Two invariants for weak exponential stability of linear time-varying differential behaviors

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Abstract

In the paper H. Bourlès, B. Marinescu, U. Oberst, 'Weak exponential stability (w.e.s.) of linear time-varying (LTV) differential behaviors', Linear Algebra and its Applications 486(2015), 1-49, we studied the problem of the title. If a finitely generated torsion module over an appropriate ring of differential operators and its associated autonomous system are regular singular the system is never w.e.s.. In contrast we computed a square complex matrix for each irregular singular module and showed that the system is w.e.s. resp. not stable if all eigenvalues of the matrix have positive real parts resp. if at least one eigenvalue has negative real part. In this supplement of the quoted paper we show that the spectrum of the matrix and the decay exponent are isomorphy invariants of the module. The proofs make essential use of results exposed in P. Maisonobe, C. Sabbah, 'D-module cohérents et holonomes', Hermann, Paris, 1993. We also complement the main w.e.s. result of our quoted paper by the case where at least one eigenvalue of the matrix is purely imaginary.

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1 Introduction

In [3] we studied the *weak exponential stability* (w.e.s.) of *linear time-varying differential* (LTV) systems [3, Def. 2.4], the varying coefficients being locally convergent *Puiseux series*. Every finitely generated left torsion module M over the appropriate integral domain of differential operators is interpreted as a system module and gives rise to a dual autonomous behavior $\mathcal{B}$. For an *irregular singular* module M we constructed a behavior isomorphism [3, Thm. 2.8]

$$\mathcal{B} \cong \left\{x \in W(\tau)^n;\ \forall t > \tau :\ x'(t) + t^{\lambda-1}(A_0 + t^{-\mu}A_1(t^{-\mu}))x(t) = 0\right\}$$
$$\text{where } 0 < \lambda, \mu \in \mathbb{Q},\ \mu^{-1} \in \mathbb{N},\ \mathbb{Z}\lambda \subseteq \mathbb{Z}\mu,\ 0 \neq A_0 \in \mathbb{C}^{n\times n},\ A_1 \in \mathbb{C} < z >^{n\times n} \quad (1)$$

for sufficiently large $\tau \geq 0$; see (69)-(71) for slight differences of (1) to its original in [3]. The ring $\mathbb{C} < z >$ is the local domain of locally convergent power series. The signal space $W(\tau) := \mathrm{C}^{\infty}(\tau, \infty)$ consists of all smooth, complex-valued functions on the open interval $(\tau, \infty) := \{t \in \mathbb{R};\ t > \tau\}$. The w.e.s. of $\mathcal{B}$, i.e., the exponential decay of its trajectories for $t \to \infty$, is determined by the spectral properties of A_0 [3, Thm. 2.8]. The system is w.e.s. with decay factors $\exp(-\alpha t^{\lambda})$ resp. not stable if all eigenvalues of A_0 have positive real part resp. at least one eigenvalue has negative real part. The proof of (1) made essential use of important results exposed in the excellent books [8] and [10].

In this supplement to [3] we show in Thm. 6.1 and in Cor. 7.2 that the number λ and the spectrum of A_0 are *isomorphy invariants* of M. Indeed λ is the largest positive slope of the *Newton polygon* of an associated differential operator and $\mathrm{spec}(A_0)$ is determined by the roots of its λ-*symbol*. Thm. 8.2 shows that the system is not w.e.s. with decay factors $\exp(-\alpha t^{\lambda})$ if all eigenvalues of A_0 have nonnegative real part and at least one is purely imaginary. This is fully analogous to the case of constant coefficients. The system may, however, be w.e.s. with a decay factor $\exp(-\alpha' t^{\lambda'}),\ 0 < \lambda' < \lambda,\ 0 < \alpha'$, as Example 8.3 demonstrates. We prove the invariance of λ by extending the important invariance result [11, Cor. 1.6.11, p. 54], [8, Prop. I.5.1.4, Def. I.5.1.5] from Laurent series to Puiseux series. Up to an addition of 1 λ is also called the *irregularity* [7, p. 15] or *nonregularity* of M [2, p. 78]. We *simplify* the proof in [8] by avoiding the *good filtrations* [8, Ex. 5.13] of differential modules and their algebraic properties and *correct* two nontrivial errors of [3], see Remarks 4.2 and 6.2 for the precise statements. Our proof gives all details whereas that of [8] gives indications only that are hard to complete for systems theorists. We emphasize, however, that the essential ideas for the generalization, in particular the use of the λ-degree and the associated graded modules, come from [8]. As far as we see [10] does not contain the result from [8, Prop. 5.1.4]. We refer to the bibliography of [3] for important references concerning exponential stability of differential systems, for instance [13] [6], [5], [1], and to [11], [8], [10], [2] for the long and extensive history of the algebraic theory of linear differential systems with varying coefficients. The results of [3] and of the present paper are constructive, cf., for instance, [4], [12], [10, Ch. 4], and can be applied to the construction of compensators [9]. The plan of the paper follows from the titles of the eight sections.

Notations and abbreviations: e.s.= exponentially stable, exponential stability, f.d.=finite-dimensional, f.g.=finitely generated, LTV=linear time-varying, resp.=respectively, $\mathrm{spec}(A_0) :=$ the spectrum or set of eigenvalues of a square complex matrix A_0, w.e.s.= weak(ly) e.s., w.l.o.g.= without loss of generality, $X^{p\times q}$=set of $p \times q$-matrices with entries in X, $X^{1\times q}$=rows, $X^q := X^{q\times 1}$=columns

2 Differential operators, modules and behaviors

We refer to our paper [3, §§1-3, pp. 1-26] for the considered systems theoretic and algebraic notions. The latter are based on notions and results exposed in [8] and [10]. The coefficient fields and associated rings of LTV differential operators are

$$\begin{array}{ccc} \mathbf{K}(m) := \mathbb{C} << z^{1/m} >> & \subset & \mathbf{K} := \bigcup_{n\in\mathbb{N},\, n\geq 1} \mathbb{C} << z^{1/n} >> \\ \bigcap & & \bigcap \\ \mathbf{A}(m) := \mathbf{K}(m)[\partial; d/dz] & \subset & \mathbf{A} := \mathbf{K}[\partial; d/dz] \end{array},\ m \geq 1,$$
$$\mathbf{A} = \mathbf{K}[\partial; d/dz] = \mathbf{K}[z\partial; zd/dz] = \mathbf{K}[-z^2\partial; -z^2 d/dz]. \tag{2}$$

The field $\mathbf{K}(1) = \mathbb{C} << z >>$ is that of locally convergent *Laurent series* whereas $\mathbf{K}$ is that of locally convergent *Puiseux series*. All representations of $\mathbf{A}$ in the last row of (2) will be used. The rings $\mathbf{A}$ and $\mathbf{A}(m)$ are (left and right) euclidean and thus principal ideal domains. The *valuation* or *order* $v : \mathbf{K} \to \mathbb{Q} \cup \{\infty\}$ is defined by $v(0) = \infty$ and

$$\begin{aligned} & v(a) := k/m \text{ if } a = \sum_{i=k}^{\infty} \alpha_i z^{i/m},\ k \in \mathbb{Z},\ \alpha_i \in \mathbb{C},\ \alpha_k \neq 0 \Longrightarrow \forall a, b \in \mathbf{K} : \\ & v(ab) = v(a) + v(b),\ v(a+b) \begin{cases} \geq \min(v(a), v(b)) \\ = \min(v(a), v(b)) & \text{if } v(a) \neq v(b) \end{cases}, \\ & \Longrightarrow \{a \in \mathbf{K};\ v(a) \geq 0\} = \bigcup \left\{ \mathbb{C} < z^{1/m} > \quad m \geq 1 \right\}. \end{aligned} \tag{3}$$

We consider f.g. *system* $\mathbf{A}$*-left modules*

$$\begin{aligned} & M = \mathbf{A}^{1\times q}/U,\ U = \mathbf{A}^{1\times p} R, \text{ where} \\ & R = \sum_{j=0}^{n} A_j(z^{1/m})(-z^2\partial)^j \in \mathbf{A}(m)^{p\times q} \subset \mathbf{A}^{p\times q},\ A_j \in \mathbb{C} << z >>^{p\times q}. \end{aligned} \tag{4}$$

We define $\sigma(R) := \rho^{-m}$ where ρ is the minimum of the convergence radii of all entries of all $A_j,\ j = 0, \cdots, n$, so that the $A_j(t^{-1/m})$ are smooth matrix functions on the open real interval $(\sigma(R), \infty) := \{t \in \mathbb{R};\ t > \sigma(R)\}$. The operators $a(z^{1/m}),\ a \in \mathbb{C} << z >>$, and $-z^2\partial$ act on $W(\tau) = \mathrm{C}^{\infty}(\tau, \infty)$ by $\circ$ via

$$\begin{aligned} & (a(z^{1/m}) \circ w)(t) := a(t^{-1/m}) w(t) \text{ if } t > \tau \geq \sigma\left(a(z^{1/m})\right) \\ & (-z^2\partial) \circ w = w' = dw/dt \Longrightarrow (z\partial) \circ w = -tw',\ \partial \circ w = -t^2 w'. \end{aligned} \tag{5}$$

The special form of this action is explained in [3, (14), Remark 2.1]. The matrix R gives rise to the family of solution spaces or *behaviors*

$$\begin{aligned} \mathcal{B}(R, \tau) &:= \{w \in W(\tau)^q;\ R \circ w = 0\} \\ &= \left\{ w \in W(\tau)^q;\ \forall t > \tau :\ \sum_j A_j(t^{-1/m}) w^{(j)}(t) = 0 \right\},\ \tau \geq \sigma(R). \end{aligned} \tag{6}$$

Since $\sigma(R)$ depends on R we define the equivalence [3, (18)] of two families

$$\begin{aligned} & (\mathcal{B}(R_i, \tau))_{\tau \geq \sigma(R_i)},\ R_i \in \mathbf{A}^{p_i \times q},\ i = 1, 2, \text{ by:} \\ & \exists \tau_0 \geq \max(\sigma(R_1), \sigma(R_2)) \forall \tau \geq \tau_0 :\ \mathcal{B}(R_1, \tau) = \mathcal{B}(R_2, \tau). \end{aligned} \tag{7}$$

If $U = \mathbf{A}^{1\times p_i} R_i,\ i = 1, 2$, the equivalence class

$$\mathcal{B}(U) := \mathrm{cl}\left((\mathcal{B}(R_1, \tau))_{\tau \geq \sigma(R_1)}\right) = \mathrm{cl}\left((\mathcal{B}(R_2, \tau))_{\tau \geq \sigma(R_2)}\right) \tag{8}$$

depends on U [3, (19), Lemma 3.7] only and not on the choice of a special R_i and is called the *behavior associated to* U or to the module M with the *given presentation* $M = \mathbf{A}^{1\times q}/U$. The assignment $M = \mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U)$ can be extended to a contravariant functor that establishes a categorical duality between f.g. $\mathbf{A}$-left modules with a given presentation $\mathbf{A}^{1\times q}/U$ and behaviors [3, Thm. 2.3]. The f.g. module M is a torsion module if and only if $\dim_{\mathbf{K}}(M) < \infty$ and is then cyclic of the form

$$M = \mathbf{A}^{1\times q}/U \cong \mathbf{A}/\mathbf{A}P,\ P = \sum_{i=0}^{n} a_i(z\partial)^i \in \mathbf{A}(m) \subset \mathbf{A} = \mathbf{K}[z\partial; zd/dz],\ a_n \neq 0, \tag{9}$$

where $P \in \mathbf{A}$ is a nonzero differential operator of degree $\deg_\partial(P) = \deg_{z\partial}(P) = \dim_{\mathbf{K}}(M)$ [8, Prop. I.4.3.3], [3, Lemma and Def. 3.15]. The associated behavior $\mathcal{B}(U)$ is then called *autonomous* and characterized by the property that for sufficiently large $\tau \geq \sigma(R)$ all trajectories $w \in \mathcal{B}(R, \tau)$ are uniquely determined by a fixed number of initial conditions [3, §3.6]. For

$$m = 1,\ \mathbf{K}(1) = \mathbb{C} << z >> \subset \mathbf{A}(1) = \mathbf{K}(1)[\partial; d/dz],\ R \in \mathbf{A}(1)^{p\times q},\ 0 \neq P \in \mathbf{A}(1) \tag{10}$$

the f.g. torsion module

$$\begin{aligned} &M(1) = \mathbf{A}(1)[\partial]^{1\times q}/\mathbf{A}(1)^{1\times p}R \cong \mathbf{A}(1)/\mathbf{A}(1)P \text{ with} \\ &M = \mathbf{K} \otimes_{\mathbf{K}(1)} M(1) \cong \mathbf{A}/\mathbf{A}P, \end{aligned} \tag{11}$$

is also called a *meromorphic connection* [8, §I.4.3]. The torsion module M from (9) and the operator $P \in \mathbf{A}(m)$ are called *regular singular* [8, Def. 5.1.1], [10, Def. 3.9], [3, §5.2] if

$$\begin{aligned} &a_n^{-1}P = \sum_{i=0}^{n} a_i a_n^{-1}(z\partial)^i \in \mathbb{C} < z^{1/m} > [z\partial] \\ &\iff \forall i < n : v(a_i/a_n) = v(a_i) - v(a_n) \geq 0 \\ &\iff \forall i < n : (v(a_n) - v(a_i))(n-i)^{-1} \leq 0, \end{aligned} \tag{12}$$

and otherwise *irregular singular*. The latter property signifies that

$$\lambda(P) := \max_{i<n,\ a_i\neq 0}(v(a_n) - v(a_i))(n-i)^{-1} > 0. \tag{13}$$

The number λ of (1) coincides with $\lambda(P)$. Notice that we used $\lambda := -\lambda(P)$ in [3, (185), Thm. 5.5, Thm. 2.8]. The behavior $\mathcal{B}(U) \cong \mathcal{B}(\mathbf{A}P)$ of a regular singular module is never weakly exponentially stable [3, Thm. 2.8,(i)], and regular singularity is therefore uninteresting for asymptotic stability.

3 The Newton polygon

We consider the torsion module $M \cong \mathbf{A}/\mathbf{A}P$ from (9) with

$$P = \sum_{i=0}^{n} a_i(z\partial)^i \in \mathbf{A}(m) \subset \mathbf{A},\ a_i \in \mathbf{K}(m),\ a_n \neq 0. \tag{14}$$

The *Newton polygon* [8, §5.1, pp. 24-28], [10, §3.3] of P is the *convex hull* $N(P)$ of the set

$$\left\{(x, y) \in \mathbb{Q}^2;\ \exists j \leq n \text{ with } a_j \neq 0,\ x \leq j,\ y \geq v(a_j)\right\} \subset N(P). \tag{15}$$

The vertices of $N(P)$ are among the points $(j, v(a_j)),\ a_j \neq 0$. The point $(n, v(a_n))$ is one vertex of $N(P)$ and the half-line $(n, v(a_n)) + \mathbb{Q}_+(0,1)$ is the unique vertical edge of $N(P)$ where $\mathbb{Q}_+ := \{\alpha \in \mathbb{Q};\ \alpha \geq 0\}$. The half-line

$$(i_{\min}, v(a_{i_{\min}})) + \mathbb{Q}_+(-1,0),\ i_{\min} = \max\{i;\ v(a_i) = \min\{v(a_j);\ 0 \leq j \leq n\}\} \tag{16}$$

with vertex $(i_{\min}, v(a_{i_{\min}}))$ is the unique horizontal edge of $N(P)$. The remaining edges are line segments of the form

$$\begin{aligned}&[(i, v(a_i)), (j, v(a_j))] := \{(x,y) = (1-t)(i, v(a_i)) + t(j, v(a_j));\ 0 \leq t \leq 1\}\\ &= \left\{(x,y) \in \mathbb{Q}^2;\ y = (v(a_j) - v(a_i))(j-i)^{-1}(x-i) + v(a_i),\ i \leq x \leq j\right\},\\ &i < j,\ v(a_i) < v(a_j) < \infty.\end{aligned} \tag{17}$$

The *slope* of this edge is $(v(a_j) - v(a_i))(j-i)^{-1} > 0$.

Corollary 3.1. *The Newton polygon has the unique vertex $(n, v(a_n))$ or no edges with positive slope if and only if P is regular singular.*

Lemma 3.2. *For an irregular singular differential operator $P \in \mathbf{A}$ as in* (14) *a positive number $\lambda \in \mathbb{Q}$ is a slope of (an edge of) $N(P)$ if and only if*

$$\begin{gathered}\exists i < j \text{ with } 0 \leq i < j \leq n,\ v(a_i) < v(a_j) < \infty\ \forall k \text{ with } 0 \leq k \leq n,\ v(a_k) < \infty :\\ \lambda i - v(a_i) = \lambda j - v(a_j) \geq \lambda k - v(a_k).\end{gathered} \tag{18}$$

The largest positive slope (of an edge of) $N(P)$ is

$$\lambda(P) \underset{(12)}{=} \max\nolimits_{i<n, a_i \neq 0}(v(a_n) - v(a_i))(n-i)^{-1}. \tag{19}$$

Proof. $\Longrightarrow$: If an edge has this slope it has, by (17), the form

$$\lambda = \frac{v(a_j) - v(a_i)}{j-i},\ i < j,\ v(a_i) < v(a_j) \Longrightarrow \lambda j - v(a_j) = \lambda i - v(a_i). \tag{20}$$

Consider the line $y = \lambda(x-i) + v(a_i)$ through $(i, v(a_i))$ and $(j, v(a_j))$. By definition of $N(P)$ and its edges the Newton polygon $N(P)$ lies above this line and hence

$$\forall k \text{ with } a_k \neq 0:\ v(a_k) \geq \lambda(k-i) + v(a_i) \Longrightarrow \lambda i - v(a_i) \geq \lambda k - v(a_k). \tag{21}$$

$\Longleftarrow$: analogous. □

Equations (1), (12) and (19) explain the significance of the Newton polygon of P for weak exponential stability of $\mathcal{B}(U)$. In the sequel we are going to show that $\lambda(P)$ and other objects are invariants of M. We introduce the set

$$S_+(P) := \{\lambda \in \mathbb{Q};\ \lambda > 0,\ \lambda \text{ is a slope of } N(P)\}. \tag{22}$$

By Cor. 3.1 the set $S_+(P)$ is empty if and only if P is regular singular.

4 The λ-degree and its associated graded ring

Assume

$$0 < \lambda, \mu \in \mathbb{Q},\ 0 < m \in \mathbb{N},\ \mathbb{Z}\lambda + \mathbb{Z}m^{-1} = \mathbb{Z}\mu \Longrightarrow \mu^{-1} \in \mathbb{N}. \tag{23}$$

Lemma 3.2 suggests to define the following λ-degree of a differential operator

$$\begin{aligned} &P=\sum_{i=0}^{n} a_i(z\partial)^i \in \mathbf{A},\ a_i\in\mathbf{K},\ a_n\neq 0: \\ &\deg_\lambda(P)=\max_i(\lambda i - v(a_i)) = \max_{i,\ a_i\neq 0}(\lambda i - v(a_i)) \in \mathbb{Q}. \end{aligned} \tag{24}$$

In the sequel we assume

$$P\in\mathbf{A}(m) \Longrightarrow \forall i\leq n \text{ with } a_i\neq 0:\ v(a_i)\in\mathbb{Z}m^{-1} \Longrightarrow \deg_\lambda(P)\in\mathbb{Z}\mu. \tag{25}$$

Assume

$$\begin{aligned} &a=z^k u\in\mathbb{C}<<z>>,\ k\in\mathbb{Z},\ u\in\mathbb{C}<z>,\ u(0)\neq 0,\ v(a(z^{1/m}))=k/m, \\ &\Longrightarrow (z\partial)(a(z^{1/m})) = km^{-1}z^{k/m}u(z^{1/m}) + z^{(k+1)/m}m^{-1}u'(z^{1/m}) \\ &\Longrightarrow v\left((z\partial)(a(z^{1/m}))\right)\begin{cases} = v\left(a(z^{1/m})\right) = k/m & \text{if } v\left(a(z^{1/m})\right)\neq 0 \\ > v\left(a(z^{1/m})\right) & \text{if } v\left(a(z^{1/m})\right) = 0 \end{cases} \\ &\Longrightarrow \forall i>0:\ v\left((z\partial)^i(a(z^{1/m}))\right)\begin{cases} = v\left(a(z^{1/m})\right) & \text{if } v\left(a(z^{1/m})\right)\neq 0 \\ > 0 & \text{if } v\left(a(z^{1/m})\right) = 0 \end{cases}. \end{aligned} \tag{26}$$

For $P=a(z\partial)^i\neq 0$ and $Q=b(z\partial)^j\neq 0$ the Leibniz formula implies

$$\begin{aligned} &PQ=\sum_{k=0}^{i}\binom{i}{k} a(z\partial)^{i-k}(b)(z\partial)^{k+j} \\ &\Longrightarrow \deg_\lambda(PQ) = \max_{k\leq i}\left(\lambda(k+j) - v\left(a(z\partial)^{i-k}(b)\right)\right) \\ &\underset{(26)}{\leq} (\lambda i - v(a)) + (\lambda j - v(b)) = \deg_\lambda(P)+\deg_\lambda(Q). \end{aligned} \tag{27}$$

For arbitrary $P,Q\in\mathbf{A}(m)$ this obviously implies

$$\begin{aligned} &\deg_\lambda(P+Q)\leq\max(\deg_\lambda(P),\deg_\lambda(Q)) \\ &\deg_\lambda(PQ)\leq\deg_\lambda(P)+\deg_\lambda(Q). \end{aligned} \tag{28}$$

The degree function $\deg_\lambda$ and (28) induce the increasing $\mathbb{Z}\mu$-*filtration* of $\mathbf{A}(m)$ by $\mathbb{C}$-subspaces $\mathbf{A}(m)_k$:

$$\begin{aligned} &\mathbf{A}(m)_k := \{P\in\mathbf{A}(m):\ \deg_\lambda(P)\leq k\}\supseteq \\ &\mathbf{A}(m)_{<k} = \mathbf{A}(m)_{k-\mu} = \{P\in\mathbf{A}(m):\ \deg_\lambda(P)<k\},\ k\in\mathbb{Z}\mu \\ &\mathbf{A}(m) = \bigcup_{k\in\mathbb{Z}\mu}\mathbf{A}(m)_k,\ \mathbf{A}(m)_k\mathbf{A}(m)_\ell\subseteq\mathbf{A}(m)_{k+\ell}. \end{aligned} \tag{29}$$

For a nonzero $P\in\mathbf{A}(m)$ and $d:=\deg_\lambda(P)\in\mathbb{Z}\mu$ its λ-*symbol* is defined as

$$\sigma_\lambda(P) := P+\mathbf{A}(m)_{<d}\in G_d := \mathbf{A}(m)_d/\mathbf{A}(m)_{<d}. \tag{30}$$

Moreover we define $\sigma_\lambda(0)=0$. With (28) we construct the $\mathbb{Z}\mu$-graded algebra

$$G := \mathrm{gr}_\lambda(\mathbf{A}(m)) := \oplus_{k\in\mathbb{Z}\mu}G_k,\ G_k := \mathbf{A}(m)_k/\mathbf{A}(m)_{<k} \tag{31}$$

with the sum resp. product, for $P_i + \mathbf{A}(m)_{<d_i} \in G_{d_i},\ i = 1, 2,$

$$\begin{aligned} &(P_1 + \mathbf{A}(m)_{<d_1}) + / * (P_2 + \mathbf{A}(m)_{<d_2}) := (P_1 + / * P_2 + \mathbf{A}(m)_{<d_1+d_2}) \\ &\sigma_\lambda(P_1) + / * \sigma_\lambda(P_2) := P_1 + / * P_2 + \mathbf{A}(m)_{<d} \in G_d,\ d := \deg_\lambda(P_1) + \deg_\lambda(P_2). \end{aligned} \tag{32}$$

We compute some λ-symbols: Any nonzero $a \in \mathbb{C} << z >>$ can be uniquely written as

$$\begin{aligned} &a = \alpha z^k + b,\ 0 \neq \alpha \in \mathbb{C}, v(a) = k,\ k < v(b) \Longrightarrow a(z^{1/m}) = \alpha z^{k/m} + b(z^{1/m}), \\ &v\left(a(z^{1/m})\right) = v\left(\alpha z^{k/m}\right) = k/m < v\left(b(z^{1/m})\right) \\ &\Longrightarrow a(z^{1/m}) \in \mathbf{A}(m),\ \deg_\lambda\left(a(z^{1/m})\right) = -k/m > \deg_\lambda\left(b(z^{1/m})\right) \\ &\Longrightarrow \sigma_\lambda\left(a(z^{1/m})\right) = \alpha z^{k/m} + \mathbf{A}(m)_{<-k/m} = \alpha \sigma_\lambda(z^{1/m})^k \in G_{-k/m} \\ &\sigma_\lambda\left(z^{1/m}\right) \in G_{-1/m}. \end{aligned} \tag{33}$$

Obviously $\deg_\lambda(z\partial) = \lambda$ and $\sigma_\lambda(z\partial) \in G_\lambda$ and hence for $a(z^{1/m})$ from (33)

$$\begin{aligned} &d := \deg_\lambda\left(a(z^{1/m})(z\partial)^i\right) = \lambda i - v(a(z^{1/m})) = \lambda i - km^{-1}, \\ &\sigma_\lambda\left(a(z^{1/m})(z\partial)^i\right) = \sigma_\lambda\left(a(z^{1/m})\right)\sigma_\lambda\left(z\partial\right)^i \\ &= \alpha\sigma_\lambda\left(z^{1/m}\right)^k \sigma_\lambda(z\partial)^i \in G_d = G_{-v(a(z^{1/m}))+\lambda i}. \end{aligned} \tag{34}$$

For a nonzero

$$\begin{aligned} &P = \sum_{i, a_i \neq 0} a_i (z\partial)^i,\ a_i \in \mathbf{K}(m),\ d := \deg_\lambda(P) = \max_{i, a_i \neq 0}(\lambda i - v(a_i)), \\ &v(a_i) = k(i)/m,\ a_i = \alpha_i z^{k(i)/m} + b_i,\ 0 \neq \alpha_i \in \mathbb{C},\ v(a_i) < v(b_i) \end{aligned} \tag{35}$$

equation (34) implies

$$\sigma_\lambda(P) = \sum_i \left\{\alpha_i \sigma_\lambda\left(z^{1/m}\right)^{k(i)} \sigma_\lambda(z\partial)^i;\ \lambda i - v(a_i) = \lambda i - k(i)m^{-1} = d\right\}. \tag{36}$$

The preceding equations suggest to introduce the (commutative) Laurent polynomial algebra and degree function

$$\begin{aligned} &L := \mathbb{C}[\xi, \xi^{-1}, \eta] = \oplus_{(k,j)\in\mathbb{Z}\times\mathbb{N}} \mathbb{C}\xi^k \eta^j, \\ &\deg_\lambda\left(\sum_{k,j} \alpha_{k,j}\xi^k\eta^j\right) := \max_{(k,j), \alpha_{k,j}\neq 0}(\lambda j - km^{-1}) \in \mathbb{Z}\mu. \end{aligned} \tag{37}$$

The degree function on $\mathbb{C}[\xi, \xi^{-1}, \eta]$ satisfies $\deg_\lambda(fg) = \deg_\lambda(f) + \deg_\lambda(g)$ and gives rise to the $\mathbb{Z}\mu$-algebra grading

$$\begin{gathered} L = \oplus_{d\in\mathbb{Z}\mu} L_d,\ L_d = \oplus_{(k,j)\in\mathbb{Z}\times\mathbb{N}}\left\{\mathbb{C}\xi^k\eta^j;\ \deg_\lambda(\xi^k\eta^j) = \lambda j - km^{-1} = d\right\} \\ L_{d_1}L_{d_2} \subseteq L_{d_1+d_2}. \end{gathered} \tag{38}$$

Lemma 4.1. *The algebra* $G := \mathrm{gr}_\lambda(\mathbf{A}(m))$ *is commutative and the map*

$$\varphi : L = \mathbb{C}[\xi, \xi^{-1}, \eta] \to G = \mathrm{gr}_\lambda(\mathbf{A}(m)),\ \xi \mapsto \sigma_\lambda(z^{1/m}),\ \eta \mapsto \sigma_\lambda(z\partial), \tag{39}$$

is an isomorphism of graded algebras ($\varphi(L_d) = G_d$*). For* $P, Q \in \mathbf{A}(m)$ *this implies*

$$\deg_\lambda(PQ) = \deg_\lambda(P) + \deg_\lambda(Q),\ \deg_\lambda(PQ - QP) < \deg_\lambda(P) + \deg_\lambda(Q). \tag{40}$$

Proof. (i) *Commutativity*: Due to equation (36) it suffices to show that $\sigma_\lambda(z^{1/m})$ and $\sigma_\lambda(z\partial)$ commute. But

$$\begin{aligned} &(z\partial)z^{1/m} = z^{1/m}(z\partial) + m^{-1}z^{1/m}, \\ &\deg_\lambda(z^{1/m}) = -1/m \underset{\lambda>0}{<} \lambda - 1/m = \deg_\lambda\left(z^{1/m}(z\partial)\right) \\ &\Longrightarrow \sigma_\lambda(z\partial)\sigma_\lambda(z^{1/m}) = \sigma_\lambda\left((z\partial)z^{1/m}\right) = \sigma_\lambda(z^{1/m})\sigma_\lambda(z\partial). \end{aligned} \tag{41}$$

(ii) Since G is commutative, $\sigma_\lambda(z^{1/m})^{-1} = \sigma_\lambda(z^{-1/m})$ is invertible and

$$\deg_\lambda(\xi) = -m^{-1} = -v(z^{1/m}) = \deg_\lambda(z^{1/m}),\ \deg_\lambda(\eta) = \deg_\lambda(z\partial) = \lambda$$

the homomorphism

$$\varphi : L \to G,\ \xi \mapsto \sigma_\lambda(z^{1/m}),\ \eta \mapsto \sigma_\lambda(z\partial),$$

is well and uniquely defined and preserves the grading. Due to (36) it is surjective.
(iii) φ *is injective*: Since $\varphi(L_d) = G_d,\ d \in \mathbb{Z}\mu$, it suffices to show that $\varphi|_{L_d}$ is injective. If $f \in L_d$ and $\varphi(f) = 0$ then f has the form

$$\begin{aligned} f &= \sum_j \left\{\alpha_j \xi^{k(j)}\eta^j;\ \lambda j - k(j)m^{-1} = d\right\} \in L_d,\ \alpha_j \in \mathbb{C},\ \text{and} \\ \varphi(f) &= \sum_j \left\{\alpha_j \sigma_\lambda\left(z^{1/m}\right)^{k(j)} \sigma_\lambda(z\partial)^j;\ \lambda j - k(j)m^{-1} = d\right\} \\ &= \sigma_\lambda(Q) = 0,\ Q := \sum_j \left\{\alpha_j z^{k(j)/m}(z\partial)^j;\ \lambda j - k(j)m^{-1} = d\right\}. \end{aligned}$$

The equation $\sigma_\lambda(Q) = 0$ implies $\deg_\lambda(Q) < d$. If there was an index j with $\alpha_j \neq 0$ this would imply

$$\deg_\lambda(Q) = \max_{j,\alpha_j\neq 0}(\lambda j - k(j)m^{-1}) = \max_{j,\alpha_j\neq 0} d = d.$$

Hence all α_j and f are zero.
(iv) In particular, G is an integral domain. If $P_1, P_2 \in \mathbf{A}(m)$ are nonzero of degree $d_i := \deg_\lambda(P_i)$ then also $0 \neq \sigma_\lambda(P_i) \in G_{d_i}$ and hence

$$\begin{aligned} &0 \neq \sigma_\lambda(P_1)\sigma_\lambda(P_2) = P_1P_2 + \mathbf{A}(m)_{<d_1+d_2} \in G_{d_1+d_2} = \mathbf{A}(m)_{d_1+d_2}/\mathbf{A}(m)_{<d_1+d_2} \\ &\Longrightarrow \deg_\lambda(P_1P_2) = d_1 + d_2. \end{aligned}$$

□

Remark 4.2. (i) In the sequel we identify

$$\mathbb{C}[\xi,\xi^{-1},\eta] = G = \mathrm{gr}_\lambda(\mathbf{A}(m)),\ \xi = \sigma_\lambda(z^{1/m}) \in G_{-1/m},\ \eta = \sigma_\lambda(z\partial) \in G_\lambda. \tag{42}$$

Notice that

$$\begin{aligned} &\partial = z^{-1}(z\partial) = (z^{1/m})^{-m}(z\partial),\ \deg_\lambda(\partial) = \lambda + 1, \\ &\Longrightarrow \sigma_\lambda(\partial) = \xi^{-m}\eta \in \mathbb{C}[\xi,\xi^{-1},\eta] = \mathbb{C}[\xi,\xi^{-1},\xi^{-m}\eta] \\ &\Longrightarrow \mathrm{gr}_\lambda(\mathbf{A}(m)) = \mathbb{C}[\sigma_\lambda\left(z^{1/m}\right), \sigma_\lambda\left(z^{1/m}\right)^{-1}, \sigma_\lambda(\partial)]. \end{aligned} \tag{43}$$

(ii) In [8, p. 25, 3. below] it is asserted that for $m = 1$ and $\lambda = \lambda_0/\lambda_1,\ \lambda_0,\lambda_1 > 0$ the graded ring $\mathrm{gr}_\lambda(\mathbf{A}(1))$ is isomorphic to the polynomial algebra $\mathbb{C} << z >> [\eta]$. According to the preceding computations this is an error.

5 Graded modules and the Newton polygon

The assumptions are those of Section 4. Moreover we assume a nonzero *irregular singular* differential operator

$$\begin{gathered} 0 \neq P = \sum_{i=0,a_i\neq 0}^{n} a_i(z\partial)^i \in \mathbf{A}(m),\ a_n \neq 0,\ d = \deg_\lambda(P),\ v(a_i) = k(i)/m \\ \max_{i<n}(v(a_n) - v(a_i)) > 0. \end{gathered} \tag{44}$$

For simplicity we write (for fixed m)

$$\mathbf{K}' := \mathbf{K}(m) = \mathbb{C} << z^{1/m} >> \subset \mathbf{A}' := \mathbf{A}(m) = \mathbf{K}'[\partial; d/dz] = \mathbf{K}'[z\partial; zd/dz]. \tag{45}$$

The filtration (29) of $\mathbf{A}'$ induces increasing $\mathbb{Z}\mu$-filtrations of $\mathbf{A}'P$ and of $\mathbf{A}'/\mathbf{A}'P$ defined by

$$\begin{aligned} &(\mathbf{A}'P)_d := \mathbf{A}'_d \bigcap \mathbf{A}'P = \mathbf{A}'_{d-\deg_\lambda(P)}P,\ d \in \mathbb{Z}\mu, \\ &(\mathbf{A}'/\mathbf{A}'P)_d = (\mathbf{A}'_d + \mathbf{A}'P)/\mathbf{A}'P \underset{\text{ident.}}{=} \mathbf{A}'_d/(\mathbf{A}'P)_d \subset \mathbf{A}'/\mathbf{A}'P, \\ &\mathbf{A}'_{d_1}(\mathbf{A}'P)_{d_2} \subseteq (\mathbf{A}'P)_{d_1+d_2},\ \mathbf{A}'_{d_1}(\mathbf{A}'/\mathbf{A}'P)_{d_2} \subseteq (\mathbf{A}'/\mathbf{A}'P)_{d_1+d_2}. \end{aligned} \tag{46}$$

In analogy to $G := \mathrm{gr}_\lambda(\mathbf{A}') = \oplus_{d\in\mathbb{Z}\mu}\mathbf{A}'_d/\mathbf{A}'_{<d}$ these filtrations give rise to the graded G-modules

$$\begin{aligned} &\mathrm{gr}_\lambda(\mathbf{A}'P) = \oplus_{d\in\mathbb{Z}\mu}\,\mathrm{gr}_\lambda(\mathbf{A}'P)_d \text{ where} \\ &\mathrm{gr}_\lambda(\mathbf{A}'P)_d := (\mathbf{A}'P)_d/(\mathbf{A}'P)_{<d} = \mathbf{A}'_{d-\deg_\lambda(P)}P/\mathbf{A}'_{<d-\deg_\lambda(P)}P \\ &\mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P) = \oplus_{d\in\mathbb{Z}\mu}(\mathbf{A}'/\mathbf{A}'P)_d/(\mathbf{A}'/\mathbf{A}'P)_{<d} \cong \oplus_d(\mathbf{A}'_d + \mathbf{A}'P)/(\mathbf{A}'_{<d} + \mathbf{A}'P). \end{aligned} \tag{47}$$

The symbols σ_λ are defined as in (30). We infer the graded G-isomorphism

$$\mathrm{gr}_\lambda(\mathbf{A}') \cong \mathrm{gr}_\lambda(\mathbf{A}'P),\ \mathrm{gr}_\lambda(\mathbf{A}')_{d-\deg_\lambda(P)} \cong \mathrm{gr}_\lambda(\mathbf{A}'P)_d,\ \sigma_\lambda(Q) \mapsto \sigma_\lambda(Q)\sigma_\lambda(P). \tag{48}$$

The isomorphism theorem induces the exact sequences

$$
\begin{aligned}
&0 \to A'_{d-\deg_\lambda(P)} \xrightarrow{\cdot P} A'_d \xrightarrow{\mathrm{can}} (A'/A'P)_d \to 0\\
&0 \to \mathrm{gr}_\lambda(\mathbf{A}') \xrightarrow{\cdot\sigma_\lambda(P)} \mathrm{gr}_\lambda(\mathbf{A}') \xrightarrow{\mathrm{can}} \mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P) \to 0\\
&\sigma_\lambda(P) \underset{(36)}{=} \sum_i \left\{\alpha_i \xi^{k(i)}\eta^i; \lambda i - v(a_i) = \deg_\lambda(P)\right\},\ v(a_i) = k(i)/m \text{ if } \alpha_i \neq 0\\
&\Longrightarrow \mathrm{gr}_\lambda(\mathbf{A}')/\,\mathrm{gr}_\lambda(\mathbf{A}')\sigma_\lambda(P) = \mathbb{C}[\xi,\xi^{-1},\eta]/\mathbb{C}[\xi,\xi^{-1},\eta]\sigma_\lambda(P) \cong \mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P).
\end{aligned} \tag{49}
$$

The Laurent polynomial ring $\mathrm{gr}_\lambda(\mathbf{A}') = \mathbb{C}[\xi,\xi^{-1},\eta]$ is factorial. Its units are the elements $\alpha\xi^k,\ 0 \neq \alpha \in \mathbb{C}, k \in \mathbb{Z}$. We identify primes that are associated or differ by a unit only. The only monomial prime of $\mathrm{gr}_\lambda(\mathbf{A}')$ is $\eta = \sigma_\lambda(z\partial)$. For a module M over a commutative ring R its *annihilator* is the ideal $\mathrm{ann}_R(M) := \{r \in R;\ rM = 0\}$. If $\mathfrak{a}$ is any ideal of R we have $\mathfrak{a} = \mathrm{ann}_R(R/\mathfrak{a})$. The *root* of $\mathfrak{a}$ is the ideal $\sqrt{\mathfrak{a}} := \left\{r \in R;\ \exists k > 0 \text{ with } r^k \in \mathfrak{a}\right\}$. The set of prime factors of $\sigma_\lambda(P)$ is denoted by

$$
\begin{aligned}
&\mathrm{Pr}(\lambda, P) := \{p;\ p \text{ prime },\ p \text{ divides } \sigma_\lambda(P)\}\\
&\Longrightarrow \mathrm{gr}_\lambda(\mathbf{A}')\sigma_\lambda(P) = \mathrm{gr}_\lambda(\mathbf{A}') \prod_{p\in \mathrm{Pr}(\lambda,P)} p^{k(p)}\\
&\qquad \subseteq \sqrt{\mathrm{gr}_\lambda(\mathbf{A}')\sigma_\lambda(P)} = \mathrm{gr}_\lambda(\mathbf{A}') \prod_{p\in \mathrm{Pr}(\lambda,P)} p,\ 0 < k(p) \in \mathbb{N}.
\end{aligned} \tag{50}
$$

Since $\sigma_\lambda(P)$ is homogeneous of degree $d = \deg_\lambda(P)$, i.e., $\sigma_\lambda(P) \in G_d$, also its prime factors in $\mathrm{Pr}(\lambda, P)$ are homogeneous. The following lemma establishes the connection of the *geometric properties* of the Newton polygon $N(P)$ with the *algebraic* properties of $\mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P)$.

Lemma 5.1. *(cf. [8, Def. I.5.1.5]) With the data and assumptions of this section the positive rational number λ is a slope of the Newton polygon $N(P)$ of the irregular singular differential operator P if and only if $\mathrm{Pr}(\lambda, P)$ contains a nonmonomial prime p, i.e., $p \neq \eta = \sigma_\lambda(z\partial)$.*

Proof. From equation (49) we have

$$
\sigma_\lambda(P) = \sum_{\ell, a_\ell \neq 0} \left\{\alpha_\ell \xi^{k(\ell)}\eta^\ell; \lambda\ell - v(a_\ell) = d\right\},\ d := \deg_\lambda(P),\ v(a_\ell) = k(\ell)/m. \tag{51}
$$

$\Longrightarrow$: Since λ is a slope we know from Lemma 3.2 that

$$
\begin{aligned}
&\exists i < j \text{ with } 0 \leq i < j \leq n,\ a_i \neq 0, a_j \neq 0, \forall \ell \text{ with } 0 \leq \ell \leq n,\ a_\ell \neq 0:\\
&\lambda i - v(a_i) = \lambda j - v(a_j) \geq \lambda\ell - v(a_\ell)\\
&\Longrightarrow d := \deg_\lambda(P) = \max\nolimits_{\ell, a_\ell \neq 0}(\lambda\ell - v(a_\ell)) = \lambda i - v(a_i) = \lambda j - v(a_j)\\
&\Longrightarrow \sigma_\lambda(P) = \alpha_i\xi^{k(i)}\eta^i + \alpha_j\xi^{k(j)}\eta^j + \sum_{\ell \neq i,j}\left\{\alpha_\ell\xi^{k(\ell)}\eta^\ell;\ \lambda\ell - v(a_\ell) = d\right\}.
\end{aligned} \tag{52}
$$

It is obvious that $\sigma_\lambda(P)$ is not of the form $\alpha\xi^k\eta^\ell$ and therefore has a nonmonomial prime.

$\Longleftarrow$: analogous. □

6 The principal invariance theorem

We are going to show that the sets $\mathrm{Pr}(\lambda, P),\ \lambda > 0$, *and* $S_+(P)$ *of positive slopes of the Newton polygon* $N(P)$ *are invariants of the module* $M \cong \mathbf{A}/\mathbf{A}P$, *cf. [11, Cor. 1.6.11] [8, Prop. I.5.1.4].*

We assume an arbitrary $\mathbf{K}$-f.d. irregular singular $\mathbf{A}$-module M and two isomorphisms

$$M \cong \mathbf{A}/\mathbf{A}P_1 \cong \mathbf{A}/\mathbf{A}P_2,\ P_i \in \mathbf{A},\ n(i) := \deg_\lambda(P_i) \in \mathbb{Q}. \tag{53}$$

Let

$$\begin{aligned} &\cdot Q_1 : \mathbf{A}/\mathbf{A}P_1 \to \mathbf{A}/\mathbf{A}P_2, f + \mathbf{A}P_1 \mapsto fQ_1 + \mathbf{A}P_2,\ d(1) := \deg_\lambda(Q_1),\\ &\cdot Q_2 : \mathbf{A}/\mathbf{A}P_2 \to \mathbf{A}/\mathbf{A}P_1,\ g + \mathbf{A}P_2 \mapsto gQ_2 + \mathbf{A}P_1,\ d(2) := \deg_\lambda(Q_2), \end{aligned} \tag{54}$$

be mutually inverse isomorphisms. We assume m sufficiently large such that all entries of the operators P_1, P_2, Q_1, Q_2 belong to $\mathbf{A}' := \mathbf{A}(m)$. The isomorphisms (54) thus imply inverse isomorphisms

$$\begin{aligned} &\cdot Q_1 : \mathbf{A}'/\mathbf{A}'P_1 \to \mathbf{A}'/\mathbf{A}'P_2, f + \mathbf{A}'P_1 \mapsto fQ_1 + \mathbf{A}'P_2,\\ &\cdot Q_2 : \mathbf{A}'/\mathbf{A}'P_2 \to \mathbf{A}'/\mathbf{A}'P_1,\ g + \mathbf{A}'P_2 \mapsto gQ_2 + \mathbf{A}'P_1. \end{aligned} \tag{55}$$

For μ from (23) and $k, \ell \in \mathbb{Z}\mu$ these inverse isomorphisms induce

$$\begin{aligned} &\cdot Q_1 : (\mathbf{A}'/\mathbf{A}'P_1)_k = (\mathbf{A}'_k + \mathbf{A}'P_1)/\mathbf{A}'P_1 \overset{\cdot Q_1}{\cong} X_k := (\mathbf{A}'_k Q_1 + \mathbf{A}'P_2)/\mathbf{A}'P_2\\ &\qquad\qquad \overset{\cdot Q_2}{\cong} (\mathbf{A}'_k Q_1 Q_2 + \mathbf{A}'P_1)/\mathbf{A}'P_1 = (\mathbf{A}'/\mathbf{A}'P_1)_k.\\ &X_k = (\mathbf{A}'_k Q_1 + \mathbf{A}'P_2)/\mathbf{A}'P_2 \subseteq (\mathbf{A}'_{k+d(1)} + \mathbf{A}'P_2)/\mathbf{A}'P_2\\ &\qquad\qquad = (\mathbf{A}'/\mathbf{A}'P_2)_{k+d(1)} =: Y_{k+d(1)}. \end{aligned} \tag{56}$$

Likewise, the isomorphism $\cdot Q_2$ induces, for all $k \in \mathbb{Z}\mu$,

$$\begin{aligned} &Y_{k+d(1)} = (\mathbf{A}'_{k+d(1)} + \mathbf{A}'P_2)/\mathbf{A}'P_2\\ &\qquad \overset{\cdot Q_2}{\cong} (\mathbf{A}'_{k+d(1)} Q_2 + \mathbf{A}'P_1)/\mathbf{A}'P_1 \subseteq (\mathbf{A}'/\mathbf{A}'P_1)_{k+d(1)+d(2)}\\ &X_{k+2d(1)+d(2)} = (\mathbf{A}'_{k+d(1)+d(2)} Q_1 + \mathbf{A}'P_2)/\mathbf{A}'P_2\\ &\qquad \overset{\cdot Q_2}{\cong} (\mathbf{A}'_{k+d(1)+d(2)} Q_1 Q_2 + \mathbf{A}'P_1)/\mathbf{A}'P_1 \underset{(54)}{=} (\mathbf{A}'/\mathbf{A}'P_1)_{k+d(1)+d(2)}\\ &\Longrightarrow \forall k \in \mathbb{Z}\mu :\ Y_{k+d(1)} \subseteq X_{k+2d(1)+d(2)},\ Y_k \subseteq X_{k+d(1)+d(2)}\\ &\Longrightarrow \text{ with } e := d(1) + d(2) \forall k \in \mathbb{Z}\mu : X_k \underset{(56)}{\subseteq} Y_{k+e},\ Y_k \subseteq X_{k+e}. \end{aligned} \tag{57}$$

Obviously the family $(X_k)_k$ is an increasing filtration of $\mathbf{A}'/\mathbf{A}'P_2$ with

$$\mathbf{A}'_k X_\ell \subseteq X_{k+\ell} \text{ and } \bigcup_\ell X_\ell = \mathbf{A}'/\mathbf{A}'P_2, \tag{58}$$

It induces the graded $\mathrm{gr}_\lambda(\mathbf{A}')$-module $\mathrm{gr}_\lambda(X) := \oplus_\ell X_\ell / X_{<\ell}$, and the isomorphism $\cdot Q_1$ induces the $\mathrm{gr}_\lambda(\mathbf{A}')$-isomorphism

$$\begin{aligned} &\cdot\sigma_\lambda(Q_1) :\ \mathbb{C}[\xi, \xi^{-1}, \eta]/\mathbb{C}[\xi, \xi^{-1}, \eta]\sigma_\lambda(P_1) \underset{(49)}{=} \mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P_1) \cong \mathrm{gr}_\lambda(X) \Longrightarrow\\ &\qquad \mathrm{ann}_{\mathrm{gr}_\lambda(\mathbf{A}')}(\mathrm{gr}_\lambda(X)) = \mathbb{C}[\xi, \xi^{-1}, \eta]\sigma_\lambda(P_1). \end{aligned} \tag{59}$$

Theorem 6.1. *(cf. [8, Prop. 5.1.4, Def. 5.1.5]) Under the preceding assumptions the following sets are invariants of $M \cong \mathbf{A}/\mathbf{A}P_1 \cong \mathbf{A}/\mathbf{A}P_2$, cf.* (22), (50)*:*

$$\begin{gathered}\sqrt{\mathrm{ann}_{\mathrm{gr}_\lambda(\mathbf{A}')}\left(\mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P_1)\right)} = \sqrt{\mathrm{ann}_{\mathrm{gr}_\lambda(\mathbf{A}')}\left(\mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P_2)\right)},\ 0<\lambda\in\mathbb{Q} \\ \mathrm{Pr}(\lambda,P_1)=\mathrm{Pr}(\lambda,P_2),\ S_+(P_1)=S_+(P_2).\end{gathered} \tag{60}$$

In particular, the largest slopes of $N(P_1)$ and $N(P_2)$ coincide.

Proof. According to (50) and Lemma 5.1 is suffices to show that

$$\mathrm{ann}_{\mathrm{gr}_\lambda(\mathbf{A}')}\left(\mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P_1)\right) = \mathrm{ann}_{\mathrm{gr}_\lambda(\mathbf{A}')}\left(\mathrm{gr}_\lambda(X)\right) \subseteq \sqrt{\mathrm{ann}_{\mathrm{gr}_\lambda(\mathbf{A}')}\left(\mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P_2)\right)}.$$

These annihilator ideals are homogeneous and therefore it suffices to show this inclusion for homogeneous elements. Let $\varphi = \sigma_\lambda(f) \in \mathrm{gr}_\lambda(\mathbf{A}')_d,\ f \in \mathbf{A}'_d \setminus \mathbf{A}'_{<d}$, and $\varphi\,\mathrm{gr}_\lambda(X) = 0$ and hence

$$\begin{aligned}&\forall \ell\in\mathbb{Z}\mu:\ \varphi(X_\ell/X_{<\ell}) = 0 \subseteq X_{\ell+d}/X_{<(\ell+d)} \Longrightarrow fX_\ell \subseteq X_{\ell+d-\mu} \\ &\underset{\text{induction}}{\Longrightarrow} \forall \ell\in\mathbb{Z}\mu\forall i\in\mathbb{N}:\ f^iX_\ell \subseteq X_{\ell+id-i\mu} \\ &\underset{(57)}{\Longrightarrow} \forall k\in\mathbb{Z}\mu\forall i\in\mathbb{N}: f^iY_k \subseteq f^iX_{k+e} \subseteq X_{k+e+id-i\mu} \subseteq Y_{k+2e+id-i\mu}.\end{aligned} \tag{61}$$

Choose

$$\begin{aligned}&i\in\mathbb{N} \text{ with } i \geq 2e\mu^{-1}+1 \iff k+2e+id-i\mu \leq k+id-\mu \\ &\Longrightarrow f^iY_k \subseteq Y_{k+2e+id-i\mu} \subseteq Y_{<(k+id)} \subseteq Y_{k+id} \\ &\Longrightarrow \varphi^i\left(Y_k/Y_{<k}\right) = 0 \subseteq Y_{k+id}/Y_{<(k+id)} \\ &\Longrightarrow \forall k\in\mathbb{Z}\mu:\ \varphi^i\,\mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P_2)_k = 0 \Longrightarrow \varphi^i\,\mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P_2) = 0 \\ &\Longrightarrow \varphi^i \in \mathrm{ann}_{\mathrm{gr}_\lambda(\mathbf{A}')}\left(\mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P_2)\right) \Longrightarrow \varphi \in \sqrt{\mathrm{ann}_{\mathrm{gr}_\lambda(\mathbf{A}')}\left(\mathrm{gr}_\lambda(\mathbf{A}'/\mathbf{A}'P_2)\right)}.\end{aligned} \tag{62}$$

□

Remark 6.2. In [8, §I.3.2.6] and with the notations of the proof of Thm. 6.1 the author uses the false implication $\varphi(X_\ell/X_{<\ell}) = 0 \Longrightarrow fX_\ell \subseteq X_{<\ell} = X_{\ell-\mu}$ instead of the *correct inclusion* $fX_\ell \subseteq X_{\ell+d-\mu}$ where $\deg_\lambda(f) = d$.

7 The invariance of $\mathrm{spec}(A_0)$

The assumptions of the preceding sections remain in force. By multiplying P with a_n^{-1} we assumme w.l.o.g. that the irregular singular differential operator P has the form

$$P = \sum_{i=0}^{n} a_i(z\partial)^i,\ a_i \in \mathbb{C} << z^{1/m} >>,\ a_n := 1\,(\Longrightarrow v(a_n) = 0)\,,\ M \cong \mathbf{A}/\mathbf{A}P. \tag{63}$$

The largest slope of the Newton polygon is

$$\lambda := \max_{i,a_i\neq 0}(v(a_n)-v(a_i))(n-i)^{-1} = -\min_{i,a_i\neq 0} v(a_i)(n-i)^{-1}. \tag{64}$$

From (23) we recall

$$0<\mu\in\mathbb{Q},\ \mu^{-1}\in\mathbb{N},\ \mathbb{Z}m^{-1}+\mathbb{Z}\lambda = \mathbb{Z}\mu. \tag{65}$$

We recall some equations from [3] that have to be changed slightly because as in [10, p. 68] we used the notation $P = (z\partial)^n + a_1(z\partial)^{n-1} + \cdots + a_n$ in [3, (181)]. Also note again that the λ from [3, (185)] differed from (64) by the sign and was negative. With $E := z^\lambda(z\partial) = z^{1+\lambda}\partial$ we proved in [3, proof of Thm. 5.4] that

$$\begin{aligned} &E^i = z^{i\lambda}\sum_{k=0}^{i}\alpha(i,k)(z\partial)^k,\ 0 \le i \le n,\ \alpha(i,k) \in \mathbb{C},\ \alpha(i,i) = 1 \\ &z^{n\lambda}P = \sum_{k=0}^{n} z^{n\lambda}a_k(z\partial)^k = \sum_{i=0}^{n} b_iE^i, \\ &b_i \in \mathbb{C} < z^\mu >= \{b \in \mathbb{C} << z^\mu >>;\ v(b) \ge 0\},\ b_n = 1. \end{aligned} \tag{66}$$

More precisely we get

$$\begin{aligned} z^{n\lambda}P &= \sum_{k=0}^{n} z^{n\lambda}a_k(z\partial)^k = \sum_{i=0}^{n} b_iE^i = \sum_{0\le k\le i\le n} b_i\alpha(i,k)z^{i\lambda}(z\partial)^k \\ &= \sum_{k=0}^{n}\left(\sum_{i=k}^{n}\alpha(i,k)z^{i\lambda}b_i\right)(z\partial)^k \Longrightarrow \forall k:\ a_k := \sum_{i=k}^{n}\alpha(i,k)z^{(i-n)\lambda}b_i \\ \Longrightarrow \forall k:\ & z^{(n-k)\lambda}a_k = \sum_{i=k}^{n}\alpha(i,k)z^{(i-k)\lambda}b_i = b_k + z^\lambda c_k, \\ &\qquad \text{with } c_k \in \mathbb{C} < z^\mu >,\ v(c_k) \ge 0,\ b_n = 1,\ c_n = 0 \\ \Longrightarrow \forall k:\ & v\left(z^{(n-k)\lambda}a_k\right) = (n-k)\lambda + v(a_k) \ge 0,\ \left(z^{(n-k)\lambda}a_k\right)(0) = b_k(0) \\ \Longrightarrow \forall k & \text{ with } a_k \ne 0:\ (n-k)\lambda + v(a_k) = 0 \iff b_k(0) \ne 0. \end{aligned} \tag{67}$$

The inequality $(n-k)\lambda + v(a_k) \ge 0$ also follows from (64):

$$\lambda \ge (v(a_n) - v(a_k))(n-k)^{-1} = -v(a_k)(n-k)^{-1}\ \ (a_n = 1,\ v(a_n) = 0)\,. \tag{68}$$

The representation $\mathbf{A}P = \mathbf{A}\sum_{i=0}^{n} b_iE^i$ gives rise to the following standard matrix A [3, (195)]

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ -b_0 & -b_1 & \cdots & \cdots & -b_{n-1} \end{pmatrix} = A_0 + z^\mu A_1(z^\mu) \text{ with } A_1(z) \in \mathbb{C} < z >^{n\times n} \\ A_0 &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ -b_0(0) & -b_1(0) & \cdots & \cdots & -b_{n-1}(0) \end{pmatrix} \in \mathbb{C}^{n\times n},\ \det(\eta\,\mathrm{id}_n - A_0) = \sum_{i=0}^{n} b_i(0)\eta^i, \end{aligned} \tag{69}$$

and ensuing $\mathbf{A}$-isomorphism [3, (197)]

$$\begin{aligned} &\mathbf{A}/\mathbf{A}P \cong \mathbf{A}^{1\times n}/\mathbf{A}^{1\times n}(E\,\mathrm{id}_n - A),\ f + \mathbf{A}P \longleftrightarrow \xi + \mathbf{A}^{1\times n}(E\,\mathrm{id}_n - A) \\ &f \in \mathbf{A},\ \xi = (\xi_0, \cdots, \xi_{n-1}) \in \mathbf{A}^{1\times n},\ f = \sum_{i=0}^{n-1}\xi E^i,\ \xi = (f, 0, \cdots, 0). \end{aligned} \tag{70}$$

Remark 7.1. In [3, Thm. 2.8, (37), (201)] we used m^{-1} instead of μ from (65) in (69). This was a slight error, but has no effect on the validity of the derivations in [3].

Recall the action $(a(z^\mu)\circ w)(t)=a(t^{-\mu})w(t)$ and $-z^2\partial\circ w=w'$. By duality the $\mathbf{A}$-isomorphism (70) induces the behavior isomorphisms [3, (198)]

$$\begin{aligned}&\mathcal{B}(P,\tau):=\{y\in W(\tau);P\circ y=0\}\cong\mathcal{B}(E\,\mathrm{id}_n-A,\tau)=\{x\in W(\tau)^n;\ E\circ x=A\circ x\}\\&\qquad=\left\{x\in W(\tau)^n;\ x'(t)+t^{\lambda-1}\left(A_0+t^{-\mu}A_1(t^{-\mu})\right)x(t)=0\right\},\ y\leftrightarrow x\\&y=x_0,\ x=(x_0,\cdots,x_{n-1})^\top=(y,E\circ y,\cdots,E^{n-1}\circ y)^\top\end{aligned}\tag{71}$$

for all sufficiently large τ. In (71) we applied

$$(E\circ x)(t)=\left(z^{\lambda+1}\partial\circ x\right)(t)=\left(-z^{\lambda-1}(-z^2\partial\circ x)\right)(t)=-t^{1-\lambda}x'(t).\tag{72}$$

Equation (71) is the precise version of (1). The spectrum $\mathrm{spec}(A_0)$ determines the w.e.s. of the behavior $\mathcal{B}$ and consists of the roots of the characteristic polynomial $\det(\eta\,\mathrm{id}_n-A_0)=\sum_{i=0}^n b_i(0)\eta^i$.

We finally relate the latter to the λ-symbol $\sigma_\lambda(P)$ from (36). Recall from Section 5 and 6 that

$$\begin{aligned}&\mathbf{A}':=\mathbf{A}(m),\ \mathrm{gr}_\lambda(\mathbf{A}')=\mathbb{C}[\xi,\xi^{-1},\eta],\ \xi=\sigma_\lambda(z^{1/m}),\ \eta=\sigma_\lambda(z\partial)\\&\mathrm{gr}_\lambda(\mathbf{A}/\mathbf{A}P)=\mathbb{C}[\xi,\xi^{-1},\eta]/\mathbb{C}[\xi,\xi^{-1},\eta]\sigma_\lambda(P).\end{aligned}\tag{73}$$

According to (50) equation (73) furnishes

$$\sqrt{\mathrm{gr}_\lambda(\mathbf{A}')\sigma_\lambda(P)}=\mathrm{gr}_\lambda(\mathbf{A}')\prod_{p\in\mathrm{Pr}(\lambda,P)}p\tag{74}$$

where $\mathrm{Pr}(\lambda,P)$ is the set of prime factors of $\sigma_\lambda(P)$. Thm. 6.1 shows that the ideal of (74), and $\mathrm{Pr}(\lambda,P)$ are isomorphy invariants of M.

The polynomial $\sigma_\lambda(P)$ is homogeneous of λ-degree

$$d:=\deg_\lambda(P)=\max_{i,a_i\neq0}\left(\lambda i-v(a_i)\right)\in\mathbb{Z}\mu.\tag{75}$$

The prime factors in $\mathrm{Pr}(\lambda,P)$ are also homogeneous. Due to (64) we have

$$\begin{aligned}&\lambda:=\max_{i,a_i\neq0}-v(a_i)(n-i)^{-1}\Longrightarrow\forall i\text{ with }a_i\neq0:\ \lambda n\geq\lambda i-v(a_i),\ d=\lambda n,\\&\qquad\Longrightarrow\forall i\text{ with }a_i\neq0:\ d=\lambda i-v(a_i)\iff(n-i)\lambda+v(a_i)=0.\end{aligned}\tag{76}$$

For all i as in the last row of (76) write

$$\begin{aligned}&v(a_i)=k(i)/m,\ a_i=\alpha_iz^{k(i)/m}+\widetilde{a}_i,\ v(a_i)=k(i)/m<v(\widetilde{a}_i)\\&\underset{(49)}{\Longrightarrow}\sigma_\lambda(P)=\sum_i\left\{\alpha_i\xi^{k(i)}\eta^i;\ \lambda i-v(a_i)=d\right\}\text{ and}\\&\qquad z^{(n-i)\lambda}a_i=\alpha_i+d_i\underset{(67)}{=}b_i+z^\lambda c_i,\ v(d_i)>0\\&\Longrightarrow\forall i\text{ with }\lambda i-v(a_i)=d:\ \alpha_i=b_i(0)\\&\Longrightarrow\det(\eta\,\mathrm{id}_n-A_0)\underset{(69)}{=}\sum_i b_i(0)\eta^i\underset{(67)}{=}\sum_i\left\{b_i(0)\eta^i;\ (n-i)\lambda+v(a_i)=0\right\}\\&\qquad\qquad=\sum_i\left\{\alpha_i\eta^i;\ \lambda i-v(a_i)=d\right\}=\sigma_\lambda(P)(1,\eta).\end{aligned}\tag{77}$$

For a complex polynomial $f(\eta)$ let $V_{\mathbb{C}}(f)$ denote the set of its zeros or roots.

Corollary 7.2. *For the irregular singular* $M \cong \mathbf{A}/\mathbf{A}P$ *and the data introduced above we have*

$$\operatorname{spec}(A_0) = V_{\mathbb{C}}(\sigma_\lambda(P)(1,\eta)) = \bigcup \{V_{\mathbb{C}}(p(1,\eta));\ p \in \Pr(\lambda, P)\}\,. \tag{78}$$

Since $\Pr(\lambda, P)$ *is an isomorphy invariant of* M *according to Thm. 6.1 so is* $\operatorname{spec}(A_0)$*. Since*

$$\begin{aligned} &d := \deg_\lambda(P) = \lambda n - v(a_n)\\ &I := \{i;\ 0 \le i \le n;\ \lambda i - v(a_i) = d\} = \left\{i;\ \lambda = (v(a_n) - v(a_i))(n-i)^{-1}\right\}\\ &\sigma_\lambda(P)(1,\eta) \underset{(36)}{=} \sum_{i\in I} \alpha_i \eta^i = \alpha_n \eta^n + \cdots,\ \operatorname{spec}(A_0) = V_{\mathbb{C}}\left(\sigma_\lambda(P)(1,\eta)\right) \end{aligned} \tag{79}$$

this corollary contains a simple algorithm to check weak exponential stability of an arbitrary autonomous differential system.

8 A supplement to weak exponential stability

The assumptions and notations of Section 7 are in force. Consider the state space system of (71). *Weak exponential stability* (w.e.s.) was introduced and studied in [3, Def. 2.4, Thm. 2.7,§4]. For $t \ge t_0 > \tau \ge 0$ in (71) let $\Phi_A(t,t_0) \in \mathrm{Gl}_n(\mathrm{C}^\infty(\tau,\infty))$ denote the transition matrix of the state space system, i.e. the unique solution of

$$\Phi_A'(t,t_0) + t^{\lambda-1}\left(A_0 + t^{-\mu}A_1(t^{-\mu})\right)\Phi_A(t,t_0) = 0,\ \Phi_A(t_0,t_0) = \mathrm{id}_n\,. \tag{80}$$

An important result is

Result 8.1. *(cf. [3, Thms. 5.7, 5.8, Ex. 5.9]) There are positive constants* $\tau, c, \alpha > 0$ *such that for all* $t \ge t_0 > \tau$

$$\|\Phi_A(t,t_0)\| \begin{cases} \le c\exp(-\alpha(t^\lambda - t_0^\lambda)) & \text{if } \forall \rho \in \operatorname{spec}(A_0):\ \Re(\rho) > 0\\ \ge c\exp(\alpha(t^\lambda - t_0^\lambda)) & \text{if } \exists \rho \in \operatorname{spec}(A_0):\ \Re(\rho) < 0 \end{cases}. \tag{81}$$

The system need not be w.e.s. if

$$\forall \rho \in \operatorname{spec}(A_0):\ \Re(\rho) \ge 0,\ \exists \rho \in \operatorname{spec}(A_0):\ \Re(\rho) = 0. \tag{82}$$

The next result improves the case of (82).

Theorem 8.2. *If the matrix* A_0 *has a purely imaginary eigenvalue there is no inequality*

$$\|\Phi_A(t,t_0)\| \le c\exp(-\alpha(t^\lambda - t_0^\lambda)),\ t \ge t_0 > \tau \ge 0,\ \alpha > 0, \tag{83}$$

for sufficiently large τ*.*
As Example 8.3 shows the system may, however, be w.e.s. with a decay factor $\exp(-\alpha' t^{\lambda'})$ *with* $\lambda' < \lambda$*,* $0 < \alpha'$*.*

Proof. Assume that

$$\|\Phi_A(t,t_0)\| \le c\exp(-\alpha(t^\lambda - t_0^\lambda)),\ t \ge t_0 > \tau \ge 0,$$

holds and define the bounded trajectory $\widetilde{\Phi}(t) := \exp(\alpha t^\lambda)\Phi_A(t, t_0)$ for $t \geq t_0 > \tau$. Differentiation of $\widetilde{\Phi}(t)$ and the differential equation (80) furnish

$$\widetilde{\Phi}'(t) + t^{\lambda-1}\left((A_0 - \alpha\lambda\,\mathrm{id}_n) + t^{-\mu}A_1(t^{-\mu})\right)\widetilde{\Phi}'(t) = 0,\ t \geq t_0 > \tau.$$

This equation has the same form as (71), but with $A_0 - \alpha\lambda\,\mathrm{id}_n$ instead of A_0. The matrix $A_0 - \alpha\lambda\,\mathrm{id}_n$ has the eigenvalue $\rho - \alpha\lambda$ with $\Re(\rho - \alpha\lambda) = -\alpha\lambda < 0$. The second inequality from (81) then implies that $\widetilde{\Phi}(t)$ is unbounded for sufficiently large τ, and this is a contradiction.

□

Example 8.3. Consider the differential operator and ensuing differential equation

$$\begin{aligned} &P = -z^2\partial + z^{-1/2}(i + z^{1/2}) = a_1(z\partial) + a_0,\ a_1 = -z,\ a_0 = z^{-1/2}(i + z^{1/2}) \\ &\Longrightarrow x'(t) + t^{1/2}(i + t^{-1/2})x(t) = x' + (it^{1/2} + 1)x = 0,\ A_0 = i. \end{aligned}$$

The maximal slope λ of the Newton polygon is $\lambda := (v(a_1) - v(a_0))(2-1)^{-1} = 1 - (-(1/2)) = 3/2$ and hence P is irregular singular. The solution of the differential equation is

$$\begin{aligned} x(t) &= \exp\left(-\int_{t_0}^{t}(ix^{1/2} + 1)dx\right)x(t_0) \\ &= \exp\left(-(2/3)i(t^{3/2} - t_0^{3/2})\right)\exp(-(t - t_0))x(t_0). \end{aligned}$$

This solution is obviously w.e.s. with decay factor $\exp(-t)$, but not with decay factor $\exp(-\alpha t^\lambda) = \exp(-\alpha t^{3/2})$ as Thm. 8.2 states.

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