)\Delta(R)=$ $=P_{hc}U(R)$ according to (45). Since P_{hc} resp. $U(t)$ are invertibe in $Gl_p(K)$ resp. $Gl_p(K\{t\})$ the same is true for the $K\{t\}$-linear map $P_{hc}U(R)$, i.e. $P(L)\Delta(R)=P_{hc}U(R)\in Gl_{K\{t\}}(\mathbf{A}^p)$. There results the chain of $K\{t\}$-linear isomorphisms

(98) $$\mathbf{A}^p \underset{\cong}{\xrightarrow{\Delta(R)}} \mathbf{B}^{G'} \underset{\cong}{\xrightarrow{P(L)|\mathbf{B}^{G'}}} \mathbf{A}^p$$ with

(99) $$(P(L)|\mathbf{B}^{G'})^{-1}=\Delta(R)(P_{hc}U(R))^{-1}=\Delta(R)U(R)^{-1}P_{hc}{}^{-1}=K(R)$$

and $\mathrm{im}(K(R))=\mathrm{im}(\Delta(R))=\mathbf{B}^{G'}$. Also the equation $P(s)K(t)=I_p$ implies $P(L)K(R)=\mathrm{id}$ and by lemma 5.9 the decomposition

(100) $$\mathbf{A}^p=\ker(P(L))\oplus\mathrm{im}(K(R))=\ker(P(L))\oplus\mathbf{B}^{G'}\ .$$

But this is the condition (5.4) of theorem 5.5 in the abstract Cauchy problem, generalized without difficulty to a ground ring K and $\mathbf{A}=\mathbf{B}\{t\}$ instead of $\mathbf{A}=F\{t\}$ as in (5.5). The theorem 5.5 and the corollary 5.11 thus yield the following result which sharpens theorem 74 under the condition of column reducedness.

(101) **Theorem** (The Cauchy problem for column reduced matrices) Assumption (66), $\mathbf{A}=\mathbf{B}\{t\}=\mathbf{B}^{\mathbb{N}^r}$. Consider a IO-system

(89) $$S=\{(uy)\in\mathbf{A}^{m+p};\ P(L)(y)=Q(L)(u)\},\ PH=Q,$$

with a column reduced, $p\times p$ - square matrix $P\in K[s]^{p,p}$ with the representations

(102) $$P=P_{hc}\Delta(s)+\tilde{P}=P_{hc}U(t)\Delta(s)$$

as in (90) and (92). Notations as above. Then:

(i) The map $P(L)|\mathbf{B}^{G'}: \mathbf{B}^{G'} \cong \mathbf{A}^p$ is a $K\{t\}$-isomorphism with inverse $K(R)=(P(L)|\mathbf{B}^{G'})^{-1}$, $K(t)=P(s)^{-1}$.

(ii) $\mathbf{A}^p=\ker(P(L))\oplus\mathbf{B}^{G'}$ with the projection $\pi:=\mathrm{id}-K(R)P(L)$ onto $\ker(P(L))$.

(iii) The Cauchy problem

$$P(L)(y)=Q(L)(u)\ ,\ y|G=x\ ,$$

for arbitrary $u\in\mathbf{A}^m$ and $x\in\mathbf{B}^G$ has the unique solution $y=\pi(x)+K(R)Q(L)(u)\in\mathbf{A}^p$.

(iv) The maps

$$\ker(P(L))\cong\mathbf{B}^G\ ,\ y=\pi(x)\longleftrightarrow x=y|G\ ,$$

$$S\cong\mathbf{A}^m\times\mathbf{B}^G\ ,\ (u,y)\longleftrightarrow (u,x),\ x=y|G\ ,\ y=\pi(x)+K(R)Q(L)(u),$$

are inverse isomorphisms.

Recall that the operators $H(R)=H(t=R)$, $H=P(s)^{-1}Q(s)$, and $K(R)Q(L)$ do not coincide in general, but $H(R)-K(R)Q(L):\mathbf{A}^p \longrightarrow \ker(P(L))$. ||

(103) **Remark:** In the preceding theorem one may assume without loss of generality that P_{hc} is the unit matrix I_p. For S can also be represented as

$$S=\{(uy)\in\mathbf{A}^{m+p}\ ;\ P_1(L)(y)=Q_1(L)(u)\}$$

with $P_1:=\Delta(s)+P_{hc}{}^{-1}\tilde{P}=P_{hc}^{-1}P=U(t)\Delta(s)$ and $Q_1:=P_{hc}^{-1}Q$. Then $K_1(t):=P_1{}^{-1}=K(t)P_{hc}$, $P_1(L)=P_{hc}{}^{-1}P(L)$ and $K_1(R)=K(R)P_{hc}$. The remaining data derived from (P_1,Q_i) coincide with those derived from (P,Q). ||

(104) **Remark:** If $K=F$ is a field the vectors $d(j)\in\mathbb{N}^r$ appearing in the representation (90) of P are in general not those derived from $\deg(F[s]^{1,p}P)$ as in (5.33). Consider, for instance, the special case $r=1$, $s=s_1$, and the example

$$P(s)=\begin{pmatrix} s^2 & 1\\ s & s\end{pmatrix}=\begin{pmatrix} s^2 & 0\\ 0 & s\end{pmatrix}+\begin{pmatrix} 0 & 1\\ s & 0\end{pmatrix}=\Delta(s)+\tilde{P}(s),\ d(1)=2, d(2)=1\ .$$

By elementary row operations P can be transformed into the unique reduced Gröbner matrix or Hermite form (see (5.72))

$$P^{rg}=\begin{pmatrix} s^3-s & 0\\ s^2 & 1\end{pmatrix}=\begin{pmatrix} s & -1\\ 1 & 0\end{pmatrix}P(s) \qquad \text{with}$$

$\deg(s^3-s,0)=(1,3)\in[2]\times\mathbb{N}$ and $\deg(s^2,1)=(2,0)\in[2]\times\mathbb{N}$, hence

$$\deg(F[s]^{1,2}P)=\deg(F[s]^{1,2}P^{rg})=\{1\}\times(3+\mathbb{N})\cup\{2\}\times(0+\mathbb{N})\neq$$
$$\neq\{1\}\times(2+\mathbb{N})\cup\{2\}\times(1+\mathbb{N})$$

But $P^{rg} = \begin{pmatrix} s^3-s & 0 \\ s^2 & 1 \end{pmatrix} = \begin{pmatrix} s^3 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -s & 0 \\ s^2 & 0 \end{pmatrix}$

is also column reduced and theorem 101 is applicable to P^{rg} instead of P. The initial conditions and other data differ, however, in these two cases. ||

The Cauchy problem for simultaneously column and row reduced matrices

The statements of (101) can be sharpened if the matrix P, $P_{hc}=I_p$ without loss of generality, is also row reduced. For the beginning consider a system S as in (73) where the matrix P is row reduced, but not necessarily column reduced, i.e. where $P \in K[s]^{p,p}$ admits a representation

(105) $$P=\Delta(s)+\tilde{P}(s)=\Delta(s)(I_p+X(t))=\Delta(s)V(t)$$

where $\Delta(s)=\mathrm{diag}(s^{d(1)},\cdots,s^{d(p)})$ and $\tilde{P}_{ij}(s)=\sum\{P_{ij}(m)s^m; m<_{cw}d(i)\}$ for all i,j or, equivalently, where $X(t):=\Delta(t)\tilde{P}(s)\in(K\{t\}_+)^{p,p}$ or $V(t)=I_p+X(t)\in Gl_p(K\{t\})$ with $V(0)=I_p$.

Again $K(t):=P(s)^{-1}=V(t)^{-1}\Delta(t)$ is proper and (74) is applicable. Under these conditions the projection π of (75) can be given in more detail. Obviously $K(R)=V(R)^{-1}\Delta(R)$. Moreover $P(s)=\Delta(s)V(t)$ and (45) imply $P(L)=\Delta(L)V(R)$. We conclude

$$\pi=id-K(R)P(L)=id-V(R)^{-1}\Delta(R)\Delta(L)V(R).$$

From (97) we derive $\Delta(R)\Delta(L)=id-\pi_G$ where $\pi_G:\mathbf{A}^p=\mathbf{B}^{[p]\times\mathbb{N}^r}\longrightarrow\mathbf{B}^G$, $y\longmapsto y|G$, is the canonical projection. Substituting this into the last expression for π yields $\pi=V(R)^{-1}\pi_G V(R)$. Moreover $V(t)=I_p+X(t)$, hence $V(R)=id+X(R)$ and $\pi=V(R)^{-1}[\pi_G+\pi_G X(R)]$. I collect these results in the following

(106) **Theorem** (The Cauchy problem for row reduced matrices) Assume in the situation of theorem 74 that P is row reduced with the representation

$$P=\Delta(s)+\tilde{P}(s)=\Delta(s)(I_p+X(t))=\Delta(s)V(t)$$

with $X(t):=\Delta(t)\tilde{P}(s)\in(K\{t\}_+)^{p,p}$ and $V(t)\in Gl_p(K\{t\}$.

Let $K(t):=P(s)^{-1}=V(t)^{-1}\Delta(t)$, hence $K(R)=V(R)^{-1}\Delta(R)$. Then the projection $\pi=id-K(R)P(L)$ onto $\ker(P(L))$ has the form

(107) $$\pi=V(R)^{-1}\pi_G V(R)=V(R)^{-1}[\pi_G+\pi_G X(R)]$$

where $\pi_G:\mathbf{A}^p\longrightarrow\mathbf{B}^G, y\longmapsto y|G$, is the canonical projection. The Cauchy problem

(108) $\quad P(L)(y)=v$ with the initial condition $(V(t)*y)|G=(V(t)*x)|G$

for arbitrarily given $v\in\mathbf{A}^P$ and $x\in\mathbf{A}^P$ has the unique solution

$y=V(R)^{-1}[\pi_G(x)+\pi_G X(R)(x)+\Delta(R)(v)]=V(t)^{-1}*[x|G+(X(t)*x)|G+\Delta(t)*v]$

Remark that the case $Q=I_p$ is sufficient for this theorem.

Proof: The equation (107) was derived above. the initial condition $\pi(y-x)=V(R)^{-1}\pi_G V(R)(y-x)=0$ from (76) is equivalent to $[V(t)*(y-x)]|G=0$ since $V(R)$ is bijective. The solution $y=\pi(x)+K(R)(v)$ of (108) from (76) has the asserted form by the preceding calculations. ||

The best result is obtained when P is both column and row reduced. The main point is that the initial condition for the Cauchy problem is the natural one.

(109) **Theorem** (The Cauchy problem for both column and row reduced matrices) Assume in the situation of theorems (101) and (106) that $P(s)\in K[s]^{P,P}$ is both column and row reduced, i.e. admits representations

$$P(s)=\Delta(s)+\tilde{P}(s)=[\,I_p+\tilde{P}(s)\Delta(t)\,]\Delta(s)=U(t)\Delta(s)$$

$$P(s)=\Delta(s)[\,I_p+\Delta(t)\tilde{P}(s)\,]=\Delta(s)V(t)$$

$$\text{with } \tilde{P}(s)\Delta(t),\ X(t):=\Delta(t)\tilde{P}(s)\in(K\{t\}_+)^{P,P},$$

$$\text{hence } U(t),V(t)\in Gl_p(K\{t\}),\ U(0)=V(0)=I_p.$$

Then the statements both of theorem 101 and of theorem 106 are true. In particular the Cauchy-problem

(110) $$P(L)(y)=v,\ y|G=x,$$

for arbitrarily given $v\in\mathbf{A}^P$ and $x\in\mathbf{B}^G$ has the unique solution

(111) $$y=V(t)^{-1}*[x+\pi_G(X(t)*x)+\Delta(t)*v].\quad ||$$

The very explicit form for y comes from the row reducedness of P as in (106). The column reducedness implies $K(t)=\Delta(t)U(t)^{-1}$, hence $K(R)=\Delta(R)U(R)^{-1}$, $im(K(R))=im(\Delta(R)=\mathbf{B}^{G'}$ and the decomposition $\mathbf{A}^P=ker(P(L))\oplus\mathbf{B}^{G'}$ from which the natural initial condition is derived as in (5.5) or (101).

The operator $\pi_G X(R)$ can be further transformed. I use the equation

$\pi_G = id - \Delta(R)\Delta(L)$, hence $\pi_G X(R) = X(R) - \Delta(R)\Delta(L)X(R)$. Since P is row reduced the matrix $\tilde{P}$ in $P = \Delta(s) + \tilde{P}(s)$ can be expressed by $\tilde{P}_{ij} = \Sigma\{P_{ij}(k)s^k; k <_{cw} d(i)\}$, hence $X(t) = \Delta(t)\tilde{P}(s)$ satisfies

$$X_{ij}(t) = t^{d(i)}\tilde{P}_{ij}(s) = \Sigma\{P_{ij}(k)t^{d(i)-k};\ k <_{cw} d(i)\},$$

$$X(R)_{ij} = X_{ij}(R) = = \Sigma\{P_{ij}(k)R^{d(i)-k};\ k <_{cw} d(i)\}$$

Furthermore

$$(\Delta(R)\Delta(L)X(R))_{ij} = R^{d(i)}L^{d(i)}X(R)_{ij} =$$

$$= R^{d(i)}\Sigma\{P_{i\,j}(k)L^{d(i)}R^{d(i)-k};\ k <_{cw} d(i)\} =$$

$$R^{d(i)}\Sigma\{P_{ij}(k)L^k\ ;\ k <_{cw} d(i)\} = \Sigma\{P_{ij}(k)R^{d(i)-k}(id - \pi_k); k <_{cw} d(i)\} =$$

$$= \Sigma\{P_{ij}(k)R^{d(i)-k}\ ;\ k <_{cw} d(i)\} - \Sigma\{P_{ij}(k)R^{d(i)-k}\pi_k\ ;\ k <_{cw} d(i)\} =$$

$$= X_{ij}(R) - \Sigma\{P_{ij}(k)R^{d(i)-k}\ \pi_k\ ;\ k <_{cw} d(i)\}$$

where I have used the equations $L^{d(i)-k}R^{d(i)-k} = id_{\mathbf{A}}$ from (28) and the projection

$$\pi_k := id - R^kL^k : \mathbf{A} = \mathbf{B}^{\mathbb{N}} \longrightarrow \mathbf{B}^{G(k)},\ a \longmapsto a|G(k),$$

with $G(k) := \{n \in \mathbb{N}^r;\ n \notin k + \mathbb{N}^r\} = \mathbb{N}^r \setminus (k + \mathbb{N}^r)$ as in (97).

The above calculations lead to the

(112) **Corollary** Assumption as in theorem 109. Then the operator $\pi_G X(R) : \mathbf{A}^p \longrightarrow \mathbf{A}^p$ has the components

$$(\pi_G X(R))_{ij} = \Sigma\{P_{ij}(k)R^{d(i)-k}\pi_k\ ;\ k <_{cw} d(i)\}$$

where $\tilde{P}_{ij} = \Sigma\{P_{i\,j}(k)s^k; k <_{cw} d(i)\}$ and

$$\pi_k := id - R^kL^k : \mathbf{A} = \mathbf{B}^{\mathbb{N}^r} \longrightarrow \mathbf{B}^{G(k)}\ ,\ a \longmapsto a|G(k)\ ,\ G(k) := \mathbb{N}^r \setminus (k + \mathbb{N}^r).$$

The unique solution y of the Cauchy problem

$$P(L)(y) = v,\ y|G = x,\ v \in \mathbf{A}^p, x \in \mathbf{B}^G,$$

has the form

$$y = (I_p + X(t))^{-1} * z, z = (z_1 \cdots z_p)^T = x + \pi_G X(R)(x) + \Delta(t) * v$$

$$z_i = x_i + \Sigma\{P_{ij}(k)t^{d(i)-k} * (x_j|G(k));\ j = 1, \cdots, p\ ,\ k <_{cw} d(i)\} + t^{d(i)} * v_i$$

for $i = 1, \cdots, p$. ||

Generalized Roesser systems

The original version of this type of systems is due to Roesser [ROE] (also compare [TZ]). The notations are those from (109) and (112). The generalized Roesser system is given by

(113) $P(s)=\Delta(s)-A, A\in K^{p,p}$, $\Delta(s)=\mathrm{diag}(s_{\mu(1)},\cdots,s_{\mu(p)})$, $1\le\mu(i)\le r$.
The vectors $d(j)\in\mathbb{N}^r$ from (90) are $d(j)=e_{\mu(j)}=(0\cdots010\cdots0)$, 1 at the $\mu(j)$.th place. The matrix P is row and column reduced with $\tilde{P}(s)=-A$. The initial set G is

$$G := \{(j,n);\ n\notin d(j)+\mathbb{N}^r\}=\{(j,n)\in[p]\times\mathbb{N}^r;\ n(\mu(j))=0\}\ .$$

The Cauchy problem $P(L)(y)=v, y|G=x$, specialises to

(114) $y_i(n(1),\cdots,n(\mu(i))+1,\cdots,n(r))=\sum\{A_{ij}y_j(n);j=1,\cdots,p\}+v_i(n),\ n\in\mathbb{N}^r$

with the initial condition

$$y_i(n)=x_i(n)\ \text{ for } n(\mu(i))=0,\ x\in\mathbf{B}^G=\prod\{\mathbf{B}^{\{n\in\mathbb{N}^r\ ;n(\mu(i))=0\}};\ i=1,\cdots,p\}.$$

The matrices $X(t)$ and $V(t)$ in (109) are given by

$$P(s)=\Delta(s)(I_p-\Delta(t)A)=\Delta(s)(I_p+X(t))=\Delta(s)V(t)$$

$$X(t)=-\Delta(t)A=-\begin{pmatrix} t_{\mu(1)}A_{1-}\\ \cdots\cdots\\ t_{\mu(p)}A_{p-}\end{pmatrix},\quad V(t)=I_p-\Delta(t)A.$$

Moreover $\mathrm{im}(X(R))\subset\mathrm{im}(\Delta(R))=\mathbf{B}^{G'}$ and hence $\pi_G X(R)(x)=X(R)(x)|G=0$ in (111). The unique solution y of the Cauchy problem is

(115) $y=(I_p-\Delta(t)A)^{-1}*(x+\Delta(t)*v)$, $\Delta(t)*v=(t_{\mu(1)}*v_1,\cdots,t_{\mu(p)}*v_p)^T$

For $r=1$ the Roesser system reduces to the one-dimensional standard system

$$y(n+1)=Ay(n)+v(n),\ n=0,1,\cdots,\ y(0)=x\in\mathbf{B}$$

with the solution $y=(I_p-tA)^{-1}*(x+t*v)$.

The above theory was developed over rings and can be applied to many new situations. Consider, for instance, the polynomial algebra $F[s_0,s_1,\cdots,s_r]$ over a field F with one distinguished indeterminate s_0 and write it as

$$F[s_0,\cdots,s_r]=K[s_0],\ K:=F[s_1,\cdots,s_r]\ .$$

In this situation the standard example (5) is applicable with $\mathbf{B}=F^{\mathbb{N}^r}=F\{t_1,\cdots,t_r\}$ and

$$\mathbf{A}=\mathbf{B}^{\mathbb{N}}=(F^{\mathbb{N}^r})^{\mathbb{N}}=F^{\mathbb{N}\times\mathbb{N}^r}=F^{\mathbb{N}^{r+1}}=F\{t_1,\cdots,t_r\}\{t_0\}=F\{t_0,t_1,\cdots,t_r\}\ .$$

whose elements are written as

$$a=(a(n);\ n\in\mathbb{N}^r)=\sum\{a(n(0),\cdots,n(r))\,t_0^{n(0)}\cdot\cdots\cdot t_r^{n(r)};n\in\mathbb{N}^r\}=$$
$$=\sum\{a(n(0),-)\,t_0^{n(0)};\ n(0)\in\mathbb{N}\}\qquad\text{with}$$

$a(n(0),-)=(a(n(0),n(1),\cdots,n(r);(n(1),\cdots,n(r))\in\mathbb{N}^r)$. The operations are those by left shifts. Then

(116) $$P := s_0 I_p - A,\ A \in K^{p,p} = F[s_1, \cdots, s_r]^{p,p}$$

gives rise to generalized Roesser system as above, and the Cauchy problem

(117) $$y(n(0)+1, n(1), \cdots, n(r)) = Ay(n) + v\ ,\ v \in \mathbf{A}^p\ ,\ y(0,-) = x \in \mathbf{B}^{\mathbb{N}^r},$$

has the unique solution $y = (I_p - t_0 A)^{-1} * (x + t_o * v)$. The main point is that the arbitrary matrix A with coefficients in the polynomial algebra $F[s_1, \cdots, s_r]$ is considered as a constant matrix with coefficients in the new ground ring K. Matrices of the type (116) lead to hyperbolic systems in the continuous case (see [TR], § 15).

(118) **Example:** Consider the easiest case

$$r = 1\ ,\ F[s_o, s_1] = F[s_1][s_o],\ p = 1\ ,\ \text{and } P = s_o - s_1,\ A = s_1\ ,$$

with $y \in F^{\mathbb{N}^2} = F\{t_o, t_1\} = F\{t_1\}\{t_o\} = (F^{\mathbb{N}})\{t_o\}, v \in F\{t_o, t_1\}$. The Cauchy problem is

(119) $$y(n(0)+1, n(1)) - y(n(0), n(1)+1) = v(n(0), n(1))$$

with the initial condition $y(0,-) = x$ or $y(0, n(1)) = x(n(1))$, where $x = (x(n(1)); n(1) \in \mathbb{N}) = \sum x(n(1)) t_1^{n(1)} \in \mathbf{B} = F^{\mathbb{N}}$ and $v = \sum\{v(n(0), n(1)) t_o^{n(0)} t_1^{n(1)}; n \in \mathbb{N}^2\}$. The solution y is $y = (1 - t_o s_1)^{-1}(x + t_o v)$. But $(1 - t_o s_1)^{-1} = \sum\{s_1^{n(0)} t_o^{n(0)}; n(0) \in \mathbb{N}\}$, hence

$$(1 - t_o s_1)^{-1} x = \sum\{s_1^{n(0)} x\ t_o^{n(0)};\ n(0) \in \mathbb{N}\} =$$
$$= \sum\{x(n(0)+n(1)) t_o^{n(0)} t_1^{n(1)}; n \in \mathbb{N}^2\}$$ or
$$((1 - t_o s_1)^{-1} x)(n(0), n(1)) = x(n(0)+n(1)).$$ Also
$$(1 - t_o s_1)^{-1} t_o v = (\textstyle\sum_k s_1^k t_o^k)(\sum_l v(l, -) t_o^{l+1}) = \sum_{k,l} s_1^k v(l,-) t_o^{k+l+1} =$$
$$= \textstyle\sum_{n(o)>o}(\sum\{s_1^k v(n(0)-k-1,-); k = 0, \cdots, n(0)-1\}) t_o^{n(0)} =$$
$$= \textstyle\sum_{n(0)>o, n(1)}(\sum\{v(n(0)-k-1; n(1)+k);\ k = 0, \cdots, n(0)-1\}) t_o^{n(0)} t_1^{n(1)}\ .$$

Comparing coefficients we obtain

(120) $$y(n(0), n(1)) = x(n(0)+n(1)) + \sum\{v(n(0)-k-1, n(1)+k); k = 0, \cdots, n(0)-1\}\ .$$

For $n(0) = 0$ the second sum is empty with value 0 and yields the initial condition $y(0, n(1)) = x(n(1))$. ||

(121) **Example :** The generalized Roesser systems over rings also include certain singular systems as in [CPK]. Consider the matrix

(122) $$P(s) := \begin{pmatrix} s_0 - 1 & -s_1 \\ s_0 s_1 - s_1 & s_0 - 1 \end{pmatrix} \in F[s_0, s_1]\ ,\ F \text{ a field}\ .$$

The equation $P(L)(y)=v$ is equivalent to the system

$$(123)\left\{\begin{array}{l} y_1(n(0)+1,n(1))-y_1(n)-y_2(n(0),n(1)+1)=v_1(n) \\ y_1(n(0)+1,n(1)+1)-y_1(n(0),n(1)+1)+y_2(n(0)+1,n(1))-y_2(n)=v_2(n) \end{array}\right., n\in\mathbb{N}^2.$$

Writing $P(s)$ as

$$P(s)=\begin{pmatrix}0&0\\1&0\end{pmatrix}s_0s_1+I_2s_0+\begin{pmatrix}0&-1\\-1&0\end{pmatrix}s_1-I_2$$

shows that the system (123) is singular since $E:=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ is not invertible . On the other side the representation

$$P(s)=\begin{pmatrix}1&0\\s_1&1\end{pmatrix}\left[s_0I_2-\begin{pmatrix}1&s_1\\0&1-s_1^2\end{pmatrix}\right]$$

proves $P(s)$ to be column reduced as a matrix in $K[s_0]^{2,2}$, $K:=F[s_1]$, $P_{hc}=\begin{pmatrix}1&0\\s_1&1\end{pmatrix}\in Gl_2(F[s])$, $d(1)=d(2)=1$. The initial condition for (123) according to (95) is

$$y(0,-)=x=\sum\{x(n(1))t_1^{n(1)}\ ;n(1)\in\mathbb{N}\}\in F\{t\}^2 \quad .$$

The Cauchy problem $P(L)(y)=v$, $y(0,-)=x$, has the same solution as

$$\left[s_0I_2-\begin{pmatrix}1&s_1\\0&1-s_1^2\end{pmatrix}\right]y=\begin{pmatrix}1&0\\-s_1&1\end{pmatrix}v\ ,\ y(0,-)=x\ ,$$

and this is of the type (113) with the unique solution

$$(124)\quad \begin{aligned} y&=\left[I_2-t_0\begin{pmatrix}1&s_1\\0&1-s_1^2\end{pmatrix}\right]^{-1}*\left[x+t_0*\begin{pmatrix}1&0\\-s_1&1\end{pmatrix}v\right]=\\ &=\sum\left\{\begin{pmatrix}1&s_1\\0&1-s_1^2\end{pmatrix}^k t_0^k*\left[x+t_0*\begin{pmatrix}1&0\\-s_1&1\end{pmatrix}v\right];\ k\in\mathbb{N}\right\}\end{aligned}$$

with

$$\begin{pmatrix}1&s_1\\0&1-s_1^2\end{pmatrix}^k=\begin{pmatrix}1&\sum\{\binom{k}{i+1}s_1^{2i+1}(-1)^i\ ;i=0,\cdots,k-1\}\\0&\sum\{\binom{k}{i}s_1^{2i}(-1)^i\ ;\ i=0,\cdots,k\}\end{pmatrix}.$$

The final solution is derived as in (118) . ||

Systems according to Fornasini/Marchesini [FM] and Baratchart [BA]

These were introduced by Fornasini and Marchesini in [FM1] and [FM2] for the two-dimensional situation and then developped in a series of papers . A survey article is [FM3] . Higher dimensional systems of this type were considered by Baratchart [BA] . The main observation is that , over a *field* F, any proper rational matrix $H\in F(s_1,\cdots,s_r)^{p,m}$ admits a representation (125) below . Compare [FM3] , proposition 2 on page 50 , for the two-dimensional case and [BA] , theorem 2 , in general . I am going to study such systems over rings under the assumption (66) . Pose $d:=(1,\cdots,1)\in\mathbb{N}^r$. For $n\leq_{cw} d$

this implies $s^n = \prod\{s_i;\ n(i)=1\}$ and $s^{d-n} = \prod\{s_i;\ n(i)=0\}$. Consider cw-unital matrices $P \in K[s]^{p,p}$ of cw-degree $d=(1,\cdots,1)$ which are of the form

(125) $$P = s^d I_p + \sum\{\ P(k)s^k\ ;\ k <_{cw} d\ \}\ ,\ P(k) \in K^{p,p}\ .$$

It is customary here (see [FM3] , [BA]) to define $A(k) := -P(d-k)$ for $0 <_{cw} k \leq_{cw} d$ such that (125) gets the form

(126) $$P = s^d I_p - \sum\{\ A(d-k)s^k;\ k <_{cw} d\ \}\ ,\ A(k) \in K^{p,p}$$

This matrix is obviously row- and column reduced as in theorem (109) with

$$\Delta(s) = s^d I_p = \operatorname{diag}(s^p,\cdots,s^p)\ ,\ \tilde{P}(s) = -\sum\{A(d-k)s^k\ ;\ k <_{cw} d\ \},\quad \text{and}$$

$$X(t) = \Delta(t)\tilde{P}(s) = -\sum\{A(k)t^k;\ 0 <_{cw} k \leq_{cw} d\}\ .$$

The initial set G is

$G = \{(i,m);\ i=1,\cdots,p,\ m(\rho)=0 \text{ for at least one } \rho\} = [p]\times\{\ \mathbb{N}^r \setminus ((1,\cdots,1)+\mathbb{N}^r)\ \}$.

By (112) the operator $\pi_G X(R)$ has the form

$$(\pi_G X(R))_{i,j} = -\sum\{\ A(d-k)_{i,j} R^{d-k}\ \pi_k\ ;0 \leq_{cw} k <_{cw} d\} \qquad \text{or}$$

$$\pi_G X(R) = -\sum\{\ A(k)R^k\ \pi_{d-k}\ ;0 <_{cw} k \leq_{cw} d\}$$

where π_{d-k} is the canonical projection of $\mathbf{B}^{\mathbb{N}^r}$ onto $\mathbf{B}^{G(d-k)}$ and $G(d-k) = \mathbb{N}^r \setminus ((d-k)+\mathbb{N}^r)$. Further

$$m \in G(d-k) \Leftrightarrow m \notin (d-k)+\mathbb{N}^r \Leftrightarrow m+k \notin d+\mathbb{N}^r \Leftrightarrow$$

$$\Leftrightarrow \text{There is a } \rho \in \{1,\cdots,r\} \text{ with } m(\rho)=k(\rho)=0\ . \qquad \text{Hence}$$

$$G(d-k) = \{\ m \in \mathbb{N}^r\ ;\ \exists\ \rho \in [r] \text{ with } m(\rho)=k(\rho)=0\ \}\ .$$

In particular $G(0) = G(d-d) = \emptyset$ and $\pi_0 = 0$. Altogether we obtain the following

(127) **Theorem :** Assumption (66) . Consider the column reduced $p\times p$- matrix

$$P := s^d I_p - \sum\{A(d-k)s^k;\ 0 \leq_{cw} k <_{cw} d\}\ ,\ A(d-k) \in K^{p,p},\ d:=(1,\cdots,1)\ ,$$

as in (126) . Define the initial set $G := [p]\times(\mathbb{N}^r \setminus (d+\mathbb{N}^r))$ as above . For $0 <_{cw} k <_{cw} d$ let be $G(d-k) := \{\ m \in \mathbb{N}\ ;\ \exists\ \rho \in [r] \text{ with } m(\rho)=k(\rho)=0\}$ and π_{d-k} the canonical projection from $\mathbf{B}^{\mathbb{N}^r}$ onto $\mathbf{B}^{G(d-k)}$. Then the Cauchy problem

$$P(L)(y) = v\ ,\ y|G = x\ ,\ v \in \mathbf{B}^{[p]\times\mathbb{N}^r},\ \ x \in \mathbf{B}^G = \mathbf{B}^{[p]\times(\mathbb{N}^r\setminus(d+\mathbb{N}^r))} \qquad \text{or}$$

$y(n+d) = \sum\{\ A(d-k)y(n+k);\ 0 \leq_{cw} k <_{cw} d\} + v(n)\ ,\ y(n) = x(n)$ for $n \in \mathbb{N}^r \setminus (d+\mathbb{N}^r)$

has the unique solution

$$y = [I_p - \sum\{A(k)t^k\ ;\ 0 <_{cw} k \leq_{cw} d\}]^{-1} * z \qquad \text{with}$$

$$z = x - \sum\{\ A(k)t^k * (x|G(d-k));\ 0 <_{cw} k <_{cw} d\ \} + t_1 \cdot \cdots \cdot t_r * v$$

where $(x|G(d-k)) := (x_1|G(d-k),\cdots,x_p|G(d-k))^T$. ||

Appendix : The abstract operator calculus

(128) **Assumption** : Let $\mathbf{D} := F[s] = F[s_1, \cdots, s_r]$ be the polynomial algebra over a field F and $\mathbf{A}$ a $F[s]$-module . ||

For the special case $\mathbf{A} = F^{\mathbb{N}^r} = F\{t\}$ I introduced the right shifts R_i in (10) . They were essentially used in the considerations of this paragraph . Here I explain without proofs what can be done in general , in particular for the continuous case of partial differential equations . The book [PR] is devoted to this type of operator calculus for $r=1$. The *symbolic calculus* with distributions according to Schwartz [SCH] , Tome II , page 32 , is different . I use the notations of this paragraph with $K = \mathbf{B} = F$ and $\mathbf{A}$ instead of $\mathbf{B}\{t\}$.

(129) **Definition** (*operator calculus*) : An operator or operational calculus on the $F[s]$-module $\mathbf{A}$ is given by r F-linear operators $R_i : \mathbf{A} \longrightarrow \mathbf{A}$, $i = 1, \cdots, r$, with the following properties : (i) The operators satisfy the commutator rules

(23) $\quad R_i R_j = R_j R_i$ and $R_i L_j = L_j R_i$ for all $i \neq j$ and $L_i R_i = \mathrm{id}_{\mathbf{A}}$ for all i .

(ii a) Any equation $(\mathrm{id} - R^m L^m) a(R) = 0$, $m \in \mathbb{N}^r$, $m \neq 0$, $a \in F[t]$, implies $a = 0$. In particular , $\mathbf{A}$ is a faithful $F[t]$-module with respect to $\cdot_R$ from (20) .

(ii b) For every $b \in F[t] \cap U(F\{t\}) = \{ b \in F[t] \,;\, b(0) \neq 0 \}$ the operator $b(R) = b(R_1, \cdots, R_r)$ in $\mathrm{End}_F(\mathbf{A})$ is bijective . ||

Then the map

(130) $F[t]_S = F(t) \cap F\{t\} = F(s) \cap F\{t\} \longrightarrow \mathrm{End}_F(\mathbf{A})$, $a(t)/b(t) \longmapsto a(R) b(R)^{-1}$, $b(0) \neq 0$,

is an injective algebra homomorphism . Furthermore the map

(131) $$\begin{aligned} F[t]_T = F(t) \cap F\{\{t\}\} = F[t]_S[s] &\longrightarrow \mathrm{End}_F(\mathbf{A}) \\ \lambda = s^m a(t) b(t)^{-1} &\longmapsto \lambda(R) := L^m a(R) b(R)^{-1} \end{aligned}$$

is well-defined and satisfies the product rule from (37) . The results (40) , (45) , (68) and (74) are valid with $F[t]_S$ resp. $F[t]_T$ instead of $F\{t\}$ resp. $F\{\{t\}\}$. The relation

(132) $$\mathbb{C}[t]_S = \mathbb{C}(t) \cap \mathbb{C}\{t\} = \mathbb{C}(s) \cap \mathbb{C}\{t\} = \mathbb{C}(t) \cap \mathbb{C}\langle t \rangle$$

shows that the preceding results and hence many results of §6 hold for the large injective $\mathbb{C}[s]$-cogenerator $\mathbb{C}\langle t \rangle$ of all *convergent* power series from (1.13) . For $\mathbf{A} = C^\infty(\mathbb{R}^r)$ one obtains an operator calculus via

(133) $(R_i f)(t) := \int_0^{t_i} f(t_1, \cdots, t_{i-1}, \tau, t_{i+1}, \cdots, t_r) d\tau$, $f \in C^\infty(\mathbb{R}^r)$ (see [BE] , Ch. V)

It is easy to see that for $a=\sum a_n t^n \in \mathbb{C}\langle t\rangle$ the sequence $a(R)(f):=\sum\{a_n(R^n f)(t);\, n\in\mathbb{N}^r\}$ converges compactly and defines a C^∞-function $a(R)(f)$ and finally a $\mathbb{C}$-algebra homomorphism

(134) $$\mathbb{C}\langle t\rangle \longrightarrow \mathrm{End}_{\mathbb{C}}(C^\infty(\mathbb{R}^r)),\ a\longmapsto a(R).$$

This implies the condition (iib) of (129) . The two conditions (i) and (iia) are easy to verify and give thus rise to an operator calculus on $C^\infty(\mathbb{R}^r)$ as asserted . The same operational calculus works for $\mathcal{O}(\mathbb{C}^r)$. It is interesting to observe that no operator calculus of the type (129) exists for distributions . More precisely I can prove

(135) **Theorem** : Assume that $R:\mathcal{D}'(\mathbb{R})\longrightarrow\mathcal{D}'(\mathbb{R})$ is an arbitrary $\mathbb{C}$-linear right inverse of the differentiation operator L on distributions in one variable . Then there is a $\lambda\in\mathbb{C}$ such that the linear equation

$$(1+\lambda t)(R)(f)=f+\lambda R(f)=1\ ,\ f\in\mathcal{D}'(\mathbb{R})\ ,$$

has either no or infinitely many solutions f . This means that $(1+\lambda t)(R)$ is not bijective and hence that (129),(iib), is not satisfied for any right inverse of L .

I proved this theorem with the help of [SCH], Tome I, Ch. II, §4, the Paley-Wiener theorem and L. Baratchart . ||

7. APPLICATIONS OF THE DUALITY THEOREM TO STANDARD QUESTIONS OF SYSTEM THEORY

(1) **Assumption** : In this whole paragraph **D** is a commutative, noetherian integral domain with quotient field **K** and **A** a large injective cogenerator over **D** . Under these conditions the duality theorem 2.56 and the theorems 2.69 and 2.94 concerning IO-structures and transfer matrices are valid. This assumption is satisfied for

$$\mathbf{D}=F[s]=F[s_1,\cdots,s_r]\subset \mathbf{K} = F(s_1,\cdots,s_r)$$

and the cogenerators **A** from examples (1.7) to (1.24)

Transfer equivalence and (minimal) realizations

(2) **Reminder** : The duality theorem 2.56 induces several lattice isomorphisms. I sum up the corresponding results of § 2. I considered two non-degenerate bilinear forms

(3) $\langle -,- \rangle : \mathbf{D}^l \times \mathbf{A}^l \longrightarrow \mathbf{A}$, $\langle q,w \rangle = \sum \{ q_i \ w_i, i=1,\cdots,l \}, q \in \mathbf{D}^l, w \in \mathbf{A}^l$ and

(4) $\cdot : \mathbf{K}^l \times \mathbf{K}^l \longrightarrow \mathbf{K}$, $x \cdot y = \sum x_i y_i = x^T y$, $x, y \in \mathbf{K}^l$.

The latter is the standard non-degenerate symmetric bilinear form (" scalar product") on $\mathbf{K}^l$. These forms induce orthogonal or polar complements

$$U^{\perp} = \{ w \in \mathbf{A}^l ;\ \langle q,w \rangle = 0 \text{ for all } q \in U \},\ U \subset \mathbf{D}^l$$

(5) $$S^{\perp} = \{ q \in \mathbf{D}^l ;\ \langle q,w \rangle = 0 \text{ for all } w \in S \},\ S \subset \mathbf{A}^l \text{ , and}$$

$$V^{\pi} = \{ x \in \mathbf{K}^l ;\ x \cdot y = 0 \text{ for all } y \in V \},\ V \subset \mathbf{K}^l \ .$$

I write π (for polar) instead of $\perp$ in the last case in order to make a notational distinction to the first two cases. The lattice $\mathbf{P}(\mathbf{D}^l)$ is that of all $\mathbf{D}$-submodules of $\mathbf{D}^l$ or the projective geometry of $\mathbf{D}^l$ (Compare (2.22)). In the same fashion $\mathbf{P}(\mathbf{K}^l)$ is the projective geometry of all $\mathbf{K}$-subspaces of $\mathbf{K}^l$. By Fact$(\mathbf{D}^l)$ I denote the lattice of all epimorphisms $\mathbf{D}^l \xrightarrow{f} M$ where two epimorphisms $f_i : \mathbf{D}^l \longrightarrow M_i$, $i=1,2$, are identified if $\ker(f_1) = \ker(f_2)$ or, in other terms, if there is an isomorphism $\varphi : M_1 \longrightarrow M_2$ such that $\varphi f_1 = f_2$. Finally $\mathbf{Pf}(\mathbf{A}^l)$ is the lattice of all subsystems of $\mathbf{A}^l$. For a lattice L I denote by L^{op} the opposite lattice, ie. L with the inverse order. The duality theorem 2.56, corollary 2.48 and standard algebra induce lattice isomorphisms

(6) $$\mathbf{P}(\mathbf{D}^l)^{op} \cong \mathrm{Fact}(\mathbf{D}^l) \cong \mathbf{Pf}(\mathbf{A}^l),\ U \longleftrightarrow M \longleftrightarrow S$$

with $M = \mathbf{D}^l / U$, $U = \ker(\mathbf{D}^l \longrightarrow M)$, $S = S(M) = S(\mathbf{D}^l / U) = U^{\perp}$, $U = S^{\perp}$, $M = M(S)$. The one-one correspondence $M \longleftrightarrow S$ is nothing else than the duality $M \longmapsto S(M) = \mathrm{Hom}_{\mathbf{D}}(M,S)$ applied to factor modules of $\mathbf{D}^l$ and thus subsystems $S(M)$ of $\mathbf{A}^l$. In particular, for a matrix $R \in \mathbf{D}^{k,l}$ there results the correspondence

(7) $$U = R^T \mathbf{D}^k \text{ , } M = \mathbf{D}^l / R^T \mathbf{D}^k \text{ , } S = \{ w \in \mathbf{A}^l ;\ Rw = 0 \} \text{ , } R \in \mathbf{D}^{k,l} \ .$$

The non-degenerate form $\cdot$ induces the standard involutive lattice isomorphism (correlation)

(8) $$\mathbf{P}(\mathbf{K}^l)^{op} \cong \mathbf{P}(\mathbf{K}^l),\ V \longmapsto V^{\pi}, (V^{\pi})^{\pi} = V, \qquad \text{in particular}$$

(9) $$(R^T \mathbf{K}^k)^{\pi} = \{ \xi \in \mathbf{K}^l ;\ R\xi = 0 \} \text{ and } R^T \mathbf{K}^k = \{ \xi \in \mathbf{K}^l ;\ R\xi = 0 \}^{\pi} \text{ , } R \in \mathbf{K}^{k,l} \ .$$

In 2.91 I have also considered the exact functor "signal flow space"

(10) $$\mathrm{Syst}(\mathbf{A}) \longrightarrow \mathrm{Modf}(\mathbf{K}),\ S \longmapsto \hat{S} \ ,$$

where for $M=\mathbf{D}^l/U$, $U=R^T\mathbf{D}^k$, S and $\hat{S}$ are given by

(11) $$S=\{w\in\mathbf{A}^l; Rw=0\}=U^{\perp}=(R^T\mathbf{D}^k)^{\perp}=\mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A}) \text{ and}$$

(12) $$\hat{S}=\{\xi\in\mathbf{K}^l; R\xi=0\}=(R^T\mathbf{K}^k)^{\pi}=(\mathbf{K}U)^{\pi}\cong \mathrm{Hom}_{\mathbf{K}}(\mathbf{K}\otimes_{\mathbf{D}}M,\mathbf{K}) .$$

These data imply in particular

(13) $$m:=\mathrm{rank}(M)=\dim_{\mathbf{K}}(\mathbf{K}\otimes_{\mathbf{D}} M)=l\text{-rank}(R)=\text{input dimension of } S.$$

On the lattice level the preceding data give rise to lattice homomorphisms

(14) $$\begin{array}{ccccccc} \mathbf{Pf}(\mathbf{A}^l)^{op} & \overset{(-)^{\perp}}{\cong} & \mathbf{P}(\mathbf{D}^l) & \xrightarrow{\mathbf{K}(?)} & \mathbf{P}(\mathbf{K}^l) & \overset{(-)^{\pi}}{\cong} & \mathbf{P}(\mathbf{K}^l)^{op} \\ S & \longleftrightarrow & U & \longmapsto & \mathbf{K}U & \longleftrightarrow & V \end{array}$$

where $S=U^{\perp}, U=S^{\perp}$, $V=(\mathbf{K}U)^{\pi}=\hat{S}$, $\mathbf{K}U=V^{\pi}$. In particular, for $R\in\mathbf{D}^{k,l}$,

$$S=\{w\in\mathbf{A}^l;\ Rw=0\},\ U=R^T\mathbf{D}^k,\ \mathbf{K}\ U = R^T\mathbf{K}^k,\ V=\{\ \xi\in\mathbf{K}^l;\ R\ \xi=0\ \}.$$

The exactness of the functor $S\longmapsto\hat{S}$ or, equivalently, of $\mathbf{K}\otimes_{\mathbf{D}}(-)$, implies that $U\longmapsto\mathbf{K}U$ is really a lattice homomorphism, i.e.

(15) $$\mathbf{K}(U_1+U_2)=\mathbf{K}\ U_1+\mathbf{K}U_2 \text{ and } \mathbf{K}(U_1\cap U_2)=\mathbf{K}U_1\cap\mathbf{K}U_2,$$

or, in other terms, the maps in (14) preserve finite infima and suprema. The module M appears in (14) as

(16) $\mathbf{K}\otimes_{\mathbf{D}} M=\mathbf{K}\otimes_{\mathbf{D}}(\mathbf{D}^l/U)=\mathbf{K}^l/\mathbf{K}U$ (with the obvious identifications).

The preceding considerations contain all the necessary ingredients for transfer equivalence and minimal realizations.

(17) **Theorem and Definition** (*transfer equivalence and transfer classes*)

Assumption (1) is in force. (a) For subsystems

$$S_i=S(\mathbf{D}^l/U_i)=\{w\in\mathbf{A}^l;\ R_iw=0\}\subset\mathbf{A}^l,\ U_i=R_i^T\mathbf{D}^{k(i)},\ R_i\in\mathbf{D}^{k(i),l}, i=1,2,$$

the following statements are equivalent:

(i) $\hat{S}_1=\hat{S}_2$ (ii) $\mathbf{K}U_1=\mathbf{K}U_2$.

(iii) $\mathbf{K}\otimes_{\mathbf{D}}(\mathbf{D}^l/U_1)=\mathbf{K}\otimes_{\mathbf{D}}(\mathbf{D}^l/U_2)$ in Fact $(\mathbf{K}^l)$, i.e. as factor modules of $\mathbf{K}^l$.

(iv) There are matrices $X_1\in\mathbf{K}^{k(2),k(1)}$ and $X_2\in\mathbf{K}^{k(1),k(2)}$ such that $R_2=X_1R_1$ and $R_1=X_2R_2$.

The relation

$$S_1\sim S_2 :\Leftrightarrow S_1 \overset{\text{transfer}}{\sim} S_2 :\Leftrightarrow \hat{S}_1=\hat{S}_2$$

is an equivalence relation on $\mathbf{Pf}(\mathbf{A}^l)$, called *transfer equivalence.* The equivalence class of S is denoted by $[S]$ and called the *transfer class* of S.

(b) The map $S \longmapsto \widehat{S}$ induces the bijection

(18) $$\mathbf{Pf}(\mathbf{A}^1)/\overset{\text{transfer}}{\sim} \cong \mathbf{P}(\mathbf{K}^1), [S] \longmapsto \widehat{S}$$

Given a subspace $V \subset \mathbf{K}^1$ any system S with $\widehat{S} = V$ is called a *realization* of V.

Proof: (a) Modulo the reminder (2) the proof is trivial. The equivalence of (i) and (ii) follows from $\widehat{S} = (\mathbf{K}\,U)^{\pi}$, $\mathbf{K}\,U = (\widehat{S})^{\pi}$, that of (ii) and (iii) from $\mathbf{K} \otimes_{\mathbf{D}} (\mathbf{D}^1/U) = \mathbf{K}^1/\mathbf{K}\,U$. The condition (ii) also signifies $R_1{}^T\mathbf{K}^{k(1)} = R_2{}^T\mathbf{K}^{k(2)}$, i.e. that $R_1{}^T$ and $R_2{}^T$ have the same column space in $\mathbf{K}^1$. But this means the existence of representations

$$R_1{}^T = R_2{}^T X_2{}^T, R_2{}^T = R_1{}^T X_1{}^T \text{ or } R_1 = X_2 R_2, R_2 = X_1 R_1.$$

(b) A $\mathbf{K}$-subspace V of $\mathbf{K}^1$ is generated by $V \cap \mathbf{D}^1$. Hence, in (14), $P(\mathbf{D}^1) \longrightarrow P(\mathbf{K}^1), U \longmapsto \mathbf{K}\,U$, and $S \longmapsto \widehat{S}$ are surjective. The homomorphism theorem applied to the surjective map $S \longmapsto \widehat{S}$ and the definition of transfer equivalence induces the bijection (18) . ||

The preceding theorem yields the bijection (18) between transfer classes of subsystems of $\mathbf{A}^1$ and $\mathbf{K}$-subspaces of $\mathbf{K}^1$. The next result clarifies the structure of a single transfer class.

(19) **Theorem:** Let $V \subset \mathbf{K}^1$ be a subspace and let the system

$$S = \{w \in \mathbf{A}^1;\ Rw = 0\} = U^{\perp},\ U = R^T\mathbf{D}^k,\ R \in \mathbf{D}^{k,1},$$

be a realization of V , i.e. $V = \widehat{S} = \{\xi \in \mathbf{K}^1;\ R\xi = 0\} = (\mathbf{K}\ U)^{\pi}$. The lattice isomorphism (6) induces a lattice isomorphism

(20) $$\{U_1 \subset \mathbf{D}^1;\ \mathbf{K}\,U_1 = V^{\pi}\}^{op} \cong [S] = \{S_1 \subset A^1;\ S_1 \overset{\text{transfer}}{\sim} S\},\ U_1 \longleftrightarrow S_1$$

where $S_1 = U_1{}^{\perp}$ and $U_1 = S_1{}^{\perp}$. In particular, both sides of (20) are lattices, i.e. closed under sum(+) and intersection ($\cap$).

Proof: It has only to be remarked that $\mathbf{K}\,U_1 = \mathbf{K}\,U = V^{\pi}$ if and only if $\widehat{S}_1 = \widehat{S} = V$, i.e. $S_1 \overset{\text{transfer}}{\sim} S$. The last statement follows from

$$\mathbf{K}(U_1 \overset{+}{\cap} U_2) = \mathbf{K}\ U_1 \overset{+}{\cap} \mathbf{K}\,U_2 = V^{\pi} \overset{+}{\cap} V^{\pi} = V^{\pi} .\ ||$$

(21) **Theorem** (*Minimal realizations*) Assumptions of the preceding theorem.

(a) The submodule $\mathbf{D}^1 \cap V^{\pi}$ is the unique largest submodule of $\mathbf{D}^1$ satisfying $V^{\pi} = \mathbf{K}(\mathbf{D}^1 \cap V^{\pi})$. Hence $S_{min} := (\mathbf{D}^1 \cap V^{\pi})^{\perp}$ is the unique minimal realization of V, i.e. the least element in the transfer class [S] of S.

(b) The following statements are equivalent:

(i) S is itself minimal, hence least in its transfer class or $S_{min}=S$.

(ii) The module $M=M(S)=\mathbf{D}^l/U=\mathbf{D}^l/R^T\mathbf{D}^k$ of S is torsionfree.

(iii) The matrix R is left factor prime in the following sense: If R admits a product representation

$$R=X_1R_1 \text{ with } R_1\in\mathbf{D}^{k(1),l},\ X_1\in\mathbf{D}^{k,k(1)},\text{ and rank }(R)=\text{rank}(R_1)$$

then there is also a matrix $X\in\mathbf{D}^{k,k(1)}$ such that $R_1=XR$.

(c) The torsionfree factor module $M/T(M)$ of M is the system module of S_{min}, i.e. the injection $S_{min}\subset S$ corresponds to the canonical map $\text{can}: M\longrightarrow M/T(M)$ under duality. Here

$$T(M):=\{x\in M;\ \exists d\in\mathbf{D}, d\neq 0,\text{ with } dx=0\}$$

is the *torsion submodule* of M. The algorithmic construction of S_{min} will be treated in theorem 24.

Proof: (a) The lattice $\{U_1\subset\mathbf{D}^l; \mathbf{K}U_1=V^\pi\}$ is a sublattice of the noetherian lattice $\mathbf{P}(\mathbf{D}^l)$ and hence itself noetherian and admits a maximal element. But in a lattice (with finite infima and suprema) a minimal resp. maximal element is automatically least resp. largest. It is obvious that $\mathbf{D}^l\cap V^\pi$ is the largest $\mathbf{D}$-submodule of $\mathbf{D}^l$ satisfying $\mathbf{K}(\mathbf{D}^l\cap V^\pi)=V^\pi$. Hence $S_{min}=(\mathbf{D}^l\cap V^\pi)^\perp$ is the unique minimal realization of $V=\widehat{S}$.

(b) For any module M the torsion submodule is obtained as

$$T(M)=\ker(\text{can}:M\to\mathbf{K}\otimes_{\mathbf{D}}M=\{x/d\ ;\ x\in M, d\in\mathbf{D}, d\neq 0\},\ x\longmapsto 1\otimes x)$$

In particular, $M=\mathbf{D}^l/U$ is torsionfree if and only if

$$\text{can}: M=\mathbf{D}^l/U\longrightarrow\mathbf{K}\otimes_{\mathbf{D}}(\mathbf{D}^l/U)=\mathbf{K}^l/\mathbf{K}U,\ \bar{x}\longmapsto\bar{x},$$

is injective or, equivalently, if $U=\mathbf{D}^l\cap\mathbf{K}U=\mathbf{D}^l\cap V^\pi$ or if and only if $S=U^\perp=(\mathbf{D}^l\cap V^\pi)^\perp=S_{min}$. But this is the equivalence of (i) and (ii).

(i), (ii)$\Rightarrow$(iii) Assume that $S=(R^T\mathbf{D}^k)^\perp$ is minimal and $R=X_1R_1$ with $\text{rank}(R)=\text{rank}(R_1)$. Obviously

$$S_1:=\left(R_1{}^T\mathbf{D}^{k(1)}\right)^\perp=\{w\in\mathbf{A}^l;\ R_1w=0\}\subset S \text{ and}$$

$$\widehat{S}_1=\{\xi\in\mathbf{K}^l;\ R_1\xi=0\}\subset\widehat{S}=\{\xi\in\mathbf{K}^l;\ R\xi=0\}\quad.$$

But $\dim(\widehat{S}_1)=l-\text{rank}(R_1)=l-\text{rank}(R)=\dim(\widehat{S})$ and hence $\widehat{S}=\widehat{S}_1=V$, i.e. S and S_1 are transfer equivalent. The inclusion $S_1\subset S$ and the minimality of S by (i) imply $S_1=S$. The factorization $R_1=XR$ follows from (2.63) and this means that R is left factor prime or (iii).

The proof of "(iii) $\Rightarrow$ (i)" is almost the same as that of "(i) $\Rightarrow$ (iii)".

(c) The canonical map $M \xrightarrow{can} M/T(U)$ induces the $\mathbf{K}$-isomorphism

(22) $$\mathbf{K}\otimes_{\mathbf{D}} M \cong \mathbf{K}\otimes_{\mathbf{D}}(M/T(M))$$

and the injection $S(M/T(M)) \subset S(M) \subset \mathbf{A}^l$. The isomorphism (22) implies by (12) that $S(M/T(M))$ and $S(M)$ are transfer equivalent. By (b) the system $S(M/T(M))$ is minimal, hence least in its transfer class. Hence $S(M/T(M)) = S_{min} \in [S]$. ||

(23) **Remark:** In the literature the left factor primeness of matrices is considered only for matrices of the special IO-type $R=(-Q,P)\in\mathbf{D}^{p,m+p}$, $\det(P)\neq 0$, with $H=P^{-1}Q$ according to (2.77), and for factorizations $R=X_1R_1$, $X_1\in\mathbf{D}^{p,p}$. Then $\det(X_1)\neq 0$ and $\mathrm{rank}(R_1)=p$ follow. The left factor primeness in the sense of (21),(b)(iii) implies $R_1=XR$, hence $R=X_1XR$ and $R_1=XX_1R$. Since R and R_1 are $p\times(m+p)$-matrices of rank p the latter equations imply $X_1X=XX_1=I_p$ or $X\in Gl_p(\mathbf{D})$. Thus in this particular case left factor primeness of R coincides with left factor coprimeness of P and Q in the sense of the literature (Compare [Bos 2], Ch. 3, or [MLK] or [BFM]). ||

The minimal realization of a given subspace $V\subset\mathbf{K}^l$ of dimension $\dim(V)=m$ can be constructively found what I am going to show now. Write V as a column space $V=X^T\mathbf{D}^n$, $X\in\mathbf{K}^{n,l}$. Without loss of generality I may assume $n=m-\dim(V)$ by choosing the columns of X^T as a basis of V. Also by multiplication of the entries of X with a common denominator I may assume that X has coefficients in $\mathbf{D}$. Hence I assume $X\in\mathbf{D}^{m,l}$ with $m=\dim(V)=\mathrm{rank}(X)$. Applying the polarity π from (8) gives

$$V^\pi=\{\xi\in\mathbf{K}^l;\ X\xi=0\} \text{ and } \mathbf{D}^l\cap V^\pi=\{\xi\in\mathbf{D}^l;\ X\xi=0\}.$$

Now *solve* the *linear system* $X\xi=0$, $X\in\mathbf{D}^{m,l}$, $\xi\in\mathbf{D}^l$, i.e. find a matrix $R_{min}:=R_1\in\mathbf{D}^{k(1),l}$ such that the columns of $R_1{}^T$ generate $V^\pi\cap\mathbf{D}^l$ over $\mathbf{D}$, i.e.

$$V^\pi\cap\mathbf{D}^l=\{\xi\in\mathbf{D}^l;\ X\xi=0\}=R_1{}^T\mathbf{D}^{k(1)}.$$

Then

$$S_{min}:=S_1:=\{w\in\mathbf{A}^l;\ R_1w=0\}=(V^\pi\cap\mathbf{D}^l)^\perp$$

is the minimal realization of V according to (21). We have thus proven the

(24) **Theorem**: (*Construction of the minimal realization*) (i) Let

$$V = X^T \mathbf{D}^n \subset \mathbf{K}^l, \quad X \in \mathbf{D}^{n,l},$$

be an arbitrary subspace where $n = m := \dim(V)$ without loss of generality. Solve the linear system $X\xi = 0$ in $\mathbf{D}^l$, i.e. find a matrix R_{min} such that

$$\mathbf{D}^l \cap V^\pi = \{\xi \in \mathbf{D}^l; X\xi = 0\} = R_{min}{}^T \mathbf{D}^{k(1)}, \quad R_{min} \in \mathbf{D}^{k(1),l}.$$

Then

$$S_{min} := \{w \in \mathbf{A}^l;\ R_{min} w = 0\} = (V^\pi \cap \mathbf{D}^l)^\perp$$

is the unique minimal realization of V. This means that $\widehat{S}_{min} = V$ and that S_{min} is least among all S with $\widehat{S} = V$.

(ii) If V is already given as

$$V = \widehat{S} = \{\xi \in \mathbf{K}^l;\ R\xi = 0\}, \quad S = \{w \in \mathbf{A}^l; Rw = 0\}, \quad R \in \mathbf{D}^{k,l},$$

then S is minimal itself, i.e. $S = S_{min}$ if and only if $R^T \mathbf{D}^k = R_1{}^T \mathbf{D}^{k(1)}$. This signifies that there is a product representation $R_{min} = YR$, $Y \in \mathbf{D}^{k(1),k}$. ||

(25) **Algorithm**: To apply the preceding theorem one has to solve a linear system $X\xi = 0$ in the form

$$\{\xi \in \mathbf{D}^l;\ X\xi = 0\} = R_{min}{}^T \mathbf{D}^{k(1)}, \quad R_{min} \in \mathbf{D}^{k(1),l},$$

and to compare submodules

$$R^T \mathbf{D}^k \subset R_{min}{}^T \mathbf{D}^{k(1)}$$

and to check their equality. For the principal case $\mathbf{D} = F[s_1, \cdots, s_r]$ (and the $F[s]$-modules $\mathbf{A}$ from (1.7) to (1.24)) these problems can be algorithmically solved by using the Gröbner basis algorithms for submodules according to [PAU], Kap. 4, and other references. Actually there are several program packages doing exactly these jobs {"Reduce" et al. }. Hence the preceding theorem furnishes an efficient method for the construction of minimal realizations. ||

The realization of transfer matrices H by IO-systems can be subsumed under the preceding considerations. Indeed, according to theorem 2.94 the IO-structures of a system $S \subset \mathbf{A}^l$ can be completely read off the subspace $V := \widehat{S} \subset \mathbf{K}^l$ of dimension $m = \dim(\widehat{S})$ = input dimension of S. After possibly permuting the coordinates of $\mathbf{K}^l$ one can write V in the form

$$V = \text{graph}(H) = \left\{ \begin{pmatrix} \xi \\ H\xi \end{pmatrix};\ \xi \in \mathbf{K}^m \right\} = \begin{pmatrix} I_m \\ H \end{pmatrix} \mathbf{K}^m, \quad H \in \mathbf{K}^{p,m}, \quad p := l - m = \text{output dimension of } S.$$

The corresponding decomposition $w=\binom{u}{y}\in\mathbf{A}^l=\mathbf{A}^{m+p}$ is then a IO-structure of S and H is the transfer matrix of S with respect to this IO-structure.

(26) **Definition** (*Realization of transfer matrices*) For a given matrix $H\in\mathbf{K}^{p,m}$ any IO-system

$S=\{\binom{u}{y}\in\mathbf{A}^{m+p};\ Py=Qu\}=\{w\in\mathbf{A}^l;\ Rw=0\},\ R=(-P,Q),\ P\in\mathbf{D}^{k,p},\ \mathrm{rank}(P)=p$

$$\text{with } PH=Q \text{ or, equivalently, } S=\mathrm{graph}(H)=\binom{I_m}{H}\mathbf{K}^m$$

is called a realization of H (with respect to the given IO-structure $\binom{u}{y}\in\mathbf{A}^{m+p}$.) ||

For a given decomposition

$$\mathbf{K}^l=\mathbf{K}^{m+p}=\mathbf{K}^m\times\mathbf{K}^p \text{ or } \mathbf{A}^l=\mathbf{A}^{m+p}=\mathbf{A}^m\times\mathbf{A}^p$$

consider the projection proj: $\mathbf{K}^{m+p}\longrightarrow\mathbf{K}^m$. This induces the K-linear map $\mathrm{proj}|V: V\longrightarrow\mathbf{K}^m$ which is an isomorphism if and only if V has the form $V=\mathrm{graph}(H)$ for some unique H. In other words, the map

(27) $$\mathbf{K}^{p,m}\longrightarrow\{V\subset\mathbf{K}^l;\ \mathrm{proj}|V: V\cong\mathbf{K}^m\},\ H\longmapsto\mathrm{graph}(H),$$

is bijective. Remark that the right side of (27) is a standard affine open subset of the Grassmann variety of all m-dimensional subspaces of $\mathbf{K}^l$ and that (27) represents its standard parametrization. From theorems 17 and 21 we conclude the following

(28) **Corollary and Definition** (*transfer equivalence for IO-systems*) Two IO-systems

$$S=\left\{\binom{u}{y}\in\mathbf{A}^{m+p};\ P_iy=Q_iu\right\},\ P_iH_i=Q_i,\ i=1,2,\ \mathrm{rank}(P_i)=p$$

with the same IO-structure (u,y) and transfer matrices $H_1, H_2\in\mathbf{K}^{p,m}$ are transfer equivalent, i.e.

$$\hat{S}_1=\mathrm{graph}(H_1)=\hat{S}_2=\mathrm{graph}(H_2) \text{ if and only if } H_1=H_2.$$

The minimal realization $S_H:=S_{min}\subset\mathbf{A}^l$ of H or graph(H) has the IO-structure $(u,y)\in\mathbf{A}^{m+p}$ and can thus be represented as

$$S_H=\{w=\binom{u}{y}\in\mathbf{A}^{m+p};\ R_Hw=0\}=\{\binom{u}{y}\in\mathbf{A}^{m+p};\ P_Hy=Q_Hu\}$$

where $R_H=(-Q_H,P_H):=R_{min}\in\mathbf{D}^{k,l},\ P_H\in\mathbf{D}^{k,p},\ \mathrm{rank}(P_H)=p$. ||

The construction of S_H can also easily be derived from (24). Indeed,

$$\mathrm{graph}(H)=\binom{I_m}{H}\mathbf{K}^m=\left(I_mH^T\right)^T\mathbf{K}^m.$$

This means that one may choose $X:=(dI_m,dH^T)\in\mathbf{D}^{m,l}$ in (24) where

$d \in \mathbf{D}, d \neq 0$, is a common denominator of the entries of H. Theorem 24 and the preceding corollary furnish the following

(29) **Corollary** (*Constructive minimal realization of transfer matrices*)
Given a matrix $H \in \mathbf{K}^{p,m}$ and a (IO-)decomposition $\mathbf{K}^{l} = \mathbf{K}^{m+p} = \mathbf{K}^{m} \times \mathbf{K}^{p}$ solve the linear system

$$(dI_m, dH^T)\binom{\xi}{\eta} = 0 \ , \ \binom{\xi}{\eta} \in \mathbf{D}^{m+p} \ ,$$

where $d \neq 0$ in $\mathbf{D}$ is an arbitrary common denominator of the entries of H in $\mathbf{K} = \mathrm{Quot}(\mathbf{D})$, i.e. find a matrix $R_H = (-Q_H, P_H) \in \mathbf{D}^{k,m+p}$ such that

$$\left\{\binom{\xi}{\eta} \in \mathbf{D}^{m+p} ; \left(dI_m, dH^T\right)\binom{\xi}{\eta} = 0\right\} = R_H{}^T \mathbf{D}^k = \begin{pmatrix} -Q_H{}^T \\ P_H{}^T \end{pmatrix} \mathbf{D}^k .$$

Then the IO-system

$$S_H = \left\{\binom{u}{y} \in \mathbf{A}^{m+p} \ ; \ P_H y = Q_H u\right\}$$

is the minimal realization of H or Graph(H). Moreover

$$\mathbf{D}^{1,k} P_H = \left\{\eta \in \mathbf{D}^{1,p} \ ; \ \eta \ H \in \mathbf{D}^{1,m}\right\}$$

In words: P_H is a "universal" matrix with coefficients in $\mathbf{D}$ such that $P_H H$ has its coefficients in $\mathbf{D}$ too .

The algorithm (25) can be applied to calculate P_H and then $Q_H = P_H H$ for $\mathbf{D} = F[s_1, \cdots, s_r]$. ||

Matrix fraction descriptions (MFD) and projectivity

The assumption (1) remains in force. I refer to [BOU 2] and [MATS] for the notions and results concerning projective modules and dimension. Remember that a module is projective if and only if it is a direct summand of a free module. The free modules $\mathbf{D}^m$ or, dually by (2.56) , the systems $\mathbf{A}^m$ are of system theoretic importance in several ways, in particular for the definition (1.35) and because they admit a matrix calculus. For instance, the isomorphism

(30) $\mathbf{D}^{1,k} = \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}^k, \mathbf{D}^l) \cong \mathrm{Hom}_{\mathbf{B}}(\mathbf{A}^l, \mathbf{A}^k), R^T \longmapsto S(R^T) = (w \longrightarrow Rw)$

from (2.57) combined with the transposition $\mathbf{D}^{k,l} \cong \mathbf{D}^{l,k}, R \longrightarrow R^T$, makes it possible to identify , and I do this,

(31) $\mathbf{D}^{k,l} = \mathrm{Hom}_{\mathbf{B}}(\mathbf{A}^l, \mathbf{A}^k), R = (w \longmapsto Rw)$ (Identification).

I do not know in the moment whether projectivity for modules or, dually, injectivity for systems (Compare (2.60)) has a similar system theoretic significance. Therefore I make the following additional

(32) **Assumption and main examples:** Every finitely generated projective $\mathbf{D}$-module is free. By the theorem of Quillen and Suslin this is the case for the polynomial rings $F[s_1,\cdots,s_r], r\geq 1$, over a field F and also for their quotient rings

$$F[s]_T := \{p/q;\ p\in F[s], q\in T\} \text{ where } T\subset F[s]\setminus\{0\}$$

is a multiplicatively closed subset. ||

This assumption signifies that each injective system S is of the form $\mathbf{A}^m, m\geq 0$, up to isomorphism. For a finitely generated $\mathbf{D}$-module M and every $k\geq 0$ there is always and trivially an exact sequence

(33) $$0\longrightarrow F_k\longrightarrow F_{k-1}\longrightarrow \cdots\longrightarrow F_0\longrightarrow F_{-1} := M\longrightarrow 0$$

with free modules $F_0,\cdots,F_{k-1}$ of finite dimension. Then M has *projective dimension* $\leq k$,

(34) $$\text{proj.dim}(M)\leq k \text{ if and only if } F_k \text{ is projective}$$

too or, equivalently by (32), free for at least one or all sequences (33). One says that the *global dimension* of $\mathbf{D}$ is $\leq k$,

(35) $$\text{gl.dim}(\mathbf{D})\leq k \text{ if and only if proj.dim}(M)\leq k \text{ for all } M\in\text{Modf}(\mathbf{D}).$$

Hilbert's syzygy theorem says that for the polynomial algebra $F[s_1,\cdots,s_r]$ and its quotient rings $F[s]_T$ as in (32). the inequality

(36) $$\text{gl.dim}(F[s_1,\cdots,s_r]_T)\leq r$$

holds. The condition "$\text{gl.dim}(\mathbf{D})\leq 1$" signifies that $\mathbf{D}$ is a principal ideal domain and holds for $F[s_1]$ in particular, i.e. for 1-d system theory.

I am going to show now that modules of small projective dimension and their systems according to the main duality theorem 2.56 have a system theoretic significance. So assume first that $M=\mathbf{D}^l/U\in\text{Modf}(\mathbf{D})$ is a module with the system $S(M)=U^{\perp}=\text{Hom}_{\mathbf{D}}(\mathbf{D}^l/U,\mathbf{A})\subset\mathbf{A}^l$ and that $\text{proj.dim}(M)\leq 1$ holds. The exact sequence $0\longrightarrow U\subset \mathbf{D}^l\xrightarrow{\text{can}} M\longrightarrow 0$ and the definition (34) imply that U is free. We thus obtain an exact sequence

(37) $$0\longrightarrow \mathbf{D}^p\xrightarrow{R^T}\mathbf{D}^l\xrightarrow{\text{can}} M\longrightarrow 0$$

$$\text{with } R\in\mathbf{D}^{p,l},\ \text{rank}(R)=p=\text{o-dim}(S),\ m:=l-p=\text{i-dim}(S).$$

The duality functor S applied to (37) yields the exact sequence of systems

(38) $0 \longrightarrow S \subset \mathbf{A}^l \xrightarrow{R} \mathbf{A}^p \longrightarrow 0$ or $S=\{w\in\mathbf{A}^l;\ Rw=0\}$ with *surjective* R. Possibly after a column permutation R can be written as $R=(-Q,P)$, $P\in\mathbf{D}^{p,p}$, $\det(P)\neq 0$, and gives rise to a IO-structure of S (Compare (2.69)) with the IO-form

$$S=\{(uy)\in\mathbf{A}^{m+p};\ Py=Qu\},\ P\in\mathbf{D}^{p,p},\ \det(P)\neq 0,$$

and the transfer matrix $H=P^{-1}Q\in\mathbf{K}^{p,m}$. Conversely, such a IO-representation or exact sequence (38) imply the exactness of (37) by duality and then $\mathrm{proj.dim}(M)\le 1$. Summing this up we obtain the following

(39) **Theorem and Definition** (Systems of projective dimension ≤ 1): Assumptions (1) and (32). Let $S\subset\mathbf{A}^l$ be a subsystem with the module $M=M(S)$, $S\cong\mathrm{Hom}_{\mathbf{D}}(M,\mathbf{A})$, input dimension m and output dimension p, hence $l=m+p$. The following statements are equivalent: (i) $\mathrm{proj.dim}\,(M)\le 1$. (ii) At least one or each IO-structure of S admits a representation

(40) $$S=\{(uy)\in\mathbf{A}^{m+p};\ Py=Qu\},\ P\in\mathbf{D}^{p,p}, \det(P)\neq 0, Q\in\mathbf{D}^{p,m}.$$

The transfer matrix of S with respect to this IO-structure is then $H=P^{-1}Q$. The latter equation is usually called a *left matrix fraction description* (MFD) of H. ||

(41) **Corollary** : Since every matrix $H\in\mathbf{K}^{p,m}$ admits a left MFD $H=P^{-1}Q$, for instance $H=(dI_p)^{-1}(dH)$ where d is a common denominator of the entries of H we conclude that every "rational" matrix $H\in\mathbf{K}^{p,m}$ admits a realization whose module is of projective dimension ≤ 1. On the other side, for a $\mathbf{D}$ which is not principal, for instance for $\mathbf{D}=F[s_1,\cdots,s_r]$, $r\ge 2$, most realizations S of H respectively their modules M are not of projective dimension ≤ 1 and do not give rise to a left MFD of H. So the MFDs of transfer matrices play their exceptional role only in the standard 1-d theory and partly in the 2-d case as is shown in the next result. ||

(42) **Theorem** (Minimal realizations for $\mathrm{gl.dim}(\mathbf{D})\le 2$). Assumptions (1) and (32). The situation is that of theorem 24. If the global dimension of $\mathbf{D}$ is ≤ 2, for instance for $\mathbf{D}=F[s_1]$ and $\mathbf{D}=F[s_1,s_2]$, then the module $\mathbf{D}^l/R_{min}{}^T\mathbf{D}^{k(1)}$, $R_{min}\in\mathbf{D}^{k(1),l}$, of the minimal realization S_{min} of $V\subset\mathbf{K}^l$ is of projective dimension ≤ 1 , i.e. one may choose $k(1)=p=l-m$, $m=\dim(V)$.

Minimal 2-d systems are treated in [BFM].

Proof: By construction (see theorem 24 and its proof) we have

$$U:=\{\xi\in \mathbf{D}^l;\ X\xi=0\}=R_{min}{}^T\mathbf{D}^{k(1)},\ X\in\mathbf{D}^{m,l}.$$

Thus U appears in the exact sequence

$$0\longrightarrow U\subset \mathbf{D}^l\xrightarrow{\ X\ }\mathbf{D}^m\xrightarrow{can}\mathbf{D}^m/X\mathbf{D}^l\longrightarrow 0$$

The condition gl.dim$(\mathbf{D})\leq 2$ and (32) imply that U is free of dimension

$$\dim(U)=l-\operatorname{rank}(X)=l-m=:p\ .$$

Choosing $R_{min}\in\mathbf{D}^{p,l}$ such that the columns of $R_{min}{}^T$ are a basis of U furnishes the desired result. ||

In the remainder of this section I assume that the equivalent conditions of theorem 39 are satisfied and that S is given as in (40). The module M of S appears in the exact sequence

$$(43)\qquad \mathbf{D}^p\xrightarrow{R^T=\begin{pmatrix}-Q^T\\ P^T\end{pmatrix}}\mathbf{D}^{m+p}\xrightarrow{can}M\longrightarrow 0\ .$$

Obviously, the matrix H admits a right MFD

$$(44)\qquad H=P^{-1}Q=YU^{-1},\ U\in\mathbf{D}^{m,m},\ \det(U)\neq 0$$

too. This equation is equivalent to

$$(45)\ R\begin{pmatrix}U\\ Y\end{pmatrix}=(-Q,P)\begin{pmatrix}U\\ Y\end{pmatrix}=0 \text{ or } (U^TY^T)R^T=0 \text{ or } R^T\mathbf{D}^p=\operatorname{im}(R^T)\subset\ker(U^TY^T).$$

By the universal property of the factor module $M=\mathbf{D}^{m+p}/R^T\mathbf{D}^p$ this is again equivalent to a factorization

$$(46)\qquad (U^TY^T):\ \mathbf{D}^{m+p}\xrightarrow{can}M\xrightarrow{(U^TY^T)_{ind}}\mathbf{D}^m$$

of the map (U^TY^T). In general $(U^TY^T)_{ind}$ is not injective or, in other terms, the inclusion $R^T\mathbf{D}^p\subset\ker(U^TY^T)$ is proper and the sequence

$$(47)\qquad 0\longrightarrow\mathbf{D}^p\xrightarrow{R^T=\begin{pmatrix}-Q^T\\ P^T\end{pmatrix}}\mathbf{D}^{m+p}\xrightarrow{(U^TY^T)}\mathbf{D}^m$$

is not exact, but just a complex. However, tensoration of (47) with $\mathbf{K}$ induces the exact sequence

$$(48)\qquad 0\longrightarrow\mathbf{K}^p\xrightarrow{\begin{pmatrix}-Q^T\\ P^T\end{pmatrix}}\mathbf{K}^{m+p}\xrightarrow{(U^TY^T)}\mathbf{K}^m\longrightarrow 0$$

or, by the duality $\operatorname{Hom}_{\mathbf{K}}(-,\mathbf{K})$, the exact sequence

$$(49)\qquad 0 \longrightarrow \mathbf{K}^m \xrightarrow{\binom{U}{Y}} \mathbf{K}^{m+p} \xrightarrow{R=(-Q,P)} \mathbf{K}^p \longrightarrow 0$$

which means that

$$(50)\qquad \mathbf{K}^m = (\mathbf{A}^m)^\wedge \underset{\cong}{\xrightarrow{\binom{U}{Y}}} \widehat{S} = \{\zeta \in \mathbf{K}^{m+p}\,;\ R\zeta = 0\}$$

is an isomorphism. Indeed, $\det(U) \neq 0$ implies $U \in Gl(\mathbf{K}^m) = Gl_m(\mathbf{K})$ and that $\binom{U}{Y} : \mathbf{K}^m \longrightarrow \mathbf{K}^{m+p}$ is injective. Moreover $R\binom{\xi}{\eta} = 0$, $\binom{\xi}{\eta} \in \mathbf{K}^{m+p}$, implies $Q\xi = P\eta$ or $\eta = P^{-1}Q\xi = H\xi = YU^{-1}\xi$ and $\binom{\xi}{\eta} = \binom{U}{Y}(U^{-1}\xi)$. But this is the exactness of (49) at $\mathbf{K}^{m+p}$. The exactness of (4 7) respectively of its dual sequence

$$(51)\qquad \mathbf{A}^m \xrightarrow{\binom{U}{Y}} \mathbf{A}^{m+p} \xrightarrow{R=(-Q,P)} \mathbf{A}^p \longrightarrow 0$$

can also be fully clarified.

(52) **Theorem:** The equivalent conditions of theorem 39 be satisfied and $H = P^{-1}Q = YU^{-1}$ be a right MFD of $H, U \in \mathbf{D}^{m,m}$, $\det(U) \neq 0$. The following assertions are equivalent.

(i) The sequences

$$(51)\qquad \mathbf{A}^m \xrightarrow{\binom{U}{Y}} \mathbf{A}^{m+p} \xrightarrow{R=(-Q,P)} \mathbf{A}^p \longrightarrow 0$$

or (47) are exact, i.e. $\binom{U}{Y}$ induces a surjection

$$\binom{U}{Y}_{ind} : \mathbf{A}^m \longrightarrow S,\ v \longmapsto \binom{Uv}{Yv},$$

or a vector $\binom{u}{y} \in \mathbf{A}^{m+p}$ belongs to S if and only if it has the form $\binom{u}{y} = \binom{U}{Y} v$, i.e. $u = Uv$, $y = Yv$ for some $v \in \mathbf{A}^m$.

(ii) S is a minimal realization of H or, in other terms by (21 b), the module $M = \mathbf{D}^{m+p}/R^T\mathbf{D}^p$ is torsionfree.

Proof: Consider the commutative diagram

$$(52a)\qquad \begin{array}{ccccc} (U^TY^T) : \mathbf{D}^{m+p} & \xrightarrow{can} & M & \xrightarrow{(U^TY^T)_{ind,1}} & \mathbf{D}^m \\ \cap & & \downarrow \varphi & & \cap \\ (U^TY^T) : \mathbf{K}^{m+p} & \xrightarrow{\mathbf{K}\otimes can} & \mathbf{K}\otimes_{\mathbf{D}} M & \xrightarrow{(U^TY^T)_{ind,2}} & \mathbf{K}^m = \mathbf{K}\otimes_{\mathbf{D}}\mathbf{D}^m, \end{array}$$

$\varphi(x) = 1 \otimes x$. The map $(U^TY^T)_{ind,1}$ is injective if and only if

$R^T D^p = \ker\left(U^T Y^T\right)$, i.e. if and only if (47) and (51) are exact or, equivalently, (i) is satisfied. The map $\left(U^T Y^T\right)_{ind,2}$ is an isomorphism due to the exactness of (48) and (49). The commutativity of the right square of (52 a) implies the equivalence of : (i) is satisfied $\Leftrightarrow \left(U^T Y^T\right)_{ind,1}$ is injective $\Leftrightarrow \varphi$ is injective $\Leftrightarrow$ M is torsionfree, i.e. (ii) is satisfied. ||

Making use of the preceding theorem I can also characterize those systems whose module is projective or, by (32), free. Freeness implies torsionsfreeness of M and minimality of S, hence one may assume that the equivalent conditions of (52) are satisfied. In particular, the matrices $P \in \mathbf{D}^{p,p}$, $\det(P) \neq 0$, and $Q \in \mathbf{D}^{p,m}$ are left factor coprime by (21 b). Similarly, I can and do assume that U and Y in (52) are right factor coprime, i.e. that $U = U_1 X$, $Y = Y_1 X$, $X \in \mathbf{D}^{m,m}$, imply $X \in Gl_m(\mathbf{D})$.

(53) **Theorem and Definition** (Projectivity and zero-primeness) The equivalent statements of (52) be satisfied. Moreover assume that U and Y are right factor coprime. The following assertion are equivalent: (i) The sequence

(54) $$0 \longrightarrow \mathbf{A}^m \xrightarrow{\binom{U}{Y}} \mathbf{A}^{m+p} \xrightarrow{(-Q,P)} \mathbf{A}^p \longrightarrow 0$$

is exact. In other words, the map $\binom{U}{Y}$ induces an isomorphism

(55) $$\binom{U}{Y}_{ind} : \mathbf{A}^m \cong S,\ v \longmapsto \binom{Uu}{Yu},$$

or any $(u,y) \in S$ can be uniquely represented as $u = Uv$, $y = Yv$ with $v \in \mathbf{A}^m$.

(ii) The module M of S is free.

(iii a) The map $R = (-Q, P) : \mathbf{D}^{m+p} \longrightarrow \mathbf{D}^p$ is surjective and hence a split epimorphism, i.e. there is a section matrix $X = \binom{X_1}{X_2}$ such that $R\binom{X_1}{X_2} = -QX_1 + PX_2 = I_p$. In the language of the literature [YG]: R is zero-prime or Q and P are zero-coprime.

(iii b) The map $R^T = \binom{-Q^T}{P^T} : \mathbf{D}^p \longrightarrow \mathbf{D}^{m+p}$ is a split monomorphism, i.e. there is a retraction matrix $(Y_1 Y_2)$ of R^T such that $(Y_1 Y_2)R^T =$
$= -Y_1 Q^T + Y_2 P^T = I_p$.

(iv a) The map $\binom{U}{Y} : \mathbf{D}^m \longrightarrow \mathbf{D}^{m+p}$ is a split monomorphism.

(iv b) The map $\left(U^T Y^T\right) : \mathbf{D}^{m+p} \longrightarrow \mathbf{D}^m$ is an epimorphism or zero-prime and then automatically split.

Proof: The equivalence (iii a) and (iii b) respectively of (iv a) and (iv b) is obtained by transposition. Remark that an epimorphism onto a free or projective module is always split by definition of projectivity, that this, however, is not true in general for a monomorphism defined on a free module.

(i) $\Leftrightarrow$ (iv b) The exactness of (51) by theorem 52 implies that the sequence (54) is exact if and only if $\begin{pmatrix} U \\ Y \end{pmatrix} : \mathbf{A}^m \longrightarrow \mathbf{A}^{m+p}$ is injective which, by the duality theorem 2.56, is the same as the surjectivity of $\left(U^T Y^T\right) : \mathbf{D}^{m+p} \longrightarrow \mathbf{D}^m$.

(i), (iv) $\Rightarrow$ (iii) By the duality theorem $\mathbf{A}^m$ is injective since $\mathbf{D}^m$ is projective. The isomorphism $S \cong \mathbf{A}^m$ from (i) implies that $S = \ker(R: \mathbf{A}^{m+p} \longrightarrow \mathbf{A}^p)$ is injective and thus a system direct summand of $\mathbf{A}^{m+p}$. This and the surjectivity of $R: \mathbf{A}^{m+p} \longrightarrow \mathbf{A}^p$ imply that R admits a system section or, by (31), that $R: \mathbf{D}^{m+p} \longrightarrow \mathbf{D}^p$ has a matrix section.

(iii) $\Rightarrow$ (ii) If $R: \mathbf{D}^{m+p} \longrightarrow \mathbf{D}^p$ admits a section or $R^T: \mathbf{D}^p \longrightarrow \mathbf{D}^{m+p}$ a retraction X^T then by (5.9)

$$\mathbf{D}^{m+p} = R^T \mathbf{D}^{m+p} \oplus \ker(X^T) \quad \text{and} \quad M = \mathbf{D}^{m+p} / R^T \mathbf{D}^p \cong \ker(X^T).$$

As a direct summand of $\mathbf{D}^{m+p}$ the submodule $\ker(X^T)$ and thus M are projective and finally free by (32).

(ii) $\Rightarrow$ (iv b) Conversely, if M is free then M is of dimension m or $M \cong \mathbf{D}^m$ since $\mathbf{K} \otimes_{\mathbf{D}} M \cong \mathbf{K}^m$. Thus (46) can be completed to a commutative diagram with vertical isomorphisms

$$\begin{tikzcd}
\left(U^T Y^T\right):\ \mathbf{D}^{m+p} \arrow[r, "can"] \arrow[d, equal] & M \arrow[r, "\left(U^T Y^T\right)_{ind}"] \arrow[d, "\cong"] & \mathbf{D}^m \arrow[d, equal] \\
\mathbf{D}^{m+p} \arrow[r, "\left(U_1^T Y_1^T\right)"] & \mathbf{D}^m \arrow[r, "X^T"] & \mathbf{D}^m
\end{tikzcd}$$

which implies $U = U_1 X$, $Y = Y_1 X$, $X \in \mathbf{D}^{m,m}$. The assumed right factor coprimeness of U and Y implies $X \in Gl_m(D)$ and hence that $\left(U^T Y^T\right) : \mathbf{D}^{m+p} \longrightarrow \mathbf{D}^m$ is an epimorphism. ||

(56) **The well-known standard example** (see [WOL], Th. 5.3.1): In one-dimensional system theory over the principal ideal domain $\mathbf{D} = F[s_1] = F[s]$ a module

$M \in \mathrm{Modf}(F[s])$ is free if and only if it is torsionfree. This implies that the equivalent statements of theorem 53 are valid if and only if those of (21 b) are true. Consider, in particular, a standard system

(57) $\quad sx=Ax+Bu$ or $(sI_n-A)x=Bu \quad , \quad A\in F^{n,n},\ B\in F^{n,m},\ x\in \mathbf{A}^n,\ u\in \mathbf{A}^m$

where x is both the output and the state vector. Theorem 53 is applicable with $p=n$, $P=sI_n-A$ and $Q=B$. The surjectivity of

$$(-Q,P)=(-B,\ sI_n-A): F[s]^{m+n} \longrightarrow F[s]^n$$

of (iii a) is equivalent to

(58) $$F[s]^n = (sI_n-A)F[s]^n + BF[s]^m \qquad \text{or}$$

$F[s]^n/(sI_n-A)F[s]^n = \mathrm{can}\,(BF[s]^m)$ where $\mathrm{can}: F[s]^n \longrightarrow F[s]^n/\mathrm{im}(sI_n-A)$ is the canonical map. But

(59) $$F[s]^n/(sI_n-A)\,F[s]^n = (F^n,A),\ \overline{e_i}=e_i,$$

where e_i denotes the standard basis of $F^n \subset F[s]^n$ and where (F^n,A) is the $F[s]$-module with the standard F-structure and $s\xi=A\xi$ for $\xi\in F^n$. But then (58) is equivalent to

(60) $$F^n = \sum\{A^iBF^m;\ i=0,1,..\} = (B,AB,\cdots,A^{n-1}B)F^{mn},$$

i.e. to the controllability of (57). ||

In [ROC], Ch. IV, theorems 6 resp. 18, the systems S characterized in theorems 52 resp. 53 are called weakly resp. strongly *controllable* in the special case of $\mathbf{A}=\mathbb{R}^{\mathbb{Z}^2}$ over $\mathbb{R}[s_1,s_1^{-1},s_2,s_2^{-1}]$. In particular, these controllable systems are minimal as in the 1-d case.

Observability of Rosenbrock systems of the type (see (2.40) and (2.41))

(61) $\quad Px=Qu,\ y=Rx+Wu,\ x\in\mathbf{A}^n,\ u\in\mathbf{A}^m,\ y\in\mathbf{A}^p,\ P\in\mathbf{D}^{k,n},\ \mathrm{rank}(P)=n$

with the associated IO-systems

$$S' = \{\begin{pmatrix}u\\x\end{pmatrix}\in\mathbf{A}^{m+n};\ Px=Qu\} \text{ and}$$

$$S := \begin{pmatrix}I_m & 0\\ W & R\end{pmatrix} S = \left\{\begin{pmatrix}u\\y\end{pmatrix}\in\mathbf{A}^{m+p};\ \exists\, x \text{ such that (61) is satisfied}\right\}$$

can be more simply defined than *controllability* and also characterized.

(62) **Definition** (Observability of Rosenbrock systems) I call (61) observable if $\begin{pmatrix}I_m & 0\\ W & R\end{pmatrix}: S'\cong S$ is an isomorphism. This means that the *state vector function* x can be reconstructed from the input u and the output y. (Compare [ROC], IV.3, Th. 20). ||

Remark that observability is a property of the equations (61) and not just of S, and signifies that $\begin{pmatrix} I_m & 0 \\ W & R \end{pmatrix}$ is injective on S' since by definition $\begin{pmatrix} I_m & 0 \\ W & R \end{pmatrix} S' = S$.

(63) **Theorem:** Assumption (1). Data as in (61). The following assertions are equivalent: (i) (61) is observable. (ii) R induces an injective map

$$\ker(P : \mathbf{A}^n \longrightarrow \mathbf{A}^k) \xrightarrow{R} \mathbf{A}^p .$$

(iii) The matrix $\begin{pmatrix} P \\ R \end{pmatrix} : \mathbf{D}^n \longrightarrow \mathbf{D}^{k+p}$ is a split monomorphism or, $(P^T R^T) : \mathbf{D}^{k+p} \longrightarrow \mathbf{D}^n$ is an epimorphism or zero-prime.

This generalizes [WOL],Th. 5.3.1(b), to the multidimensional case.

Proof: $\begin{pmatrix} I_m & 0 \\ W & R \end{pmatrix} : S' \longrightarrow S \subset \mathbf{A}^{m+p}$ is injective if and only if $\ker\begin{pmatrix} I_m & 0 \\ W & R \end{pmatrix} \cap S' = \left\{\begin{pmatrix} u \\ x \end{pmatrix} \in \mathbf{A}^{m+n} ;\ u = 0,\ Px = 0,\ Rx = 0\right\}$ is zero. But this means that

$$R : \ker(P : \mathbf{A}^n \longrightarrow \mathbf{A}^k) \longrightarrow \mathbf{A}^p \quad \text{or} \quad \begin{pmatrix} P \\ R \end{pmatrix} : \mathbf{A}^n \longrightarrow \mathbf{A}^{k+p}$$

are injective. The latter property is, by the duality theorem, the same as the surjectiviy of $(P^T R^T) : \mathbf{D}^{k+p} \longrightarrow \mathbf{D}^n$ or the split monomorphy of $\begin{pmatrix} P \\ R \end{pmatrix} : \mathbf{D}^n \longrightarrow \mathbf{D}^{k+p}$. ||

Remark that I did not use condition (32) in the preceding proof .

(64) **Historical remark:** The type of module theory which I have used in the preceding considerations has played a role in other parts of system theory too. For one-dimensional state systems à la Kalman over rings modules over noetherian integral domains were used in [RWK] and [RS] . Matrix theory over multivariate polynomial rings in context with multidimensional system theory was essential in [MLK] , [YG] and [BOS 2], Ch.3, and is still important in [BFM]. That most of these papers concern the 2-d theory only is due to theorem 42 .

Rank singularities

Assumption (1) is valid. All theorems of this section are applicable to $\mathbf{D} = F[s_1, \cdots, s_r]$. The rank singularities are considered in [BOS 2], Ch. 3, and [BFM] in context with stability and stabilization of systems. I look at these from a higher point of view in the appendix on integral representations.

Let Spec($\mathbf{D}$) be the affine spectrum of all prime ideals of $\mathbf{p}$ of $\mathbf{D}$ with its

Zariski topology. See [MATS], § 4, [BOU1], II, § 4, or [MUM] for this basic algebraic geometry. For $\mathbf{p} \in \mathrm{Spec}(\mathbf{D})$ the local ring at $\mathbf{p}$ is

$$\mathbf{D_p} := \{d/s; d \in \mathbf{D}, s \notin \mathbf{p}\} \subset \mathbf{K} := \mathrm{Quot}\ (\mathbf{D}) ,$$

its maximal ideal is $\mathbf{p_p} = \mathbf{D_p}\mathbf{p}$ and its residue field $\mathbf{D}(\mathbf{p}) := \mathbf{D_p}/\mathbf{p_p}$ with the canonical map

(65) $$\mathbf{D} \subset \mathbf{D_p} \longrightarrow \mathbf{D}(\mathbf{p}) := \mathbf{D_p}/\mathbf{p_p} ,\ d \longmapsto d(\mathbf{p}) := d + \mathbf{p_p} .$$

In the same fashion a finitely generated $\mathbf{D}$-module $M := \mathbf{D}^l / R^T \mathbf{D}^k \in \mathrm{Modf}(\mathbf{D})$ gives rise to the $\mathbf{D_p}$-module $M_\mathbf{p} = \mathbf{D_p} \otimes_\mathbf{D} M$ and the finite-dimensional vector space

(66) $$M(\mathbf{p}) := M_\mathbf{p} \big/ \mathbf{p_p} M_\mathbf{p} = \mathbf{D}(\mathbf{p}) \otimes_{\mathbf{D_p}} M_\mathbf{p} = \mathbf{D}(\mathbf{p}) \otimes_\mathbf{D} M = \mathbf{D}(\mathbf{p})^l / R(\mathbf{p})^T \mathbf{D}(\mathbf{p})^k$$

with the canonical map

(67) $$\mathrm{can} : M \longrightarrow M(\mathbf{p}),\ x \longmapsto x(\mathbf{p}) := (x/1) + \mathbf{p_p} M_\mathbf{p} = 1 \otimes x .$$

The dimension of $M(\mathbf{p})$ is $\dim(M(\mathbf{p})) = l - \mathrm{rank}(R(\mathbf{p}))$ where one obtains $R(\mathbf{p})$ from R by application of (65) to the coefficients of R. For the prime ideal $\mathbf{0}$ of $\mathbf{D}$ one has in particular

(68) $$\mathbf{D}(\mathbf{0}) = \mathbf{K} ,\ M(\mathbf{0}) = \mathbf{K} \otimes_\mathbf{D} M ,\ \dim(M(\mathbf{0})) = m = l - p,\ p := \mathrm{rank}\ (R) .$$

(69) **Theorem:** Situation as described above, in particular $p := \mathrm{rank}(R)$ and $m := l - p$. Then

$$U(M) := \{\mathbf{p} \in \mathrm{Spec}(\mathbf{D})\ ;\ M_\mathbf{p} \text{ is free}\} = \{\mathbf{p} \in \mathrm{Spec}(\mathbf{D}); \dim M(\mathbf{p}) = m\} =$$
$$= \{\mathbf{p} \in \mathrm{Spec}(\mathbf{D})\ ;\ \mathrm{rank}(R(\mathbf{p})) = \mathrm{rank}(R) = p\} ,$$

and this is an open subset of $\mathrm{Spec}(\mathbf{D})$ in the Zariski topology.

Proof: By [MATS], th. 4.10 on page 28, the set $U(M) := \{\mathbf{p} \in \mathrm{Spec}(\mathbf{D})\ ;\ M_\mathbf{p} \text{ is free}\}$ is open in $\mathrm{Spec}(\mathbf{D})$.

If $M_\mathbf{p}$ is free the identity $\mathbf{K} \otimes_{\mathbf{D_p}} M_\mathbf{p} = \mathbf{K} \otimes_\mathbf{D} M$, (66) and (68) imply $\dim(M_\mathbf{p}) = \dim(\mathbf{K} \otimes_\mathbf{D} M) = m$ and $\dim(M(\mathbf{p})) = \dim(\mathbf{D}(\mathbf{p}) \otimes_{\mathbf{D_p}} M_\mathbf{p}) = m$.

Assume conversely that $1 \otimes x_1, \cdots, 1 \otimes x_m$, $x_i \in M$, is a basis of $M(\mathbf{p}) = \mathbf{D}(\mathbf{p}) \otimes_\mathbf{D} M$. Then, by Nakayama's lemma, $1 \otimes x_1, \cdots, 1 \otimes x_m$ in $M_\mathbf{p} = \mathbf{D_p} \otimes_\mathbf{D} M$ generate $M_\mathbf{p}$ and finally the m vectors $1 \otimes x_1, \cdots, 1 \otimes x_m$ in $\mathbf{K} \otimes_\mathbf{D} M = \mathbf{K} \otimes_{\mathbf{D_p}} M_\mathbf{p}$ generate the m-dimensional $\mathbf{K}$-space $\mathbf{K} \otimes_\mathbf{D} M$, hence are a $\mathbf{K}$-basis and especially $\mathbf{K}$-linearly independent. But then the $1 \otimes x_i \in M_\mathbf{p}$ are $\mathbf{D_p}$-linearly independent too and finally a basis since they generate $M_\mathbf{p}$. ||

As in the preceding theorem one shows by Nakayama's lemma that

(70) $\dim M(\mathbf{p}) \geq \dim(\mathbf{K}\otimes_{\mathbf{D}} M) = m = l - \mathrm{rank}(R)$ and $\mathrm{rank}(R(\mathbf{p})) \leq \mathrm{rank}(R) = p$ for all $\mathbf{p} \in \mathrm{Spec}(\mathbf{D})$.

(71) **Corollary and Definition** (*rank singularities* , see [BFM]): Situation of the preceding theorem. (i) The complement of $U(M)$ is

$$RS(M) := \mathrm{Spec}(\mathbf{D}) \setminus U(M) = \{ \mathbf{p} \in \mathrm{Spec}(\mathbf{D});\ M_{\mathbf{p}} \text{ is not free}\} =$$
$$= \{\mathbf{p} \in \mathrm{Spec}(\mathbf{D}); \mathrm{rank}(R(\mathbf{p})) < \mathrm{rank}(R)\},$$

and is called the set of rank singularities of M. It is Zariski-closed in $\mathrm{Spec}(\mathbf{D})$.

(ii) The set $RS(M)$ is empty if and only if M is projective. If condition (32) is satisfied this signifies by (53) that R is left zero- prime.

(iii) If M is torsion or, in other terms, if $\mathbf{K}\otimes_{\mathbf{D}} M = 0$ and $m = \dim(\mathbf{K}\otimes_{\mathbf{D}} M) = 0$, then $RS(M)$ coincides with the support of M, $RS(M) = \mathrm{supp}(M)$.

Proof: (ii) A finitely generated module is projective if and only if it is locally free, i.e. if and only if all $M_{\mathbf{p}}$ are free (See [BOU 1], II, § 5, Th. 1).

(iii) If M is torsion then

$$U(M) = \{\mathbf{p};\ M(\mathbf{p}) = 0 \text{ or } M_{\mathbf{p}} = 0\} \text{ and } RS(M) = \{\mathbf{p};\ M_{\mathbf{p}} \neq 0\} = \mathrm{supp}(M)$$

by [BOU 1], II. 4.4. ||

(72) **Remark:** For a polynomial algebra $\mathbf{D} = F[s_1, \cdots, s_r]$ (or, more generally, any affine integral domain) one may replace $\mathrm{Spec}(\mathbf{D})$ by $\bar{F}^r$ as usual where $F \subset \bar{F}$ is an algebraic closure, for instance $\mathbb{R} \subset \mathbb{C}$. (Refer to [MUM]). Any $\xi = (\xi_1, \cdots, \xi_r) \in \bar{F}^r$ gives rise to the maximal ideal

$\mathbf{m}(\xi) := \ker(F[s] \longrightarrow \bar{F},\ s_i \longmapsto \xi_i)$ and $F \subset F[s]/\mathbf{m}(\xi) = F[s]\big(\mathbf{m}(\xi)\big) \subset \bar{F}$.

So $\mathbf{D}(\mathbf{p})$ from (65) can be replaced by $\bar{F}$ for $\mathbf{p} = \mathbf{m}(\xi)$ and $R(\mathbf{p})$ from (66) by $R(\xi)$ which is obtained from R by substituting ξ_i for s_i , $i = 1, \cdots, r$.

Similarly one defines

$$M(\xi) := \bar{F}\otimes_{F[s]} M = \bar{F}^{1} / R(\xi)^{T} \bar{F}^{k} .$$

The subsets $U(M)$ resp. $RS(M)$ are replaced by the Zariski-open resp. Zariski-closed subsets

$$U_{\bar{F}}(M) = \{\xi \in \bar{F}^r ;\ M_{\mathbf{m}(\xi)} \text{ is free }\} = \{\xi \in \bar{F}^r ;\ \mathrm{rank}(R(\xi)) = p\} ,\ p := \mathrm{rank}(R) ,$$

resp. $RS_{\bar{F}}(M) = \{\xi \in \bar{F}^r ;\ M_{\mathbf{m}(\xi)} \text{ is not free}\} = \{\xi \in \bar{F}^r ;\ \mathrm{rank}(R(\xi)) < \mathrm{rank}(R)\}$.

These considerations show that definition 71 really specializes to that of the literature [BFM]. ||

In the next theorem I make the additional

(73) **Assumption**: The ring $\mathbf{D}$ is a normal domain of finite Krull dimension, for instance $\mathbf{D}=F[s_1,\cdots,s_r]=F[s]$ or its quotient rings $F[s]_T$. ||

A ring is called *normal* if it is a noetherian, integrally closed integral domain. Refer to [MATS], § 5, for the *Krull dimension* of $\mathbf{D}$ or Spec($\mathbf{D}$) and of its closed subsets. The *codimension* of a closed subset X of Spec($\mathbf{D}$) is $\mathrm{cod}(X):=\dim(\mathbf{D})-\dim(X)$. Since $\mathbf{D}$ is normal the local rings $\mathbf{D}_\mathbf{p}$ of $\dim(\mathbf{D}_\mathbf{p})=\mathrm{height}(\mathbf{p})=1$ at the minimal non-zero prime ideals $\mathbf{p}$ are discrete valuation rings (DVR) and hence principal ([MATS], Cor. of th. 11.5 on page 82). A module $N\in \mathrm{Modf}(\mathbf{D})$ is called *almost zero* according to [BOU1], VII, § 4, if $M_\mathbf{p}=0$ for all $\mathbf{p}$ of $\mathrm{height}(\mathbf{p})=\dim(\mathbf{D}_\mathbf{p})=1$.

(74) **Theorem**: Assumption (73). For a finitely generated $\mathbf{D}$-module $M=\mathbf{D}^1/R^T\mathbf{D}^k$ the codimension $\mathrm{cod}(RS(M))$ is at least two if and only if the torsion module $T(M)$ of M is almost zero. This is the case if M is torsionfree or, equivalently by (21 b), if the associated system $S(M)$ is minimal (in its transfer class).

If the Krull dimension of $\mathbf{D}$ is at most two, for instance for $F[s_1,s_2]$, then the condition $\mathrm{cod}(RS(M))\geq 2$ signifies that $RS(M)$ is finite.

Proof: Assume first that $\mathrm{cod}(RS(M)\geq 2$, i.e. $\mathrm{height}(\mathbf{p})\geq 2$ for all $\mathbf{p}\in RS(M)$. This implies that for a height one prime ideal $\mathbf{p}$ with principal $\mathbf{D}_\mathbf{p}$ $\mathbf{p}$ is not in $RS(M)$, but in $U(M)$ and hence $M_\mathbf{p}$ is free from which $T(M_\mathbf{p})=T(M_\mathbf{p})=0$ follows. By definition this means that $T(M)$ is almost zero.

Conversely if $T(M)$ is almost zero then $T(M_\mathbf{p}) = T(M_\mathbf{p})=0$ for all $\mathbf{p}$ of height one. Over the DVR $\mathbf{D}_\mathbf{p}$ the torsionfree module $M_\mathbf{p}$ is then free and thus $\mathbf{p}\in U(M)$. Hence $RS(M)=\mathrm{Spec}(\mathbf{D})\setminus U(M)$ contains no prime ideals of height one which is equivalent to $\mathrm{cod}(RS(M))\geq 2$. ||

Appendix : Exponential solutions and integral representations

This subject is the second main theme of the books [EH] and [PAL] besides the fundamental principle used in § 2, and is rather difficult to understand in detail. It generalizes to partial differential equations the well-known easy result that the solutions of ordinary differential equations with constant coefficients are linear combinations of functions $a(t)e^{\tau t}$ with polynomials $a(t)$ and characteristic values $\tau \in \mathbb{C}$. I use Björk's presentation [BJ], Ch. 8, of this theory and indicate without any proofs and analytic details how this theory can be useful for system theory.

I consider the large injective cogenerator $\mathbf{A}=C^{\infty}(\mathbb{R}^r)$ over $\mathbb{C}[s]=\mathbb{C}[s_1,\cdots,s_r]$ from (1.18) with $(L_i a)(t)=\partial a/\partial t_i$, $L=L_t=(\partial/\partial t_1,\cdots,\partial/\partial t_r)$. For a IO-system

(75) $S=\{(uy)\in \mathbf{A}^{m+p};\ P(L)(y)=Q(L)(u)\}$, $P\in \mathbb{C}[s]^{k,p}$, $\operatorname{rank}(P)=p$, $PH=Q$,

the sequence

(76) $$0 \longrightarrow \ker(P)=\{y\in \mathbf{A}^p;\ P(L)(y)=0\} \longrightarrow S \longrightarrow \mathbf{A}^m \longrightarrow 0,$$
$$y \longmapsto (0,y)\quad,\quad (u,y)\longmapsto y$$

is exact according to (2.69) and dual to the exact sequence

(77) $0 \rightarrow \mathbb{C}[s]^m \rightarrow M(S) \rightarrow M:=M(\ker(P))=\mathbb{C}[s]^p/P^T\mathbb{C}[s]^k \rightarrow 0$

The system module

$$M=\mathbb{C}[s]^p\big/P^T\mathbb{C}[s]^k \cong \mathbb{C}[s]^{1,p}\big/\mathbb{C}[s]^{1,k}P$$

of $\ker(P)$ is torsion and has played an important role for discrete systems in § 5 starting with (5.31) and (5.32). In (5.33) $\ker(P)\cong F^G$ was interpreted as the state space of the IO-system S. The elements $y\in\ker(P)$ represent those outputs of S which correspond to zero inputs. The theory from [EH], Ch 7, [PAL], Ch 6 and [BJ],Ch. 8, allows to say something about these y.

Since M is torsion the support of M coincides with the variety of rank singularities by (71), (iii) . As subvarities of $\mathbb{C}^r$ we obtain as in (72)

(78) **Corollary and Definition:** Data of (75), $M:=\mathbb{C}[s]^p\big/P^T\mathbb{C}[s]^k$. Then

$$\operatorname{Supp}_{\mathbb{C}^r}(M)=\{\tau\in\mathbb{C}^r;\ M_{\mathfrak{m}(\tau)}\neq 0\}=\{\tau\in\mathbb{C}^r;\ \operatorname{rank}(P(\tau))<p\}=RS_{\mathbb{C}^r}(M).$$

This variety of rank singularities is called the *characteristic variety* of M or P in [BJ], 8.1.7. ||

Let $L_\tau=(\partial/\partial\tau_1,\cdots,\partial/\partial\tau_r)$ be the differential operator with respect to

$\tau \in \mathbb{C}^r$ and $\tau \cdot t = \tau_1 t_1 + \cdots + \tau_r t_r$. An easy induction on m furnishes

$$L_\tau{}^m(\tau^n e^{\tau \cdot t}) = L_t{}^n(t^m e^{\tau \cdot t}),\ m,n \in \mathbb{N}^r,$$

and implies the formula

(79) $$\left[A(\tau, L_\tau)^T (P(\tau)^T e^{\tau \cdot t})\right]^T = P(L_t)(A(\tau,t)e^{\tau \cdot t})$$

for matrices $P(s)$. resp. $A(\tau,t)$ with coefficients in $\mathbb{C}[s]$ resp. $\mathbb{C}[\tau,t]$.

(80) **Definition and Corollary** (*Exponential solution*) : Given $P \in \mathbb{C}[s]^{k,p}$, $\mathrm{rank}(P) = p$, an exponential solution of $P(L_t)y(t) = 0$ or in $\ker(P) = \{y \in \mathbf{A}^p ;\ P(L_t)(y) = 0\}$ is a solution of the form $y(t) = a(t)e^{\tau \cdot t}$, $a(t) \in \mathbb{C}[t]^p$, which, by (79), satisfies

$$P(L_t)(a(t)e^{\tau \cdot t}) = [a(L_\tau)^T (P(\tau)^T e^{\tau \cdot t})]^T = 0.$$

If, in particular, $0 \neq a \in \mathbb{C}^p$ is a non-zero constant the vector $ae^{\tau \cdot t}$ is a solution of $P(L_t)(ae^{\tau \cdot t}) = 0$ if and only if *a* is a solution of $P(\tau)a = 0$, and then $\mathrm{rank}(P(\tau)) < p$ or $\tau \in RS_{\mathbb{C}^r}(M)$. ||

The proposition 8.1.8 of [BJ] asserts that

(81) $\{\tau \in \mathbb{C}^r;\ \exists\, a(t) \neq 0$ such that $a(t)e^{\tau \cdot t}$ is a exponential solution $\} \subset$

$\subset RS(M) = \mathrm{Supp}(M)$.

The main result of [BJ], Ch. 8, after [EH] and [PAL] is the following

(82) **Integral representation theorem** ([BJ], Th. 8.1.3): Situation as explained above. Every solution $y(t) \in C^\infty(\mathbb{R}^r)^p$ of $P(\partial/\partial t_1, \cdots, \partial/\partial t_r)y(t) = 0$ is a finite sum of integral solutions of the form $\int a(\tau,t)e^{\tau \cdot t}\,d\mu(\tau)$ with the following specifications: $a(\tau,t) \in \mathbb{C}[\tau,t]$, μ is a measure on $\mathbb{C}^r$ and

(83) $P(L_t)(a(\tau,t)e^{\tau \cdot t}) = 0$ for all $\tau \in \mathrm{Supp}(\mu)$ (:= the support of μ).

There are several important convergence conditions which I omit here. ||

Condition (83) implies that

(84) $a(\tau,t)e^{\tau \cdot t}$ is an exponential solution of $P(L_t)y = 0$ if $a(\tau,-) \neq 0$ and $\tau \in \mathrm{Supp}(\mu)$

and, in particular by (81) ,

(85) $$\{\tau \in \mathbb{C}^r ;\ a(\tau,-) \neq 0,\ \tau \in \mathrm{Supp}(\mu)\} \subset \mathrm{Supp}(M) = RS(M).$$

Since the growth behaviour of $a(\tau,t)e^{\tau \cdot t}$ depends essentially on these $\tau \in RS(M)$ it is clear that the integral representation theorem will have a significance for the "stability" of solutions of $P(L_t)y = 0$. In suggestive terms (82) can be stated as follows : Every solution y of $P(L_t)y=0$ is an infinite linear combination of exponential solutions, and this obviously generalizes the corresponding easy result for ordinary differential equations.

(86) **Special case** (The simply characteristic case after [BJ], Th. 8.8.1): Assume that there is an ideal $\mathbf{I}\subset \mathbb{C}[s]$ which coincides with its radical ($q^n\in\mathbf{I}$ implies $q\in\mathbf{I}$) such that

$$P^T\mathbb{C}[s]^k = \mathbf{I}\times\cdots\times\mathbf{I} \subset \mathbb{C}[s]^p \text{ or } M=\mathbb{C}[s]^p\big/P^T\mathbb{C}[s]^k=(\mathbb{C}[s]/\mathbf{I})^p$$

Then the characteristic variety of M is the vanishing set of $\mathbf{I}$, i.e.

$$\mathrm{Supp}_{\mathbb{C}^r}(M)=V_{\mathbb{C}^r}(\mathbf{I})=\{\tau\in\mathbb{C}^r\ ;\ q(\tau)=0 \text{ for all } q\in\mathbf{I}\}\ .$$

In this case the integral representation of (82) simplifies to $y=\sum_{finite}\int a(\tau)\,e^{\tau\cdot t}\,d\mu(\tau)$ with $a(\tau)\in\mathbb{C}[\tau]^p$ satisfying $P(\tau)a(\tau)=0$ for $\tau\in\mathrm{Supp}(\mu)$ and measures μ with $\mathrm{Supp}(\mu)\subset\mathrm{Supp}(M)$. The $a(\tau)e^{\tau\cdot t}$, $\tau\in\mathrm{Supp}(\mu)$, are exponential solutions of $P(L_t)y=0$ by (80). ||

According to [EH], Ch.7, and [PAL], Ch. 6, the integral representation theorem is valid for other cogenerators **A** from theorem 2.54 but the exact meaning of these results in the context of this paper is not yet completely clear to me. For the discrete cases of $\mathbb{C}\langle t\rangle\subset\mathbb{C}\{t\}=\mathbb{C}^{\mathbb{N}^r}$ with the left shifts as $\mathbb{C}[s]$-operations the exponentials have to be replaced by $(1-\tau_1t_1)^{-1}\cdots(1-\tau_rt_r)^{-1}=\sum\{\tau^nt^n\ ;\ n\in\mathbb{N}^r\}$ since the isomorphism (4.24) sends this power series into $e^{\tau\cdot t}$.

To realize the potential of the integral representation theorem for system theory, in particular for stability and stabilization problems, much further work has to be carried out.

8. BLOCK DIAGRAMS AND THEIR ASSOCIATED SYSTEMS

The assumption (2.64)=(7.1) and the theorems in §2 and §7 based on this assumption are valid . I use the standard terminology concerning block diagrams and flow graphs coming mainly from the theory of electrical networks since Kirchhoff (around 1850) , but in the form of precise mathematical definitions and generalized to the multidimensional situation . My main references for the language were [KUO], CH.3, and [CH] . The one-dimensional analogues of the following results are also treated in [BY] under the title " interconnection of systems" and probably in [WIL2] , but this latter book is not available to me .

A *block diagram* is simply a IO-system valued graph . A finite *graph* or *diagram scheme* is a pair (V,E) of a finite set V of *vertices* or *nodes* and a finite set E of *edges* or *arrows* together with two maps $\mathrm{dom(ain)}, \mathrm{cod(omain)} : E \longrightarrow V$. As usual I write

$$e : v \longrightarrow w \text{ if } e \in E \text{ , } \mathrm{dom}(e) = v \text{ and } \mathrm{cod}(e) = w \text{ ,}$$

and call v the *domain* or *source* and w the *codomain* or *sink* of e . Remark that *loops* e with $\mathrm{dom}(e) = \mathrm{cod}(e)$ and *multiple edges* $e_1 \neq e_2$ with $\mathrm{dom}(e_1) = \mathrm{dom}(e_2)$ and $\mathrm{cod}(e_1) = \mathrm{cod}(e_2)$ are permitted .

(1) **Definition** (*block diagram of IO-systems*): A block diagram of IO-systems is given by the following data :

(i) a finite graph (V,E) (ii) a natural number $n(v)$ for every vertex $v \in V$

(iii) a IO-system

$S_e = \{(uy) \in \mathbf{A}^{n(v)+n(w)} ; P_e y = Q_e u\}$, $P_e \in \mathbf{D}^{k(e),n(w)}$, $P_e H_e = Q_e$, $\mathrm{rank}(P_e) = n(w)$ with the transfer matrix $H_e \in \mathbf{K}^{n(w),n(v)}$ for every edge $e \in E$.

The customary notation for these data is

$$\overset{u \in \mathbf{A}^{n(v)}}{\bullet\!\!-\!\!-\!\!-} \boxed{S_e \text{ or } H_e} \longrightarrow \overset{y \in \mathbf{A}^{n(w)}}{\bullet} \quad . \;||$$

The numbers $n(v)$ resp. $n(w)$ are the input resp. output dimension of the system S_e along $e : v \longrightarrow w$. The interpretation of S_e is obvious : A signal u with $n(v)$ components at the node v is transmitted through the system S_e and furnishes a $n(w)$-dimensional signal y at the node w . The transfer matrix H_e is usually called the *gain* or *transmittance* along e . If $\mathbf{D}$ is a F-algebra over a field F and if the system S_e along e is of the simple IO-form

$$S_e = \{(uy) \in \mathbf{A}^{m+p} ; y = Au\} \text{ , } A \in F^{p,m} \subset \mathbf{D}^{p,m} \text{ with } H = A \in F^{p,m} \subset \mathbf{K}^{p,m}$$

then one uses the special representation $\bullet\!\!-\!\!(A)\!\!\longrightarrow\!\bullet$ to emphasize this simplicity . An unadorned arrow $\bullet\!\!\longrightarrow\!\bullet$ stands for $\{(uy) \in \mathbf{A}^{m+m} ; y = u\}$.

A block diagram gives rise to a natural new system which is formed according to the following verbal principle : The signal at a node v is used as an input for all systems S_e starting at v . All output vectors at a node w add up to form the signal vector at w . In more mathematical terms define

(2) $$n(V):=\sum\{n(v);v\in V\}$$ and identify

$$\mathbf{A}^{n(V)}=\mathbf{A}^{\sum n(v)}=\prod\{\mathbf{A}^{n(v)};v\in V\}\ ,\ x=(x_v;\ v\in V)\ ,\ x_v\in\mathbf{A}^{n(v)}\ .$$

Call a node v *initial* if no edge ends in v , i.e. if $cod(e)\neq v$ for all $e\in E$. Then

$$In(V):=\{v\in V;v\text{ initial }\}\text{ and }Nin(V):=V\setminus In(V)$$

are the sets of initial resp. non-initial nodes . Now define the system S *associated to* the block diagram by

(3) $$S:=\{x=(x_v;v\in V)\in\mathbf{A}^{n(V)};\ (4)\text{ is satisfied }\}$$ where

the condition (4) expresses the verbal principle from above and is given by

(4) For all non-initial nodes w and all edges $e:dom(e)\longrightarrow w$ with sink w there are signal vectors $y_e\in\mathbf{A}^{n(w)}$ such that $(x_{dom(e)},y_e)\in S_e$ and $x_w=\sum\{y_e;e\in E,cod(e)=w\}$. ||

That S is really a subsystem of $\mathbf{A}^{n(V)}$ follows easily from (2.39) . The signals x_v at the initial nodes v can only come from outside . Hence

(5) $$x:=\binom{x_I}{x_N}\in\mathbf{A}^{n(V)}=\mathbf{A}^{I(V)}\times\mathbf{A}^{N(V)}$$

$$I(V):=\sum\{n(v);\ v\in In(V)\}\ ,\ \mathbf{A}^{I(V)}=\prod\{\mathbf{A}^{n(V)}\ v\in In(V)\}$$

$$N(V):=\sum\{n(v);v\in Nin(V)\}\ ,\ \mathbf{A}^{N(V)}=\prod\{\mathbf{A}^{n(V)};v\in Nin(V)\}$$

is a natural IO-structure for S from (3) . However , the decomposition (5) is not always a IO-structure in the sense of (2.69) , i.e. , in general , the signals x_v , $v\in In(V)$, cannot be chosen arbitrarily as trivial counterexamples show and as is well-known from the theory of electrical networks (see [CH]). As outputs one often chooses only certain of the vectors x_w , $w\in Nin(V)$, and obtains variants of (3) by the Rosenbrock method (see (2.38) pp. and the later examples) .

The IO-structures of S and , in particular , whether (5) is really a IO-structure , are completely determined by

$$\widehat{S}\subset\mathbf{K}^{n(V)}=\prod\{\mathbf{K}^{n(v)};v\in V\}=\{\xi=(\xi_v;v\in V);\ \xi_v\in\mathbf{K}^{n(v)}\}$$

according to (2.94) . The condition $(x_{dom(e)},y_e)\in S_e$ from (4) is replaced by

$$(\xi_{dom(e)},\eta_e)\in\widehat{S}_e=\mathrm{graph}(H_e)\text{ or }\eta_e=H_e\,\xi_{dom(e)}$$

and the condition $x_w=\sum\{y_e;e\in E,cod(e)=w\}$, $w\in Nin(V)$, by

$$\xi_w=\sum\{\eta_e;\ cod(e)=w\}=\sum\{H_e\,\xi_{cod(e)}\ ;\ e\in E,cod(e)=w\}$$

Altogether we obtain

(6) $\widehat{S}=\{(\xi_v;v\in V)\in\mathbf{K}^{n(V)};\text{For all } w\in \mathrm{Nin}(V):\xi_w=\sum\{H_e\xi_{dom(e)};e\in E,\mathrm{cod}(e)=w\}\}$

I introduce the abbreviations

$$\begin{aligned}&H_{wv}=\sum\{H_e\ ;e\in E,e:v\longrightarrow w\}\in\mathbf{K}^{n(w),n(v)}\\ (7)\qquad &H_{N,I}=(H_{wv}\ ;w\in \mathrm{Nin}(V),v\in \mathrm{In}(V))\in\mathbf{K}^{N(V),I(V)}\ \text{(block matrix)}\\ &H_{N,N}=(H_{wv}\ ;w,v\in\mathrm{Nin}(V))\in\mathbf{K}^{N(V),N(V)}\end{aligned}$$

$$\xi=\begin{pmatrix}\xi_I\\ \xi_N\end{pmatrix}\in\mathbf{K}^{I(V)+N(V)},\ \xi_I:=(\xi_v;\ v\in\mathrm{In}(V)),\xi_N:=(\xi_w\ ;w\in\mathrm{Nin}(V))$$

The defining equations $\xi_w=\sum\{H_e\xi_{cod(e)};\ e\in E,\mathrm{cod}(e)=w\}$ from (6) become

$$\xi_N=H_{N,I}\,\xi_I+H_{N,N}\,\xi_N \text{ or } (I-H_{N,N})\xi_N=H_{N,I}\,\xi_I,\ I=\mathrm{id}\in\mathbf{K}^{N(V),N(V)}$$

We conclude

(8) $\widehat{S}=\{\xi=\begin{pmatrix}\xi_I\\ \xi_N\end{pmatrix}\in\mathbf{K}^{I(V)+N(V)}=\prod\{\mathbf{K}^{n(v)}:v\in V\};\ (I-H_{N,N})\xi_N=H_{N,I}\xi_I\}$.

By theorem 2.94 $x=\begin{pmatrix}x_I\\ x_N\end{pmatrix}$ resp. $\begin{pmatrix}\xi_I\\ \xi_N\end{pmatrix}$ are a IO-structure of S if and only if the projection $\widehat{S}\longrightarrow\mathbf{K}^{I(V)}$, $\xi\longmapsto\xi_I$, is an isomorphism . But this is the same as the invertibility of $I-H_{N,N}$. We conclude

(9) **Theorem** (IO-structure for block diagrams) For a block diagram (1) and the derived data as above the signal flow space $\widehat{S}$ of the associated system S is

$$\widehat{S}=\{\xi=\begin{pmatrix}\xi_I\\ \xi_N\end{pmatrix}\in\mathbf{K}^{I(V)+N(V)};\ (I-H_{N,N})\xi_N=H_{N,I}\xi_I\}\ .$$

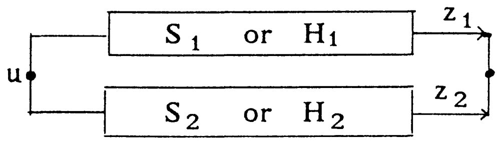
The natural decompositions $\xi=\begin{pmatrix}\xi_I\\ \xi_N\end{pmatrix}$ or $x=\begin{pmatrix}x_I\\ x_N\end{pmatrix}$ define a IO-structure of S if and only if the matrix $(I-H_{N,N})$ is invertible in $\mathrm{Gl}_{N(V)}(\mathbf{K})$, and then $H:=(I-H_{N,N})^{-1}H_{N,I}\in\mathbf{K}^{N(V),I(V)}$ is the transfer matrix of S with respect to this IO-structure . ||

(10) **Remark and Definition** (*signal flow* or *Mason graph* , see [KUO] or [CH]): The signal flow or Mason graph of a block diagram is the matrix-valued graph (V,E) where the matrix $H_e\in\mathbf{K}^{n(w),n(v)}$ is assigned to every edge $e:v\longrightarrow w$. These data are usually written as $\bullet\xrightarrow{H_e}\bullet$. The matrix-valued graph gives rise to the system of linear equations $(I-H_{N,N})\xi_N=H_{N,I}\,\xi_I$ with coefficients in **K** which defines $\widehat{S}$ according to (9) . The " *topological* or graph theoretic [CH] " calculation of $\det(I-H_{N,N})$ and $H=(I-H_{N,N})^{-1}H_{N,I}$ is one of the main objectives of the book [CH] . Hence the algorithms of this and other books on graph theory can be usefully applied to decide whether

the natural decomposition $x = \begin{pmatrix} x_I \\ x_N \end{pmatrix}$ is really a IO-structure and , if so , to calculate its transfer matrix $H = (I - H_{N,N})^{-1} H_{N,I}$. ||

(12) **Example** (*parallel composition*) :

Consider the block diagramm



with IO-systems

$S_i = \{(u_i\ z_i) \in \mathbf{A}^{m+p};\ P_i z_i = Q_i u\},\ i = 1, 2,\ P_i \in \mathbf{A}^{k(i),p}$, $\mathrm{rank}(P_i) = p$, $P_i H_i = Q_i$.

The associated system according to (3) is

$S = \{(u\,y) \in \mathbf{A}^{m+p}$; $\exists\ z_1, z_2$ such that $y = z_1 + z_2$, $P_1 z_1 = Q_1 u$, $P_2 z_2 = Q_2 u\} =$

$= \{(u\,y) \in \mathbf{A}^{m+p}$; $\exists\ z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ such that $\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} u$ and $y = (I_p I_p) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\}$.

We can apply (2.38) or (2.41). So let $X = (X_1, X_2)$ and Y be universal with $(X_1, X_2) \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} = Y(I_p\ I_p)$, i e. $X_1 P_1 = Y = X_2 P_2$. This means in other words that X_1 , X_2 are universal with $X_1 P_1 = X_2 P_2$.

From (2.41) we conclude $S = \{(u\,y) \in \mathbf{A}^{m+p}$; $X \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} u = Yy\}$ or

(13) $$S = \{(u,y) \in \mathbf{A}^{m+p};\ X_i P_i y = (X_1 Q_1 + X_2 Q_2)\, u\}$$ where

X_1, X_2 are universal with $X_1 P_1 = X_2 P_2$.

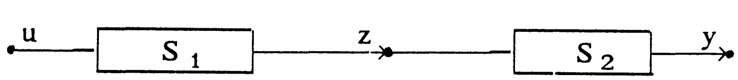
The system is the *parallel composition* of S_1 and S_2 . The transfer matrix is most easily obtained from

$\widehat{S} = \{ (\xi\,\eta) \in \mathbf{K}^{m+p}$; $\exists\ \zeta_1, \zeta_2$ such that $\eta = \zeta_1 + \zeta_2$, $\zeta_i = H_i \xi$, $i = 1,2\} =$

$= \{(\xi, \eta) \in \mathbf{K}^{m+p};\ \eta = (H_1 + H_2)\,\xi\} = \mathrm{graph}(H_1 + H_2)$.

Hence the transfer matrix of the parallel composition S of S_1 and S_2 is $H_1 + H_2$ as was to be expected . ||

(14) **Example (Series composition):** Consider the block diagram



with two IO-Systems

$S_1 = \{(u\,z) \in \mathbf{A}^{m+p(1)};\ P_1 z = Q_1 u\}$, $P_1 \in \mathbf{D}^{k(1),p(1)}$, $\mathrm{rank}(P_1) = p(1)$, $P_1 H_1 = Q_1$

$S_2 = \{(z\,y) \in \mathbf{A}^{m(2)+p};\ P_2 y = Q_2 z\}$, $P_2 \in \mathbf{D}^{k(2),p}$, $\mathrm{rank}(P_2) = p$, $P_2 H_2 = Q_2$, $m(2) = p(1)$.

The *series* or *cascade composition* $S := S_2 * S_1$ of S_1 and S_2 is not the system according to (3), but the "Rosenbrock projection-" image

(15) $$S := S_2 * S_1 := \{(u\ y) \in \mathbf{A}^{m+p};\ \exists\ z \text{ such that } P_1 z = Q_1 u\ ,\ P_2 y = Q_2 z\ \}.$$

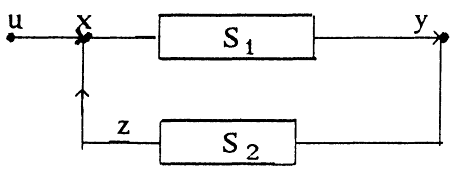
We get rid of *z* by choosing

(16) X, Y universal with $X P_1 = Y Q_2$ and obtain from (2.38)

(17) $$S_2 * S_1 = \{ (u\ y) \in \mathbf{A}^{m+p}\ ;\ Y P_2 y = X Q_1 u\ \}.$$

with the IO-structure $(u\ y)$. The transfer matrix is most easily obtained from $\widehat{S} = \{(\xi\ \eta) \in \mathbf{K}^{m+p}\ ;\ \exists\ \zeta \text{ s.t. } \zeta = H_1 \xi\ , \eta = H_2 \zeta\ \} = \{(\xi\ \eta)\ ; \eta = H_2 H_1 \xi\} = \mathrm{graph}(H_2 H_1)$. Hence the series composition $S_2 * S_1$ has the transfer matrix $H_2 H_1$ as expected and desired. ||

(18) **Example(feedback system):** A feedback block diagram is given as



with two IO-systems

$S_1 = \{(x\ y) \in \mathbf{A}^{m+p}\ ;\ P_1 y = Q_1 x\}$, $P_1 \in \mathbf{D}^{k(1),p}$, $\mathrm{rank}(P_1) = p$, $P_1 H_1 = Q_1$, $H_1 \in \mathbf{K}^{p,m}$

$S_2 = \{(y\ z) \in \mathbf{A}^{p+m}\ ;\ P_2 z = Q_2 y\}$, $P_2 \in \mathbf{D}^{k(2),m}$, $\mathrm{rank}(P_2) = m$, $P_2 H_2 = Q_2$, $H_2 \in \mathbf{K}^{m,p}$.

The arrow $\xrightarrow{u}$ represents the system $\{(u, y) \in \mathbf{A}^{m+m}\ ;\ y = u\}$. The *feedback system* is then

(19) $$S := \{\ (u\ y) \in \mathbf{A}^{m+p};\ \exists z \text{ with } x = u + z\ , P_1 y = Q_1 x\ ,\ P_2 z = Q_1 y\ \} =$$
$$= \{\ (u\ y) \in \mathbf{A}^{m+p}\ ;\ \exists z \text{ with } P_1 y - Q_1 u = Q_1 z\ ,\ P_2 z = Q_1 y\}.$$

To get rid of *z* again we choose

(20) X and Y universal with $X Q_1 = Y P_2$

and obtain from (2.38)

(21) $$S = \{\ (u\ y) \in \mathbf{A}^{m+p};\ (X P_1 - Y Q_1) y = X Q_1 u\ \}.$$ Then

(22) $$\widehat{S} = \{\ (\ \xi\ \eta\ ) \in \mathbf{K}^{m+p};\ \exists \zeta \text{ with } \eta = H_1(\ \xi + \zeta\ )\ ,\ \zeta = H_2\ \eta\ \} =$$
$$= \{\ (\ \xi\ \eta\ ) \in \mathbf{K}^{m+p}\ ;\ (I_p - H_1 H_2)\ \eta = H_1 \xi\ \}$$

As in (9) $(\xi\ \eta)$ resp. $(u\ y)$ form a IO-structure of S if and only if the matrix $I_p - H_1 H_2$ is invertible. This means that 1 is not an eigenvalue of $H_1 H_2$ or of $H_2 H_1$ (From $\eta = H_1 H_2 \eta \neq 0$ one concludes $H_2 \eta = (H_2 H_1) H_2 \eta \neq 0$ and the same with 1 and 2 interchanged) and is thus equivalent to $I_m - H_2H_1 \in Gl_m(\mathbf{K})$. If $I_p - H_1H_2 \in Gl_p(\mathbf{K})$ the transfer matrix of the feedback system S is $H := (I_p - H_1H_2)^{-1} H_1 \in \mathbf{K}^{p,m}$ as expected. There are other variants of feedback systems (see, for instance, [WOL] and [BOS], Ch.3) which can be treated in the same fashion. ||

Invertible systems

Under this heading I consider systems with (left , right) invertible transfer matrices . In the one-dimensional situation these systems were introduced and discussed in [WOL] , § 5.5 . Assumption (2.64)=(7.1) is still in force . I consider IO-systems

(23) $\qquad S=\{(uy)\in\mathbf{A}^{m+p} ; Py = Qu\}$, $P\in\mathbf{D}^{k,p}$, $\mathrm{rank}(P)=p$, $PH=Q$.

In particular , I need IO-systems whose transfer matrix is the identity and which are thus transfer equivalent to the system

$$\{(uy)\in\mathbf{A}^{m+m} ; y=u\}=\mathrm{im}(\Delta) , \Delta:\mathbf{A}^m \longrightarrow \mathbf{A}^{m+m} , u \longmapsto (u,u) ,$$

and are given as

(24) $\qquad S=\{(uy)\in\mathbf{A}^{m+m} ; Py=Pu\}$, $P\in\mathbf{D}^{k,m}$, $\mathrm{rank}(P)=m$.

(25) **Definition :** I call a system of the form (24) a quasi-identity system . ||

The projection $\mathrm{proj} : S \longrightarrow \mathbf{A}^m$, $(u,y) \longmapsto u$, has the diagonal Δ as a section or right inverse . This gives rise to the direct sum decomposition

(26) $S = \mathrm{im}(\Delta)\oplus \ker(\mathrm{proj}) \cong \mathbf{A}^m \oplus \ker(P)$, $(u,y) \longleftrightarrow (u,z)$, $y=u+z$, $z\in\ker(P)$.

(27) **Interpretation :** The idea of a quasi-identity system is that it changes an input u only "slightly" to give an output $y=u+z$ with $Pz=0$, $\mathrm{rank}(P)=m$. In the discrete case of (5.11) or (5.41) one can force $z=0$ and $y=u$ by selecting the initial condition $z|G=0$ or $y|G=u|G$. In general one would like to have that any solution z of $Pz=0$ is "small" , "stable" , "negligeable" etc . The use of the integral representation theorem 7.82 for the investigation of these questions requires more work as said at the end of §7 . ||

The data of (23) give rise to maps $\mathbf{K}^m \xrightarrow{H} \mathbf{K}^p \xrightarrow{P} \mathbf{K}^k$, $PH=Q$, where P is injective since $\mathrm{rank}(P)=p$. We conclude

(28) **Corollary :** (i) An IO-system (23) induces an isomorphism

$$P : \mathrm{im}(H) = H\mathbf{K}^m \cong \mathrm{im}(Q)=Q\mathbf{K}^m , \eta \longmapsto P\eta .$$

In particular , $\mathrm{rank}(Q)=\mathrm{rank}(H)$.

(ii) Hence $H\in\mathbf{K}^{p,m}$ is left invertible , i.e. there is a retraction $H'\in\mathbf{K}^{m,p}$ with $H'H=I_m$, if and only if $\mathrm{rank}(H)=\mathrm{rank}(Q)=m$. The latter condition signifies that the m columns of Q or H are linearly independent and implies $m\le p$. ||

(29) **Theorem and Definition** (*left invertible systems*)

(i) A IO-system S as in (23) has a left invertible transfer matrix H if and only if there is a IO-system $S'\subset\mathbf{A}^{p+m}$ such that the series composition $S'*S$ is a quasi-identity system . Such a system is called *left invertible* or *input observable* in [WOL] , p.164 , in the 1-d case .

(ii) If $H'\in\mathbf{K}^{m,p}$ is a left inverse of H , i.e. $H'H=I_m$, if

$$S'=\{(y,v)\in\mathbf{A}^{p+m}\ ;\ P'v=Q'y\}\ ,\ P'\in\mathbf{D}^{k',m}\ ,\ \mathrm{rank}(P')=m\ ,\ P'H'=Q'\ ,$$

is a realization of H' , for instance the minimal one from (7.29) , and if U and V are universal (matrices) with $UP=VQ'$ then

$$S'*S=\{(u,v)\in\mathbf{A}^{m+m}\ ;\ VP'v=VP'u\}\ ,\ VP'=UQ\ ,$$

is a quasi-identity system .

Proof : If S' has the transfer matrix H' and if $S'*S$ is a quasi-identity system with transfer matrix I_m then $H'H=I_m$ by (14) . Conversely assume the data as (ii) . The IO-representation of $S'*S$ follows from (17) . Its transfer matrix is $H'H=I_m$ and thus $S'*S$ is a quasi-identity system . ||

Remark that neither H' nor S' are unique in (29),(ii) .

(30) **Interpretation** : If the output y of S is used as an input for S' the latter system gives an output v which does not differ too much from the input u of S since $VP'(v-u)=0$, $\mathrm{rank}(VP')=m$. Or one can almost derive the input u from the output y of S , hence the designation *input observable* . As said before in (27) I cannot currently answer the question how to choose P' such that v-u is *small in a technically interesting sense* . ||

(31) **Corollary** : Given the situation of (29),(ii) , a *compensator* to observe u from y can also be constructed in the following fashion . Since $\mathrm{rank}(H)=m$ I can assume , possibly after permuting the rows of H , that H has the form $H=\begin{pmatrix}H_1\\H_2\end{pmatrix}\in\mathbf{K}^{(p-m)+m,m}$ with $H_2\in Gl_m(\mathbf{K})$. With the corresponding block decomposition $P=(P_1P_2)\in\mathbf{D}^{k,(p-m)+m}$ of P we obtain $PH=P_1H_1+P_2H_2=Q$

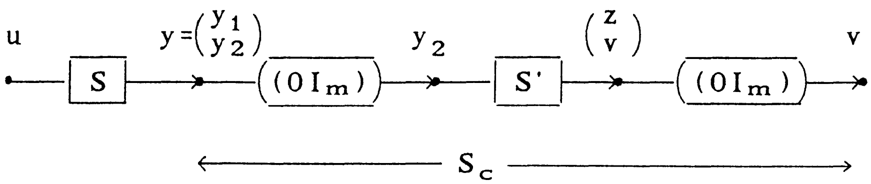
and conclude

$$(-P_1\ Q)\begin{pmatrix} H_1 H_2^{-1} \\ H_2^{-1} \end{pmatrix} = P_2\ , \quad (-P_1\ Q) \in \mathbf{D}^{k,p} \quad .$$

From this and $\operatorname{rank}(-Q,P) = \operatorname{rank}(-P_1,Q,P_2) = p$ we derive $\operatorname{rank}(-P_1,Q) = p$ and define the IO-system

$$(32) \qquad S' := \{(y_2,(z,v)) \in \mathbf{A}^{m+(p-m)+m}\ ;\ -P_1 z + Qv = P_2 y_2\}$$

with the transfer matrix $\begin{pmatrix} H_1 H_2^{-1} \\ H_2^{-1} \end{pmatrix}$. The system S" of the series composition



has the transfer matrix $(0\, I_m)\begin{pmatrix} H_1 H_2^{-1} \\ H_2^{-1} \end{pmatrix}(0\, I_m)\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = H_2^{-1} H_2 = I_m$ and is hence a quasi-identity system as desired. The trivial systems $(0\, I_m)$ simply take certain components of the inputs as the output. The system S" is given as

$$S'' = \{(uv) \in \mathbf{A}^{m+m}\ ;\ P''v = P''u\},\ \operatorname{rank}(P'') = m, \qquad \text{and as}$$

$$S'' = \{(uv) \in \mathbf{A}^{m+m}\ ;\ \exists\ y,z : Py = Qu\ ,\ -P_1 z + Qv = (0\ P_2)y\} =$$

$$= \{(uv) \in \mathbf{A}^{m+m}\ ;\ \exists \begin{pmatrix} y \\ z \end{pmatrix} :\ \begin{pmatrix} (P_1\ P_2) & 0 \\ (0\ \ P_2) & P_1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\} \ .$$

Now choose a matrix (XY) universal with $(XY)\begin{pmatrix} (P_1\ P_2) & 0 \\ (0\ \ P_2) & P_1 \end{pmatrix} = 0$, i.e. $XP_1 = YP_1 = (X+Y)P_2 = 0$, and conclude from (2.38) that

$$S'' = \{(uv) \in \mathbf{A}^{m+m}\ ;\ (X\,Y)\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0\} = \{(uv) \in \mathbf{A}^{m+m}\ ;\ -YQv = XQu\} =$$

$$= \{(uv) \in \mathbf{A}^{m+m}\ ;\ (-P'',P'')\begin{pmatrix} u \\ v \end{pmatrix} = 0\} \ .$$

The quasi-uniqueness theorem 2.63 implies $(XY)\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} = U(-P'',P'')$, hence $YQ = UP'' = -XQ$ and finally

$$(33) \quad S'' = \{(uv) \in \mathbf{A}^{m+m}\ ;\ XQv = XQu\}\ ,\ \operatorname{rank}(XQ) = m\ ,\ X \text{ universal with } XP_1 = 0\ .$$

The Y from above is not used anymore. The advantage of this construction compared to that of (29),(ii), is its simplicity where indeed S' can be trivially and S" simply derived from S. ||

Right invertible or output controllable systems can be introduced and investigated in a similar manner. Right invertibility of $H \in \mathbf{K}^{p,m}$ signifies that $\operatorname{rank}(H) = p$ or that $H : \mathbf{K}^m \longrightarrow \mathbf{K}^p$ is surjective and admits a right inverse H" with $HH'' = I_p$ and implies $p \le m$.

(34) **Theorem** (*right invertible systems*)

(i) A IO-system (23) has a right invertible transfer matrix H if and only if there is a IO-system $S'' \subset \mathbf{A}^{p+m}$ suct that $S * S''$ is a quasi-identity system . Such a system is called *right invertible* or *output controllable* in [WOL], p.164 .

(ii) If $H'' \in \mathbf{K}^{p,m}$ is a right inverse of H with the realization

$$S'' = \{(yu) \in \mathbf{A}^{p+m};\ P''u = Q''y\}\ ,\ P'' \in \mathbf{D}^{k'',m}\ ,\ \operatorname{rank}(P'') = m\ ,\ P''H'' = Q''\ ,$$

for instance the minimal one , and if Y and Z are universal with $YP'' = ZQ$ then $S * S'' = \{(yz) \in \mathbf{A}^{p+p}\ ;\ ZPz = ZPy\}$, $ZP = YQ''$ with the transfer matrix $HH'' = I_p$. ||

The designation " output controllable " is justified by the following interpretation : If a desired output y of S is taken as an input for S" then the output z of $S * S''$ or S is not exactly the desired y , but does not differ too much from it . The concluding remark of (30) applies again , however . The corollary (31) has a right inverse counterpart which I omit .

(35) **Corollary and Definition** (*Invertible systems*) : If in the situation of (29) or (34), (ii) , the matrix H is invertible , i.e. $m = p$ and $\operatorname{rank}(Q) = \operatorname{rank}(H) = m = p$, then the system $S' := \{(yu) \in \mathbf{A}^{m+m}\ ;\ Qu = Py\}$, $QH^{-1} = P$, realizes H^{-1} and

$$S' * S = \{(uv) \in \mathbf{A}^{m+m}\ ;\ Qv = Qu\} \text{ and } S * S' = \{(yz) \in \mathbf{A}^{m+m}\ ;\ Pz = Py\}$$

are quasi-identity systems . Such a S is called an invertible IO-system . ||

The question of *exact model matching* according to [WOL], p.316 , generalizes that of the right invertibility of systems . Given the IO-system S from (23) and a model system

(36) $$S_{mod} = S_2 = \{(vy) \in \mathbf{A}^{m(2)+p};\ P_2 y = Q_2 v\}\ ,\ P_2 \in \mathbf{D}^{k(2),p}\ ,$$

$$\operatorname{rank}(P_2) = p\ ,\ P_2 H_2 = Q_2, \text{ with the transfer matrix } H_{mod} = H_2$$

one looks for a " compensating " system $S_{comp} = S_1$ such that $S * S_{comp}$ is transfer equivalent to S_{mod} . This implies a factorization $HH_{comp} = H_{mod}$ or $HH_1 = H_2$. Such a factiorization exists if and only if

(37) $$H_2 \mathbf{K}^{m(2)} \subset H\mathbf{K}^m \quad \text{or} \quad \{H^T \xi = 0 \Rightarrow H_2^T \xi = 0\}\ .$$

(38) **Theorem** (*exact model matching*)

(i) Given the system S from (23) and a model system (36) there is a compensating system S_{comp} such that $S * S_{comp}$ is transfer equivalent to S_{mod} if and only if (37) is satisfied .

(ii) If $H_{mod} = HH_{comp}$, $H_{comp} \in \mathbf{K}^{m,m(2)}$, if

$S_{comp} = \{(vu) \in \mathbf{A}^{m(2)+m};\ P_1 u = Q_1 v\}$, $P_1 \in \mathbf{D}^{k(1),m}$, $\mathrm{rank}(P_1) = m$, $P_1 H_1 = Q_1$

is a realization of H_{comp} and if Y and V are universal with $YQ = VP_1$ then

$$S * S_{comp} = \{(vy) \in \mathbf{A}^{m(2)+p};\ YPy = VQ_1 v\},\ YQ = VP_1,$$

is transfer equivalent to S_{mod}. The proof is analogous to that of (29) and (34). ||

Again the preceding theorem treats only the algebraic side of the exact model matching problem and does not necessarily give a technically useful compensator.

References

[BA] Baratchart L., On discrete n-dimensional linear constant systems, in [BL], 3-8

[BE] Berg L., Introduction to the operational calculus, North-Holland 1967

[BFM] Bisiacco M., E. Fornasini, G. Marchesini, Dynamic regulation of 2d-systems: A state space approach, Linear Algebra and its Applications 122-124 (1989), 195-218

[BJ] Björk J.-E., Rings of Differential Operators, North-Holland 1979

[BL] Byrnes C.I., A. Lindquist (Eds.), Frequency Domain and State Space Methods for Linear Systems, North-Holland 1986

[BM] Byrnes C.I., C.F. Martin (Eds.), Geometrical Methods for the Theory of Linear Systems, D. Reidel 1980

[BOR] Borel A. et al., Algebraic D-modules, Academic Press 1989

[BOS1] Bose N.K. (Editor), Multidimensional systems: Theory and applications, IEEE Press 1979, New York

[BOS2] Bose N. K. (Editor), Multidimensional systems theory, D. Reidel Publishing Company 1985

[BOU 1] Bourbaki N., Commutative algebra, Hermann, Paris 1972

[BOU 2] Bourbaki N., Algèbre Ch. 10 (Algèbre homologique), Masson, Paris 1980

[BOU3] Bourbaki N., Théorie des Ensembles I, Hermann 1956

[BU] Buchberger B., Gröbner bases: An algorithmic method in polynomial ideal theory, in [BOS2], 184-232

[BY] Blomberg H., R. Ylinen, Algebraic theory for multivariable linear systems, Academic Press 1983

[CH] Chen W. K., Applied graph theory. Graphs and electrical networks, North-Holland 1976

[CPK] Conte G., A.M. Perdon, T. Kaczorek, A geometric approach to singular 2D linear systems, Proc. IFAC workshop Prague 1988

[EH] Ehrenpreis L., Fourier analysis in several complex variables, Wiley Interscience Publishers 1970

[FM1] Fornasini E., G. Marchesini, State-space realization theory of two-dimensional filters, IEEE Trans. Aut. Cont. 21 (1976), 484-492

[FM2] —, Doubly indexed dynamical systems: State-space models and structural properties, Math. Systems Theory 12 (1978), 59-72

[FM3] —, Structure and Properties of Two-Dimensional Systems, in [TZ], 37-88

[FSK] Furukawa A., T. Sasaki, H. Kobayashi, Gröbner basis of a module over $K[x_1,\cdots,x_n]$ and polynomial solutions of a system of linear equations, Proc. Symsac 1986 (Waterloo), 1986

[FU] Fuchs L., Teilweise geordnete algebraische Strukturen, Vandenhoek und Ruprecht 1966

[GA] Gabriel P., Etude infinitésimale des schémas en groupes, in Schémas en Groupes I, Lecture Notes in Mathematics 151 (1970), 476 - 562, Springer Verlag

[GR1] Grauert H., R. Remmert, Analytische Stellenalgebren, Springer 1971

[GR2] Grauert H., R. Remmert, Coherent analytic sheaves, Springer 1984

[GR3] Grauert H., R. Remmert, Theorie der Steinschen Räume, Springer 1977

[GRE] Gregor J. , Convolutional solutions of partial difference equations , preprint 1988 , to appear in MCSS

[HA] Hauger G. , Injektive Moduln über Ringen linearer Differentialoperatoren , Monatshefte Math. 86 (1978) , 189-201

[HI] Hironaka H. , Resolution of singularities of an algebraic variety over a field of characteristic zero I, II , Ann.Math. 79 (1964), 109-326

[HÖ1] Hörmander L. , The analysis of linear partial differential operators I , Springer 1983

[HÖ2] Hörmander L. , Complex analysis in several variables, North-Holland 1973

[JA] Jacobson N. , Structure of rings, American Mathematical Society Colloquium Publications 37, Providence 1968

[KAI] Kailath T. , Linear systems, Prentice-Hall 1980

[KAM] Kamen E.W. , An operator theory of linear functional differential equations , J. Diff. Equations 27 (1978), 274-297

[KAS] Kashiwara M. , Systems of microdifferential equations , Birkhäuser 1983

[KKK] Kashiwara M. , T. Kawai , T. Kimura , Foundations of algebraic analysis , Princeton University Press 1986

[KÖ] Köthe G. , Topologische lineare Räume I , Springer 1960

[KUO] Kuo B.C. , Automatic Control Systems , Prentice Hall 1987

[LJ] Lejeune-Jalabert M. , Effectivité de calcul polynomiaux , Cours de DEA 1984/85 , Université de Grenoble

[LE] Lelong P. , Fonctionelles analytiques et fonctions entières (n variables), Les Presses de l'Université de Montreal 1968

[LO] Lim J.S. , A.V. Oppenheim (Editors) , Advanced topics in signal processing, Prentice Hall 1988

[MAC1] Macaulay M. S. , The algebraic theory of modular systems , Cambridge 1916 , reprint by Stechert-Hafner 1964

[MAC2] Macaulay M. S. , Some properties of enumeration in the theory of modular systems , Proc.London Math. Soc. 26(1927) , 531-555

[MAL] Malgrange B., Systèmes différentiels à coefficients constants, in Séminaire Bourbaki Vol. 1962/1963, 246.01-246.11

[MAT] Matlis E., Injective modules over noetherian rings, Pac. J. Math. 8 (1958), 511-528

[MATS] Matsumura M. , Commutative ring theory , Cambridge University Press 1986

[MLK] Morf M. , B. Levy , S.-Y. Kung , New results on 2-d systems theory I: 2-d polynomial matrices , factorizations and coprimeness , in [BOS1] , p. 54-65

[MUM] Mumford D. , Introduction to algebraic geometry, Lecture Notes , Harvard University.

[OB1] Oberst U. , Duality theory for Grothendieck categories and linearly compact rings , J. Algebra 15 (1970), 473-542

[OB2] Oberst U. , Multidimensional constant linear systems I, II, III, Innsbruck July 1988 - January 1989, Preprints

[PAL] Palamodov V.P., Linear differential operators with constant coefficients, Springer 1970

[PAU] Pauer F. , Gröbner Basen und ihre Anwendungen , to appear in Überblicke Mathematik , Vieweg 1990

[PR] Przeworska–Rolewicz D. , Algebraic Analysis , D.Reidel 1988

[ROC] Rocha P. , Structure and Representation of 2–D Systems , Thesis , Groningen , February 1990

[ROE] Roesser R.P. , A discrete state-space model for linear image processing , IEEE Trans. Aut. Cont. 20 (1975) , 1-10

[ROO] Roos J.E. , Locally noetherian categories and generalized strictly linearly compact rings. Applications , in Lecture Notes in Mathematics 92 (1969), 197-277, Springer

[ROS] Rosenbrock H. , State-space and multivariable theory , Nelson 1970

[RS] Rouchaleau Y. , E.D. Sontag , On the existence of minimal realizations of linear dynamical systems over noetherian integral domains , J. Comp. Syst. Sci. 18 (1979) , 65-75

[RW] Rocha P. , J.C. Willems , State for 2-d systems , Lin. Alg. Appl. 122-124 (1989) , 1003-1038

[RWK] Rouchaleau Y. , B.F. Wyman , R.E. Kalman , Algebraic structure of linear dynamical systems III . Realization theory over a commutative ring , Proc. Nat. Acad. Sci. (USA) 69 (1972) , 3404-3406

[SCH] Schwartz L. , Théorie des distributions , Tome I, II , Hermann , Paris , 1957, 1959.

[TR] Treves F. , Basic linear partial differential equations , Academic Press 1975

[TZ] Tzafestas S.G. (Editor) , Multidimensional systems. Techniques and applications , M. Dekker 1986

[WIL] Willems J.C., From time-series to linear systems. Part I: Finite dimensional linear time-invariant systems , Automatica 22 (1986), 561-580

[WIL2] Willems J.C. , The analysis of feedback systems , MIT Press 1971

[WIN] Winkler F. , A recursive method for computing a Gröbner basis of a module in $K[X]^r$, AAECC-5

[WOL] Wolowich W.A. , Linear Multivariable Systems , Springer 1974

[YG] Youla D.G. , G. Gnavi , Notes on n-dimensional system theory , IEEE Trans. Circ. Syst. 26 (1979) , 105-111