The asymptotic stability of stable and time-autonomous discrete multidimensional behaviors

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Abstract

We generalize the important paper D. Napp-Avelli, P. Rapisarda, P. Rocha, 'Time-relevant stability of 2D systems', *Automatica* 47(2011), 2373-2382, to discrete time-autonomous (ta) (=time-relevant), but *not necessarily square-autonomous* behaviors in *arbitrary dimensions*. This paper and therefore also the present one were essentially influenced by the papers J. Wood, V.R. Sule, E. Rogers, 'Causal and Stable Input/Output Structures on Multidimensional Behaviors', *SIAM J. Control Optim.* 43(2005), 1493-1520, and J.-C. Willems, 'Stability and Quadratic Lyapunov Functions for nD Systems', *Proc. International Conference on Multidimensional (nD) Systems*, Aveiro, Portugal, 2007. In the present paper the discrete domain of the independent variables is the lattice of vectors of integers of arbitrary (but fixed) length whose first component is a natural number and interpreted as a discrete time instant. The stability of an autonomous behavior is defined by a spectral condition on its characteristic variety. The behavior is time-autonomous if each trajectory is determined by a fixed number of its initial values. Under a weak additional condition a discrete stable and time-autonomous behavior is asymptotically stable in the sense that under suitable initial conditions its trajectories converge to zero when the time tends to infinity. We derive algorithms for the constructive verification of the assumptions of most of our results and in particular establish a constructive normal form of ta behaviors in arbitrary dimensions. The Fourier transform on finitely generated free abelian groups plays an important part in the derivations as it already did in the quoted papers. Stability and stabilization of multidimensional discrete behaviors were previously discussed by various colleagues, for instance by Bisiacco, Bose, Fornasini, Lin, Marchesini, Pillai, Quadrat, Rogers, Shankar, Sule, Valcher, Wood, but only partly from the analytic point of view.

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1 Introduction

A discrete linear time-invariant autonomous system or behavior is *asymptotically stable* (abbreviated as "as. stable") in the sense that all its trajectories converge to zero in the positive time direction if and only if it satisfies the *spectral stability condition* that all its characterisitic values belong to the open unit disc (cf. [14, Thm. 7.2.2,(i) and proof, Ex. 7.8 on p.271]). In particular, the powers M^t of a square complex matrix converge to 0 for $t \to \infty$ if and only if the *spectrum* of M, i.e., the set of its eigenvalues, belongs to said disc, whence the term *spectral condition*. The goal of this paper is the generalization of the preceding one-dimensional notions and result to discrete autonomous systems in arbitrary dimensions. For dimension 2 this was done in the important paper [9] by Napp, Rapisarda, Rocha which was essentially influenced by the papers [25] and [24] on continuous multidimensional stability. In dimension 1 bounded input/bounded output stability of an input/output system follows from the internal as. stability of its autonomous part. In the present paper, however, multidimensional input/output stability is not addressed, but was discussed in [25], [17] and [16]. By definition a signal approximates another one if the difference signal is as. stable. Since such approximations are ubiquitous in observer and stabilizing compensator design the significance of as. stable signals and systems for one- and multidimensional systems theory is obvious.

In a higher dimensional real space there is no obvious preferred direction or cone with respect to which as. stability could be defined, but see [13], [23] and [20] for the discussion of such cones. Here as in [9] we define the as. stability of a multidimensional trajectory with respect to a distinguished independent time variable and therefore assume that the trajectories w of the considered discrete behaviors are vector functions

$$w : \mathbb{N} \times \mathbb{Z}^n \to \mathbb{C}^\ell \,(\text{columns}),\ (t,\mu) = (t,\mu_1,\cdots,\mu_n) \mapsto w(t,\mu), \tag{1}$$

where t is interpreted as a discrete time instant and the total system dimension is $1+n,\ n \geq 0$. The group $\mathbb{Z}^n$ is chosen instead of, for instance, the monoid or lattice $\mathbb{N}^n$ for the later application of the *Fourier transform*. The value of w at time t is the function

$$w(t) :\ \mathbb{Z}^n \to \mathbb{C}^\ell,\ \mu \mapsto w(t)(\mu) := w(t,\mu). \tag{2}$$

As. stability of w is then defined by the condition $\lim_{t\to\infty} w(t) = 0$ where different types of convergence can be chosen. We treat L^2-convergence as in [9] and pointwise convergence. In contrast to dimension 1 the as. stability of a multidimensional trajectory requires, besides the spectral condition, additional properties of the functions $w(t)$, for instance being square summable in the L^2-theory. Such properties are assumed as *initial conditions* on $w(t)$ for time instants $t = 0,\cdots,d-1$ and then concluded for all $w(t),\ t \in \mathbb{N}$, by means of *time-autonomy*, whence the significance of this latter property. Here a system is called time-autonomous (abbreviated as "ta") if there is a time instant $d \in \mathbb{N}$ such that all trajectories w of the system are uniquely determined by their d initial data $w(0),\cdots,w(d-1)$ (cf. [12] and its references for the history of this term and its principal contributors). In dimension 1 autonomy and ta coincide.

The multidimensional analogue of the finite set of characteristic values of a one-dimensional autonomous behavior is the *characteristic variety* $\mathrm{char}(\mathscr{B})$ of a multidimensional autonomous behavior $\mathscr{B}$. It is an algebraic subvariety of $\mathbb{C}^{1+n}$ and determines the bases of the powers of the *polynomial-exponential* trajectories of the behavior and therefore the growth and stability properties of these trajectories. In contrast to dimension 1 a multidimensional autonomous behavior $\mathscr{B}$ contains, in general, many trajectories which are not polynomial-exponential. Therefore a direct translation of properties of $\mathrm{char}(\mathscr{B})$ into properties of all trajectories of $\mathscr{B}$ is impossible. The characteristic variety has played a prominent role in *Algebraic Analysis* since the fundamental work of Ehrenpreis, Malgrange and Palamodov on linear systems of partial differential equations with constant coefficients in the 1960s and is exposed in the needed generality for the present paper in [18]. The spectral stability condition of the one-dimensional case now obtains the form

$$\begin{gathered}\mathrm{char}(\mathscr{B})\bigcap\Lambda_2=\emptyset \text{ where}\\ \Lambda_2:=\left\{(\lambda,\omega_1,\cdots,\omega_n)\in\mathbb{C}^{1+n};\ |\lambda|\geq 1,\ \forall i=1,\cdots,n:\ |\omega_i|=1\right\}\end{gathered}\tag{3}$$

is the chosen *region of instability* of this paper. In dimension 1 this region Λ_2 is the closed exterior of the unit disc. The chosen instability region generalizes that of [9] which, in turn, was influenced by the choices in [25] and [24] in the continuous case. Below in (16) we will justify this choice by an argument from [25, §3].
In the main stability result Thm. 2.3 of this paper we generalize [9, Thm. 10] to not necessarily square autonomous discrete behaviors in arbitrary dimensions and show that time-autonomy (=*time-relevance* in [9]) and spectral stability of a multidimensional autonomous behavior and a weak additional condition imply asymptotic stability under suitable initial conditions. In dimension 2 square autonomy can be assumed w.l.o.g. [22, Prop. 2.3,(i)], [9, Lines before Thm. 10]. In Example 5.13,(i), we show that in contrast to dimension 1 time-autonomy and asymptotic L^2-stability do not imply spectral stability and thus slightly correct [9, Thm. 10]. Example 5.14 contains a family of examples for our main Thm. 2.3 in dimension $3=1+2$ with behaviors that are not square-autonomous in contrast to [9, Thm. 10]. Thm. 2.1 gives a new constructive characterization of ta.
The whole paper is written in the language of multidimensional behaviors over a suitable injective cogenerator signal module and makes constant use of the ensuing duality between these behaviors and finitely generated modules over the associated operator domain (cf., for instance, [12] or [18]). The principal additional technical tool for the proof of the main result as already in [9] is the Fourier transform for the group $\mathbb{Z}^n$ which is employed like the Fourier transform on $\mathbb{R}^n$ in connection with hyperbolic and parabolic systems of partial differential equations [6, Ch. III].
Section 2 contains a summary of the paper in precise mathematical language which is then used in all subsequent sections. In Section 3 we derive new characterizations of discrete ta behaviors that improve those from [12, Thm. 3.7]. Section 4 introduces the discrete Fourier transform according to [19, §VII.1] and derives several consequences for input/output behaviors. Section 5 contains the paper's main results on asymptotic stability. In Section 6 we show the constructivity of most of the paper's results.
Stability and stabilization of multidimensional discrete systems were studied by many authors, for instance by Bose [1], Shankar/Sule [21], Lin [8], Quadrat [15]. The survey article Oberst/Scheicher [11] contains a comprehensive list of references. The analytic properties of trajectories of stable behaviors have been treated more rarely, for instance by Bisiacco/Fornasini/Marchesini [5], [3], [4], Pillai/Shankar [13], Valcher [23], Shankar [20], Wood/Sule/Rogers [25]. The paper [9] was essential for our present considerations.

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2 A mathematical summary

The main results of this paper are summed up in Thms. 2.1 and 2.3. To enable the understanding of these theorems we first introduce the required terminology which is also used in all subsequent sections.

As indicated in the Introduction we consider discrete $1+n$-dimensional behaviors over the lattice $N:=\mathbb{N}\times\mathbb{Z}^n$, $n\geq 1$, of independent variables $(t,\mu)=(t,\mu_1,\cdots,\mu_n)$ where the distinguished first component t is interpreted as discrete time. Due to its importance for the Fourier transform we use the base field $\mathbb{C}$ of complex numbers and the signal spaces

$$\begin{gathered} W_A:=\mathbb{C}^{\mathbb{Z}^n} \text{ and } W_B:=\mathbb{C}^{\mathbb{N}\times\mathbb{Z}^n}=W_A^{\mathbb{N}},\\ W_B\ni w=(w(t,\mu))_{(t,\mu)\in N}=(w(0),w(1),\cdots),\ w(t)\in W_A,\ w(t)(\mu)=w(t,\mu). \end{gathered} \tag{4}$$

The signals $w\in W_B$ are thus interpreted as time series of signals $w(t)\in W_A=\mathbb{C}^{\mathbb{Z}^n}$ at the discrete time instance $t\in\mathbb{N}$. Let $(s_0,s)=(s_0,s_1,\cdots,s_n)$ be a list of indeterminates. The associated operator $\mathbb{C}$-algebras are

$$\begin{gathered} A=\mathbb{C}[\mathbb{Z}^n]=\mathbb{C}[s,s^{-1}]=\oplus_{\mu\in\mathbb{Z}^n}\mathbb{C}s^{\mu},\ s^{-1}:=(s_1^{-1},\cdots,s_n^{-1}), \text{ and}\\ B:=\mathbb{C}[N]=A[s_0]=\mathbb{C}[s_0,s,s^{-1}]=\oplus_{t\in\mathbb{N},\mu\in\mathbb{Z}^n}\mathbb{C}s_0^ts^{\mu}. \end{gathered} \tag{5}$$

So A and B are Laurent polynomial algebras which act on W_A resp. W_B by the standard shift actions $\circ$, for instance $(s^{\mu}\circ u)(\nu)=u(\mu+\nu)$ for $u\in W_A$ and $\mu,\nu\in\mathbb{Z}^n$. There are the canonical identifications

$$A^*:=\mathrm{Hom}_{\mathbb{C}}(A,\mathbb{C})=W_A\ni u,\ u(s^{\mu})=u(\mu), \text{ and } B^*=W_B. \tag{6}$$

The modules W_A resp. W_B are *large injective cogenerators* in the category Mod_A of A-modules resp. in Mod_B and give rise to the standard strong *duality* between finitely generated (abbreviated as "f.g.") modules and behaviors, compare (7) and, for instance, [12, Sect. 2].

The signal space W_A contains the spaces $\mathscr{S}'(\mathbb{Z}^n)\supset \mathrm{L}^2(\mathbb{Z}^n)$ (cf. (36)- (37)): A signal $u=(u(\mu))_{\mu\in\mathbb{Z}^n}\in W_A$ belongs to $\mathscr{S}'(\mathbb{Z}^n)$ if $u(\mu)$ grows at most polynomially with μ and to $\mathrm{L}^2(\mathbb{Z}^n)$ if $\sum_{\mu\in\mathbb{Z}^n}|u(\mu)|^2<\infty$.

We need W_B- and W_A-behaviors. A W_B-behavior $\mathscr{B}\subseteq W_B^{\ell}$ is defined by a matrix and its associated f.g. B-modules

$$\begin{gathered} R\in B^{k\times\ell},\ U:=B^{1\times k}R:=\sum_{i=1}^{k}BR_{i-}\subseteq B^{1\times\ell},\ M:=B^{1\times\ell}/U:\\ \mathrm{sol}_{W_B}(M)\underset{\text{Malgrange 1962}}{:=}D(M):=\mathrm{Hom}_B(M,W_B)\underset{\text{Malgrange 1962}}{\cong}\\ \mathscr{B}:=\left\{w\in W_B^{\ell};\ U\circ w=0\right\}=\left\{w\in W_B^{\ell};\ R\circ w=0\right\}. \end{gathered} \tag{7}$$

Here $B^{1\times\ell}$ resp. $W_B^{\ell}:=W_B^{\ell\times 1}$ consist of row resp. column vectors and the action $\circ$ of B on W_B is extended to matrix actions

$$B^{k\times\ell}\times W_B^{\ell\times m}\to W_B^{k\times m},\ (R,X)\mapsto R\circ X,\ (R\circ X)_{ij}:=\sum_{q=1}^{\ell}R_{iq}\circ X_{qj}. \tag{8}$$

The modules U resp. M are called the *equation module* resp. the *system module* of the behavior $\mathscr{B}$. The behavior $\mathscr{B}$ is called *autonomous* if no component w_j of the trajectories $w=(w_1,\cdots,w_\ell)^\top\in\mathscr{B}$ is free, i.e., if for each $j=1,\cdots,\ell$ there is a nonzero $f_j\in B$ such that $f_j\circ w_j=0$ for all $w\in\mathscr{B}$. By duality autonomy of $\mathscr{B}$ is equivalent to the following properties: $\operatorname{rank}(R)=\ell$ or M is a *torsion module* or the annihilator ideal

$$\operatorname{ann}_B(M):=\{b\in B;\,bM=0\}\underset{\text{duality}}{=}\operatorname{ann}_B(\mathscr{B}) \tag{9}$$

is *nonzero*. For purposes of algebraic geometry we need the set

$$\Lambda_N:=\mathbb{C}\times(\mathbb{C}\setminus\{0\})^n=\{(\lambda,\omega);\;\lambda\in\mathbb{C},\;\omega\in(\mathbb{C}\setminus\{0\})^n\} \tag{10}$$

of $1+n$-dimensional complex vectors which can be substituted into Laurent polynomials $f(s_0,s)\in B$. The *variety* or *vanishing set* of an ideal $\mathfrak{b}$ of B is

$$V_{\Lambda_N}(\mathfrak{b}):=\{(\lambda,\omega)\in\Lambda_N;\,\forall g\in\mathfrak{b};\,g(\lambda,\omega)=0\}. \tag{11}$$

The *characteristic variety* [18, (51)] of an autonomous behavior $\mathscr{B}$ or its torsion system module M from (7) is the variety

$$\begin{aligned}\operatorname{char}(\mathscr{B}):=\operatorname{char}(M):=\;&\{(\lambda,\omega)\in\Lambda_N;\,\operatorname{rank}(R(\lambda,\omega))<\operatorname{rank}(R)=\ell\}=\\ &V_{\Lambda_N}(\operatorname{ann}_B(M))=V_{\Lambda_N}(\operatorname{ann}_B(\mathscr{B})).\end{aligned} \tag{12}$$

The last equation shows that it depends on $\mathscr{B}$ only and not on the special choice of R. For stability purposes we have to use the discs and tori

$$\begin{aligned}&\mathbb{D}^1:=\{z\in\mathbb{C};\;|z|<1\},\overline{\mathbb{D}}^1:=\{z\in\mathbb{C};\;|z|\le 1\},\\ &\mathbb{T}^1:=\{z\in\mathbb{C};|z|=1\}=\partial(\mathbb{D}^1),\;\mathbb{T}^n:=\left(\mathbb{T}^1\right)^n,\text{ hence}\\ &\mathbb{R}/\mathbb{Z}\cong\mathbb{T}^1,\;x+\mathbb{Z}\mapsto\exp(2\pi ix),\text{ and }\mathbb{R}^n/\mathbb{Z}^n\cong\mathbb{T}^n\end{aligned} \tag{13}$$

and the disjoint *stability decomposition*

$$\Lambda_N=\Lambda_1\uplus\Lambda_2,\;\Lambda_2:=\{\lambda\in\mathbb{C};\;|\lambda|\ge 1\}\times\mathbb{T}^n \tag{14}$$

where Λ_1 resp. Λ_2 are considered as the *stable resp. unstable region*, the instability region Λ_2 coming from (3).

Definition 2.1. Consider an autonomous behavior $\mathscr{B}$ from (7) with its torsion system module M. Then $\mathscr{B}$ and M are called Λ_1-stable if the spectral conditions

$$\operatorname{char}(\mathscr{B})\subseteq\Lambda_1\text{ or, equivalently, }\operatorname{char}(\mathscr{B})\bigcap\Lambda_2=\emptyset$$

are satisfied.

The Λ_1-stability of $\mathscr{B}$ also signifies that

$$(\lambda,\omega)\in\operatorname{char}(\mathscr{B})\text{ and }\omega\in\mathbb{T}^n\text{ imply }\lambda\in\mathbb{D}^1. \tag{15}$$

As announced in the Introduction we give an argument from [25, §3] in the continuous case, for simplicity in dimension $2=1+1$, which suggests this special choice of Λ_2 to reach asymptotic stability: A typical polynomial-exponential trajectory of a behavior is a finite sum of trajectories

$$\begin{aligned}&w(t,\mu)=w_0(t,\mu)\lambda^t\omega^\mu,\;0\ne w_0\in\mathbb{C}[t,\mu]^\ell,\\ &0\ne\lambda,\omega\in\mathbb{C},\;(\lambda,\omega)\in\operatorname{char}(\mathscr{B}),\;t\in\mathbb{N},\mu\in\mathbb{Z}.\end{aligned} \tag{16}$$

The function w_0 is polynomial in t and μ. If $|\omega|\neq 1$ then the functions $w(t)$ grow fast for $\mu\to\infty$ or $\mu\to-\infty$ if $w_0(t,-)\neq 0$ and such trajectories are undesirable, for instance by reasons of physics, and excluded by suitable initial conditions. If $|\omega|=1$ and $|\lambda|\geq 1$ then $w(t,\mu)$ does not converge to zero for $t\to\infty$ if $w_0(-,\mu)\neq 0$ and this prevents as. stability. Such trajectories are excluded too by choosing the instability region $\Lambda_2:=\left\{(\lambda,\omega)\in\mathbb{C}^2;\ |\lambda|\geq 1,\ |\omega|=1\right\}$ from (3) or (14).
The W_B-behavior $\mathscr{B}\subset W_B^\ell$ is called *time-autonomous* (ta) or *time-relevant* [9, Def. 5] if there is a time instant d such that the projection

$$\begin{gathered}\mathrm{proj}:\mathscr{B}\to\left(W_A^\ell\right)^d=W_A^{\ell d},\ w\mapsto(w(0),\cdots,w(d-1))^\top,\ \text{is injective and hence}\\ \mathrm{proj}:\mathscr{B}\cong\mathscr{B}_d:=\mathrm{proj}(\mathscr{B}).\end{gathered}\tag{17}$$

Here the image $\mathscr{B}_d:=\mathrm{proj}(\mathscr{B})\subseteq W_A^{\ell d}$ is a computable W_A-behavior. The following theorem is a summary of the Thms. 3.2, 3.4 and 6.5 where we give a new constructive characterization of ta behaviors.

Theorem 2.2. *The behavior $\mathscr{B}$ from* (7) *is ta if and only if the B-module M is f.g. as A-module or if and only if $\mathscr{B}$ is isomorphic to a behavior $\mathscr{B}'=\left\{w'\in W_B^{\ell'};\ R'\circ w'=0\right\}\subset W_B^{\ell'}$ where the matrix R' and the behavior $\mathscr{B}'$ have the following* ta normal form*:*

$$\begin{gathered}R'=\begin{pmatrix}R_0\\ s_0\,\mathrm{id}_{\ell'}-E\end{pmatrix},\ R_0\in A^{k'\times\ell'},\ E\in A^{\ell'\times\ell'},\ \exists X\in A^{k'\times k'}\ \text{with}\ R_0E=XR_0,\\ \mathscr{B}'=\left\{w'\in W_B^{\ell'};\ R'\circ w'=0\right\}=\\ =\left\{w'\in W_B^{\ell'};\ R_0\circ w'(0)=0,\ \forall t\in\mathbb{N}:\ w'(t)=E^t\circ w'(0)\right\}.\end{gathered}\tag{18}$$

Thus a trajectory $w'\in W_B^{\ell'}$ belongs to $\mathscr{B}'$ if and only if its initial value $w'(0)$ satisfies $R_0\circ w'(0)=0$ and if the remaining components $w'(t)$ are determined by the dynamic equations $w'(t)=E^t\circ w'(0)$. In other words, the projection

$$\mathscr{B}'\cong\mathscr{B}_1':=\left\{v'\in W_A^{\ell'};\ R_0\circ v'=0\right\},\ w'\mapsto w'(0),\tag{19}$$

from the W_B-behavior $\mathscr{B}'$ onto the W_A-behavior $\mathscr{B}_1'$ is an isomorphism.

The existence of X with the commutation relation $R_0E=XR_0$ ensures that the W_A-equations $R_0\circ w'(0)=0$ and $w'(t)=E^t\circ w'(0)$ imply the W_B-equation $R_0\circ w=0$.

Let $\mathrm{C}^0(\mathbb{T}^n)$ resp. $\mathscr{D}(\mathbb{T}^n)$ denote the $\mathbb{C}$-algebras of continuous resp. smooth functions on $\mathbb{T}^n$. If $g\in A$ is nonzero its set of zeros in $\mathbb{T}^n$ has measure zero (Lemma 4.2). If g has no zeros in $\mathbb{T}^n$ the function $g^{-1}:\omega\mapsto g(\omega)^{-1}$ is smooth. So many rational functions $H\in\mathbb{C}(s)$ can be considered as continuous or smooth functions on $\mathbb{T}^n$ and the same applies to matrices $H\in\mathbb{C}(s)^{p\times m}$.
The main result of the paper is the following condensed form of Thms. 5.6, 5.11, 4.7,(2) and Lemma 4.1.

Theorem and Definition 2.3. *Assume that the autonomous behavior $\mathscr{B}$ from* (7) *is Λ_1-stable and time-autonomous with the isomorphic projection* $\mathrm{proj}:\mathscr{B}\cong\mathscr{B}_d$*. In addition assume any input/output representation of $\mathscr{B}_d$ with transfer matrix $H\in\mathbb{C}(s)^{p\times m}$:*

$$\begin{gathered}\mathscr{B}_d=\left\{\begin{pmatrix}y\\ u\end{pmatrix}\in W_A^{p+m};\ P\circ y=Q\circ u\right\},\ \ell d=p+m,\\ (P,-Q)\in A^{k\times(p+m)},\ \mathrm{rank}(P)=\mathrm{rank}(P,-Q)=p,\ PH=Q.\end{gathered}\tag{20}$$

If $w \in \mathscr{B}$,

$$w(0),\cdots,w(d-1)\in\begin{cases}(i)\ \mathrm{L}^2(\mathbb{Z}^n)^\ell\\(ii)\ \mathscr{S}'(\mathbb{Z}^n)^\ell\end{cases}\quad and\ \begin{cases}(i)\ H\ is\ continuous\ on\ \mathbb{T}^n\\(ii)\ \mathrm{rank}(P(\omega))=p\ for\ \omega\in\mathbb{T}^n\end{cases}$$
$$then\ \forall t\in\mathbb{N}:\ w(t)\in\begin{cases}(i)\ \mathrm{L}^2(\mathbb{Z}^n)^\ell\\(ii)\ \mathscr{S}'(\mathbb{Z}^n)^\ell\end{cases}\quad and\ \begin{cases}(i)\ \lim_{t\to\infty}w(t)=0\in\mathrm{L}^2(\mathbb{Z}^n)^\ell\\(ii)\ \forall\mu\in\mathbb{Z}^n:\lim_{t\to\infty}w(t,\mu)=0\end{cases}, \tag{21}$$

where the limit in $\mathrm{L}^2(\mathbb{Z}^n)$ *is taken with respect to the norm* $\|u\|_2=\left(\sum_{\mu\in\mathbb{Z}^n}|u(\mu)|^2\right)^{1/2}$. *Moreover the projections*

$$\begin{cases}(i)\ \mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}\overset{\mathrm{proj}}{\longrightarrow}\mathscr{B}_d\bigcap\mathrm{L}^2(\mathbb{Z}^n)^{p+m}\overset{\mathrm{proj}}{\longrightarrow}\mathrm{L}^2(\mathbb{Z}^n)^m\\(ii)\ \mathscr{B}\bigcap\left(\mathscr{S}'(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}\overset{\mathrm{proj}}{\longrightarrow}\mathscr{B}_d\bigcap\mathscr{S}'(\mathbb{Z}^n)^{p+m}\overset{\mathrm{proj}}{\longrightarrow}\mathscr{S}'(\mathbb{Z}^n)^m\end{cases}, \tag{22}$$
$$w\mapsto(w(0),\cdots,w(d-1))^\top=\binom{y}{u}\mapsto u,$$

are bijective, in particular, the required initial conditions (i) $w(t)\in\mathrm{L}^2(\mathbb{Z}^n)^\ell$ *resp. (ii)* $w(t)\in\mathscr{S}'(\mathbb{Z}^n)^\ell$ *for* $0\le t\le d-1$ *can be satisfied by an arbitrary choice of the input component* u *in (i)* $\mathrm{L}^2(\mathbb{Z}^n)^m$ *resp. (ii)* $\mathscr{S}'(\mathbb{Z}^n)^m$.
The assertions in (21) *and* (22) *up to the second projection in* (22) *also hold if in addition to ta and* Λ_1*-stability the behavior* $\mathscr{B}$ *from* (7) *is* square-autonomous *(compare [9, Thm. 10] for* $n=1$*) with* $R\in B^{\ell\times\ell}$ *and* $\det(R)\neq 0$ *or if* $\mathscr{B}_d$ *is* strictly controllable*, i.e.,* $\mathscr{B}_d\cong W_A^m$ *for some m.*
The limit conditions in (21) *for (i)* $w\in\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}$ *resp. (ii)* $w\in\mathscr{B}\bigcap\left(\mathscr{S}'(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}$ *are called* asymptotic L^2- *resp.* pointwise stability *of* $\mathscr{B}$.

Recall that every behavior admits various input/output (abbreviated as "IO") decompositions as used in (20). At present we do not know whether the assertions of the preceding theorem hold without the assumed properties of the IO decomposition of $\mathscr{B}_d$.

3 Time-autonomous behaviors

The characterization of ta in Thms. 3.2 and 3.4 below completes that of [12, Thm. 3.7] with a substantially simpler proof. As in [12, Thm. 3.7] the following considerations hold for an arbitrary F-affine integral domain A over a field F, the injective A-cogenerator $W_A:=A^*:=\mathrm{Hom}_F(A,F)$ *with its canonical A-action, the polynomial algebra* $B:=A[s_0]$ *and the injective B-cogenerator*

$$W_B=\mathrm{Hom}_F(B,F)\underset{\text{ident.}}{=}\mathrm{Hom}_A(B,W_A)\underset{\text{ident.}}{=}W_A^{\mathbb{N}}.$$

For the modules and behavior from (7) let $\delta_j:=(0,\cdots,0,\overset{j}{1},0,\cdots,0),\ j=1,\cdots,\ell$, denote the standard basis of $\mathbb{C}^{1\times\ell}$, hence

$$M=B^{1\times\ell}/U=\sum_{t,\mu,j}\mathbb{C}(s_0^t s^\mu\delta_j+U)=\sum_{t,j}A(s_0^t\delta_j+U)=\sum_j B(\delta_j+U).$$

There are the canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}}(M,\mathbb{C})\cong\mathrm{Hom}_A(M,W_A)\cong\mathrm{Hom}_B(M,W_B)\cong\mathscr{B},\ \alpha\mapsto\varphi\mapsto\phi\mapsto w,\\ \text{with } w_j(t,\mu)=\alpha(s_0^t s^\mu\delta_j+U)=\varphi(s_0^t\delta_j+U)(\mu)=\phi(\delta_j+U)(t,\mu).\end{aligned} \tag{23}$$

We will mostly identify all isomorphic modules from (23).
In the remainder of this section we assume that $\mathscr{B}$ *from* (7) *is autonomous and that* $\mathfrak{b} := \mathrm{ann}_B(M)$ *is the nonzero annihilator of the system module* M. Then M is a f.g. $B/\mathfrak{b}$-module and

$$\overline{B} := B/\mathfrak{b} = A[\overline{s_0}] \to M^\ell,\ \overline{g} = g + \mathfrak{b} \mapsto (g\delta_1 + U, \cdots, g\delta_\ell + U) \tag{24}$$

is injective. We consider the filtration of $B = A[s_0]$ and of $B^{1\times\ell}$ by the A-submodules

$$B_d := \oplus_{q=0}^{d-1} A s_0^q,\ d \geq 0, \text{ hence } B_0 = 0,\ B_1 = A,\ B_d^{1\times\ell} = \oplus_{q=0}^{d-1} \oplus_{j=1}^{\ell} A s_0^q \delta_j. \tag{25}$$

By duality the injection $\mathrm{inj} : B_d^{1\times\ell} \subset B^{1\times\ell}$ induces the surjective projection

$$\begin{gathered}\mathrm{proj} := \mathrm{inj}^* :\ W_B^\ell = (W_A^\ell)^{\mathbb{N}} = \mathrm{Hom}_{\mathbb{C}}(B^{1\times\ell}, \mathbb{C}) = \mathrm{Hom}_A(B^{1\times\ell}, W_A) \longrightarrow \\ \left(W_A^\ell\right)^d = W_A^{\ell d} = \mathrm{Hom}_{\mathbb{C}}(B_d^{1\times\ell}, \mathbb{C}) = \mathrm{Hom}_A(B_d^{1\times\ell}, W_A),\ w \mapsto (w(0), \cdots, w(d-1))^\top.\end{gathered} \tag{26}$$

The A-modules $U_d := B_d^{1\times\ell} \bigcap U$ and $M_d := B_d^{1\times\ell}/U_d$ give rise to the dual commutative diagrams with the canonical maps

$$\begin{array}{ccc} B_d^{1\times\ell} & \stackrel{\mathrm{inj}}{\longrightarrow} & B^{1\times\ell} \\ \downarrow \mathrm{can} & & \downarrow \mathrm{can} \\ M_d & \stackrel{\mathrm{inj_{ind}}}{\longrightarrow} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} (W_A^\ell)^d = \left(B_d^{1\times\ell}\right)^* & \stackrel{\mathrm{proj}}{\longleftarrow} & (W_A^\ell)^{\mathbb{N}} = \left(B^{1\times\ell}\right)^* \\ \bigcup & & \bigcup \\ U_d^\perp = M_d^* & \stackrel{\mathrm{proj}}{\longleftarrow} & \mathscr{B} = M^* = U^\perp \end{array} \tag{27}$$

where the induced map $\mathrm{inj_{ind}}$ is injective and hence $\mathrm{proj} : \mathscr{B} \to U_d^\perp$ is surjective. Notice in particular that

$$U_1 = A^{1\times\ell} \bigcap U \subseteq U_2 \subseteq \cdots \subseteq U = \bigcup_{d=1}^{\infty} U_d \text{ and } M_1 = A^{1\times\ell}/U_1 \subseteq M_2 \subseteq \cdots \subseteq \bigcup_{d=1}^{\infty} M_d = M. \tag{28}$$

Summing up we obtain

Corollary 3.1. *For the preceding data the image of the* W_B*-behavior* $\mathscr{B} = U^\perp \subseteq W_B^\ell$ *under the projection* $\mathrm{proj} : W_B^\ell = \left(W_A^\ell\right)^{\mathbb{N}} \to \left(W_A^\ell\right)^d$ *is the* W_A*-behavior*

$$\mathscr{B}_d := \mathrm{proj}(\mathscr{B}) = \left\{(w(0), \cdots, w(d-1))^\top;\ w \in \mathscr{B}\right\} = U_d^\perp \subseteq \left(W_A^\ell\right)^d = W_A^{\ell d}.$$

Theorem 3.2. *Assume that the behavior* $\mathscr{B}$ *from* (7) *is autonomous with the derived data from* (24)-(28)*. Then the following assertions are equivalent:*

1. $\mathscr{B}$ *is time-autonomous (ta).*
2. *The f.g.* B*-module* M *is f.g. as* A*-module and hence (cf.* (23)*)*

$$\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}) = \mathrm{Hom}_A(M, W_A) = \mathrm{Hom}_B(M, W_B) = \mathscr{B}$$

   is also a W_A*-behavior.*
3. *The element* $\overline{s_0} = s_0 + \mathfrak{b} \in \overline{B} = B/\mathfrak{b} = A[\overline{s_0}]$ *is integral over* A.
4. *There is a monic (in* s_0*) polynomial* $f = s_0^d + f_{d-1}s_0^{d-1} + \cdots + f_0 \in B = A[s_0]$ *which annihilates* M *or* $\mathscr{B}$*, i.e.,* $f \in \mathfrak{b}$.

In particular, ta of $\mathscr{B}$ depends on $\mathscr{B}$ only and is independent of its matrix representation.

Proof. 1. $\Longrightarrow$ 2.: By definition and by Cor. 3.1 there is $d \in \mathbb{N}$ such that $\mathrm{proj} : \mathscr{B} \to \mathscr{B}_d$ is bijective. With (27) we conclude that also $M_d = M$ which is A-f.g..
2. $\Longrightarrow$ 1.: For A-f.g. M equation (28) implies the existence of $d \in \mathbb{N}$ with $M_d = M$ and then the isomorphism $\mathrm{proj} : \mathscr{B} \cong \mathscr{B}_d$.
2. $\Longrightarrow$ 3.: If M is A-f.g. the injection $\overline{B} = B/\mathfrak{b} \to M^\ell$ from (24) implies that $\overline{B}$ is also A-f.g. In particular $\overline{B}$ and especially $\overline{s_0}$ are integral over A.
3. $\Longrightarrow$ 4.: Since $\overline{s_0}$ is integral over A there is a monic univariate polynomial $f(x) = x_0^d + f_{d-1}x^{d-1} + \cdots + f_0 \in A[x]$ such that $f(\overline{s_0}) = 0$ or $f(s_0) = s_0^d + f_{d-1}s_0^{d-1} + \cdots + f_0 \in \mathfrak{b}$.
4. $\Longrightarrow$ 2.: If $\overline{s_0} \in \overline{B} = A[\overline{s_0}]$ is integral over A then $\overline{B}$ is a f.g. A-module. But M is a f.g. B- and $\overline{B}$-module. Together these properties imply that M is a f.g. A-module. ☐

The more difficult proof of [12, Thm. 3.7] is still useful for the actual computation of a monic polynomial $f = s_0^d + f_{d-1}s_0^{d-1} + \cdots + f_0 \in B$ with $fM = 0$, $f \circ \mathscr{B} = 0$ according to [12, Cor. 3.8].
For $w \in W_B^\ell = \left(W_A^\ell\right)^{\mathbb{N}}$ define

$$x \in \left(W_A^{\ell d}\right)^{\mathbb{N}} \text{ by } x(t) = (x_0(t), \cdots, x_{d-1}(t))^\top := (w(t), \cdots, w(t+d-1))^\top \in W_A^{\ell d} = \left(W_A^\ell\right)^d.$$
$$\text{Then } \phi : W_B^\ell \to W_B^{\ell d},\ w \mapsto x = \phi(w), \tag{29}$$

is an injective behavior morphism with the left inverse $x \mapsto w$, $w(t) := x_0(t)$. For any behavior $\mathscr{B} \subseteq W_B^\ell$ define the image $\mathscr{B}_{\mathrm{st}} := \phi(\mathscr{B})$. Then ϕ induces the isomorphism

$$\phi : \mathscr{B} \cong \mathscr{B}_{st},\ w \mapsto x,\ x_i(t) = w(t+i), \text{ and } \mathscr{B}_d = \mathscr{B}_{st,1} \subseteq W_A^{\ell d} = \left(W_A^\ell\right)^d. \tag{30}$$

Corollary 3.3. *Assume that the autonomous behavior $\mathscr{B}$ in Thm. 3.2 is ta and that $M_d = M$, $d > 0$. Then also $\mathscr{B}_{st}$ is ta and there is the commutative diagram of behavior isomorphisms*

$$\begin{array}{ccccc} \mathscr{B} & \stackrel{\mathrm{proj}}{\longrightarrow} & \mathscr{B}_d, & w \mapsto & (w(0), \cdots, w(d-1))^\top, \\ \downarrow \phi & & \| & & \| \\ \mathscr{B}_{\mathrm{st}} & \stackrel{\mathrm{proj}}{\longrightarrow} & \mathscr{B}_{\mathrm{st},1}, & x \mapsto & x(0) \end{array}$$

Thus it suffices to study ta behaviors with $\mathscr{B} \cong \mathscr{B}_1$ in the sequel.

Theorem 3.4. *The autonomous behavior $\mathscr{B} \subset W_B^\ell$ from* (7) *is ta with $\mathscr{B} \cong \mathscr{B}_1$, $w \mapsto w(0)$, if and only if there are matrices*

$$R_0 \in A^{k\times\ell},\ E \in A^{\ell\times\ell},\ X \in A^{k\times k} \textit{ and then } R := \begin{pmatrix} R_0 \\ s_0\,\mathrm{id}_\ell - E \end{pmatrix} \in B^{(k+\ell)\times\ell} \textit{ with}$$
$$R_0E = XR_0 \textit{ and } \mathscr{B} = \left\{w \in W_B^\ell;\ R \circ w = 0\right\}. \textit{ Then}$$
$$\mathscr{B}_1 = \left\{v \in W_A^\ell;\ R_0 \circ v = 0\right\} \textit{ and } \left(w \in \mathscr{B} \iff w(0) \in \mathscr{B}_1 \textit{ and } w(t) = E^t \circ w(0)\right). \tag{31}$$

Proof. $\Longrightarrow$: Let $\mathscr{B} := U^\perp$ be ta with $\mathscr{B} \cong \mathscr{B}_1$. From (28) we infer the isomorphism

$$M_1 = A^{1\times\ell}/U_1 \cong B^{1\times\ell}/U,\ U_1 := A^{1\times\ell} \bigcap U, \text{ hence } B^{1\times\ell} = A^{1\times\ell} + U.$$
$$\text{Define } E \in A^{\ell\times\ell} \text{ by } s_0\delta_j = E_{j-} + u_j,\ u_j \in U,\ j = 1, \cdots, \ell.$$

The matrix $E \in A^{\ell\times\ell}$ makes $A^{1\times\ell}$ a B-module with $s_0\cdot\xi := \xi E$. For $\xi \in U_1$ we derive

$$s_0\cdot\xi = \xi E = \sum_{j=1}^{\ell}\xi_j E_{j-} = \sum_{j=1}^{\ell}\xi_j\left(s_0\delta_j - u_j\right) = s_0\xi - \sum_{j=1}^{\ell}\xi_j u_j \in U \Longrightarrow$$
$$s_0\cdot U_1 = U_1E \subseteq U_1 \text{ and } (s_0\cdot\xi)+U = s_0(\xi+U).$$

We conclude that U_1 is a B-submodule of $A^{1\times\ell}$, that $M_1 = A^{1\times\ell}/U_1$ is a B-module with $s_0(\xi+U_1) = (s_0\cdot\xi)+U_1 = \xi E+U_1$ and that $M_1 \cong M$ is a B-isomorphism. By duality $\mathscr{B}_1$ is a B-module with $s_0\circ v = Ev$ and the projection $\mathrm{proj}: \mathscr{B}\to\mathscr{B}_1,\ w\mapsto w(0)$, is bijective and B-linear, i.e.,

$$w(t) = (s_0^t\circ w)(0) = s_0^t\circ w(0) = E^t w(0),\ w\in\mathscr{B}.$$

Let $U_1 = A^{1\times k}R_0$, $R_0\in A^{k\times\ell}$, hence $\mathscr{B}_1 = \left\{v\in W_A^\ell;\ R_0\circ v = 0\right\}$. Since $s_0\cdot U_1 = U_1E\subseteq U_1$ there is a matrix $X\in A^{k\times k}$ with $R_0E = XR_0$ and then also $R_0E^t = X^tR_0$. For $w\in\mathscr{B}$ with $w(t) = E^t\circ w(0)$ this implies (with R from (31))

$$\forall t\in\mathbb{N}:\ R_0\circ w(t) = 0 \Longrightarrow R_0\circ w = 0 \Longrightarrow \mathscr{B}\subseteq\mathscr{B}' := \left\{w\in W_B^\ell;\ R\circ w = 0\right\}.$$

If, conversely, $w'\in\mathscr{B}'$ then $R_0\circ w'(0) = 0$ and $w'(t) = E^tw'(0)$. Since $\mathscr{B}\cong\mathscr{B}_1$ there is a unique $w\in\mathscr{B}$ with

$$w(0) = w'(0) \text{ and } w(t) = E^t\circ w(0) = E^t\circ w'(0) = w'(t) \Longrightarrow w' = w\in\mathscr{B} \Longrightarrow \mathscr{B}' = \mathscr{B}.$$

$\Longleftarrow$: For

$$\mathscr{B} = \left\{w\in W_B^\ell;\ R\circ w = 0\right\} =$$
$$\left\{w\in W_B^\ell;\ \forall t\in\mathbb{N}:\ R_0\circ w(t) = 0,\ w(t+1) = E\circ w(t)\right\},\ XR_0 = R_0E,$$

the projection $\mathscr{B}\to\mathscr{B}_1 := \left\{v\in W_A^\ell;\ R_0\circ v = 0\right\}$, $w\mapsto w(0)$, is bijective with the inverse map $w(0)\mapsto w := (E^t\circ w(0))_{t\in\mathbb{N}}$. The equation $R_0E^t = X^tR_0$ is used here to conclude $R_0\circ w(t) = 0$ for all $t\in\mathbb{N}$, i.e., $R_0\circ w = 0$. ☐

The next corollary permits to decide constructively whether $\mathscr{B}$ is *square-autonomous* which, in contrast to the case $n = 1$ of [9], is rarely the case for $n > 1$. Let $R_1\in B^{k_1\times k}$ be a universal left annihilator of R so that

$$\begin{gathered} B^{1\times k_1} \xrightarrow{\circ R_1} B^{1\times k} \xrightarrow{\circ R} B^{1\times\ell} \xrightarrow{\mathrm{can}} M\to 0 \text{ is exact, hence} \\ B^{1\times k}/B^{1\times k_1}R_1 \cong U = B^{1\times k}R,\ x+B^{1\times k_1}R_1\mapsto xR. \end{gathered} \tag{32}$$

The Quillen-Suslin theorem for modules over the Laurent algebra B implies that a f.g. projective B-module U is free.

Corollary 3.5. *Assume that in* (7) *the behavior $\mathscr{B}$ is autonomous or* $\mathrm{rank}(R) = \ell$. *Then the following assertions are equivalent:*

(i) The behavior $\mathscr{B}$ is square-autonomous, i.e., $U = B^{1\times k}R = B^{1\times\ell}R'$ for some square matrix $R'\in B^{\ell\times\ell}$ with $\det(R')\neq 0$.

(ii) The module U is free, necessarily of dimension ℓ since $\mathrm{rank}(R) = \ell$, *or, equivalently, projective. In other words, this signifies that M has projective dimension at most* 1.

(iii) There is a matrix $S_1 \in B^{k\times k_1}$ *such that* $R_1S_1R_1 = R_1$.

These conditions imply: $E_1 := S_1R_1 = E_1^2$, $R_1E_1 = R_1$,

$$B^{1\times k}E_1 = B^{1\times k_1}R_1,\ B^{1\times k} = B^{1\times k}E_1 \oplus B^{1\times k}(\mathrm{id}_k - E_1) \text{ and}$$
$$B^{1\times k}(\mathrm{id}_k - E_1) \cong B^{1\times k}/B^{1\times k}E_1 = B^{1\times k}/B^{1\times k_1}R_1 \cong U.$$

The universal left annihilator R_1 *can be computed with all Computer Algebra systems. The existence of* S_1 *and of* $E_1 = S_1R_1$ *and the projectivity of* U *can be checked by means of [11, Alg. 8.2,(4)].*

Proof. $(i) \iff (ii)$: obvious.
$(ii) \iff (iii)$: The projectivity of $U \cong B^{1\times k}/B^{1\times k_1}R_1$ is equivalent to $B^{1\times k_1}R_1$ being a direct summand of $B^{1\times k}$. This, in turn, signifies that there is an *idempotent (projection)* matrix $E_1 = E_1^2 \in B^{k\times k}$ such that $B^{1\times k_1}R_1 = B^{1\times k}E_1$ or

$$E_1 = S_1R_1 \text{ for some } S_1 \text{ and } R_1 = R_1E_1 = R_1S_1R_1.$$

□

4 Application of the Fourier transform

The assumptions of Sections 1, 2 and 3 remain in force, in particular $A = \mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[s, s^{-1}]$. *The Fourier transform is an essential ingredient of [24] and [9] and therefore also of this paper. The following results on this transform are taken from [2, §II.1] and [19, §VII.1].*
The multiplicative abelian groups $\mathbb{T}^1$ and $\mathbb{T}^n$ are compact and thus [2, Ch.II] is applicable to them. The form (cf. (13))

$$\mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{T}^1,\ (\mu, \omega) \mapsto \omega^\mu = \omega_1^{\mu_1} \cdots \omega_n^{\mu_n}, \tag{33}$$

is non-degenerate and induces the isomorphism

$$\mathbb{T}^n \cong \mathrm{Hom}(\mathbb{Z}^n, \mathbb{T}^1),\ \omega \mapsto (\mu \mapsto \omega^\mu), \tag{34}$$

where $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{T}^1)$ is the group of homomorphisms or characters from $\mathbb{Z}^n$ to $\mathbb{T}^1$. We need various spaces of functions on $\mathbb{Z}^n$, i.e, of multi-sequences, and of functions on $\mathbb{T}^n$, viz.

$$\mathbb{C}^{\mathbb{Z}^n} \supset \mathscr{S}'(\mathbb{Z}^n) \supset \mathrm{L}^2(\mathbb{Z}^n),\ \mathscr{D}'(\mathbb{T}^n) \supset \mathrm{L}^2(\mathbb{T}^n) \supset \mathrm{C}^0(\mathbb{T}^n), \tag{35}$$

with the following specifications: The space $\mathscr{S}'(\mathbb{Z}^n)$ consists of sequences v of *slow or at most polynomial growth* [19, (VII,1;2)], i.e.,

$$\mathscr{S}'(\mathbb{Z}^n) := \left\{ v \in \mathbb{C}^{\mathbb{Z}^n};\ \exists M > 0, k \in \mathbb{N} \text{ with } |v(\mu)| \le M|\mu|^k \right\} \tag{36}$$

where $|\mu| := |\mu_1| + \cdots + |\mu_n|$. The space $\mathrm{L}^2(\mathbb{Z}^n)$ is given as

$$\mathrm{L}^2(\mathbb{Z}^n) := \left\{ v \in \mathbb{C}^{\mathbb{Z}^n};\ \sum_{\mu \in \mathbb{Z}^n} |v(\mu)|^2 < \infty \right\}. \tag{37}$$

The notation is that of [2, Ch.II] which is in contrast to the widely used $\ell_2(\mathbb{Z}^n)$. This is a Hilbert space with the hermitian scalar product $< u, v > := \sum_{\mu \in \mathbb{Z}^n} \overline{u(\mu)}v(\mu)$ for

$u, v \in \mathrm{L}^2(\mathbb{Z}^n)$ and the 2-norm $||v||_2 := \left(\sum_{\mu\in\mathbb{Z}^n} |v(\mu)|^2\right)^{1/2}$. Here $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. The finite powers $\mathrm{L}^2(\mathbb{Z}^n)^\ell$ inherit the Hilbert space structure. The spaces $\mathscr{S}'(\mathbb{Z}^n)$ and $\mathrm{L}^2(\mathbb{Z}^n)$ are A-submodules of $W_A = \mathbb{C}^{\mathbb{Z}^n}$. Time autonomy enters the subsequent considerations by means of the following lemma.

Lemma 4.1. *(compare [9, Prop. 8]) If $\mathscr{B}$ is ta with $\mathscr{B} \cong \mathscr{B}_d, w \mapsto (w(0), \cdots, w(d-1))^\top$, then also*

$$\mathscr{B}\bigcap\left(\mathscr{S}'(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \cong \mathscr{B}_d\bigcap\left(\mathscr{S}'(\mathbb{Z}^n)^\ell\right)^d, \ \mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \cong \mathscr{B}_d\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^d, \tag{38}$$

or, in other terms, if

$$w \in \mathscr{B} \text{ and } \forall t \le d-1: \ w(t) \in \begin{cases}\mathscr{S}'(\mathbb{Z}^n)^\ell \\ \mathrm{L}^2(\mathbb{Z}^n)^\ell\end{cases} \quad \text{then } \forall t \in \mathbb{N}: \ w(t) \in \begin{cases}\mathscr{S}'(\mathbb{Z}^n)^\ell \\ \mathrm{L}^2(\mathbb{Z}^n)^\ell\end{cases}. \tag{39}$$

Proof. It suffices to prove this for a ta behavior in the normal form of Thms. 2.2, 3.4:

$$\mathscr{B} = \left\{w \in \left(W_A^\ell\right)^{\mathbb{N}}; \ R_0 \circ w(0) = 0, \ w(t) = E^t \circ w(0)\right\} \cong$$
$$\mathscr{B}_1 = \left\{v \in W_A^\ell; \ R_0 \circ v = 0\right\}, \ w \mapsto w(0).$$

Since $\mathrm{L}^2(\mathbb{Z}^n)$ and $\mathscr{S}'(\mathbb{Z}^n)$ are A-submodules of $W_A = \mathbb{C}^{\mathbb{Z}^n}$ and $w(t) = E^t \circ w(0)$ the assertion follows. □

The space $\mathscr{D}'(\mathbb{T}^n)$ is the space of distributions on the smooth manifold $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ which is the dual space of the Fréchet space $\mathscr{D}(\mathbb{T}^n)$ of smooth functions on $\mathbb{T}^n$. The functions φ in this space are identified with the smooth functions $\widetilde{\varphi}$ on $\mathbb{R}^n$ which are $\mathbb{Z}^n$-periodic, i.e., satisfy $\widetilde{\varphi}(x+\mu) = \widetilde{\varphi}(x)$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{Z}^n$. The identification is given by

$$\widetilde{\varphi}(x_1, \cdots, x_n) := \varphi(e^{2\pi i x}), \ e^{2\pi i x} := (e^{2\pi i x_1}, \cdots, e^{2\pi i x_n}), \ \varphi(\omega) := \widetilde{\varphi}(x) \text{ if } \omega = e^{2\pi i x}. \tag{40}$$

The partial derivatives $\partial_j\varphi$, $j = 1, \cdots, n$, of $\varphi \in \mathscr{D}(\mathbb{T}^n)$ are defined by

$$\begin{gathered}(\partial_j\varphi)(e^{2\pi i x}) := \partial\varphi(e^{2\pi i x})/\partial x_j, \text{ for instance } \partial_j\omega^\mu = (2\pi i)\mu_j\omega^\mu, \\ \alpha \in \mathbb{N}^n, \ |\alpha| = \alpha_1 + \cdots + \alpha_n, \ \partial^\alpha := \partial_1^{\alpha_1} * \cdots * \partial_n^{\alpha_n}, \ \partial^\alpha\omega^\mu = (2\pi i)^{|\alpha|}\mu^\alpha\omega^\mu.\end{gathered} \tag{41}$$

In the same way the continuous functions $f \in \mathrm{C}^0(\mathbb{T}^n)$ are identified with $\mathbb{Z}^n$-periodic continuous functions on $\mathbb{R}^n$. The unique Haar measure on $\mathbb{T}^n$ with total mass 1 is denoted by $d\omega$. For $f \in \mathrm{C}^0(\mathbb{T}^n)$ it is given by

$$\int_{\mathbb{T}^n} f(\omega)d\omega = \int_{[0,1]^n} f(e^{2\pi i x})dx = \int_{x_0+[0,1]^n} f(e^{2\pi i x})dx, \ x_0 \in \mathbb{R}^n. \tag{42}$$

The space of (Borel)-measurable functions is denoted by

$$\mathscr{L}^0(\mathbb{T}^n) := \{f : \mathbb{T}^n \to \mathbb{C}; \ f \text{ (Borel)-measurable}\}. \tag{43}$$

It has the subspace $\mathscr{U}$ of almost everywhere zero functions f, i.e., for which the measure of $\{\omega \in \mathbb{T}^n; \ f(\omega) \neq 0\}$ is zero and hence $\int_{\mathbb{T}^n} f(\omega)d\omega = 0$. It gives rise to the factor space

$$\mathrm{L}^0(\mathbb{T}^n) := \mathscr{L}^0(\mathbb{T}^n)/\mathscr{U} \ni \overline{f} := f + \mathscr{U} \tag{44}$$

where $\overline{f}$ is usually denoted by f again. The space $\mathrm{L}^0(\mathbb{T}^n)$ is a $\mathbb{C}$-algebra with the multiplication $\overline{f_1}\cdot\overline{f_2} := \overline{f_1 f_2}$. An element $\overline{f}\in \mathrm{L}^0(\mathbb{T}^n)$ is invertible if and only if the zero set $V_{\mathbb{T}^n}(f) := \{\omega\in\mathbb{T}^n;\ f(\omega)=0\}$ has measure zero. The inverse $\overline{g} := \overline{f}^{-1}$ is then given by the representative

$$g\in\mathscr{L}^0(\mathbb{T}^n) \text{ with } g(\omega) := \begin{cases} f(\omega)^{-1} & \text{if } f(\omega)\neq 0 \\ 0 & \text{if } f(\omega)=0 \end{cases}. \tag{45}$$

There are the canonical inclusions

$$\mathscr{D}(\mathbb{T}^n)\subset \mathrm{C}^0(\mathbb{T}^n)\subset \mathrm{L}^0(\mathbb{T}^n). \tag{46}$$

As usual one defines

$$\mathrm{L}^2(\mathbb{T}^n) := \left\{ f\in \mathrm{L}^0(\mathbb{T}^n);\ \int_{\mathbb{T}^n} |f(\omega)|^2 d\omega < \infty \right\}. \tag{47}$$

The space $\mathrm{L}^2(\mathbb{T}^n)$ is a Hilbert space with the hermitian scalar product $< f,g > := \int_{\mathbb{T}^n} \overline{f(\omega)} g(\omega) d\omega$. The embeddings $\mathrm{C}^0(\mathbb{T}^n)\subset \mathrm{L}^2(\mathbb{T}^n)\subset \mathscr{D}'(\mathbb{T}^n)$ are the canonical ones, i.e.,

$$\mathrm{L}^2(\mathbb{T}^n)\to\mathscr{D}'(\mathbb{T}^n),\ f\mapsto \left(\varphi\mapsto \int_{\mathbb{T}^n} f(\omega)\varphi(\omega)d\omega = \int_{[0,1]^n} f(e^{2\pi i x})\varphi(e^{2\pi i x})dx\right). \tag{48}$$

The space $\mathscr{D}'(\mathbb{T}^n)$ is a $\mathscr{D}(\mathbb{T}^n)$-module via $(\varphi F)(\psi) := F(\varphi\psi)$ for $\varphi,\psi\in\mathscr{D}(\mathbb{T}^n)$ and $F\in\mathscr{D}'(\mathbb{T}^n)$ and the spaces $\mathrm{L}^2(\mathbb{T}^n)$ and $\mathrm{C}^0(\mathbb{T}^n)$ are $\mathscr{D}(\mathbb{T}^n)$-submodules of $\mathscr{D}'(\mathbb{T}^n)$ with the standard multiplication of functions. Since $\mathbb{T}^n$ is a product of infinite subsets of $\mathbb{C}^n$ the canonical map

$$A = \mathbb{C}[s,s^{-1}]\to\mathscr{D}(\mathbb{T}^n),\ f\mapsto(\omega\mapsto f(\omega)), \tag{49}$$

of a (Laurent) polynomial to its associated polynomial function is injective and therefore we may and do consider A as subalgebra of $\mathscr{D}(\mathbb{T}^n)\subset \mathrm{L}^0(\mathbb{T}^n)$. In particular, $\mathscr{D}'(\mathbb{T}^n)$ is also an A-module. The following lemma permits to consider the quotient field $\mathbb{C}(s) := \mathrm{quot}(A)$ as subfield of $\mathrm{L}^0(\mathbb{T}^n)$.

Lemma 4.2. *If $f\in\mathbb{C}[s,s^{-1}]$ is nonzero then the zero set $V_{\mathbb{T}^n}(f) := \{\omega\in\mathbb{T}^n;\ f(\omega)=0\}$ has measure zero.*

Proof. The main idea of this proof, viz. the use of the Fubini theorem, is due to our colleagues Hell and Wagner. W.l.o.g. we assume $0\neq f\in\mathbb{C}[s]$. The proof proceeds by induction on n. The beginning $n=1$ is trivial since the univariate polynomial f has only $\deg(f)$ roots. For $n>1$ we write

$$\omega = (\omega',\omega_n),\ d\omega' := d\omega_1\cdots d\omega_{n-1},\ d\omega = d\omega' d\omega_n,\ s = (s_1,\cdots,s_n) = (s',s_n),$$
$$f = f_d(s')s_n^d + \cdots + f_0(s')\in A = \mathbb{C}[s] = \mathbb{C}[s'][s_n],\ 0\neq f_d\in\mathbb{C}[s'].$$

Let c_Ω denote the indicator function of a subset Ω of $\mathbb{T}^n$, i.e.,

$$c_\Omega(\omega) = \begin{cases} 1 & \text{if } \omega\in\Omega \\ 0 & \text{if } \omega\notin\Omega \end{cases}. \text{ Then}$$

$$c(\omega',\omega_n) := c_{V_{\mathbb{T}^n}(f)}(\omega',\omega_n) = c_{V(\omega')}(\omega_n) \text{ with } V(\omega') := V_{\mathbb{T}}(f(\omega',s_n)).$$

Define $\Omega'_\infty := \left\{\omega'\in\mathbb{T}^{n-1};\ V(\omega') \text{ has infinite cardinality}\right\}$, $\mathbb{T}^{n-1} = \Omega'_\infty \uplus \Omega'_{\mathrm{fin}}$.

If $V(\omega')$ has only finitely many elements then trivially $(d\omega_n)(V(\omega')) = \int_{V(\omega')} c_{V(\omega')}(\omega_n)d\omega_n = 0$. For $\omega' \in \Omega'_\infty$ the polynomial $f(\omega', s_n)$ has infinitely many zeros and hence

$$f(\omega', s_n) = f_d(\omega')s_n^d + \cdots + f_0(\omega') = 0 \Longrightarrow f_d(\omega') = 0 \Longrightarrow \omega' \in V_{\mathbb{T}^{n-1}}(f_d) \Longrightarrow \Omega'_\infty \subseteq V_{\mathbb{T}^{n-1}}(f_d).$$

Now we use Fubini's theorem and obtain

$$(d\omega)(V_{\mathbb{T}^n}(f)) = \int_{\mathbb{T}^n} c(\omega)d\omega = \int_{\mathbb{T}^{n-1}} \left(\int_{\mathbb{T}} c(\omega', \omega_n)d\omega_n \right) d\omega' =$$
$$\int_{\mathbb{T}^{n-1}} \left(\int_{\mathbb{T}} c_{V(\omega')}(\omega_n)d\omega_n \right) d\omega' = \int_{\mathbb{T}^{n-1}} (d\omega_n)(V(\omega'))d\omega' = \int_{\Omega'_\infty} (d\omega_n)(V(\omega'))d\omega' \le$$
$$\int_{V_{\mathbb{T}^{n-1}}(f_d)} 1 d\omega' = (d\omega')\left(V_{\mathbb{T}^{n-1}}(f_d)\right) = 0$$

where the last equation follows by induction. □

Now let $0 \neq h = \frac{f}{g} \in \mathbb{C}(s) = \mathrm{quot}(A)$, $g \neq 0$, be any rational function. According to the preceding lemma the set $\{\omega \in \mathbb{T}^n;\ g(\omega) \neq 0\}$ has measure 1 and therefore g is a unit in $\mathrm{L}^0(\mathbb{T}^n)$, hence $\frac{f}{g} = fg^{-1} \in \mathrm{L}^0(\mathbb{T}^n)$ with the identification $A \subset \mathrm{C}^0(\mathbb{T}^n) \subset \mathrm{L}^0(\mathbb{T}^n)$.

Lemma 4.3. *1. The rational field $\mathbb{C}(s)$ is a subfield of $\mathrm{L}^0(\mathbb{T}^n)$, hence also $\mathbb{C}(s)^{p\times m} \subset \mathrm{L}^0(\mathbb{T}^n)^{p\times m}$.*
2. Let $H \in \mathbb{C}(s)^{p\times m} \subset \mathrm{L}^0(\mathbb{T}^n)^{p\times m}$, $\mathbb{C}(s) = \mathrm{quot}(A)$, and let $g \in A$ be the least common denominator in A of the entries H_{ij} of H, i.e.,

$$H_{ij} = \frac{f_{ij}}{g_{ij}},\ f_{ij}, g_{ij} \in A,\ g_{ij} \neq 0,\ \gcd(f_{ij}, g_{ij}) = 1,$$
$$g = \mathrm{lcm}_{ij}(g_{ij}),\ Ag = \left\{ g' \in A;\ g'H \in A^{p\times m} \right\}.$$

If $g(\omega) \neq 0$ for all $\omega \in \mathbb{T}^n$ then $H \in \mathscr{D}(\mathbb{T}^n)^{p\times m}$.
3. A matrix $P \in A^{k\times p}$ has a left inverse in $\mathrm{L}^0(\mathbb{T}^n)^{p\times k}$ if and only if $\mathrm{rank}(P) = p$. In particular, $\widehat{v} \in \mathrm{L}^0(\mathbb{T}^n)^p$ and $P\widehat{v} = 0$ then imply $\widehat{v} = 0$.
4. Consider the input/output data $(P, -Q) \in A^{k\times(p+m)}$ with $p = \mathrm{rank}(P) = \mathrm{rank}(P, -Q)$. Hence there are the unique transfer matrix *$H \in \mathbb{C}(s)^{p\times m}$ with $PH = Q$ and, by 3., $X \in \mathrm{L}^0(\mathbb{T}^n)^{p\times k}$ with $XP = \mathrm{id}_p \in \mathrm{L}^0(\mathbb{T}^n)^{p\times p}$. Then $H = XQ \in \mathrm{L}^0(\mathbb{T}^n)^{p\times m}$.*

Proof. 2. Let $G := gH \in A^{p\times m} \subset \mathscr{D}(\mathbb{T}^n)^{p\times m}$. The assumption on g implies $g^{-1} \in \mathscr{D}(\mathbb{T}^n)$ and hence $H = g^{-1}G \in \mathscr{D}(\mathbb{T}^n)^{p\times m}$.
3. $\Longleftarrow$: The rank condition signifies that P has a left inverse in $\mathbb{C}(s)^{p\times k} \subset \mathrm{L}^0(\mathbb{T}^n)^{p\times k}$.
$\Longrightarrow$: Let $X \in \mathrm{L}^0(\mathbb{T}^n)^{p\times k}$ be such a left inverse. Then $X(\omega)P(\omega) = \mathrm{id}_p$ for almost all ω and in particular for one ω_0, hence $p = \mathrm{rank}(P(\omega_0)) \le \mathrm{rank}(P) \le p$.
4. $PH = Q \in \mathbb{C}(s)^{k\times m} \subset \mathrm{L}^0(\mathbb{T}^n)^{k\times m} \Longrightarrow H = \mathrm{id}_p H = XPH = XQ \in \mathrm{L}^0(\mathbb{T}^n)^{p\times m}$. □

The next lemma refines item 3. of Lemma 4.3.

Lemma 4.4. *1. A matrix $P \in \mathscr{D}(\mathbb{T}^n)^{k\times p}$ has a left inverse in $\mathscr{D}(\mathbb{T}^n)^{p\times k}$ if and only if $\mathrm{rank}(P(\omega)) = p$ for all $\omega \in \mathbb{T}^n$. In particular, $\widehat{v} \in \mathscr{D}'(\mathbb{T}^n)^p$ and $P\widehat{v} = 0$ then imply $\widehat{v} = 0$.*
2. If in Lemma 4.3,4., the matrix $P \in A^{k\times p}$ satisfies the condition of 1., i.e., $\mathrm{rank}(P(\omega)) = p$ for all $\omega \in \mathbb{T}^n$, then $H \in \mathscr{D}(\mathbb{T}^n)^{p\times m}$.

Proof. 1. (Compare [2, §I.3.2, Remark]) The necessity is obvious. Assume, conversely, that $\mathrm{rank}(P(\omega)) = p$ for all $\omega \in \mathbb{T}^n$.
(i) $p = 1$, $P = (P_1, \cdots, P_k)^\top$: The rank condition signifies that the $P_i \in \mathscr{D}(\mathbb{T}^n)$ have no common zero. Then $Q_i(\omega) := \overline{P_i(\omega)} \in \mathscr{D}(\mathbb{T}^n)$ and

$$f(\omega) := \sum_{i=1}^{k} |P_i(\omega)|^2 = \sum_{i=1}^{k} Q_i(\omega)P_i(\omega) > 0 \Longrightarrow$$

$$f^{-1} \in \mathscr{D}(\mathbb{T}^n) \Longrightarrow \mathrm{id}_1 = 1 = \sum_{i=1}^{k} \left(f^{-1}Q_i\right) P_i = (f^{-1}Q_1, \cdots, f^{-1}Q_k)P$$

(ii) $k \geq p \geq 1$: Let $P_i, i = 1, \cdots, q$ denote the family of all $p \times p$-submatrices of P and $d_i := \det(P_i)$. For all $\omega \in \mathbb{T}^n$ the rank condition $\mathrm{rank}(P(\omega)) = p$ signifies that there is an i with $d_i(\omega) \neq 0$. Thus the d_i have no common zero in $\mathbb{T}^n$ and by the first case there is a representation $1 = \sum_{i=1}^{q} f_i d_i$, $f_i \in \mathscr{D}(\mathbb{T}^n)$. Let $P_{i,\mathrm{adj}} \in \mathscr{D}(\mathbb{T}^n)^{p\times p}$ denote the adjoint matrix of P_i with $P_{i,\mathrm{adj}}P_i = d_i\,\mathrm{id}_p$. For notational simplicity assume that the rows of P_i are the first p rows of P, i.e.

$$P = \begin{pmatrix} P_i \\ P_i' \end{pmatrix} \text{ and thus } X_iP = d_i\,\mathrm{id}_p \text{ for } X_i := (P_{i,\mathrm{adj}}, 0) \in \mathscr{D}(\mathbb{T}^n)^{p\times(p+(k-p))} \Longrightarrow$$
$$\sum_{i=1}^{q} f_iX_iP = \sum_{i=1}^{q} f_id_i\,\mathrm{id}_p = \mathrm{id}_p \Longrightarrow XP = \mathrm{id}_p \text{ with } X := \sum_{i=1}^{q} f_iX_i.$$

2. The matrix X in 1. is a left inverse of P in $\mathscr{D}(\mathbb{T}^n)^{p\times k} \subset \mathrm{L}^0(\mathbb{T}^n)^{p\times k}$. From Lemma 4.3,4., we infer $H = XQ \in \mathscr{D}(\mathbb{T}^n)^{p\times m}$ since $\mathscr{D}(\mathbb{T}^n)$ is a subalgebra of $\mathrm{L}^0(\mathbb{T}^n)$. □

The Fourier transform on $\mathbb{Z}^n$ is the isomorphism [19, Thm. VII.1.I]

$$\begin{aligned} &\mathscr{F}_{\mathbb{Z}^n} : \mathscr{S}'(\mathbb{Z}^n) \cong \mathscr{D}'(\mathbb{T}^n),\ v \mapsto \widehat{v},\ \widehat{v} := \sum_{\mu\in\mathbb{Z}^n} v(\mu)\omega^{-\mu}, \\ &\widehat{v}(\varphi) := \sum_{\mu\in\mathbb{Z}^n} v(\mu) \int_{\mathbb{T}^n} \varphi(\omega)\omega^{-\mu}d\omega,\ \varphi \in \mathscr{D}(\mathbb{T}^n). \end{aligned} \tag{50}$$

The convergence of this sum is, of course, that of distributions. Fourier's cotransform [2, Def. II.1.3] is the isomorphism

$$\overline{\mathscr{F}}_{\mathbb{Z}^n} : \mathscr{S}'(\mathbb{Z}^n) \cong \mathscr{D}'(\mathbb{T}^n),\ \overline{\mathscr{F}}_{\mathbb{Z}^n}(v)(\omega) := \sum_{\mu\in\mathbb{Z}^n} v(\mu)\omega^{+\mu}. \tag{51}$$

The Fourier transform and cotransform on the dual group $\mathbb{T}^n$ of $\mathbb{Z}^n$ are the isomorphisms

$$\begin{aligned} &\mathscr{F}_{\mathbb{T}^n}, \overline{\mathscr{F}}_{\mathbb{T}^n} : \mathscr{D}'(\mathbb{T}^n) \cong \mathscr{S}'(\mathbb{Z}^n), F \mapsto \widehat{F} := \mathscr{F}_{\mathbb{T}^n}(F), \overline{\mathscr{F}}_{\mathbb{T}^n}(F), \\ &\widehat{F}(\mu) := F(\omega^{-\mu}),\ \overline{\mathscr{F}}_{\mathbb{T}^n}(F)(\mu) := F(\omega^{\mu}). \end{aligned} \tag{52}$$

Fourier's inversion theorem [19, Thm. VII.1.I] says that the maps

$$\mathscr{F}_{\mathbb{Z}^n} : \mathscr{S}'(\mathbb{Z}^n) \rightleftarrows \mathscr{D}'(\mathbb{T}^n) : \overline{\mathscr{F}}_{\mathbb{T}^n},\ v \leftrightarrow \widehat{v},\ v(\mu) = \widehat{v}(\omega^{\mu}), \tag{53}$$

are inverses of each other. The theorems [2, Thms. II.1.1, II.1.2] show that the preceding maps induce inverse *isometries*

$$\mathscr{F}_{\mathbb{Z}^n} : \mathrm{L}^2(\mathbb{Z}^n) \rightleftarrows \mathrm{L}^2(\mathbb{T}^n) : \overline{\mathscr{F}}_{\mathbb{T}^n},\ v \leftrightarrow \widehat{v},\ v(\mu) = \widehat{v}(\omega^{\mu}). \tag{54}$$

The preceding isomorphisms are componentwise extended to *signal vectors*.

Lemma 4.5. *The Fourier isomorphism $\mathscr{F}_{\mathbb{Z}^n}:\mathscr{S}'(\mathbb{Z}^n)\cong\mathscr{D}'(\mathbb{T}^n)$ is A-linear.*

Proof. For

$$v\in\mathscr{S}'(\mathbb{Z}^n),\ \mu\in\mathbb{Z}^n,\ s^\mu\in A=\mathbb{C}[s,s^{-1}]\subset\mathscr{D}(\mathbb{T}^n),\ \omega\in\mathbb{T}^n \text{ one has}$$
$$\widehat{s^\mu\circ v}=\sum_{\nu\in\mathbb{Z}^n}(s^\mu\circ v)(\nu)\omega^{-\nu}=\sum_{\nu\in\mathbb{Z}^n}v(\mu+\nu)\omega^{-\nu}=\sum_{\nu\in\mathbb{Z}^n}v(\nu)\omega^{-(\nu-\mu)}=\omega^\mu\widehat{v}=s^\mu\widehat{v}.$$

□

Corollary 4.6. *Consider a $\mathbb{C}^{\mathbb{Z}^n}$-behavior $\mathscr{B}:=\left\{v\in\left(\mathbb{C}^{\mathbb{Z}^n}\right)^\ell;\ R\circ v=0\right\}$ defined by $R\in A^{k\times\ell}$. Then the Fourier transform induces inverse isomorphisms*

$$\mathscr{F}_{\mathbb{Z}^n}:\mathscr{B}\bigcap\mathscr{S}'(\mathbb{Z}^n)^\ell\cong\widehat{\mathscr{B}}:=\left\{\widehat{v}\in\mathscr{D}'(\mathbb{T}^n)^\ell;\ R\widehat{v}=0\right\}:\overline{\mathscr{F}}_{\mathbb{T}^n}$$
$$\mathscr{F}_{\mathbb{Z}^n}:\mathscr{B}\bigcap\mathrm{L}^2(\mathbb{Z}^n)^\ell\cong\widehat{\mathscr{B}}\bigcap\mathrm{L}^2(\mathbb{T}^n)^\ell:\overline{\mathscr{F}}_{\mathbb{T}^n}.$$

Notice that $R(\omega)\in\mathscr{D}(\mathbb{T}^n)^{k\times\ell}$, so $R(\omega)\widehat{v}$ is defined in the preceding corollary.

Theorem 4.7. *1. Consider an IO behavior*

$$\mathscr{B}:=\left\{\binom{y}{u}\in\left(\mathbb{C}^{\mathbb{Z}^n}\right)^{p+m};\ P\circ y=Q\circ u\right\},\ (P,-Q)\in A^{k\times(p+m)},$$
$$\mathrm{rank}(P)=\mathrm{rank}(P,-Q)=p,\ \textit{ with transfer matrix } H\in\mathbb{C}(s)^{p\times m},\ PH=Q,$$

with the associated inverse isomorphisms

$$\mathscr{F}_{\mathbb{Z}^n}:\mathscr{B}\bigcap\mathrm{L}^2(\mathbb{Z}^n)^{p+m}=\left\{\binom{y}{u}\in\mathrm{L}^2(\mathbb{Z}^n)^{p+m};\ P\circ y=Q\circ u\right\}\cong$$
$$\widehat{\mathscr{B}}\bigcap\mathrm{L}^2(\mathbb{T}^n)^{p+m}=\left\{\binom{\widehat{y}}{\widehat{u}}\in\mathrm{L}^2(\mathbb{T}^n)^{p+m};\ P\widehat{y}=Q\widehat{u}\right\}:\overline{\mathscr{F}}_{\mathbb{T}^n}.$$

These data imply the inverse isomorphisms

$$\widehat{\mathscr{B}}\bigcap\mathrm{L}^2(\mathbb{T}^n)^{p+m}\cong\left\{\widehat{u}\in\mathrm{L}^2(\mathbb{T}^n)^m;\ H\widehat{u}\in\mathrm{L}^2(\mathbb{T}^n)^p\right\},\ \binom{\widehat{y}}{\widehat{u}}\longleftrightarrow\widehat{u},\ \widehat{y}=H\widehat{u},$$

and then also the injective, but generally not surjective projection

$$\mathscr{B}\bigcap\mathrm{L}^2(\mathbb{Z}^n)^{p+m}\to\mathrm{L}^2(\mathbb{Z}^n)^m,\ \binom{y}{u}\mapsto u.$$

2. (i) If H is continuous, i.e., $H\in\mathrm{C}^0(\mathbb{T}^n)^{p\times m}$, then there are the canonical isomorphisms

$$\mathscr{B}\bigcap\mathrm{L}^2(\mathbb{Z}^n)^{p+m}\cong\mathrm{L}^2(\mathbb{Z}^n)^m\cong\widehat{\mathscr{B}}\bigcap\mathrm{L}^2(\mathbb{T}^n)^{p+m}\cong\mathrm{L}^2(\mathbb{T}^n)^m,\ \binom{y}{u}\leftrightarrow u\leftrightarrow\binom{\widehat{y}}{\widehat{u}}\leftrightarrow\widehat{u}.$$

(ii) If P has a left inverse in $\mathscr{D}(\mathbb{T}^n)^{p\times k}$ and hence $H\in\mathscr{D}(\mathbb{T}^n)^{p\times m}$ (cf. Lemma 4.4) then likewise

$$\mathscr{B}\bigcap\mathscr{S}'(\mathbb{Z}^n)^{p+m}\cong\mathscr{S}'(\mathbb{Z}^n)^m\cong\widehat{\mathscr{B}}\cong\mathscr{D}'(\mathbb{T}^n)^m.$$

In both cases the inverse isomorphisms are given by $\widehat{u}\mapsto\binom{\widehat{y}}{\widehat{u}}=\binom{H\widehat{u}}{\widehat{u}}$. Notice that $\mathrm{L}^2(\mathbb{T}^n)$ resp. $\mathscr{D}'(\mathbb{T}^n)$ are $\mathrm{C}^0(\mathbb{T}^n)$- resp. $\mathscr{D}(\mathbb{T}^n)$-modules so that $H\widehat{u}$ is well-defined in both cases.

3. A behavior $\mathscr{B}\subseteq\left(\mathbb{C}^{\mathbb{Z}^n}\right)^\ell$ is autonomous if and only if $\mathscr{B}\bigcap\mathrm{L}^2(\mathbb{Z}^n)^\ell=0$.

4. If $\mathscr{B}=D(M)\subseteq\left(\mathbb{C}^{\mathbb{Z}^n}\right)^\ell$ is any behavior with its largest controllable subbehavior

$$\mathscr{B}_{\mathrm{cont}}=D(M/\mathrm{tor}(M))\text{ then }\mathscr{B}\bigcap\mathrm{L}^2(\mathbb{Z}^n)^\ell=\mathscr{B}_{\mathrm{cont}}\bigcap\mathrm{L}^2(\mathbb{Z}^n)^\ell.$$

Proof. 1. According to Lemma 4.3, 3. and 4., there is a left inverse $X \in \mathrm{L}^0(\mathbb{T}^n)^{p\times k}$ of P $(XP = \mathrm{id}_p)$ and then $H = XQ \in \mathrm{L}^0(\mathbb{T}^n)^{p\times m}$. Assume

$$\begin{pmatrix}\widehat{y}\\ \widehat{u}\end{pmatrix} \in \widehat{\mathscr{B}}\bigcap \mathrm{L}^2(\mathbb{T}^n)^{p+m} \Longrightarrow P\widehat{y} = Q\widehat{u} \in \mathrm{L}^2(\mathbb{T}^n)^k \Longrightarrow$$
$$\widehat{y} = XP\widehat{y} = XQ\widehat{u} = H\widehat{u} \in \mathrm{L}^0(\mathbb{T}^n)^p \Longrightarrow H\widehat{u} = \widehat{y} \in \mathrm{L}^2(\mathbb{T}^n)^p.$$

This proves that the projection $\begin{pmatrix}\widehat{y}\\ \widehat{u}\end{pmatrix} \mapsto \widehat{u}$ maps $\widehat{\mathscr{B}}\bigcap \mathrm{L}^2(\mathbb{T}^n)^{p+m}$ into the given space on the right and is injective. If, conversely, $\widehat{u} \in \mathrm{L}^2(\mathbb{T}^n)^m$ and $\widehat{y} := H\widehat{u} \in \mathrm{L}^2(\mathbb{T}^n)^p$ then

$$\begin{pmatrix}\widehat{y}\\ \widehat{u}\end{pmatrix} \in \mathrm{L}^2(\mathbb{T}^n)^{p+m} \text{ and } P\widehat{y} = PH\widehat{u} = Q\widehat{u} \in \mathrm{L}^0(\mathbb{T}^n)^k \bigcap \mathrm{L}^2(\mathbb{T}^n)^k = \mathrm{L}^2(\mathbb{T}^n)^k.$$

Hence $\begin{pmatrix}\widehat{y}\\ \widehat{u}\end{pmatrix} \in \widehat{\mathscr{B}}$ and the projection maps this onto $\widehat{u}$, i.e., the projection is surjective.
2. (i) If H is continuous and $\widehat{u} \in \mathrm{L}^2(\mathbb{T}^n)^m$ then always $H\widehat{u} \in \mathrm{L}^2(\mathbb{T}^n)^p$. The isomorphism from 1. gets the simpler form $\widehat{\mathscr{B}}\bigcap \mathrm{L}^2(\mathbb{T}^n)^{p+m} \cong \mathrm{L}^2(\mathbb{T}^n)^m$ which implies $\mathscr{B}\bigcap \mathrm{L}^2(\mathbb{Z}^n)^{p+m} \cong \mathrm{L}^2(\mathbb{Z}^n)^m$.
(ii) The proof is analogous, but uses Lemma 4.4 instead of Lemma 4.3,(3), for the cancelation of P in $P\widehat{y} = Q\widehat{u} = PH\widehat{u}$.
3. An autonomous behavior $\mathscr{B} \subset W_B^\ell$ is an IO behavior with

$$m = 0,\ Q = 0 \in A^{k\times 0} = 0,\ H = 0 \in \mathbb{C}(s)^{p\times 0} = 0.$$

Part 1. implies an embedding $\mathscr{B}\bigcap \mathrm{L}^2(\mathbb{Z}^n)^{p+m} \to \mathrm{L}^2(\mathbb{Z}^n)^m = 0$, hence $\mathscr{B}\bigcap \mathrm{L}^2(\mathbb{Z}^n)^\ell$=0. Assume conversely that $\mathscr{B}$ is not autonomous, i.e., $m > 0$ in (1) for any IO representation

$$\mathscr{B} := \left\{ \left(\begin{smallmatrix} y\\ u\end{smallmatrix}\right) \in \left(\mathbb{C}^{\mathbb{Z}^n}\right)^{p+m};\ P\circ y = Q\circ u \right\},\ \ell = p+m,\ m > 0,\ (P, -Q) \in A^{k\times(p+m)}, \Longrightarrow$$
$$\mathscr{B}\bigcap \mathrm{L}^2(\mathbb{Z}^n)^\ell \cong \widehat{\mathscr{B}}\bigcap \mathrm{L}^2(\mathbb{T}^n)^\ell.$$

Let $H = g^{-1}G$ with $G \in A^{p\times m}$ and $0 \neq g \in A \subset \mathbb{C}(s)\bigcap \mathscr{D}(\mathbb{T}^n)$. Choose any nonzero $\widehat{v} \in \mathrm{L}^2(\mathbb{T}^n)^m$ and pose $\widehat{u} := g\widehat{v} \in \mathrm{L}^2(\mathbb{T}^n)^m$. Then $\widehat{u} = g\widehat{v} \neq 0$ and

$$\widehat{y} := H\widehat{u} = Gg^{-1}g\widehat{v} = G\widehat{v} \in \mathrm{L}^2(\mathbb{T}^n)^p \underset{1.}{\Longrightarrow} 0 \neq \begin{pmatrix}\widehat{y}\\ \widehat{u}\end{pmatrix} \in \widehat{\mathscr{B}}\bigcap \mathrm{L}^2(\mathbb{T}^n)^\ell \Longrightarrow \mathscr{B}\bigcap \mathrm{L}^2(\mathbb{Z}^n)^\ell \neq 0.$$

4. Choose an epimorphism $A^{1\times q} \to \mathrm{tor}(M)$ so that $D(\mathrm{tor}(M))$ is an autonomous subbehavior of $\left(\mathbb{C}^{\mathbb{Z}^n}\right)^q$. The exact module and induced dual behavior sequences

$$0 \to \mathrm{tor}(M) \to M \xrightarrow{\mathrm{can}} M/\mathrm{tor}(M) \to 0 \text{ and } 0 \to \mathscr{B}_{\mathrm{cont}} \subseteq \mathscr{B} \xrightarrow{\mathrm{can}} D(\mathrm{tor}(M)) \to 0$$

induce the exact sequence

$$0 \to \mathscr{B}_{\mathrm{cont}}\bigcap \mathrm{L}^2(\mathbb{Z}^n)^\ell \subseteq \mathscr{B}\bigcap \mathrm{L}^2(\mathbb{Z}^n)^\ell \longrightarrow D(\mathrm{tor}(M))\bigcap \mathrm{L}^2(\mathbb{Z}^n)^q \underset{3.}{=} 0 \Longrightarrow$$
$$\mathscr{B}_{\mathrm{cont}}\bigcap \mathrm{L}^2(\mathbb{Z}^n)^\ell = \mathscr{B}\bigcap \mathrm{L}^2(\mathbb{Z}^n)^\ell.$$

□

Example 4.8. In contrast to Thm. 4.7,(3), the behavior $\left\{v \in \mathbb{C}^{\mathbb{Z}};\ (s_1 - 1)\circ v = 0\right\} = \mathbb{C}(\cdots, 1, 1, 1, \cdots)$ is autonomous and contained in $\mathscr{S}'(\mathbb{Z})$, but not zero since $s_1 - 1$ is not invertible in $\mathscr{D}(\mathbb{T}^1)$.

For a matrix

$$R = R_d(s)s_0^d + \cdots + R_0(s) \in B^{k\times\ell} = A[s_0]^{k\times\ell} \text{ with associated behavior}$$
$$\mathscr{B} := \left\{w \in W_B^\ell;\ R\circ w = 0\right\} = \left\{w \in W_B^\ell;\ \forall t \in \mathbb{N}:\ R_d \circ w(t+d) + \cdots + R_0 \circ w(t) = 0\right\}$$

the inverse Fourier isomorphisms induce B-isomorphisms

$$\begin{gathered}\mathscr{B}\bigcap\left(\mathscr{S}'(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \cong \widehat{\mathscr{B}} := \left\{\widehat{w} \in \left(\mathscr{D}'(\mathbb{T}^n)^\ell\right)^{\mathbb{N}}:\ \forall t \in \mathbb{N}:\ R_d\widehat{w}(t+d) + \cdots + R_0\widehat{w}(t) = 0\right\}\\ \mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \cong \widehat{\mathscr{B}}\bigcap\left(\mathrm{L}^2(\mathbb{T}^n)^\ell\right)^{\mathbb{N}} :=\\ \left\{\widehat{w} \in \left(\mathrm{L}^2(\mathbb{T}^n)^\ell\right)^{\mathbb{N}}:\ \forall t \in \mathbb{N}:\ R_d\widehat{w}(t+d) + \cdots + R_0\widehat{w}(t) = 0\right\}.\end{gathered} \tag{55}$$

Theorem 4.9. *1. Assume*

$$(P,-Q) \in A^{k\times(\ell+\ell)},\ \mathrm{rank}(P) = \mathrm{rank}(P,-Q) = \ell,\ R := Ps_0 - Q \in B^{k\times\ell},$$
$$H \in \mathbb{C}(s)^{\ell\times\ell} \subset \mathrm{L}^0(\mathbb{T}^n)^{\ell\times\ell} \text{ with } PH = Q$$

and consider the behavior

$$\mathscr{B} := \left\{w \in W_B^\ell;\ R\circ w = 0\right\} = \left\{w \in W_B^\ell;\ \forall t \in \mathbb{N}:\ P\circ w(t+1) = Q\circ w(t)\right\}.$$

Assume in addition that (i) H is continuous resp. that (ii) P has a left inverse in $\mathscr{D}(\mathbb{T}^n)^{\ell\times k}$ and thus H is smooth (cf. Lemma 4.4). Then the projections

$$\begin{cases}(i)\ \mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \to \mathrm{L}^2(\mathbb{Z}^n)^\ell\\ (ii)\ \mathscr{B}\bigcap\left(\mathscr{S}'(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \to \mathscr{S}'(\mathbb{Z}^n)^\ell\end{cases},\ w \mapsto w(0),$$

are bijective. The inverse isomorphisms are given by $w(0) \mapsto w$ with $\widehat{w(t)} = H^t\widehat{w(0)}$.
2. Under the assumptions of 1. there is the direct decomposition

$$\left\{w \in \mathscr{B};\ w(0) \in \mathrm{L}^2(\mathbb{Z}^n)^\ell\right\} = \left(\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}\right) \oplus \{w \in \mathscr{B};\ w(0) = 0\}.$$

Unless the autonomous behavior $\mathscr{C} := \left\{v \in \left(\mathbb{C}^{\mathbb{Z}^n}\right)^\ell;\ P\circ v = 0\right\}$ is zero or, equivalently, P has a left inverse in $A^{p\times k}$ the second summand $\{w \in \mathscr{B};\ w(0) = 0\}$ is infinite-dimensional.

Proof. 1. We give the proof for $\mathrm{L}^2(\mathbb{Z}^n)$ and continuous H, the other case is analogous. We use $\widehat{\mathscr{B}}$ from (55) and the commutative diagram with vertical isomorphisms

$$\begin{array}{ccll}\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} & \overset{\mathrm{proj}}{\longrightarrow} & \mathrm{L}^2(\mathbb{Z}^n)^\ell, & w \mapsto w(0),\\ \downarrow \mathscr{F}_{\mathbb{Z}^n}^{\mathbb{N}} & & \downarrow \mathscr{F}_{\mathbb{Z}^n} & \\ \widehat{\mathscr{B}}\bigcap\left(\mathrm{L}^2(\mathbb{T}^n)^\ell\right)^{\mathbb{N}} & \overset{\mathrm{proj}}{\longrightarrow} & \mathrm{L}^2(\mathbb{T}^n)^\ell, & \widehat{w} \mapsto \widehat{w}(0),\end{array}$$

Therefore it suffices to prove the bijectivity of the projection in the second row. According to Thm. 4.7, 1., the equation $P\widehat{w}(t+1) = Q\widehat{w}(t)$ is equivalent to $\widehat{w}(t+1) = H\widehat{w}(t)$, hence $\widehat{w}(t) = H^t\widehat{w}(0) \in \mathrm{L}^0(\mathbb{T}^n)^\ell$. This proves that the projection $\widehat{w} \mapsto \widehat{w}(0)$ is injective. Since H is assumed continuous so are all powers H^t. Hence if $\widehat{w}(0) \in \mathrm{L}^2(\mathbb{T}^n)^\ell$ then

$$\forall t \in \mathbb{N}: \ \widehat{w}(t) := H^t\widehat{w}(0) \in \mathrm{L}^2(\mathbb{T}^n)^\ell \text{ and } P\widehat{w}(t+1) = PH\widehat{w}(t) = Q\widehat{w}(t) \Longrightarrow \widehat{w} \in \widehat{\mathscr{B}}.$$

This shows that the projection is surjective as asserted.
2. Let $w' \in \mathscr{B}$ satisfy $w'(0) \in \mathrm{L}^2(\mathbb{Z}^n)^\ell$ and, according to 1., let $w \in \mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}$ be the unique trajectory with $w(0) = w'(0)$. Then $w' - w \in \mathscr{B}$ with $(w'-w)(0) = 0$ and $w' = w + (w' - w)$ is the asserted sum representation. Assume $w \in \mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}$ and $w(0) = 0$. Then $P \circ w(t+1) = Q \circ w(t)$ and especially $P \circ w(1) = Q \circ 0 = 0$ or $w(1) \in \mathscr{C}$. Since the autonomous behavior $\mathscr{C}$ has no square summable trajectories by Thm. 4.7, 3., we infer $w(1) = 0$. Inductively one proves $w(t) = 0$ for all $t \in \mathbb{N}$. This shows that the sum in 2. is direct.
Since $(P, -Q)$ defines an IO behavior the equation $P \circ y = Q \circ u$ has a solution $y \in \left(\mathbb{C}^{\mathbb{Z}^n}\right)^\ell$ for each $u \in \left(\mathbb{C}^{\mathbb{Z}^n}\right)^\ell$. One constructs trajectories $w \in \mathscr{B}$ with $w(0) = 0$ inductively by

$$P \circ w(t+1) = Q \circ w(t) \text{ for } t \geq 1,\ P \circ w(1) = 0$$

where each $w(t)$ is unique up to a summand in $\mathscr{C}$ only. If the latter is nonzero then $\{w \in \mathscr{B};\ w(0) = 0\}$ is infinite-dimensional □

For the stability theory in Section 5 we apply Thm. 4.9 in the following special situation. Consider a nonzero polynomial

$$f = f_d(s)s_0^d + \cdots + f_0(s) \in B = A[s_0] \text{ with } f_d(\omega) \neq 0 \text{ for } \omega \in \mathbb{T}^n \text{ and define}$$
$$P := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & f_d \end{pmatrix},\ Q := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -f_0 & -f_1 & -f_2 & \cdots & -f_{d-2} & -f_{d-1} \end{pmatrix} \in A^{d\times d}. \tag{56}$$

It is obvious that P has a left inverse in $\mathscr{D}(\mathbb{T}^n)^{d\times d}$ and that

$$H := P^{-1}Q = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -f_0/f_d & -f_1/f_d & -f_2/f_d & \cdots & -f_{d-2}/f_d & -f_{d-1}/f_d \end{pmatrix} \in \mathscr{D}(\mathbb{T}^n)^{d\times d}. \tag{57}$$

The polynomial f resp. the matrices $(P, -Q)$ give rise to the behaviors

$$\begin{aligned} \mathscr{B} &:= \{w \in W_B;\ f \circ w = 0\} = \{w \in W_B;\ \forall t \in \mathbb{N}:\ f_d \circ w(t+d) + \cdots + f_0 \circ w(t) = 0\} \\ \mathscr{B}_{st} &:= \left\{x \in W_B^d;\ (Ps_0 - Q) \circ x = 0\right\} = \left\{x \in W_B^\ell;\ \forall t \in \mathbb{N}:\ P \circ x(t+1) = Q \circ x(t)\right\} \end{aligned} \tag{58}$$

and the standard isomorphisms

$$\mathscr{B} \cong \mathscr{B}_{st},\ w \leftrightarrow x,\ x(t) = (w(t), \cdots, w(t+d-1))^\top,\ w(t) = x_0(t),$$
$$\text{and also } \mathscr{B}\bigcap\mathscr{S}'(\mathbb{Z}^n)^{\mathbb{N}} \cong \mathscr{B}_{st}\bigcap\left(\mathscr{S}'(\mathbb{Z}^n)^d\right)^{\mathbb{N}},\ \mathscr{B}\bigcap\mathrm{L}^2(\mathbb{Z}^n)^{\mathbb{N}} \cong \mathscr{B}_{st}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^d\right)^{\mathbb{N}}. \tag{59}$$

Corollary 4.10. *For the data from* (56)-(59) *the projections*

$$\mathscr{B}\bigcap\mathrm{L}^2(\mathbb{Z}^n)^{\mathbb{N}} \to \mathrm{L}^2(\mathbb{Z}^n)^d,\ w \mapsto (w(0), \cdots, w(d-1))^\top,$$
$$\mathscr{B}\bigcap\mathscr{S}'(\mathbb{Z}^n)^{\mathbb{N}} \to \mathscr{S}'(\mathbb{Z}^n)^d,\ w \mapsto (w(0), \cdots, w(d-1))^\top,$$

are bijective. The inverse isomorphism is given by

$$\widehat{x}(t)=H^t\widehat{x}(0) \text{ where } x(t):=(w(t),\cdots,w(t+d-1))^\top,$$
$$\widehat{x}(t)=(\widehat{w(t)},\cdots,\widehat{w(t+d-1)})^\top,\ \widehat{x}(0)=(\widehat{w(0)},\cdots,\widehat{w(d-1)})^\top.$$

5 Asymptotic stability

In this section we connect the first four sections whose assumptions remain in force. In the sequel we have to consider complex matrices $M\in\mathbb{C}^{\ell\times\ell}$, their spectrum $\operatorname{spec}(M):=\{\lambda\in\mathbb{C};\ \operatorname{rank}(\lambda\operatorname{id}_\ell-M)<\ell\}$ and the spectral radius $\rho(M):=\max_{\lambda\in\operatorname{spec}(M)}|\lambda|$. We use that the function $M\mapsto\rho(M)$ is continuous [7, Cor. 4.2.2]. We also apply the discs $\mathbb{D}^1$ and $\overline{\mathbb{D}}^1$ from (13).

Corollary 5.1. *Data and assumptions from Thm. 4.9 with continuous H.*
1. If the behavior $\mathscr{B}$ is Λ_1-stable (compare Def. 2.1 and (15)*) then* $\operatorname{spec}(H(\omega))\subset\mathbb{D}^1:=\{z\in\mathbb{C};\ |z|<1\}$ *for all $\omega\in\mathbb{T}^n$. If, conversely,* $\operatorname{spec}(H(\omega))\subset\mathbb{D}^1$ *for* almost all $\omega\in\mathbb{T}^n$ *and hence* $\operatorname{spec}(H(\omega))\subset\overline{\mathbb{D}}^1$ *for* all $\omega\in\mathbb{T}^n$ *then almost all elements of* $\operatorname{char}(\mathscr{B})$ *belong to Λ_1.*
2. If P has a left inverse in $\mathscr{D}(\mathbb{T}^n)^{\ell\times k}$ or, equivalently, $\operatorname{rank}(P(\omega))=\ell$ *for all $\omega\in\mathbb{T}^n$ then $\mathscr{B}$ is Λ_1-stable if and only if* $\operatorname{spec}(H(\omega))\subset\mathbb{D}^1$ *for all $\omega\in\mathbb{T}^n$. This is applicable in Cor. 4.10.*

Proof. We use Lemma 4.3, 3., viz. that

$$\operatorname{rank}(P(\omega))=\operatorname{rank}(P)=\ell \text{ for almost all } \omega\in\mathbb{T}^n \text{ and}$$
$$P(\omega)H(\omega)=Q(\omega),\ P(\omega)(\lambda\operatorname{id}_\ell-H(\omega))=\lambda P(\omega)-Q(\omega)=R(\lambda,\omega).$$

1. $\Longrightarrow$: For $\omega\in\mathbb{T}^n$ and $\lambda\in\operatorname{spec}(H(\omega))$ the last equation implies

$$\operatorname{rank}(R(\lambda,\omega))\le\operatorname{rank}(\lambda\operatorname{id}_\ell-H(\omega))<\ell\Longrightarrow(\lambda,\omega)\in\operatorname{char}(\mathscr{B})\subseteq\Lambda_1\Longrightarrow|\lambda|<1.$$

$\Longleftarrow$: For almost all $\omega\in\mathbb{T}^n$ the relations

$$\operatorname{rank}(P(\omega))=\ell,\ \text{hence}\ \operatorname{rank}(R(\lambda,\omega))=\operatorname{rank}(\lambda\operatorname{id}_\ell-H(\omega)),\ \text{and}\ \operatorname{spec}(H(\omega))\subset\mathbb{D}^1$$

hold. If for *such* ω the pair (λ,ω) is contained in $\operatorname{char}(\mathscr{B})$ then $\lambda\in\operatorname{spec}(H(\omega))$ and thus $|\lambda|<1$ and $(\lambda,\omega)\in\Lambda_1$. The continuity of $\rho(H(\omega))$ and the density of subsets of measure 1 of $\mathbb{T}^n$ imply $\rho(H(\omega))\le 1$ for all $\omega\in\mathbb{T}^n$ and hence $\operatorname{spec}(H(\omega))\subset\overline{\mathbb{D}}^1$.
2. In this case $\operatorname{rank}(R(\lambda,\omega))=\operatorname{rank}(\lambda\operatorname{id}_\ell-H(\omega))$ holds for all $\omega\in\mathbb{T}^n$. □

Consider, more generally, any compact parameter space Ω and a continuous function $E:\Omega\to\mathbb{C}^{\ell\times\ell}$, i.e., $E\in\mathrm{C}^0(\Omega)^{\ell\times\ell}$, for instance H from Cor. 5.1.

Let $\|-\|$ denote any norm on $\mathbb{C}^\ell$ and also the associated matrix norm on $\mathbb{C}^{\ell\times\ell}$:

$$\|M\|:=\max_{x\in\mathbb{C}^\ell,x\neq 0}\|Mx\|/\|x\| \text{ with } \|Mx\|\le\|M\|\|x\| \text{ and } \|M_1M_2\|\le\|M_1\|\|M_2\|. \tag{60}$$

The Jordan decomposition of a complex matrix shows that for any $\rho_1>\rho(M)$ there is a constant $a_1\ge 1$ such that $\|M^t\|\le a_1\rho_1^t$ for all $t\in\mathbb{N}$. Then for any matrix $\Delta\in\mathbb{C}^{\ell\times\ell}$ the following inequality holds [7, §4.2.4, Ex. 9]:

$$\|(M+\Delta)^t\|\le a_1\left(\rho_1+a_1\|\Delta\|\right)^t,\ t\in\mathbb{N}. \tag{61}$$

The following lemma and corollary are the basic tool for the proof of asymptotic L^2-stability.

Lemma 5.2. *Let $\rho_0 := \max_{\omega\in\Omega}\rho(E(\omega))$ be the maximum of the continuous function $\omega \mapsto \rho(E(\omega))$ on the compact space Ω and choose $\rho_0 < \rho_2$ arbitrarily. Then there is $a_2 \geq 1$ such that for all $\omega \in \Omega$ and $t \in \mathbb{N}$: $\|E(\omega)^t\| \leq a_2\rho_2^t$.*

Proof. Choose

$$\rho_0 < \rho_1 < \rho_2 \Longrightarrow \forall\omega\in\Omega\exists a(\omega)\geq 1\forall t\in\mathbb{N}:\ \|E(\omega)^t\| \leq a(\omega)\rho_1^t.$$

Since $E : \Omega \to \mathbb{C}^{\ell\times\ell}$ is continuous there is, for each $\omega\in\Omega$, an open set

$$\begin{gathered}U(\omega)\subset\Omega \text{ with } \omega\in U(\omega) \text{ and } \forall\omega'\in U(\omega): \|E(\omega')-E(\omega)\| \leq a(\omega)^{-1}(\rho_2-\rho_1)\\ \underset{(61)}{\Longrightarrow} \forall\omega'\in U(\omega)\forall t\in\mathbb{N}:\ \|E(\omega')^t\| = \|\left(E(\omega)+(E(\omega')-E(\omega))\right)^t\| \leq\\ a(\omega)\left(\rho_1+a(\omega)\|E(\omega')-E(\omega)\|\right)^t \leq\\ a(\omega)\left(\rho_1+a(\omega)a(\omega)^{-1}(\rho_2-\rho_1)\right)^t = a(\omega)\rho_2^t \Longrightarrow \|E(\omega')^t\| \leq a(\omega)\rho_2^t.\end{gathered}$$

Since Ω is compact there are points ω_i, $i = 1,\cdots,q$, such that $\Omega = \bigcup_{i=1}^q U(\omega_i)$. Let $a_2 := \max_{i=1,\cdots,q} a(\omega_i)$. The preceding inequality implies the asserted inequality

$$\forall\omega'\in\Omega\forall t\in\mathbb{N}:\ \|E(\omega')^t\| \leq a_2\rho_2^t.$$

□

In the next lemma we use the 2-norm $\|-\|_2$ on $\mathbb{C}^\ell$ and $\mathbb{C}^{\ell\times\ell}$.

Corollary 5.3. *Assume in Lemma 5.2 that for all $\omega\in\Omega$ the spectrum of $E(\omega)$ is contained in the open unit disc, i.e., $\rho(E(\omega)) < 1$, and hence $\rho_0 := \max_{\omega\in\Omega}\rho(E(\omega)) < 1$. Then for any ρ_2 with $\rho_0 < \rho_2 < 1$ there is a constant $a_2 \geq 1$ such that $\|E(\omega)^t\|_2 \leq a_2\rho_2^t$ for all $\omega\in\Omega$ and $t\in\mathbb{N}$. In particular, $E(\omega)^t$ converges to zero,* uniformly *in $\omega\in\Omega$, and $\|E(\omega)^t\|_2$ is uniformly bounded. For all $\widehat{u}\in \mathrm{L}^2(\mathbb{T}^n)^\ell$ this implies $E^t\widehat{u}\in\mathrm{L}^2(\mathbb{T}^n)^\ell$ and $\lim_{t\to\infty}\|E^t\widehat{u}\|_2 = 0$.*

Proof. Only the last statement has to be shown. For $\omega\in\mathbb{T}^n$ we have

$$\begin{gathered}\|E(\omega)^t\widehat{u}(\omega)\|_2^2 \leq \|E(\omega)^t\|_2^2\|\widehat{u}(\omega)\|_2^2 \leq a_2^2\rho_2^{2t}\|\widehat{u}(\omega)\|_2^2 \Longrightarrow\\ \|E^t\widehat{u}\|_2^2 = \int_{\mathbb{T}^n}\|E(\omega)^t\widehat{u}(\omega)\|_2^2 d\omega \leq a_2^2\rho_2^{2t}\int_{\mathbb{T}^n}\|\widehat{u}(\omega)\|_2^2 d\omega = a_2^2\rho_2^{2t}\|\widehat{u}\|_2^2 \underset{t\to\infty}{\longrightarrow} 0.\end{gathered}$$

□

Definition 5.4. A behavior $\mathscr{B}\subseteq\left(\left(\mathbb{C}^{\mathbb{Z}^n}\right)^\ell\right)^{\mathbb{N}}$ is called L^2-*stable* if $\lim_{t\to\infty}\|w(t)\|_2 = 0$ for all $w\in\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}$.

Lemma 5.5. *(cf. [24, Sect. VII]) Under the assumptions of Thm. 4.9 with continuous H or of Cor. 4.10 a Λ_1-stable behavior is L^2-stable.*

Proof. We consider the case of Thm. 4.9, that of Cor. 4.10 is then a direct consequence. By assumption $H\in\mathrm{C}^0(\mathbb{T}^n)^{\ell\times\ell}$. Since $\mathscr{B}$ is Λ_1-stable we infer from Cor. 5.1 that $\mathrm{spec}(H(\omega))\subset\mathbb{D}^1$ for all $\omega\in\mathbb{T}^n$. With the notations from Thm. 4.9 the trajectories in $w\in\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}$ satisfy $\widehat{w(t)} = H^t\widehat{w(0)}$. With Cor. 5.3 applied to H we conclude $\lim_{t\to\infty}\|\widehat{w(t)}\|_2 = 0$. Since $\mathscr{F}_{\mathbb{Z}^n} : \mathrm{L}^2(\mathbb{Z}^n)\cong\mathrm{L}^2(\mathbb{T}^n)$ is an isometry we infer $\lim_{t\to\infty}\|w(t)\|_2 = 0$.

□

The following theorem establishes the first half of Thm. 2.3.

Theorem 5.6. *Assume that the behavior* $\mathscr{B} \subset \left((\mathbb{C}^{\mathbb{Z}^n})^\ell\right)^{\mathbb{N}}$ *is* Λ_1*-stable and time-autonomous with* $\mathscr{B} \cong \mathscr{B}_d,\ w \mapsto (w(0), \cdots, w(d-1))^\top$ *(cf. Cor. 3.1). Assume in addition that the behavior* $\mathscr{B}_d \subseteq (W_A^\ell)^d$ *admits an input/output (IO) representation with continuous transfer matrix (cf. Thm. 4.7, 2.), viz.*

$$\begin{gathered}\mathscr{B}_d = \left\{ \binom{y}{u} = (w(0), \cdots, w(d-1))^\top \in W_A^{p+m};\ P \circ y = Q \circ u \right\},\ \ell d = p + m, \\ (P, -Q) \in A^{k \times (p+m)},\ \operatorname{rank}(P) = \operatorname{rank}(P, -Q) = p,\ PH = Q,\ H \in \mathrm{C}^0(\mathbb{T}^n)^{p \times m}.\end{gathered} \tag{62}$$

If $w \in \mathscr{B}$ *and* $w(0), \cdots, w(d-1) \in \mathrm{L}^2(\mathbb{Z}^n)^\ell$ *then* $w(t) \in \mathrm{L}^2(\mathbb{Z}^n)^\ell$ *for all* $t \in \mathbb{N}$ *and* $\lim_{t\to\infty} \|w(t)\|_2 = 0$, *in particular* $\mathscr{B}$ *is* L^2*-stable. Moreover the projections*

$$\begin{gathered}\mathscr{B} \bigcap \left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \cong \mathscr{B}_d \bigcap \left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^d \cong \mathrm{L}^2(\mathbb{Z}^n)^m, \\ w \mapsto (w(0), \cdots, w(d-1))^\top = \binom{y}{u} \mapsto u\end{gathered} \tag{63}$$

are bijective.

Proof. The isomorphisms (63) follow from Lemma 4.1 and Thm. 4.7,(2).
The properties of $\mathscr{B}$ are preserved under behavior isomorphisms. According to Cor. 3.3 and Thm. 3.4 we assume wlog that $d = 1$ and that $\mathscr{B}$ is given as

$$\mathscr{B} = \left\{ w \in \left((C^{\mathbb{Z}^n})^\ell\right)^{\mathbb{N}};\ R \circ w = 0 \right\},\ R = \begin{pmatrix} R_0 \\ s_0\, \mathrm{id}_\ell - E \end{pmatrix},\ R_0 \in A^{k \times \ell},\ E \in A^{\ell \times \ell},\ R_0 E = X R_0.$$

Hence $\mathscr{B}_1 = \left\{ v \in \left(\mathbb{C}^{\mathbb{Z}^n}\right)^\ell;\ R_0 \circ v = 0 \right\}$ and any trajectory $w \in \left(W_B^\ell\right)^{\mathbb{N}}$ belongs to

$\mathscr{B} \bigcap \left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}$ if and only if $w(0) \in \mathrm{L}^2(\mathbb{Z}^n)^\ell$, $R_0 \circ w(0) = 0$ and $w(t) = E^t \circ w(0)$.

Moreover (63) furnishes

$$\begin{gathered}\mathscr{B} \bigcap \left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \cong \mathscr{B}_1 \bigcap \mathrm{L}^2(\mathbb{Z}^n)^\ell = \left\{ v \in \mathrm{L}^2(\mathbb{Z}^n)^\ell;\ R_0 \circ v = 0 \right\} \cong \\ \widehat{\mathscr{B}} \bigcap \left(\mathrm{L}^2(\mathbb{T}^n)^\ell\right)^{\mathbb{N}} = \left\{ \widehat{w} \in \left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}};\ R_0 \widehat{w}(0) = 0,\ \widehat{w}(t) = E^t \widehat{w}(0) \right\} \cong \\ \widehat{\mathscr{B}_1} \bigcap \mathrm{L}^2(\mathbb{T}^n)^\ell = \left\{ \widehat{v} \in \mathrm{L}^2(\mathbb{T}^n)^\ell;\ R_0 \widehat{v} = 0 \right\},\ w \leftrightarrow w(0) \leftrightarrow \widehat{w} \leftrightarrow \widehat{w(0)},\ \text{with } \widehat{w(t)} = E^t \widehat{w(0)}.\end{gathered} \tag{64}$$

By assumption the behavior $\mathscr{B}_1$ has an IO decomposition with continuous transfer matrix $H \in \left(\mathbb{C}(s) \bigcap \mathrm{C}^0(\mathbb{T}^n)\right)^{p \times m}$, i.e.,

$$\ell = p + m,\ R_0 = (P, -Q) \in A^{k \times (p+m)},\ \operatorname{rank}(P) = \operatorname{rank}(P, -Q) = p,\ PH = Q.$$

By Thm. 4.7,(2), this decomposition induces the inverse isomorphisms

$$\mathrm{L}^2(\mathbb{T}^n)^m \cong \widehat{\mathscr{B}_1} \bigcap \mathrm{L}^2(\mathbb{T}^n)^{p+m},\ \widehat{u} \leftrightarrow \binom{H\widehat{u}}{\widehat{u}},\ \text{and also } \mathrm{C}^0(\mathbb{T}^n)^m \cong \widehat{\mathscr{B}_1} \bigcap \mathrm{C}^0(\mathbb{T}^n)^{p+m}.$$

Since $\mathscr{B}_1 = \left\{ v \in W_A^\ell;\ R_0 \circ v = 0 \right\}$ and $R_0 E = X R_0$ the matrix $E \in A^{\ell \times \ell} \subseteq \mathrm{C}^0(\mathbb{T}^n)^{\ell \times \ell}$ maps $\widehat{\mathscr{B}_1} \bigcap \mathrm{C}^0(\mathbb{T}^n)^{p+m}$ into itself. The second preceding isomorphism gives rise to a unique matrix

$$E'\in \mathrm{C}^0(\mathbb{T}^n)^{m\times m} \text{ with } E\left(\begin{smallmatrix}H\\ \mathrm{id}_m\end{smallmatrix}\right)=\left(\begin{smallmatrix}H\\ \mathrm{id}_m\end{smallmatrix}\right)E' \Longrightarrow$$
$$\forall\omega\in\mathbb{T}^n:\ E(\omega)\begin{pmatrix}H(\omega)\\ \mathrm{id}_m\end{pmatrix}=\begin{pmatrix}H(\omega)\\ \mathrm{id}_m\end{pmatrix}E'(\omega) \text{ and } \forall t\in\mathbb{N}:\ E^t\left(\begin{smallmatrix}H\\ \mathrm{id}_m\end{smallmatrix}\right)=\left(\begin{smallmatrix}H\\ \mathrm{id}_m\end{smallmatrix}\right)(E')^t \Longrightarrow$$
$$\forall\widehat{u}\in \mathrm{L}^2(\mathbb{T}^n)^m,\ \forall t\in\mathbb{N}:\ E^t\left(\begin{smallmatrix}H\\ \mathrm{id}_m\end{smallmatrix}\right)\widehat{u}=\left(\begin{smallmatrix}H\\ \mathrm{id}_m\end{smallmatrix}\right)(E')^t\widehat{u}.$$

We check the assumptions of Cor. 5.3 for E': Let $\omega\in\mathbb{T}^n$, $\lambda\in\operatorname{spec}(E'(\omega))$ and $0\neq v\in\mathbb{C}^m\subset \mathrm{C}^0(\mathbb{T}^n)^m$ a nonzero eigenvector, $E'(\omega)v=\lambda v$. Hence also $v':=\begin{pmatrix}H(\omega)\\ \mathrm{id}_m\end{pmatrix}v$ is nonzero. The preceding equations imply

$$E(\omega)v'=E(\omega)\begin{pmatrix}H(\omega)\\ \mathrm{id}_m\end{pmatrix}v=\begin{pmatrix}H(\omega)\\ \mathrm{id}_m\end{pmatrix}E'(\omega)v=\lambda\begin{pmatrix}H(\omega)\\ \mathrm{id}_m\end{pmatrix}v=\lambda v'\Longrightarrow$$
$$(\lambda\,\mathrm{id}_\ell-E(\omega))\,v'=0\Longrightarrow\begin{pmatrix}(\mathrm{id}_p,-H(\omega))\\ \lambda\,\mathrm{id}_\ell-E(\omega)\end{pmatrix}v'=0\underset{R_0(\omega)=(P(\omega),-Q(\omega))=P(\omega)(\mathrm{id}_P,-H(\omega))}{\Longrightarrow}$$
$$\begin{pmatrix}R_0(\omega)\\ \lambda\,\mathrm{id}_\ell-E(\omega)\end{pmatrix}v'=0\underset{v'\neq 0}{\Longrightarrow}\operatorname{rank}\begin{pmatrix}R_0(\omega)\\ \lambda\,\mathrm{id}_\ell-E(\omega)\end{pmatrix}<\ell\Longrightarrow(\lambda,\omega)\in\operatorname{char}(\mathscr{B})\subset\Lambda_1\Longrightarrow|\lambda|<1.$$

Here we used the Λ_1-stability of $\mathscr{B}$. We have thus shown that for all $\omega\in\mathbb{T}^n$ the spectrum $\operatorname{spec}(E'(\omega))$ is contained in the open unit disc and conclude with Cor. 5.3 that $\lim_{t\to\infty}\|(E')^t\widehat{u}\|_2=0$ for all $\widehat{u}\in \mathrm{L}^2(\mathbb{T}^n)^m$. Since H is continuous this implies the L^2-convergence

$$\lim_{t\to\infty}E^t\left(\begin{smallmatrix}H\\ \mathrm{id}_m\end{smallmatrix}\right)\widehat{u}=\lim_{t\to\infty}\left(\begin{smallmatrix}H\\ \mathrm{id}_m\end{smallmatrix}\right)(E')^t\widehat{u}=\left(\begin{smallmatrix}H\\ \mathrm{id}_m\end{smallmatrix}\right)\lim_{t\to\infty}(E')^t\widehat{u}=\left(\begin{smallmatrix}H\\ \mathrm{id}_m\end{smallmatrix}\right)0=0.$$

This convergence and the isomorphisms

$$\mathrm{L}^2(\mathbb{T}^n)^m\overset{\left(\begin{smallmatrix}H\\ \mathrm{id}_m\end{smallmatrix}\right)}{\longrightarrow}\widehat{\mathscr{B}_1}\bigcap \mathrm{L}^2(\mathbb{T}^n)^{p+m}\overset{\overline{\mathscr{F}}_{\mathbb{T}^n}}{\longrightarrow}\mathscr{B}_1\bigcap \mathrm{L}^2(\mathbb{Z}^n)^{p+m},\ \widehat{u}\mapsto\left(\begin{smallmatrix}H\widehat{u}\\ \widehat{u}\end{smallmatrix}\right)=\widehat{w(0)}\mapsto w(0),$$

imply

$$\lim_{t\to\infty}E^t\circ w(0)=0 \text{ for all } w(0)\in\mathscr{B}_1\bigcap \mathrm{L}^2(\mathbb{Z}^n)^{p+m}$$

and by means of (64) also the L^2-stability of $\mathscr{B}$, i.e.,

$$\lim_{t\to\infty}w(t)=0 \text{ for all } w\in\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}.$$

□

Corollary 5.7. *In the preceding theorem and its proof consider the controllable part* $\mathscr{B}_{1,\mathrm{cont}}$ *of* $\mathscr{B}_1$*, given by*

$$\mathscr{B}_{1,\mathrm{cont}}:=D(M_1/\operatorname{tor}(M_1))=\left\{v\in W_A^\ell;\ R_{0,\mathrm{cont}}\circ v=0\right\} \text{ where}$$
$$R_{0,\mathrm{cont}}\in A^{k_{\mathrm{cont}}\times\ell},\ \operatorname{tor}(M_1)=A^{1\times k_{\mathrm{cont}}}R_{0,\mathrm{cont}}/A^{1\times\ell}R_0\subseteq M_1=A^{1\times\ell}/A^{1\times k}R_0, \text{ hence}$$
$$R_0=YR_{0,\mathrm{cont}} \text{ for some } Y.$$

Then the behavior

$$\mathscr{B}':=\left\{w\in W_B^\ell;\ R'\circ w=0\right\} \text{ with } R':=\begin{pmatrix}R_{0,\mathrm{cont}}\\ s_0\,\mathrm{id}_\ell-E\end{pmatrix}$$

is ta with $\mathscr{B}_1'=\mathscr{B}_{1,\mathrm{cont}}$ *and contained in* $\mathscr{B}$*, hence* $\operatorname{ann}_B(\mathscr{B})\subseteq\operatorname{ann}_B(\mathscr{B}')$ *and* $\operatorname{char}(\mathscr{B}')\subseteq\operatorname{char}(\mathscr{B})$. *The input/output decompositions of* $\mathscr{B}_1$ *and* $\mathscr{B}_{1,\mathrm{cont}}$ *and the associated transfer matrices coincide. If* $\mathscr{B}'$ *instead of* $\mathscr{B}$ *satisfies the assumptions of Thm. 5.6 then* $\mathscr{B}'$ *and* $\mathscr{B}$ *are* L^2*-stable. Hence one may assume wlog that* $\mathscr{B}_1$ *is controllable.*

Proof. The A-linear map

$$\circ E : M_1 = A^{1\times\ell}/A^{1\times k}R_0 \to M_1,\ \xi + A^{1\times k}R_0 \mapsto \xi E + A^{1\times k}R_0,\ R_0E = XR_0,$$

maps $\mathrm{tor}(M_1)$ into itself and therefore there is a matrix X' with $R_{0,\mathrm{cont}}E = X'R_{0,\mathrm{cont}}$. This signifies that $\mathscr{B}'$ is indeed ta with $\mathscr{B}'_1 = \mathscr{B}_{1,\mathrm{cont}}$. Thm. 4.7,(4), and the ta of $\mathscr{B}$ and $\mathscr{B}'$ furnish

$$\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} \cong \mathscr{B}_1\bigcap \mathrm{L}^2(\mathbb{Z}^n)^\ell = \mathscr{B}_{1,\mathrm{cont}}\bigcap \mathrm{L}^2(\mathbb{Z}^n)^\ell \cong \mathscr{B}'\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}},\ \text{hence}$$
$$\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}} = \mathscr{B}'\bigcap\left(\mathrm{L}^2(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}$$

But $\mathscr{B}'$ is L^2-stable by Thm. 5.6 and hence so is $\mathscr{B}$. □

Corollary 5.8. *Assume as in Thm. 5.6 that the behavior $\mathscr{B}$ is Λ_1-stable and ta with $\mathscr{B} \cong \mathscr{B}_d$. Assume in addition that*

1. *$\mathscr{B}$ is* square-autonomous *[9, Thm. 10] or that*
2. *$\mathscr{B}_d$ is strictly controllable or, equivalently, its module M_d is A-free.*

If $w \in \mathscr{B}$ and $w(t) \in \mathrm{L}^2(\mathbb{Z}^n)^\ell$ for $t \le d-1$ then $w(t) \in \mathrm{L}^2(\mathbb{Z}^n)^\ell$ for all $t \in \mathbb{N}$ and $\lim_{t\to\infty}\|w(t)\|_2 = 0$. *In particular, $\mathscr{B}$ is* L^2*-stable.*

Proof. 1. Square-autonomy signifies that $\mathscr{B} = \left\{w \in \left((\mathbb{C}^{\mathbb{Z}^n})^{\mathbb{N}}\right)^\ell;\ R\circ w = 0\right\}$ with a *square matrix* $R \in B^{\ell\times\ell}$ and $f := \det(R) \neq 0$. Let $M := B^{1\times\ell}/B^{1\times\ell}R$ be the module of $\mathscr{B}$. The characteristic variety of $\mathscr{B}$ or M is

$$\mathrm{char}(\mathscr{B}) = \{(\lambda,\omega) \in \Lambda_N;\ \mathrm{rank}(R(\lambda,\omega)) < \ell\} = V_{\Lambda_N}(\det(R)) = V_{\Lambda_N}(f) \subseteq \Lambda_1.$$

The behavior $\mathscr{B}_f := \left\{w \in \left(\mathbb{C}^{\mathbb{Z}^n}\right)^{\mathbb{N}};\ f\circ w = 0\right\}$ has the characteristic variety $V_{\Lambda_N}(f)$ and is thus Λ_1-stable too.
Since $\mathscr{B}$ is ta Thm. 3.2,(4), furnishes a monic (in s_0) polynomial $g = s_0^d + g_{d-1}(s)s_0^{d-1} + \cdots + g_0(s) \in B$ which annihilates $\mathscr{B}$ and M, hence

$$gM = 0 \Longrightarrow gB^{1\times\ell} \subseteq B^{1\times\ell}R \Longrightarrow \exists X \text{ with } g\,\mathrm{id}_\ell = XR \Longrightarrow g^\ell = \det(X)\det(R) = \det(X)f.$$

Since g^ℓ is monic in s_0 the leading coefficient of f is a unit in $A = \mathbb{C}[s,s^{-1}]$, i.e., of the form αs^μ, $0 \neq \alpha \in \mathbb{C}$, $\mu \in \mathbb{Z}^n$ and hence f has the form from (56). From Lemma 5.5 we infer that $\mathscr{B}_f$ is L^2-stable.
With the adjoint matrix R_{adj} of R one gets

$$f\,\mathrm{id}_\ell = \det(R)\,\mathrm{id}_\ell = R_{\mathrm{adj}}R \Longrightarrow fB^{1\times\ell} \subseteq B^{1\times\ell}R \Longrightarrow$$
$$fM = 0 \Longrightarrow f\circ\mathscr{B} = 0 \Longrightarrow \mathscr{B} \subseteq \mathscr{B}_f^\ell.$$

Since $\mathscr{B}_f$ is L^2-stable so is $\mathscr{B}$. If $w \in \mathscr{B}$ and $w(0),\cdots,w(d-1) \in \mathrm{L}^2(\mathbb{Z}^n)^\ell$ then $w \in \left(\mathrm{L}^2(\mathbb{Z}^n)^{\mathbb{N}}\right)^\ell$ by Lemma 4.1 and hence $\lim_{t\to\infty}\|w(t)\|_2 = 0$ due to the L^2-stability.
2. The assumption in 2. signifies that with the notations of the proof of Thm. 5.6 there is a matrix $X \in A^{m\times\ell}$ such that

$$\phi : A^{1\times m} \cong M_1 = A^{1\times\ell}/U_1,\ \xi \mapsto \phi(\xi) = \xi X + U_1,\ \text{and hence } X\circ : \mathscr{B}_1 \cong W_A^m.$$

Let $E' \in A^{m\times m}$ be the unique matrix with $\phi(\circ E') = (\circ E)_{\mathrm{ind}}\phi$ and hence $XE\circ v = E'X\circ v$ for $v \in \mathscr{B}_1$. Define the behavior

$$\mathscr{B}' := \left\{ w' \in (W_A^m)^{\mathbb{N}};\ (s_0 \,\mathrm{id}_m - E')\circ w' = 0 \right\} =$$
$$\left\{ (W_A^m)^{\mathbb{N}};\ \forall t \in \mathbb{N}:\ w'(t+1) = E'\circ w'(t) \right\},\ \text{hence } \mathscr{B} \cong \mathscr{B}', w \mapsto w',\ w'(t) = X\circ w(t).$$

Since the isomorphic behaviors $\mathscr{B}$ and $\mathscr{B}'$ are Λ_1-stable and ta and since $\mathscr{B}'$ is obviously square-autonomous the L^2-stability of the two behaviors follows from part 1. $\square$

Although the Fourier transform is an isometry only on the L^2-spaces special forms of convergence to zero can also be shown for the signal space $\mathscr{S}'(\mathbb{Z}^n)$. For this purpose one has to sharpen Cor. 5.3. As usual we define the differential operators $\Delta^{(\alpha)} = (\alpha!)^{-1}\partial^\alpha :\ \mathscr{D}(\mathbb{T}^n) \to \mathscr{D}(\mathbb{T}^n),\ \alpha \in \mathbb{N}^n$. These are extended to matrices componentwise. For $\varphi, \psi \in \mathscr{D}(\mathbb{T}^n)$ the Leibniz formula furnishes $\Delta^{(\alpha)}(\varphi\psi) = \sum_{\beta+\gamma=\alpha}\Delta^{(\beta)}(\varphi)\Delta^{(\gamma)}(\psi)$ which is also valid for matrix functions $\phi, \Psi \in \mathscr{D}(\mathbb{T}^n)^{\ell\times\ell}$. For matrix functions $E, \phi_1, \cdots, \phi_t \in \mathscr{D}(\mathbb{T}^n)^{\ell\times\ell}$, $t \geq 1$, induction yields the equation

$$\begin{aligned}\Delta^{(\alpha)}(\phi_1 * \cdots * \phi_t) &= \sum_{\beta_1+\cdots+\beta_t=\alpha}\Delta^{(\beta_1)}(\phi_1) * \cdots * \Delta^{(\beta_t)}(\phi_t),\ \beta_j = (\beta_{j1}, \cdots, \beta_{jn}),\\ \text{and } \Delta^{(\alpha)}(E^t) &= \sum_{\beta_1+\cdots+\beta_t=\alpha}\Delta^{(\beta_1)}(E) * \cdots * \Delta^{(\beta_t)}(E),\end{aligned} \tag{65}$$

Lemma 5.9. *1. Assume $E \in \mathscr{D}(\mathbb{T}^n)^{\ell\times\ell}$ and that $\mathrm{spec}(E(\omega)) \subset \mathbb{D}^1$ for all $\omega \in \mathbb{T}^n$, Then all derivatives $\partial^\alpha\left(E(\omega)^t\right)$, $\alpha \in \mathbb{N}^n$, or, equivalently, all its entries converge to 0 uniformly in $\omega \in \mathbb{T}^n$. In other words: E^t converges to 0 in the topology of $\mathscr{D}(\mathbb{T}^n)^{\ell\times\ell}$.*
2. If $w \in \left(\mathscr{S}'(\mathbb{Z}^n)^\ell\right)^{\mathbb{N}}$ satisfies $\widehat{w(t)} = E^t\widehat{w(0)}$ then $\lim_{t\to\infty} w(t,\mu) = 0$ for all $\mu \in \mathbb{Z}^n$. In general, this convergence is not uniform in $\mu \in \mathbb{Z}^n$.

Proof. 1. According to Lemma 5.2 we choose $a_2 \geq 1$ and ρ_j with $0 < \rho_2 < \rho_3 < 1$ such that $\|E(\omega)^t\| \leq a_2\rho_2^t$ for all $t \in \mathbb{N}$. Define $a_3 := 1 + \max_{\beta\leq\alpha}\max_{\omega\in\mathbb{T}^n}\|\Delta^{(\beta)}(E)(\omega)\|$ where $\beta \leq \alpha$ is the componentwise order. Consider one summand $\phi := \Delta^{(\beta_1)}(E) * \cdots * \Delta^{(\beta_t)}(E)$ in the second sum of (65). Let $1 \leq j_1 < \cdots < j_q \leq t$ be the set of indices $j \leq t$ with $\beta_j \neq 0$. Then this summand has the form

$$\begin{gathered}\phi = E^{j_1-1} * \Delta^{(\beta_1)}(E) * E^{j_2-j_1-1} * \Delta^{(\beta_2)}(E) * \cdots * E^{j_q-j_{q-1}-1} * \Delta^{(\beta_q)}(E) * E^{t-j_q} \Longrightarrow\\ \|\phi\| \leq a_2\rho_2^{j_1-1} * a_3 * \cdots * a_2\rho_2^{j_q-j_{q-1}-1} * a_3 * a_2\rho_2^{t-j_q} = a_2^{q+1}a_3^q\rho_2^{t-q}.\ \text{Moreover}\\ \beta_1 + \cdots + \beta_t = \alpha \Longrightarrow |\beta_1| + \cdots + |\beta_t| = |\alpha| \Longrightarrow q \leq |\alpha| \Longrightarrow\\ \|\phi\| \leq a_2^{|\alpha|+1}a_3^{|\alpha|}\rho_2^{t-|\alpha|} \Longrightarrow\\ \|\Delta^{(\alpha)}(E^t)\| \leq a_2^{|\alpha|+1}a_3^{|\alpha|}\rho_2^{t-|\alpha|}\sharp\{\beta = (\beta_1, \cdots, \beta_t);\ \beta_1 + \cdots + \beta_t = \alpha\}.\end{gathered}$$

But the number of elements of the last set is

$$\begin{gathered}p(t) := \sharp\{\beta = (\beta_1, \cdots, \beta_t);\ \beta_1 + \cdots + \beta_t = \alpha\} = \prod_{j=1}^n \binom{\alpha_j+t-1}{\alpha_j} \Longrightarrow\\ \|\Delta^{(\alpha)}(E^t)\| \leq p(t)a_2^{|\alpha|+1}a_3^{|\alpha|}\rho_2^{t-|\alpha|}.\end{gathered}$$

Since $p(t)$ is a polynomial function, since $\Delta^{(\alpha)} = \alpha!^{-1}\partial^\alpha$ and since $\rho_2 < \rho_3 < 1$ there is a constant $a_4(\alpha) \geq 1$ such that $\|\partial^\alpha(E^t)(\omega)\| \leq a_4(\alpha)\rho_3^t$ for all $\omega \in \mathbb{T}^n$ and this

proves the assertion.
2. By assumption

$$\widehat{w(t)} = E^t \widehat{w(0)}, \text{ hence } \forall j = 1, \cdots, \ell : \ \widehat{w_j(t)} = \sum_{m=1}^{\ell} (E^t)_{jm} \widehat{w_m(0)} \text{ and}$$

$$\forall j = 1, \cdots, \ell, \ \forall \varphi \in \mathscr{D}(\mathbb{T}^n) : \ \widehat{w_j(t)}(\varphi) = \sum_{m=1}^{p} \widehat{w_m(0)} \left((E^t)_{jm} \varphi \right)$$

Since the $(E^t)_{jm}$ converge to 0 in $\mathscr{D}(\mathbb{T}^n)$ by part 1. so do the $(E^t)_{jm}\varphi$. Since the $\widehat{w_m(0)} \in \mathscr{D}'(\mathbb{T}^n)$ are continuous on $\mathscr{D}(\mathbb{T}^n)$ we infer $\lim_{t\to\infty} \widehat{w_j(t)}(\varphi) = 0$. For $\varphi := \omega^\mu$, $\mu \in \mathbb{Z}^n$, we obtain $\widehat{w_j(t)}(\varphi) = w_j(t)(\mu) = w_j(t,\mu)$ and hence $\lim_{t\to\infty} w(t)(\mu) = 0$. □

Definition 5.10. The autonomous behavior $\mathscr{B} \subseteq W_B^\ell = (W_A^\ell)^{\mathbb{N}}$ is called *pointwise stable* if

$$\forall w \in \mathscr{B} \bigcap \left(\mathscr{S}'(\mathbb{Z}^n)^\ell \right)^{\mathbb{N}} \forall \mu \in \mathbb{Z}^n : \ \lim_{t\to\infty} w(t,\mu) = 0.$$

This limit is not uniform in the $\mu \in \mathbb{Z}^n$ in general.

Theorem 5.11. *Lemma 5.4, Thm. 5.6 and Corollary 5.8 remain valid for pointwise stability if in the relevant IO representations $(P, -Q)$ the matrix P has a left inverse with entries in $\mathscr{D}(\mathbb{T}^n)$ and hence the transfer matrix H is smooth.*

Proof. By means of Lemma 5.9 the proofs are analogous to those for L^2-stability. □

The next results consider the problem whether the converse of Lemma 5.5 holds, i.e., whether under the assumptions of Thms. 4.9 or Cor. 4.10 L^2-stability implies Λ_1-stability.

Theorem 5.12. *(cf. [24, Sect. VII]) (i) In the situation of Thm. 4.9 assume that $\mathscr{B}$ is L^2-stable. Then for* almost all $\omega \in \mathbb{T}^n$ *the spectrum* $\mathrm{spec}(H(\omega))$ *is contained in the open disc $\mathbb{D}^1$. By Cor. 5.1* $\mathrm{spec}(H(\omega))$ *is then contained in the closed disc $\overline{\mathbb{D}}^1$ for all $\omega \in \mathbb{T}^n$ and almost all $(\lambda, \omega) \in \mathrm{char}(\mathscr{B})$ belong to Λ_1.*
(ii) If in (i) $\ell = 1$ and thus $H \in \mathrm{C}^0(\mathbb{T}^n)$ the necessary condition for L^2-stability of $\mathscr{B}$ in (i) is also sufficient.

Proof. (i) From Thm. 4.9 we know $\widehat{w(t)} = H^t \widehat{w(0)}$ for $w \in \mathscr{B}$, $w(0) \in \mathrm{L}^2(\mathbb{Z}^n)^\ell$. Since $\mathscr{F}_{\mathbb{Z}^n}$ is an isometry on the L^2-spaces and $\lim_{t\to\infty} \|w(t)\|_2 = 0$ we infer

$$\lim_{t\to\infty} \|\widehat{w(t)}\|_2^2 = 0 \Longrightarrow \forall w(0) \in \mathrm{L}^2(\mathbb{Z}^n)^\ell : \ \lim_{t\to\infty} \int_{\mathbb{T}^n} \|H(\omega)^t \widehat{w(0)}(\omega)\|_2^2 d\omega = 0.$$

Let ε_m, $m = 1, \cdots, \ell$, denote the standard basis of $\mathbb{C}^\ell$ and $\delta_0 := (\delta_{0\mu})_{\mu\in\mathbb{Z}^n} \in \mathbb{C}^{\mathbb{Z}^n}$. Then

$$w(0) := \delta_0 \varepsilon_m \in \mathrm{L}^2(\mathbb{Z}^n)^\ell, \ \widehat{w(0)} = \widehat{\delta_0} \varepsilon_m = \varepsilon_m, \ \widehat{w(t)} = H^t \widehat{w(0)} = (H^t)_{-m},$$

$$\lim_{t\to\infty} \|\widehat{w(t)}\|_2^2 = \lim_{t\to\infty} \int_{\mathbb{T}^n} \sum_{j=1}^{\ell} |(H(\omega)^t)_{jm}|^2 d\omega = 0 \Longrightarrow$$

$$\forall j, m = 1, \cdots, \ell : \ \lim_{t\to\infty} \int_{\mathbb{T}^n} |(H(\omega)^t)_{jm}|^2 d\omega = 0,$$

i.e., the entries $(H(\omega)^t)_{jm}$ of H^t converge to zero in $\mathrm{L}^2(\mathbb{T}^n)$. A non-trivial standard result of probability theory states the following: An L^2-convergent (more generally: L^p-convergent, $p \geq 1$) sequence of random variables has an almost sure convergent subsequence. More precisely, let $(\Omega, \mathscr{A}, P)$ be a probability space. Given random variables

$$f, f_1, f_2, \dots \in \mathrm{L}^2(\Omega, \mathscr{A}, P), \text{ with } \lim_{t\to\infty} \int_\Omega |f(\omega) - f_t(\omega)|^2 dP(\omega) = 0 \text{ then}$$
$$\exists \Omega_1 \in \mathscr{A} \exists 1 \leq t(1) < t(2) < \cdots \text{ such that } P(\Omega_1) = 1 \text{ and } \forall \omega \in \Omega_1 : \lim_{k\to\infty} f_{t(k)}(\omega) = f(\omega).$$

We apply this consecutively to all sequences $(H(\omega)^t)_{jm}$ and obtain a subset $\Omega_1 \subset \mathbb{T}^n$ of measure 1 and an increasing sequence $1 \leq t(1) < t(2) < \cdots$ such that $\lim_{k\to\infty}(H(\omega)^{t(k)})_{jm} = 0$ for all j, m and $\omega \in \Omega_1$ and hence also $\lim_{k\to\infty} H(\omega)^{t(k)} = 0$ for $\omega \in \Omega_1$. For each $\omega \in \Omega_1$ let $\lambda(\omega)$ be one eigenvalue with $|\lambda(\omega)| = \rho(H(\omega))$ and $0 \neq v(\omega) \in \mathbb{C}^\ell$ a corresponding eigenvector. Then

$$\forall \omega \in \Omega_1 : 0 = \lim_{k\to\infty} H(\omega)^{t(k)} v(\omega) = \lim_{k\to\infty} \lambda(\omega)^{t(k)} v(\omega) \Longrightarrow$$
$$\lim_{k\to\infty} \lambda(\omega)^{t(k)} = 0 \Longrightarrow \rho(H(\omega)) = |\lambda(\omega)| < 1.$$

(ii) Let $(\forall' \omega \in \mathbb{T}^n)$ signify (*for all* $\omega \in \mathbb{T}^n$ *up to a set of measure zero*). If $\ell = 1$ and thus $H \in \mathrm{C}^0(\mathbb{T}^n)$ we obviously have $\|H(\omega)\| = \rho(H(\omega)) = |H(\omega)|$. By assumption

$$\forall' \omega \in \mathbb{T}^n : \ |H(\omega)| < 1 \text{ and } \forall \omega \in \mathbb{T}^n : \ |H(\omega)| \leq 1 \Longrightarrow$$
$$\forall' \omega \in \mathbb{T}^n : \ \lim_{t\to\infty} H(\omega)^{2t} = 0 \text{ and}$$
$$\forall \omega \in \mathbb{T}^n : \ |\widehat{w(t)}(\omega)|^2 = |H(\omega)^{2t} \widehat{w(0)}(\omega)|^2 \leq |\widehat{w(0)}(\omega)|^2.$$

With Lebesgues' dominated convergence theorem we conclude

$$\lim_{t\to\infty} \|\widehat{w(t)}\|_2^2 = \lim_{t\to\infty} \int_{\mathbb{T}^n} |\widehat{w(t)}(\omega)|^2 d\omega = 0.$$

□

Example 5.13. The following two examples show that under the assumptions of Thm. 4.9 L^2-stability does not imply Λ_1-stability in contrast to the assertion in [9, Thm. 10] and that the necessary condition of Thm. 5.12,(i), for L^2-stability is not sufficient for $\ell > 1$. In Thm. 4.9 choose $n = 1$.
(i) Let $\ell := 1,\ p(s_1) := 1,\ q(s_1) = \frac{1}{2}(s_1 + 1)$, hence $h = p^{-1}q = q$ and consider

$$\mathscr{B} := \left\{ w \in \left(\mathbb{C}^{\mathbb{Z}}\right)^{\mathbb{N}};\ (s_0 - q(s_1)) \circ w = 0 \text{ or } w(t+1) = \frac{1}{2}(s_1 + 1) \circ w(t) \right\}.$$

We infer

$$\operatorname{spec}(h(\omega)) = \{q(\omega)\} = \left\{ \frac{1}{2}(\omega + 1) \right\},\ \omega = e^{2\pi i x} \in \mathbb{T}^1. \text{ But}$$
$$|q(\omega)|^2 = \frac{1}{4}(\omega + 1)(\overline{\omega} + 1) = \frac{1}{2}(1 + \cos(2\pi x)) \Longrightarrow$$
$$\forall \omega \in \mathbb{T}^1 : \ |q(\omega)| \leq 1 \text{ and } (|q(\omega)| = 1 \iff \omega = 1).$$

By Thm. 5.12,(ii), this behavior is L^2-stable, but not Λ_1-stable since

$$\operatorname{spec}(h(1)) = \{1\} \Longrightarrow (1,1) \in \operatorname{char}(\mathscr{B}) \setminus \Lambda_1.$$

(ii) With the notations from Thm. 4.9 and from (i) choose

$$k = \ell = 2,\ P := \mathrm{id}_2,\ Q := \begin{pmatrix} q & 1 \\ 0 & q \end{pmatrix},\ H := P^{-1}Q = Q \Longrightarrow$$

$$\mathscr{B} := \left\{ w \in \left((\mathbb{C}^{\mathbb{Z}})^2\right)^{\mathbb{N}};\ (\mathrm{id}_2 s_0 - Q) \circ w = 0 \text{ or } w(t+1) = Q(s_1) \circ w(t) \right\},$$

$$Q^t = \begin{pmatrix} q^t & tq^{t-1} \\ 0 & q^t \end{pmatrix},\ |\left(Q(\omega)^t\right)_{12}|^2 = t^2 |q(\omega)|^{2(t-1)} = t^2 \left(\frac{1}{2}(1+\cos(2\pi x))\right)^{t-1}.$$

It is obvious that $\operatorname{spec}(H(\omega)) = \operatorname{spec}(Q(\omega)) = \{q(\omega)\}$. With (i) this implies that $\mathscr{B}$ satisfies the necessary condition of Thm. 5.12,(i), for L^2-stability. Now consider the function

$$f(x) := \frac{1}{2}(1+\cos(2\pi x)) - (1-\pi x) \text{ with } f(0) = 0,\ f'(x) = -\pi \sin(2\pi x) + \pi \geq 0$$

$$\Longrightarrow \forall x \text{ with } 0 \leq x \leq \pi^{-1} : |q(e^{2\pi i x})|^2 = \frac{1}{2}(1+\cos(2\pi x)) \geq 1 - \pi x \geq 0 \Longrightarrow$$

$$\forall x \text{ with } 0 \leq x \leq \pi^{-1} \forall t \geq 1 : |q(e^{2\pi i x})|^{2(t-1)} \geq (1-\pi x)^{t-1} \geq 0 \Longrightarrow$$

$$\int_0^1 |\left(Q(e^{2\pi i x})^t\right)_{12}|^2 dx \geq t^2 \int_0^{\pi^{-1}} (1-\pi x)^{t-1} dx = t^2 (\pi t)^{-1} = \pi^{-1} t \underset{t\to\infty}{\longrightarrow} \infty.$$

According to the proof of Thm. 5.12,(i), L^2-stability of $\mathscr{B}$ implies L^2-convergence to 0 of all entries of $H(\omega)^t = Q(\omega)^t$. The preceding calculation shows that this does not hold for the entry $(Q(\omega)^t)_{12}$, hence $\mathscr{B}$ is not L^2-stable.
This example also shows that the the analogue of Willems' conjecture in [24, Sect. VII, (13)] for the continuous case does not hold in the discrete case.

Example 5.14. We construct a whole family of examples for Thm. 2.3 in the case $n = 2$ and $\ell = 2$ where the behavior $\mathscr{B}$ is *not* square-autonomous in contrast to [9, Thm. 10]. Consider nonzero Laurent polynomials $P, Q \in A = \mathbb{C}[s_1, s_1^{-1}, s_2, s_2^{-1}]$ such that $P(\omega) \neq 0$ for $\omega \in \mathbb{T}^2$, hence $P^{-1} \in \mathscr{D}(\mathbb{T}^2)$, for instance $P = s_1 - 2$ and $Q = s_2 - 1$. Define $R_0 := (P, -Q) \in A^{1\times 2}$. The ring A is factorial, so the greatest common divisor $g := \gcd(P, Q) \in A$ of P and Q in A exists. The factoriality of A also implies that

$$\ker\left((P,-Q)\circ : A^2 \to A\right) = A\left(g^{-1}(Q,P)\right)^{\top}. \tag{66}$$

This enables to determine all matrices $E \in A^{2\times 2}$ as needed in Thm. 3.4, viz.

$$\begin{gathered} \left\{E \in A^{2\times 2};\ \exists a \in A \text{ with } R_0 E = a R_0\right\} = \\ \left\{E(a,b,c) := \begin{pmatrix} a+bQg^{-1} & cQg^{-1} \\ bPg^{-1} & a+cPg^{-1} \end{pmatrix};\ a,b,c \in A\right\}. \end{gathered} \tag{67}$$

According to Thm. 3.4 any $E = E(a,b,c)$ as in (67) gives rise to the matrix

$$R := R(a,b,c) := \begin{pmatrix} (P,-Q) \\ s_0\,\mathrm{id}_2 - E(a,b,c) \end{pmatrix} \in B^{3\times 2} \tag{68}$$

and the time-autonomous behavior

$$\begin{gathered} \mathscr{B} := \left\{w \in W_B^2;\ R \circ w = 0\right\},\ W_A := \mathbb{C}^{\mathbb{Z}^2},\ W_B := W_A^{\mathbb{N}}, \text{ with} \\ \mathscr{B}_1 = \operatorname{im}\left(\mathscr{B} \to W_A^2,\ w \mapsto w(0)\right) = \left\{\begin{pmatrix} y \\ u \end{pmatrix} \in W_A^2;\ P \circ y = Q \circ u\right\} \\ \text{and transfer function } H(\omega) = P(\omega)^{-1} Q(\omega) \in \mathscr{D}(\mathbb{T}^2). \end{gathered} \tag{69}$$

The three 2×2-subdeterminants of R are

$$\begin{gathered}
d_1 := \det\begin{pmatrix} P & -Q \\ s_0-a-bQg^{-1} & -cQg^{-1}\end{pmatrix} = Q\det\begin{pmatrix} P & -1 \\ s_0-a-bQg^{-1} & -cg^{-1}\end{pmatrix} = \\
Q\left(s_0-a-(bQ+cP)g^{-1}\right), \\
d_2 := \det\begin{pmatrix} P & -Q \\ -bPg^{-1} & s_0-a-cPg^{-1}\end{pmatrix} = P\det\begin{pmatrix} 1 & -Q \\ -bg^{-1} & s_0-a-cPg^{-1}\end{pmatrix} = \\
P\left(s_0-a-(bQ+cP)g^{-1}\right), \\
d_3 := \det(s_0\,\mathrm{id}_2 - E).
\end{gathered} \tag{70}$$

Hence the characteristic variety of $\mathscr{B}$ is

$$\begin{gathered}
\mathrm{char}(\mathscr{B}) = \{(\lambda,\omega)\in\Lambda_N;\ \mathrm{rank}(R(\lambda,\omega))<2\} = \\
\{(\lambda,\omega)\in\Lambda_N;\ \forall i=1,2,3:\ d_i(\lambda,\omega)=0\}.
\end{gathered} \tag{71}$$

To check the Λ_1-stability condition $\mathrm{char}(\mathscr{B})\bigcap\Lambda_2=\emptyset$ choose $(\lambda,\omega)\in\mathrm{char}(\mathscr{B})\bigcap\left(\mathbb{C}\times\mathbb{T}^2\right)$. The conditions $d_i(\lambda,\omega)=0$ for $i=1,2$ and the assumption $P(\omega)\neq 0$ imply

$$\begin{gathered}
\lambda = a+(bQ+cP)g^{-1} \text{ with } a:=a(\omega), b:=b(\omega), \ldots \text{ and} \\
d_3(\lambda,\omega) = \det\begin{pmatrix} \lambda-a-bQg^{-1} & -cQg^{-1} \\ -bPg^{-1} & \lambda-a-cPg^{-1}\end{pmatrix} = \\
\det\begin{pmatrix} cPg^{-1} & -cQg^{-1} \\ -bPg^{-1} & bQg^{-1}\end{pmatrix} = bc\det\begin{pmatrix} Pg^{-1} & -Qg^{-1} \\ -Pg^{-1} & Qg^{-1}\end{pmatrix} = 0.
\end{gathered} \tag{72}$$

We conclude that

$$\begin{gathered}
\mathrm{char}(\mathscr{B})\bigcap\left(\mathbb{C}\times\mathbb{T}^2\right) = \\
\left\{(a(\omega)+b(\omega)(Q/g)(\omega)+c(\omega)(P/g)(\omega),\omega);\ \omega\in\mathbb{T}^2\right\}. \\
\text{Hence } \mathrm{char}(\mathscr{B})\bigcap\Lambda_2=\emptyset \iff \rho := \rho(a,b,c) := \\
\max_{\omega\in\mathbb{T}^2}\left(|a(\omega)+b(\omega)(Q/g)(\omega)+c(\omega)(P/g)(\omega)|\right) < 1.
\end{gathered} \tag{73}$$

Thus the time-autonomous system $\mathscr{B}$ from (69) is Λ_1-stable if and only if $\rho<1$. If this is the case and since $P(\omega)\neq 0$ on $\mathbb{T}^2$ Thm. 2.3 furnishes the L^2- and pointwise stability of $\mathscr{B}$, and moreover the following canonical isomorphisms

$$\begin{gathered}
\mathscr{B}\bigcap\left(\mathrm{L}^2(\mathbb{Z}^2)^2\right)^{\mathbb{N}} \cong \mathscr{B}_1\bigcap \mathrm{L}^2(\mathbb{Z}^n)^2 \cong \mathrm{L}^2(\mathbb{Z}^n) \text{ resp.} \\
\mathscr{B}\bigcap\left(\mathscr{S}'(\mathbb{Z}^2)^2\right)^{\mathbb{N}} \cong \mathscr{B}_1\bigcap \mathscr{S}'(\mathbb{Z}^n)^2 \cong \mathscr{S}'(\mathbb{Z}^n) \\
w\leftrightarrow w(0)=\left(\begin{smallmatrix} y \\ u\end{smallmatrix}\right)\leftrightarrow u,\ w(t)=E^t\circ w(0),\ \widehat{y}=H(\omega)\widehat{u}, \\
H(\omega)=P(\omega)^{-1}Q(\omega).
\end{gathered} \tag{74}$$

Since $d_3(\lambda,\omega)=\det(\lambda\,\mathrm{id}_2-E(\omega))=0$ the value $\lambda=a+b(Q/g)+c(P/g)$ with $a:=a(\omega)$ etc is one eigenvalue of $E(\omega)$. The second eigenvalue λ' of $E(\omega)$ satisfies

$$\begin{gathered}
\lambda+\lambda' = a+b(Q/g)+c(P/g)+\lambda' = \mathrm{trace}(E(\omega)) = \\
=2a+b(Q/g)+c(P/g), \text{ hence } \lambda'=a, \text{ indeed } d_3(a,\omega)=0.
\end{gathered} \tag{75}$$

If $\rho(a,b,c)\geq 1$ for the chosen (a,b,c) any $r>\rho(a,b,c)$ gives rise to the vector $r^{-1}(a,b,c)$ with

$$\begin{aligned}&\rho\left(r^{-1}(a,b,c)\right)<1 \text{ and thus to the matrix}\\ &R:=R(r^{-1}(a,b,c)):=\begin{pmatrix}(P,-Q)\\ s_0\,\mathrm{id}_2-E(r^{-1}(a,b,c))\end{pmatrix}\end{aligned}\tag{76}$$

whose associated time-autonomous behavior $\mathscr{B}$ from (69) is Λ_1-, L^2- and pointwise stable. A special example is obtained by

$$\begin{gathered}P:=s_1-2,\ Q:=s_2-1,\ g=\gcd(P,Q)=1,\\ r:=5/2<a:=3,\ b:=c:=1,\\ \rho:=\max_{\omega\in\mathbb{T}^2}\left(|3+(\omega_2-1)+(\omega_1-2)|\right)=2<r<a,\\ E=\frac{2}{5}\begin{pmatrix}s_2+2 & s_2-1\\ s_1-2 & s_1+1\end{pmatrix},\ R:=\begin{pmatrix}s_1-2 & -(s_2-1)\\ s_0-(2/5)(s_2+2) & -(2/5)(s_2-1)\\ -(2/5)(s_1-2) & s_0-(2/5)(s_1+1)\end{pmatrix}.\end{gathered}\tag{77}$$

The spectrum of the corresponding $E(\omega)$ is

$$\begin{gathered}\mathrm{spec}(E(\omega))=\left\{\lambda,\lambda'\right\} \text{ with}\\ \lambda=(2/5)(\omega_2+\omega_1),\ |\lambda|<1,\ \lambda'=a/r=6/5>1.\end{gathered}$$

Thus no $E(\omega)$ is asymptotically stable and the time-autonomous behavior

$$\begin{aligned}\mathscr{B}':=&\left\{w\in\left(\left(\mathbb{C}^{\mathbb{Z}^2}\right)^2\right)^{\mathbb{N}};\ (s_0\,\mathrm{id}_2-E)\circ w=0\right\}=\\ &\left\{w\in\left(\left(\mathbb{C}^{\mathbb{Z}^2}\right)^2\right)^{\mathbb{N}};\ w(t)=E^t\circ w(0)\right\}\end{aligned}\tag{78}$$

is neither Λ_1-, nor L^2-nor pointwise stable whereas its time-autonomous subbehavior $\mathscr{B}$ has all these properties.
Assume that $g=\gcd(P,Q)=1$. Then the sequence

$$0\to A\overset{\circ(P,-Q)}{\longrightarrow}A^{1\times 2}\overset{\binom{Q}{P}}{\longrightarrow}A \text{ is exact,}\tag{79}$$

hence the system module $M_1:=A^{1\times 2}/A(P,-Q)$ of the W_A-behavior $\mathscr{B}_1$ is contained in A (up to isomorphism). This implies that M_1 is torsionfree or, equivalently, that $\mathscr{B}_1$ is controllable.
The A-module M_1 is projective and thus free (Theorem of Quillen-Suslin for Laurent polynomials) and hence $\mathscr{B}_1$ is strictly controllable if and only if the matrix $(P,-Q)$ has a right inverse or, equivalently, $A=AP+AQ$, i.e., if P and Q are coprime. If M_1 is not free or, equivalently, $\mathscr{B}_1$ is not strictly controllable, for instance for $P=s_1-2$ and $Q=s_2+1$, then Cor. 5.8,(2), is not applicable.
A computer calculation according to Cor. 3.5 shows that the behavior $\mathscr{B}$ defined by the data in (77) is not square-autonomous and that hence Cor. 5.8,(1), (cf. [9]) is neither applicable.

6 Constructions

All properties in context with time-autonomy can be checked constructively as will be shown below.

Result 6.1. *([18, Sect. 6]) Consider* $\Lambda_N = \Lambda_1 \uplus \Lambda_2$ *from* (14)*, ideals* $\mathfrak{a} \subseteq A = \mathbb{C}[s, s^{-1}]$*,* $\mathfrak{b} \subseteq B = A[s_0]$ *and the data from Thm. 2.3, i.e., the f.g. torsion B-module M with its nonzero annihilator* $\mathfrak{b}_M := \mathrm{ann}_B(M)$ *and associated behavior* $\mathscr{B}$*. The time-autonomy of the behavior* $\mathscr{B}$ *can be decided by means of [12, Cor. 3.8]. One can decide constructively whether* $V_{\Lambda_N}(\mathfrak{b}) \bigcap \Lambda_2 = \emptyset$ *and whether* $V_{(\mathbb{C}\setminus\{0\})^n}(\mathfrak{a}) \bigcap \mathbb{T}^n = \emptyset$*. This implies that the* Λ_1*-stability of* $\mathscr{B}$*, i.e. the condition* $V_{\Lambda_N}(\mathfrak{b}_M) \bigcap \Lambda_2 = \emptyset$*, and the condition* $\mathrm{rank}(P(\omega)) = p$ *for* $\omega \in \mathbb{T}^n$ *with its ensuing smoothness of H can also be checked. Whether the more general continuity of H on* $\mathbb{T}^n$ *in condition (i) of Thm. 5.6 can be tested is presently open. Square-autonomy of* $\mathscr{B}$ *can be checked via Cor. 3.5 and the strict controllability of* $\mathscr{B}_d$*, i.e., the freeness of* M_d*, by standard methods. These computations also furnish a different constructive version of [9, Thm. 10, Sect. 5].*

In the remainder we are going to discuss a constructive version of Thm. 2.2 and have to use the Gröbner basis algorithm for polynomial modules for this purpose as already in [12, Thms. 3.6, 3.7]. In deviation from the data of Section 3 we first use the following data:

$$A := \mathbb{C}[s] = \mathbb{C}[s_1, \cdots, s_n] \subset B = A[s_0] = \mathbb{C}[s_0, \cdots, s_n],\ W_A := \mathbb{C}^{\mathbb{N}^n},\ W_B = W_A^{\mathbb{N}} = \mathbb{C}^{\mathbb{N}\times\mathbb{N}^n}. \tag{80}$$

Below we will extend the considerations to the data of Section 3. We need the following standard results concerning *Gröbner bases* (GB) [12, Sect. 3]. Let

$$\begin{gathered} R \in B^{k\times\ell},\ \mathrm{rank}(R) = \ell,\ U := B^{1\times k}R,\ M = B^{1\times\ell}/U,\ \mathscr{B} := U^\perp \subseteq W_B^\ell, \\ \text{and } [\ell] := \{1, \cdots, \ell\},\ [\ell]\times\mathbb{N}^n \underset{\text{ident.}}{=} [\ell]\times\{0\}\times\mathbb{N}^n \subset [\ell]\times\mathbb{N}\times\mathbb{N}^n = [\ell]\times\mathbb{N}^{1+n} \ni \\ (i,\mu_0,\mu) = (i,\mu_0,\mu_1,\cdots,\mu_n),\ (i,\mu_0,\mu) + (\nu_0,\nu) := (i,\mu_0+\nu_0,\mu+\nu). \end{gathered} \tag{81}$$

We use an arbitrary term (well-)order on $[\ell]\times\mathbb{N}^n$ and the induced lexicographic term order on $[\ell]\times\mathbb{N}\times\mathbb{N}^n$ defined by

$$(i,\mu_0,\mu) < (j,\nu_0,\nu) :\Longleftrightarrow \mu_0 < \nu_0 \text{ or } \mu_0 = \nu_0 \text{ and } (i,\mu) < (j,\nu). \tag{82}$$

These term orders induce a degree function

$$\begin{gathered} \deg : B^{1\times\ell}\setminus\{0\} \to [\ell]\times\mathbb{N}\times\mathbb{N}^n \text{ with } \deg(s_0^{\mu_0}s^\mu\delta_i) = (i,\mu_0,\mu), \\ \deg(\xi) = (i, \deg_{s_0}(\xi), \mu) \text{ and } \deg(U) = \{\deg(\xi);\ 0 \neq \xi \in U\} = \deg(U) + \mathbb{N}^{1+n}. \end{gathered} \tag{83}$$

With $\Gamma_B := ([\ell]\times\mathbb{N}\times\mathbb{N}^n)\setminus\deg(U)$ one obtains the direct decompositions

$$\begin{gathered} B^{1\times\ell} = \oplus_{(i,\mu_0,\mu)\in\Gamma_B} F s_0^{\mu_0}s^\mu\delta_i \oplus U \text{ and} \\ B^{1\times\ell}/U = \oplus_{(i,\mu_0,\mu)\in\Gamma_B} F\overline{s_0^{\mu_0}s^\mu\delta_i},\ \overline{s_0^{\mu_0}s^\mu\delta_i} := s_0^{\mu_0}s^\mu\delta_i + U. \end{gathered} \tag{84}$$

Result 6.2. *([12, Thm. 3.6]) The behavior* $\mathscr{B}$ *is ta if and only if the set* $\deg(U)$ *contains an element* $(j, d(j), 0)$ *for every* $j = 1, \cdots, \ell$*, or, equivalently,*

$$\begin{gathered} \deg(U) \supseteq \uplus_{j=1}^{\ell}\{j\}\times(d(j)+\mathbb{N})\times\mathbb{N}^n,\ d := \max_j d(j) \Longrightarrow \\ \Gamma_B \subseteq [\ell]\times\{0,\cdots,d-1\}\times\mathbb{N}^n \underset{(84)}{\Longrightarrow} B^{1\times\ell} = \sum_{j,\mu_0<d} A s_0^{\mu_0}\delta_j + U = B_d^{1\times\ell} + U,\ M_d = M, \end{gathered}$$

where we use the notations from (25)-(28)*. Hence ta can be checked by a simple application of the GB algorithm for polynomial modules.*

We now assume that ta of $\mathscr{B}$ has been constructively established. Via (30) we replace $\mathscr{B}$ by the isomorphic behavior $\mathscr{B}_{\rm st} = \phi(\mathscr{B})$ with the notations from (29) and (30). The equations of the image behavior $\mathscr{B}_{\rm st}$ can be computed by standard procedures. For notational simplicity we now assume that already $\mathscr{B}_1 = \mathscr{B}$ and $M_1 = A^{1\times\ell}/U_1 = B^{1\times\ell}/U$ where $B_1 = A$ and $U_1 = A^{1\times\ell}\bigcap U$. Let G be the reduced GB with respect to the term order from (82). Again standard arguments furnish $G_0 := G\bigcap A^{1\times\ell} = \left\{g\in G;\ \deg_{s_0}(g)=0\right\}$ and that G_0 is a reduced GB of U_1 w.r.t the chosen term order on $[\ell]\times\mathbb{N}^n$, hence $U_1 = \sum_{g\in G_0} Ag$.

Lemma 6.3. *Assume the data from* (80)-(84) *and that $\mathscr{B}$ is a time-autonomous behavior with $\mathscr{B}\cong\mathscr{B}_1$. For every $j=1,\cdots,\ell$ there is a unique $g_j\in G$ with* $\deg(g_j)=(j,1,0)$ *and $G = G_0 \uplus \left\{g_j;\ j=1,\cdots,\ell\right\}$. The g_j are of the form $g_j = s_0\delta_j - E_{j-}$ where $E\in A^{\ell\times\ell}$. Thus G gives rise to the matrices and representations*

$$R_0 := (g)_{g\in G_0}\in A^{G_0\times\ell}\ \textit{and}\ R := (g)_{g\in G} = \begin{pmatrix} R_0 \\ s_0\,\mathrm{id}_\ell - E\end{pmatrix}\in B^{G\times\ell}$$
$$U_1 = \sum_{g\in G_0} Ag = A^{1\times G_0}R_0,\ U = \sum_{g\in G} Bg = B^{1\times G}R,$$
$$\mathscr{B} = U^\perp = \left\{w\in W_B^\ell;\ R\circ w = 0\right\}\cong\mathscr{B}_1 = U_1^\perp = \left\{v\in W_A^\ell;\ R_0\circ v = 0\right\},\ w\mapsto w(0).$$

Proof. We apply (84) to U_1 and U. Define

$$\Gamma_A := ([\ell]\times\{0\}\times\mathbb{N}^n)\setminus\deg(U_1)\subset\Gamma_B := ([\ell]\times\mathbb{N}\times\mathbb{N}^n)\setminus\deg(U). \text{ The isomorphism}$$
$$\oplus_{(j,0,\mu)\in\Gamma_A} F\overline{s^\mu\delta_j} = M_1\cong M = \oplus_{(j,\mu_0,\mu)\in\Gamma_B} F\overline{s_0^{\mu_0}s^\mu\delta_j},\ \overline{s^\mu\delta_j}\mapsto\overline{s^\mu\delta_j},$$

implies $\Gamma_A = \Gamma_B$. Let $g = s_0^{\mu_0}s^\mu\delta_j+\cdots\in G\setminus G_0$, i.e., $\mu_0>0$. Then $(j,\mu_0-1,\mu)\in\Gamma_B = \Gamma_A$ and hence $\mu_0-1=0$, $\mu_0=1$. Since $\mathscr{B}$ is ta there is a $g'\in G$ with $\deg(g') = (j,d(j),0)$ where $d(j)\le 1$ as just shown, hence $\deg(g)\in\deg(g')+\mathbb{N}^{1+n}$. For reduced GB this implies $g=g'$ and $\deg(g) = (j,1,0)$. This proves the unique existence of the $g_j\in G$ with $\deg(g_j) = (j,1,0)$. All other $g\in G$ have $\deg(g) = (j,0,\mu)$ and thus belong to G_0, hence $G = G_0\uplus\left\{g_j;\ j=1,\cdots,\ell\right\}$. Since $s_0\delta_j$ is the leading term of g_j the representation $g_j = s_0\delta_j - E_{j-}$ with $\deg_{s_0}(E_{j-}) = 0$ or $E_{j-}\in A^{1\times\ell}$ is obvious.
The asserted representations for $U, U_1, \mathscr{B} = U^\perp$ and $\mathscr{B}_1 = U_1^\perp$ are direct consequences of the just derived special form of the GB G. □

We now return to the data from Section 3 with its *Laurent* polynomial algebras $A = \mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[s,s^{-1}]$ and $B = A[s_0]$ and corresponding signal spaces $W_A = \mathbb{C}^{\mathbb{Z}^n}$ and $W_B = W_A^{\mathbb{N}}$. The following reduction to the polynomial case originated in Zerz' PhD-thesis and was also used in [12, Sections 2,3]. With an additional list $s' = (s'_1,\cdots,s'_n)$ of indeterminates we define the polynomial algebras and algebra surjections

$$\begin{aligned} &A' := \mathbb{C}[\mathbb{N}^{2n}] = \mathbb{C}[\mathbb{N}^n\times\mathbb{N}^n] = \mathbb{C}[s,s']\subset B' := A'[s_0] = \mathbb{C}[s_0,s,s'],\\ &\varphi: A'\to A,\ s,s'\mapsto s,s^{-1},\ \phi: B'\to B,\ s_0,s,s'\mapsto s_0,s,s^{-1},\\ &\text{with kernels}\ker(\varphi) = \sum_{i=1}^n A'(s_is'_i-1),\ \ker(\phi) = B'\ker(\varphi) = \sum_{i=1}^n B'(s_is'_i-1).\end{aligned}\tag{85}$$

The corresponding signal spaces with their canonical actions are

$$W_{A'} = (A')^* = \mathbb{C}^{\mathbb{N}^n\times\mathbb{N}^n}\ \text{and}\ W_{B'} = W_{A'}^{\mathbb{N}} = \mathbb{C}^{\mathbb{N}\times\mathbb{N}^n\times\mathbb{N}^n}.\tag{86}$$

The injective dual maps of φ and ϕ are

$$D(\varphi)=\varphi^*:F^{\mathbb{Z}^n}\to F^{\mathbb{N}^n\times\mathbb{N}^n},\ u\mapsto u',\ u'(\mu,\nu)=u(\mu-\nu),\ (\mu,\nu)\in N^{2n}=\mathbb{N}^n\times\mathbb{N}^n,$$
$$D(\phi):F^{\mathbb{N}\times\mathbb{Z}^n}\to F^{\mathbb{N}\times\mathbb{N}^n\times\mathbb{N}^n},\ w\mapsto w',\ w'(t)(\mu,\nu)=w(t)(\mu-\nu),\ t\in\mathbb{N}. \tag{87}$$

The componentwise extensions of φ and ϕ to f.g. free modules and of $D(\varphi)$ and $D(\phi)$ to finite products are denoted by the same letters. For $d,\ell>0$ and the notations from (25) there result the commutative diagrams

$$\begin{array}{ccccccccc} (B')^{1\times\ell} & \xrightarrow{\phi} & B^{1\times\ell} & & W^\ell_{B'} & \overset{D(\phi)}{\longleftarrow} & W^\ell_B & \ni & w\\ \bigcup & & \bigcup & \text{and} & \downarrow \mathrm{proj}' & & \downarrow \mathrm{proj} & & \downarrow\\ (B'_d)^{1\times\ell} & \xrightarrow{\varphi} & B_d^{1\times\ell} & & (W^\ell_{A'})^d & \overset{D(\varphi)}{\longleftarrow} & (W^\ell_A)^d & \ni & (w(0),\cdots,w(d-1))^\top \end{array} \tag{88}$$

where the horizontal maps ϕ,φ are surjective and their duals $D(\phi)=\phi^*, D(\varphi)=\varphi^*$ are injective. Now consider the matrix, modules and behavior

$$\widetilde{R}\in B^{k\times\ell},\ \mathrm{rank}(\widetilde{R})=\ell,\ U:=B^{1\times k}\widetilde{R},\ M=B^{1\times\ell}/U,\ \mathscr{B}:=U^\perp. \tag{89}$$

By multiplying $\widetilde{R}$ with a suitable $s^\mu,\ \mu\in\mathbb{N}^n$, we assume wlog that $\widetilde{R}\in\mathbb{C}[s_0,s]^{k\times\ell}\subset(B')^{k\times\ell}$. The preceding commutative diagrams induce modules, behaviors and maps

$$U':=\phi^{-1}(U)=(B')^{1\times k}\widetilde{R}+\sum_{i,j}\left\{B'(s_is'_i-1)\delta_j;1\le i\le n,\ 1\le j\le\ell\right\},$$
$$U_d=U\bigcap B_d^{1\times\ell},\ U'_d=U'\bigcap(B'_d)^{1\times\ell}=\varphi^{-1}(U_d),$$
$$\phi_{\mathrm{ind}}:M'=(B')^{1\times\ell}/U'\cong M=B^{1\times\ell}/U,\ \varphi_{\mathrm{ind}}:M'_d=(B'_d)^{1\times\ell}/U'_d\cong M_d=B_d^{1\times\ell}/U_d \tag{90}$$

and the commutative diagram with horizontal isomorphims and vertical surjections

$$\begin{array}{ccc} \mathscr{B}':=D(\phi)(\mathscr{B})=(U')^\perp & \overset{D(\phi)}{\longleftarrow} & \mathscr{B}=U^\perp\\ \downarrow\mathrm{proj}' & & \downarrow\mathrm{proj}\\ \mathscr{B}'_d=(U'_d)^\perp=\mathrm{proj}'(\mathscr{B}') & \overset{D(\varphi)}{\longleftarrow} & \mathscr{B}_d=U_d^\perp=\mathrm{proj}(\mathscr{B}) \end{array}. \tag{91}$$

All data in the preceding diagrams can be computed from those of $\mathscr{B}$ via Gröbner bases.

Corollary 6.4. *The behavior $\mathscr{B}$ is time-autonomous with $\mathscr{B}\cong\mathscr{B}_d$ if and only if $\mathscr{B}'$ is time-autonomous with $\mathscr{B}'\cong\mathscr{B}'_d$, and this latter property can be constructively checked with Result 6.2.*

Proof. According to diagram (91) the surjective projection $\mathrm{proj}:\mathscr{B}\to\mathscr{B}_d$ is also injective if and only if this holds for the projection $\mathrm{proj}':\mathscr{B}'\to\mathscr{B}'_d$. The injectivity of these projections signifies ta. ☐

We now assume that time-autonomy of $\mathscr{B}$ has been verified by the preceding corollary. After application of Cor. 3.3 we assume wlog that $\mathscr{B}$ is ta with $\mathscr{B}\cong\mathscr{B}_1,\ w\mapsto w(0)$, and then also $\mathscr{B}'\cong\mathscr{B}'_1$. The structure of $\mathscr{B}'_1$ is known from Lemma 6.3 which implies the following

Theorem 6.5. *Consider the data from* (85)-(91) *and assume that the time-autonomy of $\mathscr{B}$ with $\mathscr{B}\cong\mathscr{B}_1,\ w\mapsto w(0)$, and hence $\mathscr{B}'\cong\mathscr{B}'_1,\ w'\mapsto w'(0)$, has been established.*

Let $G'\subset (B')^{1\times\ell}$ be the reduced Gröbner basis of U' from Lemma 6.3 applied to the polynomial algebra $B'=\mathbb{C}[s_0,s,s']$, thus

$$G'=G'_0\uplus\left\{g'_j;\ j=1,\cdots,\ell\right\},\ G'_0=\left\{g'\in G';\ \deg_{s_0}(g')=0\right\},$$
$$g'_j=s_0\delta_j-E'_{j-},\ E'\in (A')^{\ell\times\ell}.\ \textit{Then}$$
$$\mathscr{B}_1=U_1^{\perp}=\left\{v\in W_A^{\ell};\ R_0\circ v=0\right\},\ \mathscr{B}=U^{\perp}=\left\{w\in W_B^{\ell};\ R\circ w=0=0\right\}\ \textit{where}$$
$$R_0:=(\varphi(g'))_{g'\in G'_0}\in A^{G'_0\times\ell},\ R=(\phi(g'))_{g'\in G'}=\begin{pmatrix}R_0\\ s_0\,\mathrm{id}_\ell-E\end{pmatrix}\in B^{G'\times\ell},\ E:=\varphi(E').$$

These are the representations of $\mathscr{B}$ from Thms. 2.2 and 3.4 which are used in Thms. 5.6 and 5.11.

Proof. The result follows directly from the explicit form of $U'_1=(B')^{1\times G'}R'$ according to Lemma 6.3 since

$$U=\phi\phi^{-1}(U)=\phi(U')=\phi((B')^{1\times G'}R')=B^{1\times G'}R,\ R=\phi(R').$$

□

7 Added in proof on June 26, 2013

We show that the assumptions on the IO decomposition of $\mathscr{B}_d$ in Thm. and Def. 2.3 are not necessary for asymptotic stability. We need the characteristic variety or *variety of rank singularities* of a *non-autonomous* behavior [10, Cor. and Def. 7.71, Remark 7.72]. For the data from (7) and (10) it is given as

$$\mathrm{char}(\mathscr{B}):=\left\{(\lambda,\omega)\in\Lambda_N;\ \mathrm{rank}(R(\lambda,\omega))<\mathrm{rank}(R)\right\}. \tag{92}$$

Notice that $\mathrm{rank}(R)<\ell$ unless $\mathscr{B}$ is autonomous and that the last two equalities of (12) only hold for autonomous $\mathscr{B}$. Analogously the characteristic variety of a W_A-behavior

$$\begin{aligned}\mathscr{B}_0=\left\{v\in W_A^{\ell};\ R_0\circ v=0\right\},\ R_0\in A^{k\times\ell},\ \text{is}\\ \mathrm{char}(\mathscr{B}_0):=\left\{\omega\in(\mathbb{C}\setminus\{0\})^n;\ \mathrm{rank}(R_0(\omega))<\mathrm{rank}(R_0)\right\}.\end{aligned} \tag{93}$$

Then $\mathrm{char}(\mathscr{B}_0)=\emptyset$ if and only if the module $A^{1\times\ell}/A^{1\times k}R_0$ is projective [10, Cor. and Def. 7.71] and thus free (Quillen-Suslin), i.e., if $\mathscr{B}_0$ is *strictly controllable*.

Theorem 7.1. *A spectrally stable and time-autonomous behavior $\mathscr{B}\subseteq W_B^{\ell}$ with* $\mathrm{proj}:\mathscr{B}\cong\mathscr{B}_d$ *is* L^2*- and pointwise stable if (i)* $\mathrm{char}(\mathscr{B}_d)\bigcap\mathbb{T}^n=\emptyset$ *or if (ii) $\mathscr{B}$ has the ta normal form from Thm. 2.2 with $k=1$:*

$$\begin{aligned}\mathscr{B}=\left\{w\in\left(W_A^{\ell}\right)^{\mathbb{N}};\ R_0\circ w(0)=0,\ \forall t\in\mathbb{N}:\ w(t)=E^t\circ w(0)\right\}\cong\\ \mathscr{B}_1=\left\{v\in W_A^{\ell};\ R_0\circ v=0\right\},\ 0\neq R_0\in A^{1\times\ell},\ E\in A^{\ell\times\ell}.\end{aligned} \tag{94}$$

The proof is a variant of that of Thm. 2.3. Notice that R_0 is assumed as just one row in (94). The condition $\mathrm{char}(\mathscr{B}_d)\bigcap\mathbb{T}^n=\emptyset$ generalizes the condition $\mathrm{rank}(P(\omega))=p$ for $\omega\in\mathbb{T}^n$ of Thm. 2.3 which therefore is not necessary for asymptotic stability. Neither is condition (i) since in case (ii) $\mathrm{char}(\mathscr{B}_1)\bigcap\mathbb{T}^n=\{\omega\in\mathbb{T}^n;\ R_0(\omega)=0\}$. We conjecture that in general asymptotic stability of a time-autonomous behavior requires *some additional* condition besides spectral stability.

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