The Computation of Purity Filtrations over Commutative Noetherian Rings of Operators and their Application to Behaviors *

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Abstract

Due to the theoretical work and computer implementations of, for instance, Barakat, Quadrat and Robertz(2012) and their coauthors the theory of finitely generated (f.g.) modules over *non-commutative* regular noetherian rings of partial differential operators with variable coefficients like the Weyl algebras and over other similar rings has become constructive in recent years. In particular these authors compute the *purity or grade filtration* of a f.g. module by homological means and discuss its significance for the associated behavior. Pommaret and Quadrat noted this significance already in 1999. In this note it is shown that over an arbitrary *commutative* noetherian ring of operators the purity filtration of a finitely generated module can be easily computed by means of the the primary decomposition of its zero submodule and indeed, with smaller complexity, by inductively computing equidimensional parts. In most books on *Constructive Commutative Algebra* the connection between this primary decomposition, the equidimensional parts and the purity filtration is *implicitly* stated for *cyclic modules*. For many *commutative* rings of operators the standard signal modules are injective cogenerators. In this case the purity filtration of the module gives rise to a corresponding filtration of the dual behavior, and the primary decomposition induces additional sum representations of the pure dimensional factors of this filtration. For *non-commutative* rings of operators the standard signal modules are in general neither injective nor cogenerators, and for such signal modules the usefulness of the purity filtration of the module for the determination of the behavior and its structural properties is not obvious. It is also shown by a counter-example that dimensional purity of the module or behavior does not imply dimensional purity of the initial conditions according to Riquier of the associated homogeneous Cauchy problem.

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1 Introduction

Due to the theoretical work and computer implementations of, for instance, Barakat [1],[2], Quadrat and Robertz [17],[19], [16] and their coauthors the theory of finitely generated (f.g.) modules over *non-commutative* regular noetherian rings of partial differential operators with variable coefficients like the Weyl algebras and over other similar rings has become constructive in recent years. A larger part of this work develops constructive *Homological Algebra*. One interest of these computations lies in their application to the analysis and synthesis of multidimensional linear systems or behaviors. The papers [2] and [16] especially compute the *purity or grade filtration* of a f.g. module by homological means and discuss its systems theoretic significance. Björk introduced this filtration in his important book [3, Thm. 2.4.15] by means of spectral sequences and the Ext-functors and discussed its properties in [3, §2.7]. For commutative pure-dimensional regular rings the filtration goes back to Roos [20]. Pommaret and Quadrat [15, Def. 12, Thm. 3] observed the systems theoretic significance of Björk's results. In his paper [16] Quadrat uses, extends and simplifies Björk's theory and makes it constructive.

The present paper uses arbitrary *commutative* noetherian rings A of operators only and elaborates [11, Ex. III.3, Thm. IV.1]. Its main goal is to show that in this commutative case different purity filtrations of a f.g. module M with dual behavior $\mathscr{B}$ can be easily computed by means of the primary decomposition of the zero submodule of M and indeed, *with smaller complexity*, by inductively computing *equidimensional parts* (cf. Thm. 3.2). In the commutative case there are different methods and implementations for the computation of primary decompositions and equidimensional parts. The SINGULAR library mprimdec.lib [7, Remark 4.1.7] enables the computation of the primary decomposition of modules as used in Thm. 3.2, some implementations work for ideals resp. cyclic modules only. Two methods to compute the equidimensional part of a cyclic module are described in [7, §4.4] (cf. also [9, pp. 315-320]), viz. that by the purity filtration according to [6] on the first lines of p.260 and that in [7, Lemma 4.4.7, Alg. 4.4.9]. No comparison is made between the complexity and the speed of the algorithms of these two methods in [7]. I have to leave this comparison and the application of the algorithms in Sections 3 and 4 to experts in *Computer Algebra* and, in particular, to the colleagues quoted above and to the reviewers of this paper. The primary decomposition of M also furnishes the primary decompositions of the pure dimensional factors of the purity filtration of M (cf. Thm. 3.2) and the corresponding sum representations of their dual behaviors (cf. (42)) which cannot be obtained by homological means alone. One such representation was already derived in [24, Thm. 7.1]. For cyclic modules $A/\mathfrak{a}$ the purity filtration and its connection with the primary decomposition and its equidimensional parts was already discussed in [22, §3.2] and in [7, §4.4] with a different terminology.

Over the standard *commutative* rings A of operators there are many natural signal A-modules $\mathscr{F}$ (with action $a \circ w$ for $a \in A$ and $w \in \mathscr{F}$) which are injective cogenerators. This signifies that the functor $M \mapsto \mathscr{B} := \mathrm{Hom}_A(M, \mathscr{F})$ from f.g. modules M to their associated behavior $\mathscr{B}$ preserves and reflects exact sequences and thus establishes a strong duality between modules and behaviors. In particular, for such signal modules module filtrations of M and behavior filtrations of $\mathscr{B}$ are in one-one correspondence (Section 4). We describe various ways to compute a behavior by means of its subbehaviors in the filtration, in particular in Thm. 4.2 from [17, Thm. 7,§5] and [16, Thm. 7, §4] with a different proof.

For most *non-commutative* multidimensional rings of operators the standard signal

modules are neither injective nor cogenerators. Module filtrations give still rise to behavior filtrations, but their connection is weaker. Thm. 4.2 gives partial answers for non-injective signal modules. Quadrat performed relevant computations in [16, §4]. If the signal module is *not* injective the usefulness of the purity filtration for the determination of the associated behavior and its structural properties is not obvious.
Already Riquier [18] solved the *initial value or Cauchy problem* for an *analytic* n-dimensional behavior $\mathscr{B} = \mathrm{Hom}_A(M, \mathscr{F})$ (in a different language, of course) and showed that every trajectory in $\mathscr{B}$ is uniquely determined by a finite family of initial functions u_σ, $\sigma \in \Sigma$, where u_σ depends on $n(\sigma) \le n$ independent variables. This suggests the natural question whether all initial functions of a pure d-dimensional behavior $\mathscr{B}$ depend on $n(\sigma) = d$ independent variables. Compare the respective indications in [16, Introduction, second paragraph]. We show in Section 5 that the answer to this question is negative.

2 Basic data

The following notions and results from *Commutative Algebra* are standard [4], [10]. Let A be a *commutative* noetherian ring, $\mathrm{spec}(A)$ resp. $\max(A)$ the sets of its prime resp. maximal ideals and Mod_A the category of A-modules. For an A-module M and a prime ideal $\mathfrak{p} \in \mathrm{spec}(A)$ we have to consider the local noetherian ring $A_\mathfrak{p} = \left\{\frac{a}{s};\ a \in A, s \in A \setminus \mathfrak{p}\right\}$ and the corresponding $A_\mathfrak{p}$-module $M_\mathfrak{p}$. The *support* resp. *associator* of M is

$$\begin{aligned} \mathrm{supp}(M) := \{\mathfrak{p} \in \mathrm{spec}(A);\ M_\mathfrak{p} \neq 0\} \supset \\ \mathrm{ass}(M) := \{\mathfrak{p} \in \mathrm{spec}(A);\ A/\mathfrak{p} \subseteq M \text{ (up to isomorphism)}\}. \text{ Then} \\ \mathrm{supp}(M) = \{\mathfrak{q} \in \mathrm{spec}(A);\ \exists \mathfrak{p} \in \mathrm{ass}(M) \text{ with } \mathfrak{p} \subseteq \mathfrak{q}\}. \end{aligned} \tag{1}$$

If $M = A/\mathfrak{a}$ is a cyclic module then

$$\mathrm{supp}(A/\mathfrak{a}) = V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{spec}(A);\ \mathfrak{a} \subseteq \mathfrak{p}\}. \tag{2}$$

If M is finitely generated (f.g.) with *annihilator* ideal

$$\mathrm{ann}_A(M) := \{a \in A;\ aM = 0\} \text{ then } \mathrm{supp}(M) = V(\mathrm{ann}_A(M)). \tag{3}$$

The support $\mathrm{supp}(M)$ is *closed under specialization*, i.e., if

$$\mathfrak{p}, \mathfrak{q} \in \mathrm{spec}(A),\ \mathfrak{p} \subseteq \mathfrak{q} \text{ and } \mathfrak{p} \in \mathrm{supp}(M) \text{ then also } \mathfrak{q} \in \mathrm{supp}(M). \tag{4}$$

The *(Krull) dimension* of the A-module M is

$$\begin{aligned} \dim(M) := \sup\{n \in \mathbb{N};\ \exists \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n,\ \mathfrak{p}_i \in \mathrm{supp}(M)\} \in \mathbb{N} \uplus \{\infty\}, \text{ especially} \\ \dim(A/\mathfrak{p}) = \sup\{n \in \mathbb{N};\ \exists \mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n,\ \mathfrak{p}_i \in \mathrm{spec}(A)\},\ \mathfrak{p} \in \mathrm{spec}(A), \\ \dim(M) \underset{(1)}{=} \sup_{\mathfrak{p} \in \mathrm{ass}(M)} \dim(A/\mathfrak{p}). \end{aligned} \tag{5}$$

The Krull dimension of the *ring* A is that of A as A-module with $\mathrm{supp}(A) = \mathrm{spec}(A)$ and may be infinite. The Krull dimension

$$\dim(A_\mathfrak{p}) = \sup\{n \in \mathbb{N};\ \exists \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p},\ \mathfrak{p}_i \in \mathrm{spec}(A)\},\ \mathfrak{p} \in \mathrm{spec}(A), \tag{6}$$

of the local noetherian ring $A_{\mathfrak{p}}$ is finite and is also called the *height* or *codimension* of the prime ideal $\mathfrak{p}$. The inequality $\dim(A/\mathfrak{p})+\dim(A_{\mathfrak{p}})\le\dim(A)$, $\mathfrak{p}\in\operatorname{spec}(A)$, holds obviously. If $\mathfrak{p},\mathfrak{q}\in\operatorname{spec}(A)$, $\mathfrak{p}\subseteq\mathfrak{q}$ then $\dim(A_{\mathfrak{p}})\le\dim(A_{\mathfrak{q}})$. The *codimension* of any nonzero module ${}_AM$ is

$$\begin{gathered}\operatorname{cd}(M):=\min\nolimits_{\mathfrak{p}\in\operatorname{supp}(M)}\dim(A_{\mathfrak{p}})\underset{(1)}{=}\min\nolimits_{\mathfrak{p}\in\operatorname{ass}(M)}\dim(A_{\mathfrak{p}}),\ \text{hence}\\ h(\mathfrak{p}):=\operatorname{cd}(A/\mathfrak{p})=\dim(A_{\mathfrak{p}})\ \text{for}\ \mathfrak{p}\in\operatorname{spec}(A),\\ (\text{height of}\ \mathfrak{a}):=h(\mathfrak{a}):=\operatorname{cd}(A/\mathfrak{a})\ \text{for}\ \mathfrak{a}\subsetneq A,\ h(\mathfrak{a})+\dim(A/\mathfrak{a})\le\dim(A),\\ \dim(M)+\operatorname{cd}(M)\le\dim(A).\end{gathered}\tag{7}$$

The codimension of the zero submodule is defined to be ∞. According to Wood, Rogers and Owens (1998, §7) the codimension $\operatorname{cd}(M)$ of a f.g. A-module M is called the *autonomy degree* of the associated behavior $\mathscr{B}:=\operatorname{Hom}_A(M,\mathscr{F})$ where ${}_A\mathscr{F}$ is assumed to be one of the standard injective cogenerator signal modules, for instance from (34) and (35) below.
A submodule Q of a f.g. A-module M is called $\mathfrak{p}$-*primary* if $\mathfrak{p}\in\operatorname{spec}(A)$ and $\operatorname{ass}(M/Q)=\{\mathfrak{p}\}$. An arbitrary submodule N of a f.g. module M admits a *primary decomposition of N in M*, viz.

$$N=\bigcap_{i\in I}Q_i,\ I\ \text{finite},\ \operatorname{ass}(M/Q_i)=\{\mathfrak{p}_i\},\ \mathfrak{p}_i\in\operatorname{spec}A.\tag{8}$$

Lemma and Definition 2.1. ([4, §IV.2, Def. 3, Prop. 4 and Cor. p.]) *For the primary decomposition* (8) *the following properties are equivalent:*

1. *(i) For all $j\in I:\ N\subsetneq\bigcap_{i\in I,\,i\neq j}Q_i$ or $\bigcap_{i\in I,\,i\neq j}Q_i\not\subseteq Q_j$.*
   (ii) The $\mathfrak{p}_i$, $i\in I$, are pairwise distinct.
2. *The $\mathfrak{p}_i$, $i\in I$, are pairwise distinct and belong to* $\operatorname{ass}(M/N)$.
3. *The finite sets I and* $\operatorname{ass}(M/N)$ *have the same number of elements.*

If these properties are satisfied the primary decomposition (8) *is called* reduced *or* irredundant.

Lemma 2.2. ([4, Thm. IV.2.1, Prop. IV.2.5]) *Every submodule N of a finitely generated (f.g.) module M admits a reduced primary decomposition $N=\bigcap_{\mathfrak{p}\in\operatorname{ass}(M/N)}Q(\mathfrak{p})$ with* $\operatorname{ass}(M/Q(\mathfrak{p}))=\{\mathfrak{p}\}$ *for all* $\mathfrak{p}\in\operatorname{ass}(M/N)$. *For a minimal* $\mathfrak{p}\in\operatorname{ass}(M/N)$ *the component $Q(\mathfrak{p})$ is uniquely determined by*

$$Q(\mathfrak{p})=\{x\in M;\ \exists s\in A\setminus\mathfrak{p}\ \textit{with}\ sx\in N\}.\tag{9}$$

A *Serre subcategory* $\mathfrak{C}$ of Mod_A is a class of modules which is closed under taking isomorphic copies, submodules, factor modules, extensions and direct sums; compare [21, §3] for a short introduction into this notion, its history and its basic properties. For a given Serre category $\mathfrak{C}$ every A-module has a largest submodule $\operatorname{Ra}_{\mathfrak{C}}(M)$ in $\mathfrak{C}$ which is called the $\mathfrak{C}$-*radical* of M. Serre categories are in one-one-correspondence with subsets $\mathfrak{P}_1\subseteq\operatorname{spec}(A)$ which are closed under specialization (cf. (4)). We write

$$\operatorname{spec}(A)=\mathfrak{P}_1\biguplus\mathfrak{P}_2.\tag{10}$$

The correspondence is given by

$$\begin{gathered}\mathfrak{P}_1:=\{\mathfrak{p}\in\operatorname{spec}(A);\ A/\mathfrak{p}\in\mathfrak{C}\},\\ \mathfrak{C}=\{C\in\operatorname{Mod}_A;\ \operatorname{supp}(C)\subseteq\mathfrak{P}_1\}=\{C\in\operatorname{Mod}_A;\ \operatorname{ass}(C)\subseteq\mathfrak{P}_1\}=\\ \{C\in\operatorname{Mod}_A;\ \forall\mathfrak{p}\in\mathfrak{P}_2:\ C_{\mathfrak{p}}=0\},\ (\operatorname{Ra}_{\mathfrak{C}}(M)=0\iff\operatorname{ass}(M)\subseteq\mathfrak{P}_2).\end{gathered}\tag{11}$$

If M is f.g. its radical $\mathrm{Ra}_{\mathfrak{C}}(M)$ can be computed via the primary decomposition of 0 in M from Lemma 2.2 [11, Thm. IV.1], [13, Result 2.8], [21, Alg. 3.1]:

$$\begin{gathered} 0=\bigcap_{\mathfrak{p}\in\mathrm{ass}(M)} Q(\mathfrak{p})=U_1\bigcap U_2,\ U_i:=\bigcap\left\{Q(\mathfrak{p});\ \mathfrak{p}\in\mathrm{ass}(M)\bigcap\mathfrak{P}_i\right\}, \\ \mathrm{ass}(M/U_i)=\mathrm{ass}(M)\bigcap\mathfrak{P}_i,\ M/U_1\in\mathfrak{C},\ U_2=\mathrm{Ra}_{\mathfrak{C}}(M),\ \mathrm{Ra}_{\mathfrak{C}}\left(M/\mathrm{Ra}_{\mathfrak{C}}(M)\right)=0. \end{gathered} \tag{12}$$

Since $\mathrm{Ra}_{\mathfrak{C}}$ determines $\mathfrak{C}=\{C\in\mathrm{Mod}_A;\ C=\mathrm{Ra}_{\mathfrak{C}}(C)\}$ the *primary decomposition almost tautologically dominates Serre categories* as one reviewer has put it. The same reviewer calls the resulting diagonal monomorphism $M\to M/U_1\times M/U_2$ in (12) the *subdirect decomposition adapted* to the Serre category.

3 The purity filtration

The assumptions of the preceding section are in force. For an exact sequence of A-modules

$$\begin{gathered} 0\to M'\to M\to M''\to 0 \text{ one concludes } \mathrm{supp}(M)=\mathrm{supp}(M')\bigcup\mathrm{supp}(M''),\ \text{hence} \\ \mathrm{cd}(M)=\inf_{\mathfrak{p}\in\mathrm{supp}(M)}\dim(A_{\mathfrak{p}})=\min\left(\mathrm{cd}(M'),\mathrm{cd}(M'')\right). \end{gathered} \tag{13}$$

If $M=\bigcup_{i\in I}M_i$ is a directed union of submodules one likewise infers

$$\mathrm{supp}(M)=\bigcup_{i\in I}\mathrm{supp}(M_i)\ \text{and}\ \mathrm{cd}(M)=\inf_{i\in I}\mathrm{cd}(M_i). \tag{14}$$

These two equations imply that for every $d\in\mathbb{N}$ the class

$$\mathfrak{C}_d:=\{C\in\mathrm{Mod}_A;\ \mathrm{cd}(C)\geq d\},\ d\in\mathbb{N}, \tag{15}$$

is a Serre category. The corresponding decomposition (11) is

$$\begin{gathered} \mathrm{spec}(A)=\mathfrak{P}_{d,1}\biguplus\mathfrak{P}_{d,2}= \\ \{\mathfrak{p}\in\mathrm{spec}(A);h(\mathfrak{p})=\dim(A_{\mathfrak{p}})\geq d\}\biguplus\{\mathfrak{p}\in\mathrm{spec}(A);h(\mathfrak{p})<d\} \end{gathered} \tag{16}$$

There are the obvious inclusions

$$\begin{gathered} \mathfrak{C}_0=\mathrm{Mod}_A\supseteq\mathfrak{C}_1\supseteq\cdots\supseteq\mathfrak{C}_d\supseteq\mathfrak{C}_{d+1}\supseteq\cdots \\ \mathrm{Ra}_0(M):=M\supseteq\cdots\supseteq\mathrm{Ra}_d(M):=\mathrm{Ra}_{\mathfrak{C}_d}(M)\supseteq\mathrm{Ra}_{d+1}(M)=\mathrm{Ra}_{d+1}(\mathrm{Ra}_d(M))\supseteq\cdots. \end{gathered} \tag{17}$$

In [15, Def. 12] $\mathrm{Ra}_d(M)$ is denoted by $\mathrm{tor}_{d-1}(M)$. From (12) we infer

$$\begin{gathered} \mathrm{Ra}_d(M)=\bigcap\left\{Q(\mathfrak{p});\ \mathfrak{p}\in\mathrm{ass}(M)\bigcap\mathfrak{P}_{d,2}\right\}=\bigcap\{Q(\mathfrak{p});\ \mathfrak{p}\in\mathrm{ass}(M),\ h(\mathfrak{p})<d\}, \\ \mathrm{cd}\left(\mathrm{Ra}_d(M)\right)\geq d,\ \mathrm{ass}(M/\mathrm{Ra}_d(M))=\mathrm{ass}(M)\bigcap\mathfrak{P}_{d,2}=\{\mathfrak{p}\in\mathrm{ass}(M);\ h(\mathfrak{p})<d\}, \\ \mathrm{Ra}_{d+1}(M)=\mathrm{Ra}_d(M)\bigcap\bigcap\{Q(\mathfrak{p});\ \mathfrak{p}\in\mathrm{ass}(M),\ h(\mathfrak{p})=d\}= \\ \bigcap\left\{\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p});\ \mathfrak{p}\in\mathrm{ass}(M),\ h(\mathfrak{p})=d\right\}. \end{gathered} \tag{18}$$

Corollary 3.1. *Let M be nonzero and f.g. For the data from* (12) *and* (18) *let*

$$m := \mathrm{cd}(M) = \min_{\mathfrak{p}\in\mathrm{ass}(M)} h(\mathfrak{p}) \leq p := \max_{\mathfrak{p}\in\mathrm{ass}(M)} h(\mathfrak{p}). \text{ Then}$$
$$M = \mathrm{Ra}_0(M) = \cdots = \mathrm{Ra}_m(M) \supsetneq \mathrm{Ra}_{m+1}(M) \supseteq \cdots \supseteq \mathrm{Ra}_p(M) \supsetneq \mathrm{Ra}_{p+1}(M) = 0.$$

Theorem and Definition 3.2. *Let ${}_AM$ be finitely generated, $d \geq 0$ and*

$$0 = \bigcap_{\mathfrak{p}\in\mathrm{ass}(M)} Q(\mathfrak{p}),\ \mathrm{ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\} \tag{19}$$

a reduced primary decomposition of 0 *in M. Then*

1. *The intersection*

$$\begin{aligned}\mathrm{Ra}_d(M) = \bigcap\{Q(\mathfrak{p});\ \mathfrak{p}\in\mathrm{ass}(M),\ h(\mathfrak{p}) < d\} \text{ with}\\ \mathrm{ass}\,(M/\mathrm{Ra}_d(M)) = \{\mathfrak{p}\in\mathrm{ass}(M);\ h(\mathfrak{p}) < d\}\end{aligned} \tag{20}$$

   is a reduced primary decomposition of $\mathrm{Ra}_d(M)$ *in M.*

2. *The intersection*

$$\begin{aligned}\mathrm{Ra}_{d+1}(M) = \bigcap\left\{\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p});\ \mathfrak{p}\in\mathrm{ass}(M),\ h(\mathfrak{p}) = d\right\} \text{ with}\\ \mathrm{ass}\,(\mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M)) = \{\mathfrak{p}\in\mathrm{ass}(M);\ h(\mathfrak{p}) = d\}\end{aligned} \tag{21}$$

   is a reduced primary decomposition of $\mathrm{Ra}_{d+1}(M)$ *in* $\mathrm{Ra}_d(M)$. *In particular,* $\mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M)$ *is pure d-codimensional, i.e., it is either zero or all its associated primes $\mathfrak{p}$ have the same codimension $d = \dim(A_{\mathfrak{p}})$ and are therefore minimal in* $\mathrm{ass}\,(\mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M))$. *This implies*

$$\forall\mathfrak{p}\in\mathrm{ass}(M) \text{ with } \dim(A_{\mathfrak{p}}) = d:$$
$$\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p}) = \mathrm{Ra}_d(Q(\mathfrak{p})) = \{x\in\mathrm{Ra}_d(M);\ \exists s\in A\setminus\mathfrak{p} \text{ with } sx\in\mathrm{Ra}_{d+1}(M)\}. \tag{22}$$

   While the component $Q(\mathfrak{p})$, $\mathfrak{p}\in\mathrm{ass}(M)$, $h(\mathfrak{p}) = d$, is not unique in general its radical $\mathrm{Ra}_d(Q(\mathfrak{p}))$ *is by* (22). *The filtration $M = \mathrm{Ra}_0(M) \supseteq \mathrm{Ra}_1(M) \supseteq \cdots$ is called the* purity filtration *of M.*

3. *In particular, if* $\mathrm{cd}(M) = m$ *and thus $M = \mathrm{Ra}_m(M)$ then*

$$\begin{aligned}M = \mathrm{Ra}_m(M) \supsetneq \mathrm{Ra}_{m+1}(M) = \bigcap\{Q(\mathfrak{p});\ h(\mathfrak{p}) = m\} \text{ with}\\ \mathrm{ass}\,(M/\mathrm{Ra}_{m+1}(M)) = \{\mathfrak{p}\in\mathrm{ass}(M);\ h(\mathfrak{p}) = m\}.\end{aligned} \tag{23}$$

   Therefore $\mathrm{Ra}_{m+1}(M)$ *resp.* $M/\mathrm{Ra}_{m+1}(M) = \mathrm{Ra}_m(M)/\mathrm{Ra}_{m+1}(M)$ *are reasonably called the* equidimensional part *resp.* factor *of M.*

4. *Since for all $d\in\mathbb{N}$*

$$\mathrm{Ra}_{d+1}(M) = \mathrm{Ra}_{d+1}\left(\mathrm{Ra}_d(M)\right)\begin{cases} = \mathrm{Ra}_d(M) & \text{if } \mathrm{cd}\left(\mathrm{Ra}_d(M)\right) > d\\ \subsetneq \mathrm{Ra}_d(M) & \text{if } \mathrm{cd}\left(\mathrm{Ra}_d(M)\right) = d\end{cases},$$

   i.e., $\mathrm{Ra}_{d+1}(M)$ *either coincides with* $\mathrm{Ra}_d(M)$ *or is its equidimensional part, equation* (23) *enables the computation of the purity filtration by inductively computing equidimensional parts.*

Proof. 1. By (19) the $Q(\mathfrak{p})$ are $\mathfrak{p}$-primary. From (18) we have $\mathrm{ass}(M/\mathrm{Ra}_d(M)) = \{\mathfrak{p} \in \mathrm{ass}(M);\ h(\mathfrak{p}) < d\}$. With Lemma 2.1,(3), we conclude that (20) is indeed a reduced primary decomposition.
2. The inclusion

$$\mathrm{Ra}_d(M)/(\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p})) \cong (\mathrm{Ra}_d(M)+Q(\mathfrak{p}))/Q(\mathfrak{p}) \subset M/Q(\mathfrak{p})$$

implies $\mathrm{ass}\,(\mathrm{Ra}_d(M)/(\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p}))) \subseteq \{\mathfrak{p}\}$. So $\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p})$ either coincides with $\mathrm{Ra}_d(M)$ or is $\mathfrak{p}$-primary in it. If there is no $\mathfrak{p} \in \mathrm{ass}(M)$ with $h(\mathfrak{p}) = d$ then $\mathrm{Ra}_d(M) = \mathrm{Ra}_{d+1}(M)$ and the primary decomposition is empty. So assume that $\mathfrak{p}_1 \in \mathrm{ass}(M)$ and $h(\mathfrak{p}_1) = d$. We show $\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p}_1) \subsetneq \mathrm{Ra}_d(M)$ by contradiction: Assume

$$\begin{gathered}
\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p}_1) = \mathrm{Ra}_d(M) \text{ or } \mathrm{Ra}_d(M) \subseteq Q(\mathfrak{p}_1) \Longrightarrow \\
\mathrm{Ra}_{d+1}(M) = \mathrm{Ra}_d(M)\bigcap\bigcap\{Q(\mathfrak{p});\ \mathfrak{p} \in \mathrm{ass}(M),\ h(\mathfrak{p}) = d\} = \\
\mathrm{Ra}_d(M)\bigcap\bigcap\{Q(\mathfrak{p});\ \mathfrak{p} \in \mathrm{ass}(M),\ \mathfrak{p} \neq \mathfrak{p}_1,\ h(\mathfrak{p}) = d\} = \\
\bigcap\{Q(\mathfrak{p});\ \mathfrak{p} \in \mathrm{ass}(M),\ \mathfrak{p} \neq \mathfrak{p}_1,\ h(\mathfrak{p}) < d+1\}.
\end{gathered}$$

But this is a contradiction to the reduced primary decomposition (20) of $\mathrm{Ra}_{d+1}(M)$ in M. So the decomposition (21) is a primary decomposition with distinct primes and satisfies condition 1.(ii) from Lemma 2.1. Likewise one sees that also condition 1.(i) of Lemma 2.1 is satisfied and that hence (21) is a reduced primary decomposition with

$$\mathrm{ass}\,(\mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M)) = \{\mathfrak{p} \in \mathrm{ass}(M);\ h(\mathfrak{p}) = d\}.$$

3. and 4. are special cases of 2.

□

Equation (22) shows that the primary decomposition of the zero submodule of the pure d-codimensional factor $\mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M)$ and the ensuing behavior decomposition in (42) below can be computed from the equidimensional part $\mathrm{Ra}_{d+1}(M)$ of $\mathrm{Ra}_d(M)$ and $\mathrm{ass}\,(\mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M))$, the modules $Q(\mathfrak{p})$ with $h(\mathfrak{p}) = d$ are not needed for this purpose.

Example 3.3. Let A be an n-variate polynomial algebra over a field, hence $\dim(A) = n$. This is a *Cohen-Macaulay (CM) ring* and *equidimensional* [10, §17], i.e., for every maximal ideal $\mathfrak{m} \in \max(A)$ the ring $A_\mathfrak{m}$ is a CM ring of dimension n, i.e.

$$\begin{gathered}
\forall \mathfrak{m} \in \max(A):\ \mathrm{depth}(A_\mathfrak{m}) = \dim(A_\mathfrak{m}) = \dim(A) \text{ and then also} \\
\forall \mathfrak{p} \in \mathrm{spec}(A):\ \mathrm{depth}(A_\mathfrak{p}) = \dim(A_\mathfrak{p}),\ h(\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim(A) \\
\forall M \in \mathrm{Mod}_A,\ M \neq 0:\ \mathrm{cd}(M) + \dim(M) = \dim(A).
\end{gathered} \tag{24}$$

For the Serre categories $\mathfrak{C}_d$ this implies

$$\begin{gathered}
\mathfrak{C}_0 = \mathrm{Mod}_A \supseteq \cdots \supseteq \mathfrak{C}_d = \{C \in \mathrm{Mod}_A;\ \mathrm{cd}(C) \geq d\} = \\
\{C \in \mathrm{Mod}_A;\ \dim(C) \leq \dim(A) - d\} \supseteq \cdots \supseteq \mathfrak{C}_{\dim(A)+1} = 0,\ 0 \leq d \leq \dim(A).
\end{gathered} \tag{25}$$

If $\mathfrak{a} = \sum_{i=1}^m A f_i$ then $h(\mathfrak{a}) \leq m$ or $\dim(A/\mathfrak{a}) \geq \dim(A) - m$ [10, Thm. 13.5]. Moreover the *unmixedness theorem* holds for A [10, Thm. 17.6]: If $\mathfrak{a} = \sum_{i=1}^m A f_i$ and $h(\mathfrak{a}) = m$ then $A/\mathfrak{a}$ is pure m-dimensional.

Remark 3.4. Assume that A is affine over a field F, i.e., of the form $A=F[s]/I$ where $F[s]=F[s_1,\cdots,s_n]$ is an n-variate polynomial algebra and I any ideal. We furnish data referring to $F[s]$- resp. A-modules with an index $F[s]$ resp. A if necessary. We identify A-modules N with $F[s]$-modules M annihilated by I by

$$\begin{gathered}M=N \text{ as abelian groups, } (f+I)x=fx,\ f\in F[s],\ f+I\in A,\ x\in M=N. \text{ Then}\\ \mathrm{Mod}_A=\left\{M\in\mathrm{Mod}_{F[s]};\ IM=0\right\},\\ V(I)=\{\mathfrak{p}\in\mathrm{spec}(F[s]);\ I\subseteq\mathfrak{p}\}\cong\mathrm{spec}(A),\ \mathfrak{p}\mapsto\mathfrak{p}/I,\\ \mathrm{supp}({}_{F[s]}M)\subseteq V(I) \text{ for } M\in\mathrm{Mod}_A,\\ \mathrm{supp}({}_{F[s]}M)\cong\mathrm{supp}({}_AM),\ \mathfrak{p}\mapsto\mathfrak{p}/I,\ \mathrm{ass}({}_{F[s]}M)\cong\mathrm{ass}({}_AM),\ \mathfrak{p}\mapsto\mathfrak{p}/I,\\ \dim(F[s]/\mathfrak{p})=\dim\left(A/(\mathfrak{p}/I)\right),\ \mathfrak{p}\in V(I),\ \dim({}_{F[s]}M)=\dim({}_AM) \text{ if } IM=0.\end{gathered} \tag{26}$$

The heights of the prime ideals $\mathfrak{p}\in V(I)$ and $\mathfrak{p}/I$ are related in the following fashion:

$$\begin{gathered}A_{\mathfrak{p}/I}=(F[s]/I)_{\mathfrak{p}/I}=F[s]_{\mathfrak{p}}/I_{\mathfrak{p}}\Longrightarrow\\ h(\mathfrak{p}/I)=\dim\left(F[s]_{\mathfrak{p}}/I_{\mathfrak{p}}\right)=h(\mathfrak{p})-h(I_{\mathfrak{p}})\le h(\mathfrak{p})-h(I) \text{ since}\\ h(I_{\mathfrak{p}})=\min\{h(\mathfrak{p}_1);\ \mathfrak{p}_1\in\mathrm{spec}(F[s]) \text{ minimal with } I\subseteq\mathfrak{p}_1\subseteq\mathfrak{p}\}\\ h(I)=\min\{h(\mathfrak{p}_1);\ \mathfrak{p}_1\in\mathrm{spec}(F[s]) \text{ minimal with } I\subseteq\mathfrak{p}_1\}.\end{gathered} \tag{27}$$

If $V_{F[s]}(I)$ is not pure-dimensional, i.e., if there are prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ which are minimal in $V_{F[s]}(I)$ with $h(I)=h(\mathfrak{p}_2)<h(\mathfrak{p}_1)$ then for $M:=F[s]/\mathfrak{p}_1$ one gets

$$\begin{gathered}\mathrm{ass}_{F[s]}(M)=\{\mathfrak{p}_1\},\ \mathrm{ass}_A(M)=\{\mathfrak{p}_1/I\},\text{ hence}\\ \mathrm{cd}_A(M)=\dim(A_{\mathfrak{p}_1/I})=0<h(\mathfrak{p}_1)-h(I)=\\ \dim(F[s]_{\mathfrak{p}_1})-h(I)=\mathrm{cd}_{F[s]}(M)-h(I).\end{gathered}$$

Therefore there is no simple relation between $\mathrm{cd}_A(M)$ and $\mathrm{cd}_{F[s]}(M)$ for $M\in\mathrm{Mod}_A$ in general. If A is a domain or, equivalently, I is a prime ideal then

$$\begin{gathered}h(\mathfrak{p}/I)=h(\mathfrak{p})-h(I) \text{ and } \mathrm{cd}({}_AM)=\mathrm{cd}\left({}_{F[s]}M\right)-h(I)\ge 0 \text{ if } IM=0,\\ \mathrm{Ra}_{F[s],d}(M)=\mathrm{Ra}_{A,d-h(I)}(M).\end{gathered} \tag{28}$$

Hence for an affine domain $A=F[s]/I$ the purity filtration of A-module ${}_AM$ coincides with that of ${}_{F[s]}M$ up to a translation of the numbering.
For an arbitrary affine F-algebra $A=F[s]/I$ it is preferable to consider the increasing sequence of Serre categories

$$\mathfrak{C}'_{-1}=0\subseteq\mathfrak{C}'_0\subseteq\cdots\subseteq\mathfrak{C}'_d:=\{C\in\mathrm{Mod}_A;\ \dim(C)\le d\}\subseteq\cdots\subseteq\mathfrak{C}'_{\dim(A)}=\mathrm{Mod}_A \tag{29}$$

with their increasing sequence of radicals

$$\mathrm{Ra}'_{-1}(M)=0\subseteq\mathrm{Ra}'_0(M)\subseteq\cdots\subseteq\mathrm{Ra}'_d(M):=\mathrm{Ra}_{\mathfrak{C}'_d}(M)\subseteq\cdots\subseteq\mathrm{Ra}'_{\dim(A)}(M)=M. \tag{30}$$

The analogue of Theorem 3.2 holds. The reduced primary decomposition (19) gives rise, for $0\le d\le\dim(M)$, to the reduced primary decompositions

$$\begin{gathered}\mathrm{Ra}'_d(M)=\bigcap\{Q(\mathfrak{p});\ \mathfrak{p}\in\mathrm{ass}(M),\ \dim(A/\mathfrak{p})>d\}\\ \mathrm{Ra}'_{d-1}(M)=\bigcap\left\{\mathrm{Ra}'_d(M)\bigcap Q(\mathfrak{p});\ \mathfrak{p}\in\mathrm{ass}(M),\ \dim(A/\mathfrak{p})=d\right\}.\end{gathered} \tag{31}$$

In particular, $\mathrm{Ra}'_d(M)/\mathrm{Ra}'_{d-1}(M)$ is pure d-dimensional, i.e.,

$$\dim(A/\mathfrak{p}) = d \text{ for all } \mathfrak{p} \in \mathrm{ass}\left(\mathrm{Ra}'_d(M)/\mathrm{Ra}'_{d-1}(M)\right). \tag{32}$$

According to (26) the equality $\dim({}_{F[s]}M) = \dim({}_AM)$ holds for each A-module M. This implies $\mathrm{Ra}'_d({}_AM) = \mathrm{Ra}'_d({}_{F[s]}M)$ for all d. If ${}_AM$ is a f.g. A-module and

$$0 = \bigcap \left\{Q(\mathfrak{p}); \mathfrak{p} \in \mathrm{ass}_{F[s]}(M)\right\},\ \mathrm{ass}_{F[s]}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}, \tag{33}$$

is a reduced primary decomposition of 0 in M as $F[s]$-module then $IQ(\mathfrak{p}) = 0$, $Q(\mathfrak{p}) \in \mathrm{Mod}_A$ and $\mathrm{ass}({}_AM) = \left\{\mathfrak{p}/I;\ \mathfrak{p} \in \mathrm{ass}_{F[s]}(M\right\}$ and, hence (33) is also a reduced primary decomposition of 0 in M as A-module. The purity filtrations (30) for ${}_AM$ as $F[s]$- and as A-modules coincide and require only the primary decomposition of f.g. polynomial modules over $F[s]$.

4 Application to behaviors

The assumptions of Thm. 3.2 are in force. In addition we assume an injective cogenerator ${}_A\mathscr{F}$. The module $\mathscr{F}$ is interpreted as a space of *signals* on which the ring A of *operators* acts via $a \circ w$, $a \in A, w \in \mathscr{F}$. The discrete resp. continuous standard cases are the following: The ring A is the polynomial algebra $A := F[s] := F[s_1, \cdots, s_n]$ over a field F. In the discrete case $\mathscr{F}$ is the space of n-variate sequences

$$\begin{aligned} \mathscr{F} := F^{\mathbb{N}^n} := \left\{w = (w(\mu))_{\mu \in \mathbb{N}^n} : \mathbb{N}^n \to F,\ \mu \mapsto w(\mu)\right\} \text{ with the shift action} \\ (s^\nu \circ w)(\mu) := w(\mu + \nu),\ \mu, \nu \in \mathbb{N}^n,\ w \in F^{\mathbb{N}^n}. \end{aligned} \tag{34}$$

In the continuous case F is the field $\mathbb{R}$ of real resp. $\mathbb{C}$ of complex numbers and the signal space $\mathscr{F}$ is chosen as the space of smooth functions in $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$ or of distributions, i.e.,

$$\begin{aligned} \mathscr{F} := \mathrm{C}^\infty(\mathbb{R}^n, F) \text{ or } \mathscr{F} := \mathscr{D}'(\mathbb{R}^n, F) \text{ with the action by partial differentiation} \\ s_i \circ w := \partial_i w := \partial w/\partial t_i,\ i = 1, \cdots, n,\ w \in \mathscr{F}. \end{aligned} \tag{35}$$

The action of a matrix $R \in A^{k\times\ell}$ on $w \in \mathscr{F}^\ell$(columns) is given by

$$R \circ w := \left(\sum_{j=1}^{\ell} R_{ij} \circ w_j\right)_{i=1,\cdots,k} \in \mathscr{F}^k. \tag{36}$$

Consider an arbitrary f.g. A-module M with its associated behavior $\mathscr{B}$, viz.

$$\begin{gathered} R \in A^{k\times\ell},\ U := A^{1\times k}R \subseteq A^{1\times\ell},\ M := A^{1\times\ell}/U \\ \mathscr{B} := U^\perp := \left\{w \in \mathscr{F}^\ell;\ \forall \xi \in U:\ \xi \circ w = 0\right\} = \\ \left\{w \in \mathscr{F}^\ell;\ R \circ w = 0\right\} \underset{\text{Malgrange 1962}}{\cong} \mathrm{Hom}_A(M, \mathscr{F}),\ w \longleftrightarrow \varphi \\ \varphi(\delta_j + U) = w_j,\ \delta_j = (0, \cdots, 0, \overset{j}{1}, 0, \cdots, 0),\ j = 1, \cdots, \ell. \end{gathered} \tag{37}$$

We will always identify $U^\perp = \mathrm{Hom}_A(A^{1\times\ell}/U, \mathscr{F})$. Since $\mathscr{F}$ is an injective cogenerator the functor $\mathrm{Hom}_A(-, \mathscr{F})$ preserves and reflects exact sequences, in particular $M = 0$

if and only if $\mathrm{Hom}_A(M,\mathscr{F})=0$. For a submodule M' of M there result the two exact sequences with the canonical injection inj and surjection can:

$$\begin{gathered}0\to M'\overset{\mathrm{inj}}{\longrightarrow}M\overset{\mathrm{can}}{\longrightarrow}M'':=M/M'\to 0\\ 0\leftarrow \mathrm{Hom}_A(M',\mathscr{F})\overset{\mathrm{Hom}(\mathrm{inj},\mathscr{F})}{\longleftarrow}\mathrm{Hom}_A(M,\mathscr{F})\overset{\mathrm{Hom}(\mathrm{can},\mathscr{F})}{\longleftarrow}\mathrm{Hom}_A(M'',\mathscr{F})\leftarrow 0,\\ \text{hence the identifications }\mathrm{Hom}_A(M'',\mathscr{F})=\left\{\varphi\in\mathrm{Hom}_A(M,\mathscr{F});\ \varphi(M')=0\right\},\\ \mathrm{Hom}_A(M,\mathscr{F})/\mathrm{Hom}_A(M/M',\mathscr{F})=\mathrm{Hom}_A(M',\mathscr{F}),\ \varphi+\mathrm{Hom}_A(M/M',\mathscr{F})=\varphi|M'.\end{gathered}\tag{38}$$

Define

$$m:=\mathrm{cd}(M)=\min_{\mathfrak{p}\in\mathrm{ass}(M)}\dim(A_{\mathfrak{p}})\text{ and }p:=\max_{\mathfrak{p}\in\mathrm{ass}(M)}\dim(A_{\mathfrak{p}}).\tag{39}$$

The purity filtration $M=\mathrm{Ra}_m(M)\supset\cdots\supset\mathrm{Ra}_{p+1}(M)=0$ from Thm. 3.2 and Cor. 3.1 induces the filtration of subbehaviors

$$\begin{gathered}0=\mathscr{B}_0=\cdots\mathscr{B}_m\subsetneq\mathscr{B}_{m+1}\subseteq\cdots\subseteq\mathscr{B}_d:=\mathrm{Hom}_A(M/\mathrm{Ra}_d(M),\mathscr{F})\subseteq\mathscr{B}_{d+1}\subseteq\cdots\\ \mathscr{B}_p\subsetneq\mathscr{B}_{p+1}=\mathscr{B}:=\mathrm{Hom}_A(M,\mathscr{F})\text{ with}\\ \mathscr{B}_{d+1}/\mathscr{B}_d\underset{\text{ident.}}{=}\mathrm{Hom}_A\left(\mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M),\mathscr{F}\right).\end{gathered}\tag{40}$$

The reduced primary decomposition (21) and application of the exact functor $\mathrm{Hom}_A(-,\mathscr{F})$ to the ensuing diagonal monomorphisms or *subdirect decompositions*

$$\begin{gathered}M/\mathrm{Ra}_{d+1}(M)\to M/\mathrm{Ra}_d(M)\times\prod_{\mathfrak{p}\in\mathrm{ass}(M),\,h(\mathfrak{p})=d}M/Q(\mathfrak{p})\text{ and}\\ \mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M)\to\prod_{\mathfrak{p}\in\mathrm{ass}(M),\,h(\mathfrak{p})=d}\mathrm{Ra}_d(M)/\left(\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p})\right)\end{gathered}\tag{41}$$

furnish the decompositions

$$\begin{gathered}\mathscr{B}_{d+1}:=\mathscr{B}_d+\sum_{\mathfrak{p}\in\mathrm{ass}(M),\,h(\mathfrak{p})=d}\mathscr{B}_{\mathfrak{p}}\text{ with }\mathscr{B}_{\mathfrak{p}}=\mathrm{Hom}_A(M/Q(\mathfrak{p}),\mathscr{F})\\ \mathscr{B}_{d+1}/\mathscr{B}_d=\sum_{\mathfrak{p}\in\mathrm{ass}(M),\,h(\mathfrak{p})=d}(\mathscr{B}_d+\mathscr{B}_{\mathfrak{p}})/\mathscr{B}_d,\\ (\mathscr{B}_d+\mathscr{B}_{\mathfrak{p}})/\mathscr{B}_d=\mathrm{Hom}_A(\mathrm{Ra}_d(M)/(\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p})),\mathscr{F}),\\ \mathscr{B}=\sum_{\mathfrak{p}\in\mathrm{ass}(M)}\mathscr{B}_{\mathfrak{p}}.\end{gathered}\tag{42}$$

The last sum representation coincides with that in [24, Thm. 7.1]. The preceding data have the following matrix descriptions in which all matrices can be computed in the standard cases:

$$\begin{gathered}M=A^{1\times\ell}/U,\ U=A^{1\times k}R,\ R\in A^{k\times\ell},\ \mathscr{B}=\left\{w\in\mathscr{F}^{\ell};\ R\circ w=0\right\},\\ \mathrm{Ra}_d(M)=A^{1\times k_d}R_d/U,\ R_d\in A^{k_d\times\ell},\ A^{1\times k_d}R_d\supseteq U,\\ M/\mathrm{Ra}_d(M)\underset{\text{ident.}}{=}A^{1\times\ell}/A^{1\times k_d}R_d,\ \mathscr{B}_d=\left\{w\in\mathscr{F}^{\ell};\ R_d\circ w=0\right\},\\ Q(\mathfrak{p})=A^{1\times k_{\mathfrak{p}}}R_{\mathfrak{p}}/U,\ R_{\mathfrak{p}}\in A^{k_{\mathfrak{p}}\times\ell},\ A^{1\times k_{\mathfrak{p}}}R_{\mathfrak{p}}\supseteq U,\ \mathscr{B}_{\mathfrak{p}}=\left\{w\in\mathscr{F}^{\ell};\ R_{\mathfrak{p}}\circ w=0\right\}.\end{gathered}\tag{43}$$

If in addition matrices $P_d \in A^{m_d \times k_d}$ and $P_{\mathfrak{p}} \in A^{m_{\mathfrak{p}} \times k_d}$ are computed such that the maps

$$\begin{aligned} A^{1\times k_d}/A^{1\times m_d}P_d &\cong A^{1\times k_d}R_d/A^{1\times k_{d+1}}R_{d+1} \cong \mathrm{Ra}_d(M)/\mathrm{Ra}_{d+1}(M), \\ \xi + A^{1\times m_d}P_d &\mapsto \xi R_d + A^{1\times k_{d+1}}R_{d+1} \mapsto (\xi R_d + U) + \mathrm{Ra}_{d+1}(M) \\ A^{1\times k_d}/A^{1\times m_{\mathfrak{p}}}P_{\mathfrak{p}} &\cong \mathrm{Ra}_d(M)/\left(\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p})\right), \\ \xi + A^{1\times m_{\mathfrak{p}}}P_{\mathfrak{p}} &\mapsto (\xi R_d + U) + \left(\mathrm{Ra}_d(M)\bigcap Q(\mathfrak{p})\right) \end{aligned} \tag{44}$$

are isomorphisms then also

$$\begin{aligned} R_d\circ : \mathscr{B}_{d+1}/\mathscr{B}_d &\cong \left\{v \in \mathscr{F}^{k_d};\ P_d \circ v = 0\right\} = \\ \sum_{\mathfrak{p}\in \mathrm{ass}(M),\, h(\mathfrak{p})=d} &\left\{v \in \mathscr{F}^{k_d};\ P_{\mathfrak{p}} \circ v = 0\right\},\ w + \mathscr{B}_d \mapsto R_d \circ w, \end{aligned} \tag{45}$$

is an isomorphism.

Remark 4.1. Every f.g A-module M admits a filtration [4, Thms. IV.1.1, IV.1.2]

$$\begin{aligned} &M_0 := 0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_{r-1} \subsetneq M_r = M \text{ with} \\ &M_i/M_{i-1} \cong A/\mathfrak{p}_i,\ \mathfrak{p}_i \in \mathrm{spec}(A),\ i = 1,\cdots,r, \text{ hence} \\ &\qquad \mathrm{ass}(M) \subseteq \{\mathfrak{p}_1,\cdots,\mathfrak{p}_r\} \subseteq \mathrm{supp}(M). \end{aligned} \tag{46}$$

Due to (1) the minimal prime ideals in the three sets of the last row coincide. The filtration induces the behavior filtration

$$\begin{aligned} 0 = \mathscr{B}_r := \mathrm{Hom}_A(M/M_r,\mathscr{F}) \subsetneq \mathscr{B}_i := \mathrm{Hom}_A(M/M_i,\mathscr{F}) \subsetneq \\ \mathscr{B}_{i-1} \subsetneq \cdots \subsetneq \mathscr{B}_0 = \mathrm{Hom}_A(M,\mathscr{F}) \text{ with} \\ \mathscr{B}_{i-1}/\mathscr{B}_i \cong \mathrm{Hom}_A(M_i/M_{i-1},\mathscr{F}) \cong \mathrm{Hom}_A(A/\mathfrak{p}_i,\mathscr{F}) = \mathfrak{p}_i^{\perp} \subsetneq \mathscr{F}. \end{aligned} \tag{47}$$

which can also be used to compute $\mathscr{B}$ by means of simpler subfactors, see Thm. 4.2 for the details. The actual computation of this filtration also requires computation of primary decompositions. Note that $A/\mathfrak{p}$ of codimension $h(\mathfrak{p})$ is the simplest type of a pure codimensional module.

If in the situation of (38) the module $\mathscr{F}$ is not injective it is still possible, at least partially, to describe $\mathrm{Hom}_A(M,\mathscr{F})$ with data from M' and $M'' = M/M'$. The following considerations derive a result by Quadrat and Robertz [17, Thm. 7,§5], [16, Thm. 7, §4] with a different proof and are applicable to arbitrary filtrations. We use the seminal *Homological Algebra* textbook [5] instead of the Baer extensions in the quoted papers. Assume that free resolutions

$$\begin{aligned} \cdots \to A^{1\times s(2)} \xrightarrow{{}_2R'} A^{1\times s(1)} \xrightarrow{R':={}_1R'} A^{1\times s(0)} \xrightarrow{\phi'} M' \to 0 \\ \cdots \to A^{1\times p(2)} \xrightarrow{{}_2R''} A^{1\times p(1)} \xrightarrow{R'':={}_1R''} A^{1\times p(0)} \xrightarrow{\phi''} M'' \to 0 \end{aligned} \tag{48}$$

are given. In Thm. 4.2 these are needed up to $A^{1\times s(1)}$ resp. $A^{1\times p(1)}$ only. The ${}_iR'$, ${}_iR''$ etc. are matrices of appropriate sizes and act on *row* vectors by multiplication on the *right*, ϕ' and ϕ'' are linear maps and the (infinite) sequences of modules in (48) are exact. In [5, Prop. V.2.2] a resolution of the first short exact sequence in (38) is *constructed*, i.e., a further free resolution

$$\begin{gathered}\cdots \longrightarrow A^{1\times(p(1)+s(1))} \overset{R:={}_1R}{\longrightarrow} A^{1\times(p(0)+s(0))} \overset{\phi}{\longrightarrow} M \to 0 \text{ with } \phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\\ {}_iR \in A^{(p(i)+s(i))\times(p(i-1)+s(i-1))},\ R :={}_1 R = \begin{pmatrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{pmatrix} \in A^{(p(1)+s(1))\times(p(0)+s(0))},\end{gathered} \tag{49}$$

such that the following diagram is commutative:

$$\begin{array}{ccccccc} & \cdots & & \cdots & & \cdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & A^{1\times s(1)} & \overset{(0,\mathrm{id}_{s(1)})}{\longrightarrow} & A^{1\times(p(1)+s(1))} & \overset{\begin{pmatrix}\mathrm{id}_{p(1)}\\ 0\end{pmatrix}}{\longrightarrow} & A^{1\times p(1))} & \to 0\\ & \downarrow R' & & \downarrow R & & \downarrow R'' & \\ 0\to & A^{1\times s(0)} & \overset{(0,\mathrm{id}_{s(0)})}{\longrightarrow} & A^{1\times(p(0)+s(0))} & \overset{\begin{pmatrix}\mathrm{id}_{p(0)}\\ 0\end{pmatrix}}{\longrightarrow} & A^{1\times p(0))} & \to 0\\ & \downarrow \phi' & & \downarrow \phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} & & \downarrow \phi'' & \\ 0\to & M' & \overset{\mathrm{inj}}{\longrightarrow} & M & \overset{\mathrm{can}}{\longrightarrow} & M''=M/M' & \to 0\\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array} \tag{50}$$

The commutativity of this diagram immediately implies

$$R = \begin{pmatrix}R'' & R_{12}\\ 0 & R'\end{pmatrix},\ \phi_2 = \phi',\ \mathrm{can}\,\phi_1 = \phi''. \tag{51}$$

The last equation also shows how to construct ϕ [5, Prop. V.2.2]: If $\delta_j,\ j=1,\cdots,p(0)+s(0)$, resp. $\delta''_j,\ j=1,\cdots,p(0)$, denote the standard bases of $A^{1\times(p(0)+s(0))}$ resp. $A^{1\times p(0)}$ then

$$\begin{gathered}\phi(\delta_j)\in M \text{ with } \phi(\delta_j)+M' = \phi''(\delta''_j)\in M/M' \text{ for } j=1,\cdots,p(0) \text{ and}\\ \phi(\delta_j)=\phi'(\delta_j) \text{ for } j=p(0)+1,\cdots,p(0)+s(0).\end{gathered} \tag{52}$$

The snake lemma [5, Lemma III.3.3] implies the exactness of the induced sequence

$$0\to\ker(\phi')\to\ker(\phi)\to\ker(\phi'')\to 0$$

from which the matrix ${}_1R = R$ is constructed like ϕ. The matrices ${}_iR,\ i=2,3,\cdots$, are constructed inductively in the same fashion.

Theorem 4.2. ([17, Thm. 7,§5],[16, Thm. 7, §4]) *With the data from above the map* $\mathrm{Hom}(\phi,\mathscr{F})$ *induces the isomorphism*

$$\begin{gathered}\mathrm{Hom}(\phi,\mathscr{F}):\mathrm{Hom}_A(M,\mathscr{F})\cong\mathscr{B}:=\left\{w=\begin{pmatrix}w''\\ w'\end{pmatrix}\in\mathscr{F}^{p(0)+s(0)};\ R\circ w=0\right\}=\\ \left\{w=\begin{pmatrix}w''\\ w'\end{pmatrix}\in\mathscr{F}^{p(0)+s(0)};\ R''\circ w''+R_{12}\circ w'=0,\ R'\circ w'=0\right\}.\end{gathered} \tag{53}$$

If $\mathscr{F}$ *is injective application of the* exact *functor* $\mathrm{Hom}_A(-,\mathscr{F})$ *to the commutative diagram* (50) *with exact rows and columns furnishes the following diagram with exact* *rows and vertical isomorphisms*

$$\begin{array}{ccccccc}
0 \leftarrow & \mathrm{Hom}_A(M',\mathscr{F}) & \stackrel{\mathrm{Hom}(\mathrm{inj},\mathscr{F})}{\longleftarrow} & \mathrm{Hom}_A(M,\mathscr{F}) & \stackrel{\mathrm{Hom}(\mathrm{can},\mathscr{F})}{\longleftarrow} & \mathrm{Hom}_A(M/M',\mathscr{F}) & \leftarrow 0 \\
 & \downarrow \mathrm{Hom}(\phi',\mathscr{F}) & & \downarrow \mathrm{Hom}(\phi,\mathscr{F}) & & \downarrow \mathrm{Hom}(\phi',\mathscr{F}) & \\
0 \leftarrow & \mathscr{B}' & \stackrel{(0,\mathrm{id}_{s(0)})\circ}{\longleftarrow} & \mathscr{B} & \stackrel{\begin{pmatrix}\mathrm{id}_{p(0)}\\ 0\end{pmatrix}\circ}{\longleftarrow} & \mathscr{B}'' & \leftarrow 0 \\
 & w' & \leftarrow & \begin{pmatrix}w''\\ w'\end{pmatrix},\ \begin{pmatrix}w''\\ 0\end{pmatrix} & \leftarrow & w'' &
\end{array}$$

$$\textit{with } \mathscr{B}' := \left\{w' \in \mathscr{F}^{s(0)},\ R' \circ w' = 0\right\},\ \mathscr{B}'' := \left\{w'' \in \mathscr{F}^{p(0)};\ R'' \circ w'' = 0\right\}. \tag{54}$$

In particular, for every solution w' of $R' \circ w' = 0$ or trajectory $w' \in \mathscr{B}'$ there is a trajectory $w = \begin{pmatrix}w''\\ w'\end{pmatrix} \in \mathscr{B}$ or a solution w'' of the inhomogeneous *equation [16, (88)]*

$$R'' \circ w'' = -R_{12} \circ w', \tag{55}$$

and all trajectories of $\mathscr{B}$ resp. of $\mathrm{Hom}_A(M,\mathscr{F})$ are obtained in this form w resp. as $\mathrm{Hom}(\phi,\mathscr{F})^{-1}(w)$.
If $\mathscr{F}$ is not injective the projection $(0,\mathrm{id}_{s(0)})\circ : \mathscr{B} \to \mathscr{B}'$ or $\mathrm{Hom}(\mathrm{inj},\mathscr{F})$ are not surjective in general, (55) *may be unsolvable for certain computed $w' \in \mathscr{B}'$ and the successive solution method of first w' and then w'' is not applicable. In other words, the filtration $0 \subset \mathrm{Hom}_A(M/M',\mathscr{F}) \subset \mathrm{Hom}_A(M,\mathscr{F})$ still holds, but does not help very much.*
Like [5, Prop. V.2.2] and [17, Thm. 7,§5] this theorem holds for arbitrary, not necessarily commutative, noetherian rings of operators.

Proof. We identify $\mathrm{Hom}_A(A^{1\times(p(0)+s(0))},\mathscr{F}) = \mathscr{F}^{p(0)+s(0)}$ (columns). For all signal modules $\mathscr{F}$ the functor $\mathrm{Hom}_A(-,\mathscr{F})$ is left exact. We apply it to the exact sequence

$$A^{1\times(p(1)+s(1))} \stackrel{R=\begin{pmatrix}R'' & R_{12}\\ 0 & R'\end{pmatrix}}{\longrightarrow} A^{1\times(p(0)+s(0))} \stackrel{\phi}{\longrightarrow} M \to 0$$

and obtain the exact sequence

$$0 \to \mathrm{Hom}_A(M,\mathscr{F}) \stackrel{\mathrm{Hom}(\phi,\mathscr{F})}{\longrightarrow} \mathrm{Hom}_A(A^{1\times(p(0)+s(0))},\mathscr{F}) = \mathscr{F}^{p(0)+s(0)} \stackrel{R\circ}{\longrightarrow} \mathscr{F}^{p(1)+s(1)}$$

which in turn implies the asserted isomorphism (53). The remaining assertions are obvious.

□

The interesting Example 6 from [16] is constructed over the *commutative* polynomial algebra $\mathbb{Q}[\partial_1,\partial_2,\partial_3]$ for which there are, for instance, the injective cogenerator signal modules (34) and (35).

5 Initial conditions according to Riquier

We consider the signal module $\mathscr{F} = F^{\mathbb{N}^n}$ from equation (34) and use the terminology of the Gröbner basis theory. A subset $N \subseteq \mathbb{N}^n$ is called an *order ideal* if $N + \mathbb{N}^n = N$. These order ideals are in bijective correspondence with the monomial ideals $F[N] = \oplus_{\nu\in N} F s^\nu \subseteq A = F[s]$ generated by the monomials s^ν, $\nu \in N$. Let $\leq$ be any *term* *(well)-order* on $\mathbb{N}^n$ and $\deg : F[s] = F[\mathbb{N}^n] \to \mathbb{N}^n \uplus \{-\infty\}$ the corresponding polynomial degree function. Any ideal $\mathfrak{a} \subseteq A$ gives rise to its behavior $\mathfrak{a}^\perp := \left\{w \in F^{\mathbb{N}^n};\ \mathfrak{a} \circ w = 0\right\}$ according to (37) and the degree set $N := \deg(\mathfrak{a}) := \{\deg(f);\ 0 \neq f \in \mathfrak{a}\}$ which is an order ideal. The complement $\Gamma := \mathbb{N}^n \setminus \deg(\mathfrak{a})$ of the degree set is called the *initial region* of $\mathfrak{a}^\perp$ for the chosen term order. Then

$$\mathfrak{a}^\perp \xrightarrow{\cong} F^\Gamma,\ w \mapsto w|\Gamma := (w(\mu))_{\mu \in \Delta}, \tag{56}$$

is an isomorphism [12, Thm. 5]. In other words: For each *initial data* $u = (u(\mu))_{\mu \in \Gamma} \in F^\Gamma$ the *initial value or Cauchy problem*

$$\mathfrak{a} \circ w = 0,\ w|\Gamma = u, \tag{57}$$

has a unique solution w in $\mathfrak{a}^\perp$.
According to Riquier [18] the initial region has an additional structure which was discussed in [12, Algorithm 9]: For every subset $S \subseteq [n] := \{1, \cdots, n\}$ with complement $S' = [n] \setminus S$ we identify

$$\mathbb{N}^S := \left\{\mu \in \mathbb{N}^n;\ \forall i \in S' :\ \mu_i = 0\right\} \subset \mathbb{N}^n. \tag{58}$$

Then there are a finite subset $\Sigma \subset \Gamma$ and subsets $S(\sigma) \subseteq [n]$, $\sigma \in \Sigma$, such that the initial region Γ has the disjoint decomposition

$$\Gamma = \biguplus_{\sigma \in \Sigma} \left(\sigma + \mathbb{N}^{S(\sigma)}\right) \text{ with } \dim(A/\mathfrak{a}) = \max_{\sigma \in \Sigma} \sharp(S(\sigma)), \tag{59}$$

where $\sharp(S(\sigma))$ denotes the number of elements of $S(\sigma)$. In the same algorithm an analogous decomposition of $\deg(\mathfrak{a})$ and corresponding results for submodules instead of ideals were established. Slightly different decompositions into cones with a different algorithm are derived in [14, Lemma 4, Lemma 9] and attributed to Janet [8] who, in his own words, had only simplified Riquier's results. In *Commutative Algebra* Riquier's disjoint decomposition (59) of Γ resp. the corresponding decomposition of $A/\mathfrak{a}$ according to Macaulay are also called a *Stanley decomposition* [22, pp.24-26]. Combining (56) and (59) we obtain the isomorphism

$$\mathfrak{a}^\perp \cong \prod_{\sigma \in \Sigma} F^{\mathbb{N}^{S(\sigma)}},\ w \mapsto (u_\sigma)_{\sigma \in \Sigma},\ u_\sigma(\mu) := w(\sigma + \mu),\ \mu \in \mathbb{N}^{S(\sigma)}. \tag{60}$$

The functions u_σ depend on the $\sharp(S(\sigma))$ *independent variables* $\mu_i \in \mathbb{N}$, $i \in S(\sigma)$.
As mentioned in the Introduction the equations (59) and (60) suggest the following question: Do all sets $S(\sigma)$ have the same cardinality $m := \dim(A/\mathfrak{a})$ if $A/\mathfrak{a}$ is pure m-dimensional, i.e., if $\dim(A/\mathfrak{p}) = m$ for all $\mathfrak{p} \in \operatorname{ass}(A/\mathfrak{a})$? We are going to show by an example that the answer to this question is negative, i.e., the dimensional purity of $A/\mathfrak{a}$ does not imply that of the initial data u_σ, $\sigma \in \Sigma$.
To construct a counter-example we choose $n = 4$. Let $\varepsilon_i \in \mathbb{N}^4$, $i = 1, \cdots, 4$, denote the standard basis of $\mathbb{Z}^4$. We consider the order ideals, initial regions and associated

monomial ideals

$$
\begin{gathered}
N_1 := (\varepsilon_1+\mathbb{N}^4)\bigcup(\varepsilon_2+\mathbb{N}^4) = \left\{\mu\in\mathbb{N}^4;\ \mu_1\ge 1 \text{ or } \mu_2\ge 1\right\},\\
\Gamma_1 := \mathbb{N}^4\setminus N_1 = \left\{\mu\in\mathbb{N}^4;\ \mu_1=\mu_2=0\right\} = \{(0,0)\}\times\mathbb{N}^{\{3,4\}} = (0,0,0,0)+\mathbb{N}^{\{3,4\}}\\
N_2 := (\varepsilon_3+\mathbb{N}^4)\bigcup(\varepsilon_4+\mathbb{N}^4) = \left\{\mu\in\mathbb{N}^4;\ \mu_3\ge 1 \text{ or } \mu_4\ge 1\right\},\\
\Gamma_2 := \mathbb{N}^4\setminus N_2 = \left\{\mu\in\mathbb{N}^4;\ \mu_3=\mu_4=0\right\} = \mathbb{N}^{\{1,2\}}\times\{(0,0)\} = (0,0,0,0)+\mathbb{N}^{\{1,2\}}\\
\mathfrak{p}_1 := F[N_1] = As_1+As_2,\ \deg(\mathfrak{p}_1)=N_1,\ \mathfrak{p}_2 := F[N_2] = As_3+As_4,\ \deg(\mathfrak{p}_2)=N_2\\
\mathfrak{a} := \mathfrak{p}_1\bigcap\mathfrak{p}_2,\ N := \deg(\mathfrak{a}) = N_1\bigcap N_2,\\
\Gamma := \mathbb{N}^4\setminus N = \Gamma_1\bigcup\Gamma_2 =\\
\Gamma_1\biguplus(\Gamma_2\bigcap N_1) = \left((0,0,0,0)+\mathbb{N}^{\{3,4\}}\right)\biguplus\left((\varepsilon_1+\mathbb{N}^{\{1,2\}})\cup(\varepsilon_2+\mathbb{N}^{\{1,2\}})\right) =\\
\left((0,0,0,0)+\mathbb{N}^{\{3,4\}}\right)\biguplus\left(\left(\varepsilon_1+\mathbb{N}^{\{1,2\}}\right)\biguplus\left(\varepsilon_2+\mathbb{N}^{\{2\}}\right)\right).
\end{gathered}
\tag{61}
$$

The ideals $\mathfrak{p}_i$ are monomial prime ideals and $\dim(A/\mathfrak{p}_i)=2$. By Lemma 2.1 the intersection $\mathfrak{a}=\mathfrak{p}_1\bigcap\mathfrak{p}_2$ is the unique primary decomposition of $\mathfrak{a}$ with $\operatorname{ass}(A/\mathfrak{a})=\{\mathfrak{p}_1,\mathfrak{p}_2\}$ and in particular $A/\mathfrak{a}$ is pure 2-dimensional. The isomorphisms $\mathfrak{p}_i^\perp\cong F^{\Gamma_i}\cong F^{\mathbb{N}^2}$ show that every signal in $\mathfrak{p}_i^\perp$ depends on one initial function in two variables. We are going to show that Γ does not admit a disjoint decomposition (59) with $\sharp(S(\sigma))=2,\ \sigma\in\Sigma$, and assume for an indirect proof that such a decomposition is given. The last equality in (61) implies

$$
\begin{gathered}
\forall\sigma\in\Sigma: ((\sigma_1,\sigma_2)=0 \text{ or } (\sigma_3,\sigma_4)=0) \text{ and}\\
S(\sigma)=\begin{cases}\{1,2\} & \text{if } \sigma=(\sigma_1,\sigma_2,0,0)\neq 0\\ \{3,4\} & \text{if } \sigma=(0,0,\sigma_3,\sigma_4)\neq 0 \text{ or } \sigma=(0,0,0,0)\end{cases}.
\end{gathered}
\tag{62}
$$

But

$$
\begin{gathered}
0:=(0,0,0,0),\ \varepsilon_1\in\Gamma\Longrightarrow\exists\sigma_1,\sigma_2\in\Sigma \text{ with } 0\in\sigma_1+\mathbb{N}^{S(\sigma_1)},\ \varepsilon_1\in\sigma_2+\mathbb{N}^{S(\sigma_2)}\Longrightarrow\\
\sigma_1=0,\sigma_2=\varepsilon_1,\ S(0)=\{3,4\},\ S(\varepsilon_1)=\{1,2\}\Longrightarrow\\
\Gamma\underset{(61)}{=}(0+\mathbb{N}^{S(0)})\biguplus(\varepsilon_1+\mathbb{N}^{S(\varepsilon_1)})\biguplus(\varepsilon_2+\mathbb{N}^{\{2\}})\Longrightarrow\\
\varepsilon_2+\mathbb{N}^{\{2\}} = \biguplus_{\sigma\in\Sigma,\ \sigma\neq 0,\varepsilon_1}(\sigma+\mathbb{N}^{S(\sigma)}).
\end{gathered}
\tag{63}
$$

Since all $S(\sigma)$ are assumed to have two elements the last equality in (63) is impossible and furnishes the asserted contradiction. Summing up we obtain

Corollary 5.1. *The ideal* $\mathfrak{a}=\mathfrak{p}_1\bigcap\mathfrak{p}_2$ *from* (61) *has the following properties: It is pure* 2*-dimensional, i.e.,* $\dim(A/\mathfrak{p})=2$ *for all* $\mathfrak{p}\in\operatorname{ass}(A/\mathfrak{a})$*. Since* $\mathfrak{a}$ *is monomial the degree set* $\deg(\mathfrak{a})$ *is unique and does not depend on the chosen term well-order. For every Riquier decomposition* (59) $\Gamma=\biguplus_{\sigma\in\Sigma}\left(\sigma+\mathbb{N}^{S(\sigma)}\right)$ *of the initial region* $\Gamma:=\mathbb{N}^n\setminus\deg(\mathfrak{a})$ *of the associated behavior* $\mathfrak{a}^\perp$ *there is a set* $S(\sigma)$ *with less than* 2 *elements, i.e., there is at least one initial function* $u_\sigma\in F^{\mathbb{N}^{S(\sigma)}}$ *according to* (60) *which depends on at most one independent variable.*

References

[1] M. Barakat, 'Spectral Filtrations via Generalized Morphisms', arXiv/org/abs/:0904.0240 (April 30, 2009)

[2] M. Barakat, 'Purity Filtration and the Fine Structure of Autonomy', *Proceedings of the 19th International Symposium on the Mathematical Theory of Networks and Systems, MTNS 2010, Budapest*, 1657-1661

[3] J.-E. Björk, *Rings of Differential Operators*, North-Holland, Amsterdam, 1979

[4] N. Bourbaki, *Commutative Algebra*, Hermann, Paris, 1972

[5] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956

[6] D. Eisenbud, C. Huneke, W. Vasconcelos, 'Direct methods for primary decompostion', *Invent. Math.* 110 (1992), 207-235

[7] G.-M. Greuel, G. Pfister, *A* **Singular** *Introduction to Commutative Algebra*, Springer, Berlin, 2002

[8] M. Janet, 'Les modules des formes algébriques et la théorie générale des systèmes différentiels, *Ann. Sci. École Norm. Sup.* 41(1924), 27-65

[9] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra 2*, Springer, Berlin, 2005

[10] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986

[11] U. Oberst, 'The significance of Gabriel localization for stability and stabilization of multidimensional input/output behaviors', *Proceedings of the 19th International Symposium on the Mathematical Theory of Networks and Systems MTNS 2010*, Budapest, 1651-1655

[12] U. Oberst, F. Pauer, 'The Constructive Solution of Linear Systems of Partial Difference and Differential Equations with Constant Coefficients', *Multidim. Syst. Signal Proc.* 12(2001), 253-308

[13] U. Oberst, M. Scheicher, 'Time-autonomy and time-controllability of discrete multidimensional behaviours', *Intern. J. Control* 85(2012), 990-1009

[14] W. Plesken, D. Robertz, 'Janet's approach to presentations and resolutions for polynomials and linear pdes', *Arch. Math.* 84(2005, 22-37

[15] J.F. Pommaret, A. Quadrat, 'Algebraic analysis of linear multidimensional control systems', *IMA J. Math. Control and Inform.* 16(1999), 275-297

[16] A. Quadrat, 'Grade Filtration of Linear Functional Systems', Acta Appl. Math. DOI 10.1007/s10440-012-9791-2 (2013), pp.1-62

[17] A. Quadrat, D. Robertz, 'Baer's extension problem for multidimensional linears systems', *Proceedings of the 18th International Symposium on the Mathematical Theory of Networks and Systems MTNS 2008*, Blacksburg (VA), pp.1-12

[18] C. Riquier, *Les Systèmes d'Équations aux Dérivées Partielles*, Gautiers-Villars, Paris, 1910

[19] D. Robertz, 'Formal Algorithmic Elimination for PDEs', Habilitationsschrift, RWTH Aachen, 2012

[20] J.-E. Roos, 'Bidualité et structure des foncteurs dérivés de $\varprojlim$ dans la catégorie des modules sur un anneau régulier', *C.R. Acad. Sci. Paris* 254(1962), 1556-1558

[21] M. Scheicher, U. Oberst, 'Multidimensional stability by Serre categories and the construction and parametrization of observers via Gabriel localizations', *SIAM J. Control and Optimization* 51(2013), 1873-1908

[22] W. Vasconcelos, *Computational Methods in Commutative Algebra and Algebraic Geometry*, Springer, Berlin, 1998

[23] Wood, J., Rogers, E., Owens, D.H. (1998), 'A formal theory of matrix primeness', Math. Control Signals Systems 11, 40-78

[24] J. Wood, U. Oberst, E. Rogers, D.H. Owens, 'A Behavioral Approach to the Pole Structure of One-Dimensional and Multidimensional Linear Systems', *SIAM J. Control Optim.* 38(2000), 627-661