Weak exponential stability of linear time-varying differential behaviors

Linear Algebra and its Applications 486(2015), 523-571

http://dx.doi.org/10.1016/j.laa.2015.08.034

H. Bourlès [*], B. Marinescu [†], U. Oberst[‡]

September 12, 2015

Abstract

We develop a new approach to exponential stability of linear time-varying (LTV) differential behaviors that is analogous to that in our paper on exponential stability of discrete LTV behaviors (to appear in SIAM J. Control Optim. 2015). Stability theory for differential state space systems with smooth coefficients is an important subject in the literature. For differential LTV behaviors with arbitrary smooth coefficients there is no reasonable stability theory. Therefore we restrict the smooth varying coefficients to functions that are defined by means of locally convergent Puiseux series. All rational functions are of this type. We introduce a new kind of behaviors and prove a module-behavior duality for these. We define a new notion of weak exponential stability (w.e.s.) of a behavior B and its associated finitely generated (f.g.) module M and show that the w.e.s. modules and behaviors are closed under isomorphisms, subobjects, factor objects and extensions. The standard uniform exponential stability of state space equations is not preserved under behavior isomorphisms and unsuitable for a behavioral theory. In the main result we assume a nonzero f.g. torsion module M and its associated autonomous behavior B. Such a module may be regular or irregular singular according to the Galois theory of differential equations. If it is nonzero and regular singular it is never w.e.s.. For irregular singular M we characterize w.e.s. of most B algebraically and constructively.

AMS-classification: 93D20, 93C15, 93B25, 34D05, 34D20
Key-words: exponential stability, differential behavior, time-varying, duality, Serre category

1 Introduction

In this paper we develop a new approach to *exponential stability* of behaviors that are described by *linear time-varying (LTV) differential* equations. The method is the ana-

[*]SATIE, ENS Cachan/CNAM, 61 Avenue President Wilson, F-94230, Cachan, France. email: henri.bourles@satie.ens-cachan.fr

[†]École Centrale de Nantes, B.P. 92101, 1, rue de la Noë, FR-44321 Nantes Cedex 3, France. email: bogdan.marinescu@irccyn.ec-nantes.fr

[‡]Institut für Mathematik, Universität Innsbruck, Technikerstrasse 13, A-6020 Innsbruck, Austria. email: ulrich.oberst@uibk.ac.at

logue of that in our paper [5] on exponential stability for discrete behaviors and differs from that in [4, Ch. 6]. Stability and stabilization for LTV *differential state space systems* is an important subject in the literature, see for instance the books [21, Chs. 6,7,8, pp. 99-141] and [12, Ch. 3, pp. 193-368] and their comprehensive bibliographies and the recent papers [19], [11], [1], [2]. The main theorems of the present paper are Thms. 2.3, 2.7 and 2.8 and are exposed in Section 2.
We use the differential field $\mathbf{K}$ of locally convergent Laurent series in $z^{1/m},\ m \geq 1$. The elements of $\mathbf{K}$ are of the form $a(z^{1/m}) = \sum_{i=k}^{\infty} a_i z^{i/m},\ k \in \mathbb{Z}$, and give rise to smooth functions $f(t) = a(t^{-1/m})$ that are defined on intervals $(\tau, \infty) \subset \mathbb{R}$ for sufficiently large $\tau \geq 0$. These $f(t)$ are the coefficient functions of the considered LTV differential systems. The field $\mathbf{K}$ contains the fields of rational and even of meromorphic functions, but only their germs at 0 are used for defining $f(t)$. With its standard derivation d/dz the field $\mathbf{K}$ gives rise to the noncommutative algebra $\mathbf{A} = \mathbf{K}[\partial; d/dz]$ of differential operators that is a principal ideal domain. We introduce a new kind of behaviors and prove a categorical duality between these behaviors and finitely generated (f.g.) $\mathbf{A}$-left modules, cf. Thm. 2.3. We define a new notion of *weak exponential stability* (w.e.s.) of a behavior $\mathcal{B}$ and its associated f.g. module M by analytic conditions on the behavior's trajectories, cf. Def. 2.4. In contrast to the standard *uniform exponential stability* (u.e.s.) [21, Def. 6.5] w.e.s. is preserved by isomorphisms and characterized by exponential decay factors $\exp(-\alpha t^{\mu})$ with $\alpha, \mu > 0$ instead of the standard factors $\exp(-\alpha t)$. The w.e.s. modules and behaviors form *Serre categories*, i.e., are closed under isomorphisms, subobjects, factor objects and extensions, cf. Thm. 2.7. In the main result Thm. 2.8 we assume a nonzero f.g. torsion module M and its associated autonomous behavior $\mathcal{B}$. Such a module may be *regular or irregular singular* [16], [20], cf. Section 5. If it is nonzero and regular singular it is never w.e.s.. If it is irregular singular we construct a complex matrix A_0 from M and show that M and $\mathcal{B}$ are w.e.s. if the eigenvalues of A_0 have positive real parts and are not w.e.s. if at least one eigenvalue of A_0 has a negative real part. If the eigenvalues of A_0 have nonnegative real parts and at least one of them is purely imaginary then M and $\mathcal{B}$ need not be w.e.s..
We refer to [5, Section 4] where it is shown (in the analogous discrete situation) that the module-behavior duality of Thm. 2.3 implies the standard consequences for differential LTV behaviors like *Ehrenpreis' fundamental principle, Willems' elimination, controllability and input/output decompositions.*

Remark 1.1. (cf. [5, Remark 1.1]) (i) *Difficulties with arbitrary analytic coefficients*: Consider the differential equation $\cos^2(t)w'(t) - w(t) = 0,\ t \in \mathbb{R}$, with its nonzero solutions $w(t) = c\exp(\tan(t)),\ c \neq 0$, and singularities in $(n + \frac{1}{2})\pi,\ n \in \mathbb{Z}$, where the system explodes. For such systems stability of any kind cannot be defined. The singularities lie in the infinite discrete set D of zeros of $\cos(t)$. In [14], [22] these singularities are omitted from the time domain, i.e., the signals are considered as smooth functions on $\mathbb{R} \setminus D$. The mathematical problems with the singularities are thus circumvented, but not the engineering ones because a time axis with infinitely many gaps has no engineering significance. This suggests to use coefficient functions that have no zeros for $t \to \infty$.
(ii) *Difficulties with smooth coefficients*: Differential rings of general smooth functions are, in general, neither integral domains nor noetherian and this is inherited by the associated rings of differential operators. Such rings have only weak and unconstructive algebraic properties. In particular, they do not admit a module-behavior duality and algebraic algorithms for the solution of systems theoretic problems. This sug-

gests to choose coefficient rings of analytic functions. The best algebraic properties are obtained if the coefficients form a differential field and the associated ring of differential operators is a principal ideal domain. The field of meromorphic functions [13], [22],[14] is unsuitable due to (i).
(iii) *Puiseux series*: It turns out that the field $\mathbf{K}$ of locally convergent Puiseux series and the derived coefficient functions have all required properties. Moreover there is a substantial *Algebraic theory of differential equations* [16], [20] that is used to derive the algebraic characterization of exponential stability in this paper. Similar coefficient domains were already considered in cf. [4, §§5.4, 6.2, 6.3].
(iv) *Constructivity*: Due to, for instance, [20, Ch. 4] and [6] the algebraic derivations of this paper are constructive if one replaces the base field $\mathbb{C}$ by the field $\mathbb{Q}(i)$ as always in numerical computations. We hope that experts in *Computer Algebra* will implement our results and algorithms.
(v) *Lyapunov theory*: This is implicitly used in the proof of Thm. 5.7 by means of the quoted Result 5.6 from [21, Thms. 7.4, 8.6] and explicitly in the proof of Thm. 5.8.
(vi) *Difference to [4]*: In this paper we do not and do not have to use the product decomposition of a differential operator in $\mathbf{A}$ into linear factors. This does not always exist. In analogy to [5] we also use new behaviors that enable the definition of weak exponential stability in generalization of the standard uniform exponential stability [21, Def. 6.5].

The Sections 3, 4 resp. 5 are devoted to the proofs of the main Thms. 2.3, 2.7 resp. 2.8. The proofs in Sections 3 and 4 are analogous to those in the discrete case [5] and only their essentially different parts are carried out in detail. In Section 3.6 we also define and characterize autonomy of a behavior as usual.
Continuous LTV systems and their stability from the engineering point of view have been treated in the books [21] and [4] and, for instance, in the papers [13], [9], [22], [14], [15], [19], [11], [1], [2].
Notations and abbreviations: $\mathbb{C}_{+(-)} := \{z \in \mathbb{C};\ \Re(z) > (<) 0\}$, e.s.= exponentially stable, f.d.=finite-dimensional, f.g.=finitely generated, p.g.f.= polynomial growth function, resp.=respectively, $\operatorname{spec}(A_0)$:=set of eigenvalues of a square complex matrix A_0, u.e.s.= uniformly e.s., w.e.s.= weakly e.s., w.l.o.g.=without loss of generality, $X^{p\times q}$=set of $p \times q$-matrices with entries in X, $X^{1\times q}$=rows, $X^q := X^{q\times 1}$=columns , $X^{\bullet\times\bullet} := \bigcup_{p,q\geq 0} X^{p\times q}$

2 Exposition of the main results

In this Section we give sufficient details for the main results in order that these can be understood without the proofs in the following sections.

Any linear time-varying continuous systems theory requires the choice of several data: the time axis, the algebra $\mathbf{K}$ of *coefficient functions*, the associated *algebra* $\mathbf{A}$ *of differential operators* and its finitely generated (f.g.) modules, the module of signals and the associated solution spaces of linear differential systems, called *behaviors*. The algebraic properties of these data are determined by those of $\mathbf{K}$ and the signal space. In this paper we make the following choices:
As time axes of variable length we use the open intervals $(\tau, \infty) := \{t \in \mathbb{R};\ t > \tau\}$ with $\tau \geq 0$. Since we are going to study the behavior of trajectories $w(t)$ for $t \to \infty$ the restriction to $\tau \geq 0$ is no loss of generality. The consideration of different initial times τ is required by the time-variance of the systems. As signal spaces on (τ, ∞) we

take the $\mathbb{C}$-spaces

$$W(\tau) = \mathrm{C}^\infty(\tau, \infty) \text{ or } W(\tau) := \mathcal{D}'(\tau, \infty) \tag{1}$$

of complex-valued smooth functions or distributions on the interval.
As coefficient field we choose the algebraic closure $\mathbf{K} := \bigcup_{m \geq 1} \mathbb{C} << z^{1/m} >>$ of the field $\mathbb{C} << z >>$ of locally convergent Laurent series in the variable z where $\mathbb{C} << z^{1/m} >>$ is the field of Laurent series in the variable $z^{1/m}$, cf. Section 3.1 and Result 3.2. The nonzero elements of $\mathbb{C} << z^{1/m} >$ have the form

$$\begin{gathered} f = a(z^{1/m}) = \sum_{i=k}^{\infty} a_i z^{i/m},\ a = \sum_{i=k}^{\infty} a_i z^i \in \mathbb{C} << z >>, \\ \text{with } k \in \mathbb{Z},\ \ a_k \neq 0,\ \sigma(a) := \left(\limsup_{i \geq 0} \sqrt[i]{|a_i|} \right) < \infty. \end{gathered} \tag{2}$$

By standard complex variable theory the number $\rho(a) := \sigma(a)^{-1}$ is the convergence radius of a and the function $a(z)$ is holomorphic in the annulus

$$\{z \in \mathbb{C};\ 0 < |z| < \rho(a)\} \tag{3}$$

and hence defines the smooth function

$$f(t) := a(t^{-1/m}) := \sum_{i=k}^{\infty} a_i t^{-i/m} \in \mathrm{C}^\infty(\sigma(f), \infty),\ f = a(z^{1/m}),\ \sigma(f) := \sigma(a)^m. \tag{4}$$

Notice that for $f = a(z^{1/m}) \in \mathbf{K}$ and $t > \sigma(f)$ we write $f(t) = a(t^{-1/m})$ and *not* $f(t^{-1})$ or $f(t^{-1/m})$. The function $a(z)$ is holomorphic in 0 too if and only if $k \geq 0$ or $a(z)$ is a locally convergent *power series*, and then $f(t) = a(t^{-1/m})$ is bounded on each closed interval $[\tau, \infty),\ \tau > \sigma(f)$. These functions $f(t)$ are the coefficient functions in our differential systems and are defined only on the interval $(\sigma(f), \infty)$ depending on f. Examples for such coefficient functions are rational functions

$$f(t) \in \mathbb{C}(t) = \mathbb{C}(t^{-1}),\ t^{1/3}\cos(t^{-1/2}),\ t^2 \exp(t^{-1}),\ \text{but not } \cos(t). \tag{5}$$

The field $\mathbf{K}$ is a *differential field* and equipped with the standard $\mathbb{C}$-linear derivation

$$\begin{gathered} d/dz : \mathbf{K} \to \mathbf{K},\ a(z^{1/m}) \to da(z^{1/m})/dz = m^{-1} z^{\frac{1}{m}-1} a'(z^{1/m}) \text{ where} \\ (-)' : \mathbb{C} << z >> \to \mathbb{C} << z >>,\ a = \sum_{i=k}^{\infty} a_i z^i \mapsto a'(z) := \sum_{i=k}^{\infty} i a_i z^{i-1}. \end{gathered} \tag{6}$$

(cf. (58)) and gives rise to the noncommutative skew-polynomial $\mathbb{C}$-algebra of differential operators [17, §1.2], cf. Section 3.2,

$$\begin{gathered} \mathbf{A} := \mathbf{K}[\partial; d/dz] = \oplus_{j=0}^{\infty} \mathbf{K}\partial^j \ni f = \sum_{j=0}^{\infty} f_j \partial^j \\ \text{with the multiplication for } a, b \in \mathbf{K},\ i, j \in \mathbb{N} : \\ (a\partial^i)(b\partial^j) = \sum_{k=0}^{i} \binom{i}{k} a \frac{d^{i-k} b}{dz^{i-k}} \partial^{k+j},\ \partial b = b\partial + db/dz, \end{gathered} \tag{7}$$

By definition almost all, i.e., up to finitely many, coefficients $f_j \in \mathbf{K}$ of f are zero. Notice that $d/dz : \mathbf{K} \to \mathbf{K}$ is a map whereas ∂ denotes an indeterminate. The algebraic properties of $\mathbf{A}$ and its f.g. modules are well-known: It is a left and right principal ideal domain, hence noetherian, [17, Th. 1.2.9, §5.7] and simple (cf. Lemma 3.5), i.e., 0 and the whole ring are its only two-sided ideals. The module structure will be used to study that of the associated behaviors. For every nonzero $a \in \mathbf{K}$ also $a(z)d/dz : \mathbf{K} \to \mathbf{K},\ b \mapsto adb/dz$, is a derivation and therefore

$$\begin{aligned} \mathbf{A} = \mathbf{K}[\partial; d/dz] = \mathbf{K}[a\partial; a(z)d/dz] = \oplus_{j\in\mathbb{N}}\mathbf{K}(a\partial)^j, \text{ especially} \\ \mathbf{A} = \mathbf{K}[\partial; d/dz] = \mathbf{K}[z\partial; zd/dz] = \mathbf{K}[-z^2\partial; -z^2d/dz] = \oplus_{j\in\mathbb{N}}\mathbf{K}(-z^2\partial)^j. \end{aligned} \tag{8}$$

For

$$\begin{aligned} & f = \sum_j f_j\partial^j \in \mathbf{A} \text{ and } R = (R_{\mu\nu})_{\mu\nu} = \sum_j R_j\partial^j \in \mathbf{A}^{p\times q} \text{ define} \\ & \sigma(f) := \max\{\sigma(f_j);\ j \in \mathbb{N}\},\ \sigma(R) := \max\{\sigma(R_{\mu\nu});\ \mu \le p, \nu \le q\} \\ & \Longrightarrow \forall j \in \mathbb{N}:\ f_j(t) \in \mathrm{C}^\infty(\sigma(f),\infty),\ R_j(t) \in \mathrm{C}^\infty(\sigma(R),\infty)^{p\times q}. \end{aligned} \tag{9}$$

For every $\tau \ge 0$ we obtain the subalgebras

$$\begin{array}{ccc} \mathbf{K}(\tau) := \left\{f = a(z^{1/m}) \in \mathbf{K};\ \tau \ge \sigma(f) = \sigma(a)^m\right\} & \subset & \mathbf{A}(\tau) = \mathbf{K}(\tau)[\partial; d/dz] \\ \bigcap & & \bigcap \\ \mathbf{K} = \bigcup_{\tau'\ge 0}\mathbf{K}(\tau') & \subset & \mathbf{A} = \bigcup_{\tau'\ge 0}\mathbf{A}(\tau') \end{array}. \tag{10}$$

Likewise the derivative

$$d/dt : \mathbf{C}(\tau) := \mathrm{C}^\infty(\tau,\infty) \to \mathbf{C}(\tau),\ g \mapsto g' = dg/dt, \tag{11}$$

is a derivation and gives rise to the skew-polynomial algebra of differential operators resp. the standard module action

$$\begin{aligned} \mathbf{B}(\tau) := \mathbf{C}(\tau)[\partial_t; d/dt] = \oplus_{j=0}^\infty\mathbf{C}(\tau)\partial_t^j \text{ resp. } \circ:\ \mathbf{B}(\tau) \times W(\tau) \to W(\tau) \text{ with} \\ \partial_t \circ w = w',\ (g\circ w)(t) = g(t)w(t),\ g \in \mathbf{C}(\tau),\ w \in W(\tau). \end{aligned} \tag{12}$$

Again ∂_t is an indeterminate. As indicated in the Introduction the algebraic properties of the algebra $\mathbf{B}(\tau)$, its f.g. modules and the signal module ${}_{\mathbf{B}(\tau)}W(\tau)$ are weak. Therefore we replace $\mathbf{B}(\tau)$ by $\mathbf{A}(\tau)$ as follows: According to Lemma 3.4 and (69) below the map

$$\begin{array}{cccc} \Phi: & \mathbf{A}(\tau) = \mathbf{K}(\tau)[-z^2\partial; -z^2d/dz] & \to & \mathbf{B}(\tau) = \mathbf{C}(\tau)[\partial_t; d/dt] \\ & \sum_{j=0}^\infty a_j(z^{1/m})(-z^2\partial)^j & \mapsto & \sum_{j=0}^\infty a_j(t^{-1/m})\partial_t^j, \\ & \Phi(-z^2\partial) = \partial_t,\ \Phi(a(z^{1/m})) = a(t^{-1/m}) & & \end{array} \tag{13}$$

is an algebra monomorphism and induces the action

$$\begin{aligned}
&\mathbf{A}(\tau)\times W(\tau)\to W(\tau),\ (f,w)\mapsto f\circ w:=\Phi(f)\circ w,\ \text{for } t>\tau,\\
&f=\sum_{j=0}^{\infty}a_j(z^{1/m})(-z^2\partial)^j\in\mathbf{A}(\tau)=\mathbf{K}(\tau)[-z^2\partial;-z^2d/dz],\ w\in W(\tau)\ \text{with}\\
&(-z^2\partial)\circ w=\partial_t\circ w=w',\ (-z^2\partial)^j\circ w=w^{(j)}=d^jw/dt^j,\\
&(a(z^{1/m})\circ w)(t)=a(t^{-1/m})w(t),\ (\partial\circ w)(t)=-t^2w',\\
&(f\circ w)(t)=\sum_{j=0}^{\infty}a_j(t^{-1/m})w^{(j)},
\end{aligned} \tag{14}$$

that makes $W(\tau)$ a left $\mathbf{A}(\tau)$-module.

Remark 2.1. The simpler map

$$\begin{aligned}
\Phi_1:\mathbf{A}(\tau)=\mathbf{K}(\tau)[\partial;d/dz]\to\mathbf{B}(\tau),\ &\sum_{j=0}^{\infty}a_j(z^{1/m})\partial^j\mapsto\sum_{j=0}^{\infty}a_j(t^{-1/m})\partial_t^j,\\
\text{with } \Phi_1(\partial)=\partial_t,\ &\Phi_1(a(z^{1/m}))=a(t^{-1/m}),
\end{aligned} \tag{15}$$

is $\mathbb{C}$-linear, but not an algebra homomorphism since, for instance,

$$\Phi_1(\partial z)=\Phi_1(z\partial+1)=t^{-1}\partial_t+1\neq t^{-1}\partial_t-t^{-2}=\partial_t t^{-1}=\Phi_1(\partial)\Phi_1(z).$$

The action $f\circ_1 w:=\Phi_1(f)\circ w$ with $\partial\circ_1 w=w'$ can be defined, but does not make $W(\tau)$ an $\mathbf{A}(\tau)$-left module and is indeed useless.

More generally, a matrix $R=\sum_j R_j(-z^2\partial)^j\in\mathbf{A}(\tau)^{p\times q}\subset\mathbf{A}^{p\times q}$ acts on a column vector $w\in W(\tau)^q$ via

$$R\circ w:=\sum_j R_j(t)w^{(j)},\ R=\sum_j R_j(-z^2\partial)^j\in\mathbf{A}(\tau)^{p\times q} \tag{16}$$

and gives rise to the solution spaces or *behaviors*

$$\begin{aligned}
\mathcal{B}(R,\tau)&:=\{w\in W(\tau)^q;\ R\circ w=0\}\\
&=\left\{w\in W(\tau)^q;\ \sum_j R_j(t)w^{(j)}=0\right\},\ \tau\geq\sigma(R).
\end{aligned} \tag{17}$$

Since $\mathbf{A}$ is not commutative the behavior $\mathcal{B}(R,\tau)$ is only a $\mathbb{C}$-space and not an $\mathbf{A}(\tau)$-module.
The dependence of the admissible τ on the defining matrix R suggests to consider behavior families $(\mathcal{B}(R,\tau))_{\tau\geq\tau_0},\ \tau_0\geq\sigma(R)$, especially $(\mathcal{B}(R,\tau))_{\tau\geq\sigma(R)}$. For the comparison of different such families we introduce the equivalence relation and equivalence classes

$$\begin{aligned}
&\forall R_k\in\mathbf{A}^{p_k\times q},\ \tau_k\geq\sigma(R_k),\ k=1,2:\\
&(\mathcal{B}(R_1,\tau))_{\tau\geq\tau_1}\equiv\ \text{(equivalent)}\ (\mathcal{B}(R_2,\tau))_{\tau\geq\tau_2}\\
&:\iff\exists\tau_3\geq\max(\tau_1,\tau_2)\forall\tau\geq\tau_3:\ \mathcal{B}(R_1,\tau)=\mathcal{B}(R_2,\tau),\\
&\mathrm{cl}\left((\mathcal{B}(R),\tau)_{\tau\geq\sigma(R)}\right):=\text{equivalence class of } (\mathcal{B}(R,\tau))_{\tau\geq\sigma(R)}.
\end{aligned} \tag{18}$$

To study the equivalence classes means to study the behaviors $\mathcal{B}(R_1,\tau)$ for large $\tau \geq \tau_3$, the transient behavior in the interval $(\sigma(R_1),\tau_3)$ is neglected. This is appropriate for stability questions where the properties of the trajectories $w(t) \in \mathcal{B}(R_1,\tau)$ for $t \to \infty$ are investigated. If $U \subseteq \mathbf{A}^{1\times q}$ is any (always f.g.) submodule and generated by the rows of a matrix $R \in \mathbf{A}^{p\times q}$, i.e., $U = \mathbf{A}^{1\times p}R$, then

$$\mathcal{B}(U) := \mathrm{cl}\left((\mathcal{B}(R,\tau))_{\tau\geq\sigma(R)}\right) \tag{19}$$

depends on U only and not on the special choice of R, cf. Lemma 3.7, and is called the *behavior associated to* U. Note that $W(\tau)$ is an $\mathbf{A}(\tau)$-, but not an $\mathbf{A}$-module and that $\mathcal{B}(U)$ is not isomorphic to $\mathrm{Hom}_{\mathbf{A}}\left(\mathbf{A}^{1\times q}/U, W\right)$ for any natural $\mathbf{A}$-signal module W. But in Section 3.3 we construct an abelian category $\mathfrak{B}$ that contains the objects $\mathcal{B}(U)$ and also suitable morphisms between these behaviors. The category of the objects $\mathcal{B}(U)$ and the behavior morphisms is the abelian *category* $\mathcal{B}eh$ *of behaviors*.

Example 2.2. We have to use the linear *state space systems* [21, p. 13]

$$\begin{aligned} &x' = F(t)x + u,\ F = A(z^{1/m}) \in \mathbf{K}^{n\times n},\ x,u \in W(\tau)^n \text{ where} \\ &A \in \mathbb{C} << z >>^{n\times n},\ F(t) = A(t^{-1/m}),\ \tau \geq \sigma(F) = \sigma(A)^m. \end{aligned} \tag{20}$$

The equation $x'(t) = F(t)x(t)$ with given initial condition $x(t_0),\ t_0 > \tau$, has the unique smooth solution

$$\begin{aligned} &x(t) = \Phi(t,t_0)x(t_0) \in \mathrm{C}^\infty(\tau,\infty)^n,\ \Phi(-,t_0) \in \mathrm{Gl}_n\left(\mathrm{C}^\infty(\tau,\infty)\right) \text{ where} \\ &\frac{d\Phi(t,t_0)}{dt} = F(t)\Phi(t,t_0),\ \Phi(t_0,t_0) = \mathrm{id}_n,\ \Phi(t,t_0)^{-1} = \Phi(t_0,t), \end{aligned} \tag{21}$$

and $\Phi(t,t_0)$ is called the *transition matrix* [21, Thm. 3.3]. There results the $\mathbb{C}$-isomorphism

$$\mathcal{B}(-z^2\partial\,\mathrm{id}_n - F,\tau) = \{x \in W(\tau)^n;\ x'(t) = F(t)x(t)\} \cong \mathbb{C}^n,\ x \mapsto x(t_0), \tag{22}$$

where $x(t) = \Phi(t,t_0)x(t_0)$. For $\tau_1 \geq \tau_0$ the isomorphism (22) induces the *restriction isomorphism*

$$\mathrm{res}: \mathcal{B}(-z^2\partial\,\mathrm{id}_n - F,\tau_0) \to \mathcal{B}(-z^2\partial\,\mathrm{id}_n - F,\tau_1),\ x \mapsto x|_{(\tau_1,\infty)}. \tag{23}$$

The distributional solutions of $x' = F(t)x + u,\ x,u \in \mathcal{D}'(\tau,\infty)$, are

$$x = \Phi(t,t_0)x_1 \text{ with } x_1' = \Phi(t,t_0)^{-1}u = \Phi(t_0,t)u. \tag{24}$$

If u is continuous the solution x is given by

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^t \Phi(t,s)u(s)ds. \tag{25}$$

The duality between f.g. modules and behaviors gets the following form: Let ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ be the abelian category of f.g. $\mathbf{A}$-modules M with a *given finite system of generators* or, equivalently, a *given representation* $M = \mathbf{A}^{1\times q}/U$ as factor of a free module $\mathbf{A}^{1\times q}$ by a submodule U. The morphisms of ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ are just the $\mathbf{A}$-linear maps. Fliess [8] calls a *module* M with the additional structure $M = \mathbf{A}^{1\times q}/U$ a *linear dynamic* or *LTV system*. In analogy to the discrete case [5, Cor. 2.7] we extend the assignment $M = \mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U)$ to a contravariant functor

$$
{}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}} \to \mathcal{B}eh, \begin{cases} M = \mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U) \\ \left(\varphi : \mathbf{A}^{1\times q_1}/U_1 \to \mathbf{A}^{1\times q_2}/U_2\right) \mapsto (\mathcal{B}(\varphi) : \mathcal{B}(U_2) \to \mathcal{B}(U_1)) \end{cases} . \tag{26}
$$

The first main theorem of this paper is

Theorem 2.3. *(cf. [5, Thm. 1.3], Sections 3.4, 3.5) The functor* (26) *is a duality (contravariant equivalence). More precisely the following properties hold:*

1. *It transforms exact sequences of modules into exact sequences of behaviors.*
2. *For all* $\mathbf{A}^{1\times q_1}/U_1, \mathbf{A}^{1\times q_2}/U_2 \in \mathbf{Mod}_{\mathbf{A}}^{\mathrm{fg}}$ *there is the* $\mathbb{C}$*-linear isomorphism*
$$
\mathrm{Hom}_{\mathbf{A}}(\mathbf{A}^{1\times q_1}/U_1, \mathbf{A}^{1\times q_2}/U_2) \cong \mathrm{Hom}_{\mathcal{B}eh}(\mathcal{B}(U_2), \mathcal{B}(U_1)),\ \varphi \mapsto \mathcal{B}(\varphi). \tag{27}
$$
3. *For all* $U_1, U_2 \subseteq \mathbf{A}^{1\times q}$*:*
$$
U_1 \subseteq U_2 \iff \mathcal{B}(U_2) \subseteq \mathcal{B}(U_1),\ \textit{especially}\ U_1 = U_2 \iff \mathcal{B}(U_2) = \mathcal{B}(U_1). \tag{28}
$$

We define *weak exponential stability* (w.e.s.) of $\mathcal{B}(U)$ from (19): For $\tau \geq 0$ a function $\varphi \in \mathbb{C}^{[\tau,\infty)}$ on the closed interval $[\tau, \infty)$ is called a function of *at most polynomial growth* (p.g.f.) if

$$
\exists c \geq 1 \exists p \in \mathbb{N} \forall t \geq \tau :\ |\varphi(t)| \leq ct^p. \tag{29}
$$

Every coefficient function (3)

$$
\begin{aligned} f(t) = a(t^{-1/m}) = t^{-k/m}u(t^{-1/m}) \text{ where} \\ f = a(z^{1/m}) \in \mathbf{K},\ a = z^k u \in \mathbb{C} << z >>,\ u \in \mathbb{C} < z >, \end{aligned} \tag{30}
$$

is a p.g.f. on every closed interval $[\tau, \infty)$, $\tau > \sigma(f) = \sigma(a)^m = \sigma(u)^m$ since $\tau^{-1/m} < \rho(u) = \rho(a) = \sigma(a)^{-1}$ and $u(z)$ is continuous and bounded in the compact disc $\left\{z \in \mathbb{C};\ |z| \leq \tau^{-1/m}\right\}$. *The p.g.f. property of the coefficient functions is essential for the derivations of this paper.* We call the p.g.f. φ positive, $\varphi > 0$, if $\varphi(t) > 0$ for all $t \geq \tau$.

On all finite-dimensional vector spaces $\mathbb{C}^q$ we use the maximum norm

$$
\begin{aligned} &\forall v = (v_1, \cdots, v_q)^\top \in \mathbb{C}^q :\ \|v\| := \max\{|v_i|;\ i = 1, \cdots, q\}, \\ &\forall A \in \mathbb{C}^{q\times q} :\ \|A\| := \max\{\|Av\|;\ v \in \mathbb{C}^q,\ \|v\| = 1\} \\ &\Longrightarrow \forall v \in \mathbb{C}^q :\ \|Av\| \leq \|A\|\|v\|. \end{aligned} \tag{31}
$$

In the following considerations $W(\tau) = \mathrm{C}^\infty(\tau, \infty)$ is the space of smooth functions so that for $w \in W(\tau)^q$ and $t \in (\tau, \infty)$ the norm $\|w(t)\|$ is defined. We define

$$
\begin{aligned} \partial(m) = (1, -z^2\partial, (-z^2\partial)^2, \cdots, (-z^2\partial)^{m-1})^\top \in \mathbf{A}^m,\ m \in \mathbb{N}, \\ (\partial(m) \circ w)(t) = (w(t), w'(t), \cdots, w^{(m-1)}(t))^\top \in (W(\tau)^q)^m,\ w \in W(\tau)^q. \end{aligned} \tag{32}
$$

Definition 2.4. The behavior $\mathcal{B}(U) = \mathrm{cl}\left((\mathcal{B}(R,\tau))_{\tau\geq\tau_0}\right)$ from (19) is called *weakly exponentially stable* (w.e.s.) if

$$
\begin{aligned} &\exists \tau_1 \geq \tau_0 \exists d \in \mathbb{N} \exists \alpha, \mu > 0 \forall m \in \mathbb{N} \exists \text{ p.g.f. } \varphi_m \in \mathbb{C}^{[\tau_1,\infty)} \text{ with } \varphi_m > 0 \\ &\qquad \forall t \geq t_0 > \tau \geq \tau_1 \forall w \in \mathcal{B}(R,\tau) : \\ &\|w^{(m)}(t)\| \leq \varphi_m(t_0) \exp\left(-\alpha(t^\mu - t_0^\mu)\right) \|(\partial(d) \circ w)(t_0)\|. \end{aligned} \tag{33}
$$

The condition in (33) can be equivalently replaced by a different p.g.f. $\psi_m > 0$ and

$$\|(\partial(m) \circ w)(t)\| \leq \psi_m(t_0) \exp\left(-\alpha(t^\mu - t_0^\mu)\right) \|(\partial(d) \circ w)(t_0)\|. \tag{34}$$

The behavior is called *exponentially stable* (e.s.) if (33) and (34) hold with $\mu = 1$. State space equations $x'(t) = F(t)x(t)$ with continuous $F(t) \in \mathrm{C}^0(\tau, \infty)^{n \times n}$ are called *uniformly exponentially stable* (u.e.s.) [21, Def. 6.5] if there are $c \geq 1$ and $\alpha > 0$ such that

$$\forall t \geq t_0 > \tau \forall x \in \mathrm{C}^1(\tau, \infty)^n \text{ with } x' = Fx : \ \|x(t)\| \leq ce^{-\alpha(t-t_0)}\|x(t_0)\|. \tag{35}$$

A w.e.s. behavior is, of course, *asymptotically stable* in the sense that for all trajectories $w \in \mathcal{B}(R, \tau)$ and all m the limit $\lim_{t\to\infty} w^{(m)}(t)$ exists and is zero. It is always autonomous (see Section 3.6), but the trajectories w are not uniquely determined by $w(t_0)$ alone, but only by the *initial vector*

$$(\partial(d) \circ w)(t_0) = (w(t_0), w'(t_0), \cdots, w^{(d-1)}(t_0))^\top \tag{36}$$

with fixed d. The number d is not unique and called a *memory-size* of the autonomous behavior. The *initial time* $t_0 > \tau$ in addition to τ is needed because τ does not belong to the open interval (τ, ∞) on which w is defined.
Recall that a behavior is called *stable* if its trajectories are bounded for $t \to \infty$.

Remark 2.5. Note that w.e.s., e.s. and u.e.s. are defined by *analytic properties of the trajectories* of the system and *not algebraically*. The w.e.s. differs from u.e.s. in the following aspects:

1. The factor $\varphi_m(t_0)$ is not constant, but a p.g.f. of the initial time t_0.
2. The trajectories decrease according to the exponential factor $\exp(-\alpha t^\mu)$ where $\mu > 0$ is any positive real number. According to whether $\mu < 1$ or $\mu > 1$ the decay is slower or faster than the usual exponential decay with the factor $\exp(-\alpha t)$.
3. The exponential decay of all derivatives of the trajectories is required.
4. The initial condition is $(\partial(d) \circ w)(t_0)$ and not $w(t_0)$.
5. All conditions are required for sufficiently large $\tau \geq \tau_1$ and not for $\tau = \tau_1 = 0$.
6. W.e.s. and e.s. are preserved by behavior isomorphisms (cf. Lemma 4.11). This implies that for any f.g. module $M = \mathbf{A}^{1\times q}/U$ the w.e.s. or e.s. of $\mathcal{B}(U)$ depends on M only and not on the special representation $M = \mathbf{A}^{1\times q}/U$. In contrast we show in Example 4.10 that *stability* and *u.e.s.* are not preserved by general behavior isomorphisms, compare [21, p. 107].

Definition 2.6. A f.g. $\mathbf{A}$-left module M is called w.e.s. resp. e.s. if for one (or all) representation(s) $M = \mathbf{A}^{1\times q}/U$ the behavior $\mathcal{B}(U)$ is w.e.s. resp. e.s..

U.e.s. state space systems are also e.s.. In the simplest case of state space equations $w' = Aw$ with a *constant* matrix $A \in \mathbb{C}^{n\times n}$ w.e.s. of the corresponding behavior signifies that A is *asymptotically stable*, i.e., $\mathrm{spec}(A) \subset \mathbb{C}_-$.

Theorem 2.7. *The weakly exponentially stable f.g.* $\mathbf{A}$*-modules and hence also the w.e.s. behaviors form a Serre category. This signifies that for an exact sequence of f.g.* $\mathbf{A}$*-left modules* $0 \to M' \to M \to M'' \to 0$ *the module* M *is w.e.s. if and only if* M' *and* M'' *are.*
Equivalently [5, Cor. 1.9], the series connection of two input/output (IO) behaviors is w.e.s. if and only if its two building blocks are .
Likewise the exponentially stable (e.s.) f.g. $\mathbf{A}$*-modules form a Serre category.*

Here an IO behavior is called w.e.s. if its autonomous part is w.e.s.. The proof of this result uses analytic tools essentially. Its proof for LTI behaviors is carried out algebraically and is much simpler.
As in the LTI case the behavior $\mathcal{B}(U)$ with $U = \mathbf{A}^{1\times p}R$ and $R \in \mathbf{A}^{p\times q}$ is autonomous if and only $M := \mathbf{A}^{1\times q}/U$ is a torsion module or $\mathrm{rank}(R) = q$ or $\dim_{\mathbf{K}}(M) < \infty$, cf. Lemma and Definition 3.15. This torsion module may be *regular* or *irregular singular* [16, pp. 1-67], [20, Ch. 3], cf. Sections 5.2, 5.3. Let $W(\tau) := \mathrm{C}^{\infty}(\tau, \infty)$ be the $\mathbf{A}(\tau)$-module of smooth signals.

Theorem 2.8. *Assume* $R \in \mathbf{A}^{p\times q}$*,* $\mathrm{rank}(R) = q$*,* $U := \mathbf{A}^{1\times p}R$*,* $M := \mathbf{A}^{1\times q}/U$*,* $d := \dim_{\mathbf{K}}(M) < \infty$ *and the associated autonomous behavior* $\mathcal{B} := \mathcal{B}(U)$*.*
(i) If M *is nonzero and regular singular then* $\mathcal{B}$ *is never weakly exponentially stable.*
(ii) If M *is irregular singular there are* $\tau_1 > \sigma(R)$ *such that for* $\tau \geq \tau_1$

$$\begin{gathered}\mathcal{B}(R,\tau) \cong \mathcal{C}(\tau) := \\ \left\{x \in W(\tau)^d;\ x'(t) + t^{-1-\lambda}\left(A_0 + t^{-1/m}A_1(t^{-1/m})\right)x(t) = 0\right\} \qquad (37) \\ \text{where } 0 > \lambda \in \mathbb{Q},\ 0 < m \in \mathbb{N},\ \ A_0 \in \mathbb{C}^{d\times d},\ A_1 \in \mathbb{C} < z >^{d\times d}\end{gathered}$$

and where $A_1(t^{-1/m})$ *is bounded on* $[\tau_1, \infty)$*.*
(iii) (a) If in (ii) all eigenvalues of A_0 *have positive real parts or, equivalently,* $\mathrm{spec}(A_0) \subset \mathbb{C}_+$*, then there are* $c, \alpha > 0$ *such that for all* $t \geq t_0 > \tau \geq \tau_1$ *and* $x \in \mathcal{C}(\tau)$ *the following inequality holds:*

$$\|x(t)\| \leq c \exp\left(-\alpha(t^{-\lambda} - t_0^{-\lambda})\right)\|x(t_0)\|. \qquad (38)$$

In particular, $\mathcal{B}(U)$ *and thus* M *are weakly exponentially stable.*
(b) If in (ii) at least one eigenvalue of A_0 *has negative real part then* $\mathcal{B}(U)$ *is not w.e.s..*
(c) If in (ii) all eigenvalues of A_0 *have nonnegative real parts and at least one eigenvalue is purely imaginary then* $\mathcal{B}(U)$ *need not be w.e.s., cf. Ex. 5.9.*

Remark 2.9. Define $F(t) := t^{-1-\lambda}\left(A_0 + t^{-1/m}A_1(t^{-1/m})\right)$ in Thm. 2.8. The condition $\mathrm{spec}(A_0) \subset \mathbb{C}_+$ from (iii)(a) implies $\mathrm{spec}(F(t)) \subset \mathbb{C}_+$ for sufficiently large τ and all $t \geq \tau$. But it is well-known that for arbitrary smooth coefficient matrices $F(t)$ this so-called *pointwise-in-time* or *frozen-time* condition is not sufficient for exponential stability of the state space system [21, Ex. 8.1], [4, Ex. 964]. This is no contradiction to Thm. 2.8,(iii), since the matrices $F(t)$ in the theorem have a special form.

Remark 2.10. *Open problems concerning Thm. 2.8*: The characterization of w.e.s. of $\mathcal{B}(U)$ in the case that all eigenvalues of A_0 have nonnegative real parts and at least one is purely imaginary is open.

3 Linear time-varying (LTV) behaviors

3.1 The construction of $\mathbb{K}$

For the readers' convenience we describe the construction of $\mathbf{K}$ *as colimit (see, for instance, the Wikipedia article on Puiseux series) and derive several of its properties from this construction. The algebraically closed field* $\bigcup_{m\geq 1}\mathbb{C}((z^{1/m}))$ *of* formal *Laurent series in* $z^{1/m}$ *is discussed in [20, Ch. 3].*
The ring $\mathbb{C}[[z]]=\mathbb{C}^{\mathbb{N}}$ of formal power series $a=\sum_{i=0}^{\infty}a_iz^i=(a_i)_{i\in\mathbb{N}}$ is a *discrete valuation domain* (DVD), i.e., a local principal ideal domain with the unique maximal ideal $\mathbb{C}[[z]]z$ or, equivalently, the unique prime element z, up to units, and the residue field $\mathbb{C}=\mathbb{C}[[z]]/\mathbb{C}[[z]]z$. The group of units is

$$\mathrm{U}(\mathbb{C}[[z]])=\left\{u=\sum_{i=0}^{\infty}u_iz^i\in\mathbb{C}[[z]];\ u_0\neq 0\right\}. \tag{39}$$

Its quotient field is the field $\mathbb{C}((z)):=\mathrm{quot}(\mathbb{C}[[z]])=\mathbb{C}[[z]][z^{-1}]$ of *formal Laurent series*. Its nonzero elements have the unique form

$$a=\sum_{i=k}^{\infty}a_iz^i=z^ku,\ k\in\mathbb{Z},\ u=\sum_{i=0}^{\infty}u_iz^i\in\mathrm{U}(\mathbb{C}[[z]]),\ u_i=a_{k+i},\ u_0=a_k\neq 0.$$

$$\text{Define } v:\mathbb{C}((z))\to\mathbb{Z}\uplus\{\infty\},\ a\mapsto v(a)=\begin{cases}k & \text{if } a\neq 0\\ \infty & \text{if } a=0\end{cases}$$
$$\Longrightarrow \mathbb{C}[[z]]=\{a\in\mathbb{C}((z));\ v(a)\geq 0\}\supset\mathbb{C}[[z]]z=\{a\in\mathbb{C}((z));\ v(a)>0\}. \tag{40}$$

The function v is the associated *discrete valuation* [3, §VI.3]. The ring of *locally convergent* power series is the subdomain

$$\begin{aligned}\mathbf{L}_1&:=\mathbb{C}<z>:=\left\{a=\sum_{i=0}^{\infty}a_iz^i\in\mathbb{C}[[z]];\ \sigma(a):=\limsup_{i\geq 0}\sqrt[i]{|a_i|}<\infty\right\}\subset\\ \mathbf{K}_1&:=\mathbb{C}<<z>>=\mathrm{quot}\left(\mathbb{C}<z>\right)=\mathbb{C}<z>[z^{-1}]\\ &=\{0\}\biguplus\left\{a=z^ku\in\mathbb{C}((z)),\ k\in\mathbb{Z},\ u=\sum_{i=0}^{\infty}u_iz^i,\ u_0\neq 0,\ \sigma(a):=\sigma(u)<\infty\right\}.\end{aligned} \tag{41}$$

Then $\mathbf{L}_1$ is also a DVD and $v|_{\mathbb{C}<<z>>}$ is a discrete valuation with the analogous properties.

Remark 3.1. (*Colimit*) Let $(I,\leq)$ be a directed ordered set, i.e., for $i,j\in I$ there is a $k\in I$ with $k\geq i,j$. A directed system of sets over I is a family $S=(S_i,\varphi_{ji}:S_i\to S_j)_{j\geq i\in I}$ of sets S_i and maps φ_{ji} such that $\varphi_{ii}=\mathrm{id}_{S_i}$ for $i\in I$ and $\varphi_{ki}=\varphi_{kj}\varphi_{ji}$ for $k\geq j\geq i$. The *colimit* or *direct limit* of S is the set

$$\begin{aligned}&\mathrm{colim}_I\,S=\mathrm{colim}_{i\in I}\,S_i:=C:=\{(i,s_i);\ i\in I,\ s_i\in S_i\}/\equiv\ \ni\mathrm{cl}(i,s_i)\text{ where}\\ &(i,s_i)\equiv(j,s_j):\Longleftrightarrow\ \exists k\geq i,j:\ \varphi_{ki}(s_i)=\varphi_{kj}(s_j)\end{aligned} \tag{42}$$

is an equivalence relation $\equiv$ and $\mathrm{cl}(i,s_i)$ denotes the equivalence class. There are canonical maps

$$\forall i\in I:\ \varphi_i:S_i\to C,\ s_i\mapsto\mathrm{cl}(i,s_i),\ \text{with } \forall j\geq i:\ \varphi_j\varphi_{ji}=\varphi_i. \tag{43}$$

with the following *universal property*: If maps $\psi_i : S_i \to C',\ i \in I$, satisfy $\psi_j\varphi_{ji} = \psi_i$ for $j \geq i$ then there is a unique map $\psi : C \to C'$ with $\psi_i = \psi\varphi_i$ for all i, viz. $\psi(\mathrm{cl}(i, s_i)) = \psi_i(s_i)$. If the maps φ_{ji} are injective then so are the φ_i. In this case we obtain bijections $\varphi_i : S_i \cong \varphi_i(S_i)$, $\varphi_i(S_i) \subseteq \varphi_j(S_j)$ for $j \geq i$ and $C = \bigcup_{i\in I}\varphi_i(S_i)$. If $S = (S_i, \varphi_{ji})_{j\geq i\in I}$ and $S' = (S_i, \varphi'_{ji})_{j\geq i\in I}$ are two directed systems a morphism from S to S' is a family $\psi = (\psi_i)_{i\in I}$ of maps $\psi_i : S_i \to S'_i$ such that $\psi_j\varphi_{ji} = \varphi'_{ji}\psi_i$ for all $j \geq i$. With the componentwise composition of these morphisms the directed systems form the category $\mathcal{S}et^I$ where $\mathcal{S}et$ is the category of sets. Via the universal property of colim_I the morphism ψ induces the map $\mathrm{colim}_I\,\psi = \mathrm{colim}_{i\in I}\,\psi_i$:

$$\mathrm{colim}_I\,\psi :\ \mathrm{colim}_I\,S \to \mathrm{colim}_I\,S',\ \mathrm{cl}(i, s_i) \mapsto \mathrm{cl}(i, \psi_i(s_i)). \tag{44}$$

and then the covariant functor

$$\mathrm{colim}_I : \mathcal{S}et^I \to \mathcal{S}et,\ S \mapsto \mathrm{colim}_I\,S,\ \psi \mapsto \mathrm{colim}_I\,\psi. \tag{45}$$

The preceding assertions hold likewise for categories of sets with an algebraic structure, for instance for ${}_{\mathbb{C}}\mathbf{Mod}^I$. The category ${}_{\mathbb{C}}\mathbf{Mod}^I$ is abelian where the kernel, cokernel, image etc. are formed componentwise, for instance

$$\begin{aligned} \ker\left(\psi : S \to S'\right) &= \left(\ker(\psi_i), \varphi_{ji}|_{\ker(\psi_i)}\right)_{j\geq i} \\ \mathrm{im}\left(\psi : S \to S'\right) &= \left(\mathrm{im}(\psi_i), \varphi'_{ji}|_{\mathrm{im}(\psi_i)}\right)_{j\geq i}. \end{aligned} \tag{46}$$

The functor $\mathrm{colim}_I :_{\mathbb{C}} \mathbf{Mod}^I \to_{\mathbb{C}} \mathbf{Mod}$ is exact.

We apply the colimit construction to the construction of $\mathbf{K}$. Let $\mathbb{N}^* = \{1, 2, \cdots\}$ be the multiplicative monoid of positive integers. We consider it as an ordered set with the order relation

$$m|n :\Longleftrightarrow m \text{ divides } n \iff \frac{n}{m} \in \mathbb{Z},\ m, n \in \mathbb{N}^*. \tag{47}$$

We define the directed system of DVD $(\mathbf{L}_m; \varphi_{nm})_{m|n\in\mathbb{N}^*}$ and its colimit by

$$\begin{gathered} \mathbf{L}_m := \mathbb{C} < z >,\ \varphi_{nm} : \mathbb{C} < z > \to \mathbb{C} < z >,\ \sum_{i=0}^{\infty} a_i z^i \mapsto \sum_{i=0}^{\infty} a_i z^{in/m}, \\ \varphi_m : \mathbb{C} < z > \to \mathbf{L} := \mathrm{colim}_{m\in\mathbb{N}^*}\,\mathbf{L}_m,\ z_m := \varphi_m(z) = \mathrm{cl}(m, z). \end{gathered} \tag{48}$$

It is obvious that the φ_{nm} and therefore the φ_m are injective $\mathbb{C}$-algebra homomorphisms, hence

$$\begin{gathered} \begin{array}{ccccc} \mathbb{C} < z > & \overset{\varphi_m}{\cong} & \varphi_m(\mathbb{C} < z >), & z \ \mapsto & z_m = \varphi_m(z) \\ a = \sum_i a_i z^i & \mapsto & a(z_m) := \sum_i a_i z_m^i := \varphi_m(a), & & \end{array} \\ \mathbb{C} < z > \cong \mathbb{C} < z_m > = \varphi_m(\mathbb{C} < z >),\ \mathbf{L} = \bigcup_{m\in\mathbb{N}^*} \mathbb{C} < z_m > . \end{gathered} \tag{49}$$

The equations

$$\begin{gathered} \forall m|n : \varphi_{nm}(z) = z^{n/m},\ \varphi_m = \varphi_n\varphi_{nm}, \varphi_m(z) = z_m \text{ imply} \\ z_m = z_n^{n/m},\ z \underset{\text{ident.}}{=} z_1 = z_m^m. \end{gathered} \tag{50}$$

We therefore define and identify

$$\begin{aligned} &z^{1/m} := z_m,\ \mathbb{C} < z > \cong \mathbf{L}_m \underset{\text{ident.}}{=} \mathbb{C} < z^{1/m} >,\ a(z) \mapsto a(z^{1/m}),\\ &\mathbb{C} < z > \underset{\text{ident.}}{=} \mathbb{C} < z_1 > = \mathbf{L}_1 \subset \mathbf{L} = \bigcup_{m \in \mathbb{N}^*} \mathbf{L}_m = \bigcup_{m \in \mathbb{N}^*} \mathbb{C} < z^{1/m} > . \end{aligned} \tag{51}$$

Notice that $z^{1/m}$ is introduced algebraically and interpreted as an indeterminate and *not* as the holomorphic function $\exp\left(m^{-1}\ln(z)\right)$ in the sliced plane. As directed union of the DVD $\mathbf{L}_m = \mathbb{C} < z^{1/m} >$ also $\mathbf{L}$ is a domain and has a quotient field $\mathbf{K} := \operatorname{quot}(\mathbf{L})$. From $\operatorname{quot}(\mathbb{C} < z >) = \mathbb{C} << z >>$ we derive

$$\begin{aligned} &\forall m \in \mathbb{N}^* :\ \mathbf{K}_m := \operatorname{quot}(\mathbb{C} < z^{1/m} >) = \mathbb{C} < z^{1/m} > [(z^{1/m})^{-1}] =: \mathbb{C} << z^{1/m} >>\\ &\qquad \subset \mathbf{K} := \bigcup_{m \in \mathbb{N}^*} \mathbf{K}_m,\\ &\forall m|n :\ \mathbf{K}_m \subseteq \mathbf{K}_n,\ \mathbb{C} << z >> \underset{\text{ident.}}{=} \mathbf{K}_1 \subset \mathbf{K}_m \subset \mathbf{K}. \end{aligned} \tag{52}$$

The nonzero elements of $\mathbf{K}$ thus have the form (2):

$$a(z^{1/m}) = z^{k/m} u(z^{1/m}),\ k \in \mathbb{Z}, a = z^k u \in \mathbb{C} << z >>,\ u \in \mathrm{U}(\mathbb{C} < z >). \tag{53}$$

The discrete valuation v from (40) induces discrete valuations

$$\begin{aligned} &v_m := m^{-1} v : \mathbb{C} << z >> \to \mathbb{Q} \uplus \{\infty\},\ v_1 = v,\ \text{with } \forall m|n \forall k :\\ &\qquad v_m(z^k) = k/m = knm^{-1}/n = v_n(z^{kn/m}) \Longrightarrow v_m = v_n \varphi_{nm}. \end{aligned} \tag{54}$$

Hence the v_m induce the surjective valuation (again denoted by v)

$$\begin{aligned} &v : \mathbf{K} \to \mathbb{Q} \uplus \{\infty\},\ v(z^{1/m}) = v_m(z) = 1/m,\ v\left(z^{k/m} u(z^{1/m})\right) = k/m \Longrightarrow\\ &\mathbf{L} = \{a \in \mathbf{K};\ v(a) \geq 0\} \supset \mathfrak{m}_{\mathbf{L}} := \{a \in \mathbf{K};\ v(a) > 0\} = \bigcup_{m \in \mathbb{N}^*} \mathbb{C} < z^{1/m} > z^{1/m}. \end{aligned} \tag{55}$$

The ring $\mathbf{L}$ is a valuation ring and $\mathfrak{m}_{\mathbf{L}}$ is its unique nonzero prime ideal [3, ch. VI, §4.5, Prop. 6]. Since $\mathbf{L}$ is the directed union of the DVD $\mathbb{C} < z^{1/m} >$ all generators of a f.g. ideal of $\mathbf{L}$ are contained in some DVD $\mathbb{C} < z^{1/m} >$ and therefore the ideal is principal. This signifies that $\mathbf{L}$ is a Bézout domain and integrally closed [3, Ex. VII.1.20]. If M is a f.g. $\mathbf{L}$-module with torsion submodule $\operatorname{tor}(M)$ then $M/\operatorname{tor}(M)$ is free and therefore $\operatorname{tor}(M)$ is a direct summand of M [3, Ex. VII.2.12], [4, Thm. 654]. The direct decomposition

$$\begin{aligned} &\mathbb{C} < z > = \mathbb{C} \oplus \mathbb{C} < z > z \ni a = a_0 + z a_1(z),\ a_0 \in \mathbb{C},\ a_1 \in \mathbb{C} < z >,\ \text{implies}\\ &\mathbf{L} = \mathbb{C} \oplus \mathfrak{m}_{\mathbf{L}} \ni f = a(z^{1/m}) = f_0 + z^{1/m} f_1,\ f_0 := a_0,\ f_1 := a_1(z^{1/m}). \end{aligned} \tag{56}$$

Result 3.2. *([18]) The field* $\mathbf{K} = \bigcup_{m \in \mathbb{N}^*} \mathbb{C} << z^{1/m} >>$ *is the algebraic closure of* $\mathbf{k} = \mathbb{C} << z >>$. *This result is very interesting, but not needed in the present paper.*

The standard $\mathbb{C}$-linear derivation

$$d/dz : \mathbf{K}_1 = \mathbb{C} << z >> \to \mathbb{C} << z >>,\ a = \sum_{i=k}^{\infty} a_i z^i \mapsto a' = \sum_{i=k}^{\infty} a_i i z^{i-1}, \tag{57}$$

can be uniquely extended to a $\mathbb{C}$-linear derivation $d/dz : \mathbf{K} \to \mathbf{K}$ [20, p.5]. Its restriction to $\mathbf{K}_m = \mathbb{C} << z^{1/m} >>$ coincides with the $\mathbb{C}$-linear derivation (cf. (6))

$$\begin{aligned} &\delta_m : \mathbf{K}_m \to \mathbf{K}_m,\ a(z^{1/m}) \mapsto \delta_m\left(a(z^{1/m})\right) := m^{-1} z^{(1/m)-1} a'(z^{1/m}), \\ &\text{since } \forall k \in \mathbb{Z}:\ \delta_m(z^k) = \delta_m((z^{1/m})^{km}) = m^{-1} z^{(1/m)-1} km (z^{1/m})^{km-1} = \\ &\qquad = kz^{k-1} = dz^k/dz. \end{aligned} \tag{58}$$

The differential field $(\mathbf{K}, d/dz)$ is the coefficient field in this paper.
We finally show that the number $\sigma(f),\ f \in \mathbf{K}$, from (4) does not depend on the choice of m with $f \in \mathbb{C} << z^{1/m} >>$. Let indeed $m|n$,

$$\begin{aligned} &a = \sum_i a_i z^i, b = \sum_j b_j z^j \in \mathbb{C} << z >> \text{ and } f = a(z^{1/m}) = b(z^{1/n}) \in \mathbf{K} \\ &\Longrightarrow a(z^{1/m}) = \sum_i a_i z^{inm^{-1}/n} = \sum_j b_j z^{j/n} \Longrightarrow b_j = \begin{cases} a_i & \text{if } j = inm^{-1} \\ 0 & \text{if } j \notin \mathbb{Z}nm^{-1} \end{cases} \\ &\Longrightarrow \limsup_j |b_j|^{j^{-1}} = \limsup_i |a_i|^{(inm^{-1})^{-1}} = \left(\limsup_i |a_i|^{i^{-1}}\right)^{m/n} \\ &\Longrightarrow \sigma(b(z^{1/n})) \underset{(2)}{=} \left(\limsup_j |b_j|^{j^{-1}}\right)^n = \left(\limsup_i |a_i|^{i^{-1}}\right)^m \underset{(2)}{=} \sigma(a(z^{1/m})). \\ &\text{Thus } \forall f \in \mathbf{K} = \bigcup_{m \in \mathbb{N}^*} \mathbb{C} << z^{1/m} >>:\ \sigma(f) := \sigma(b(z^{1/n})) = \sigma(a(z^{1/m})) \end{aligned} \tag{59}$$

is defined independently of the choice of m with $f \in \mathbb{C} << z^{1/m} >>$. In particular, for all $\tau > 0$

$$\mathbf{K}(\tau) := \{f \in \mathbf{K};\ \tau \geq \sigma(f)\} \subset \mathbf{K} \tag{60}$$

is a well-defined subset of $\mathbf{K}$. It is easily seen that $\mathbf{K}(\tau)$ is a differential subalgebra of $(\mathbf{K}, d/dz)$, i.e., $d/dz\mathbf{K}(\tau) \subseteq \mathbf{K}(\tau)$, cf. (10). We thus obtain the algebra monomorphism (3):

$$\mathbf{K}(\tau) \to \mathbf{C}(\tau) = \mathrm{C}^\infty(\tau, \infty),\ f = a(z^{1/m}) \mapsto f(t) := a(t^{-1/m}). \tag{61}$$

Corollary 3.3. *For any nonzero $f = a(z^{1/m}) \in \mathbf{K}$ there is a $\tau_0 > \sigma(f)$ such $f(t^{-1}) = a(t^{-1/m})$ has no zero in $[\tau_0, \infty)$.*

Proof. w.l.o.g. assume $a(z) = \sum_{i=0}^\infty a_i z^i,\ a_0 = a(0) \neq 0$. Since $a(z)$ is holomorphic and thus continuous in 0 there is $\rho_0 > 0$ such that $|a(z)| \geq |a_0|/2$ for $|z| \leq \rho_0$. Then $\tau_0 := \rho_0^{-m}$ furnishes the desired property. □

3.2 Differential operators

To justify the homomorphism (13) *we explain the universal property of rings of differential operators.*
Let (R, δ) be any commutative differential $\mathbb{C}$-algebra with a $\mathbb{C}$-linear derivation $\delta : R \to R$, $\delta(rs) = \delta(r)s + r\delta(s)$. It gives rise to the skew-polynomial algebra [17, p.15]

$$R[\partial; \delta] = \oplus_{j \in \mathbb{N}} R\partial^j \ni f = \sum_j f_j \partial^j \text{ with } \partial r = r\partial + \delta(r). \tag{62}$$

This construction was applied in (7), (8) (10), (12). Unless $\delta = 0$ the algebra $R[\partial;\delta]$ is noncommutative. For $\mathbb{C}$-algebras $\mathbf{A},\mathbf{B}$ let $\mathrm{Al}_{\mathbb{C}}(\mathbf{A},\mathbf{B})$ denote the set of $\mathbb{C}$-algebra homomorphisms from $\mathbf{A}$ to $\mathbf{B}$.

Lemma 3.4. *Consider $R[\partial;\delta]$ and a further $\mathbb{C}$-algebra $\mathbf{B}$. Then there is the canonical bijection*

$$\begin{gathered}\mathrm{Al}_{\mathbb{C}}\left(R[\partial;\delta],\mathbf{B}\right) \cong \left\{(\varphi,\Delta) \in \mathrm{Al}_{\mathbb{C}}\left(R,\mathbf{B}\right)\times\mathbf{B};\ \forall r\in R:\ \Delta\varphi(r) = \varphi(r)\Delta + \varphi(\delta(r))\right\},\\ \Phi \mapsto (\Phi|_R, \Phi(\partial)).\end{gathered} \tag{63}$$

In other terms: If φ and Δ are given and satisfy the equation $\Delta\varphi(r) = \varphi(r)\Delta + \varphi(\delta(r)),\ r\in R$, there is a unique algebra homomorphism
$\Phi : R[\partial;\delta] \to \mathbf{B}$ with $\Phi|R = \varphi$ and $\Phi(\partial) = \Delta$, viz.

$$\Phi : R[\partial;\delta] \to \mathbf{B}:\ f = \sum_j f_j\partial^j \mapsto \Phi(f) = \sum_j \varphi(f_j)\Delta^j. \tag{64}$$

We apply Lemma 3.4 to

$$\begin{aligned}&R[\partial;\delta] := \mathbf{K}(\tau)[-z^2\partial;-z^2d/dz],\ \mathbf{B} := \mathbf{B}(\tau) = \mathbf{C}(\tau)[\partial_t;d/dt],\\ &\varphi : \mathbf{B}(\tau) \to \mathbf{C}(\tau),\ a(z^{1/m}) \mapsto a(t^{-1/m}),\ \ \Phi(-z^2\partial) = \partial_t =: \Delta\end{aligned} \tag{65}$$

and thus have to show that

$$\partial_t a(t^{-1/m}) = a(t^{-1/m})\partial_t + \varphi\left(-z^2da(z^{1/m})/dz\right). \tag{66}$$

But

$$\begin{aligned}\partial_t a(t^{-1/m}) &= a(t^{-1/m})\partial_t + da(t^{-1/m})/dt\\ &= a(t^{-1/m})\partial_t - m^{-1}t^{-(\frac{1}{m}+1)}a'(t^{-1/m}) \in \mathbf{C}(\tau)[\partial;d/dt]\end{aligned} \tag{67}$$

and

$$\begin{aligned}&da(z^{1/m})/dz = m^{-1}z^{(1/m)-1}a'(z^{1/m})\\ &\Longrightarrow -z^2da(z^{1/m})/dz = -m^{-1}z^{\frac{1}{m}+1}a'(z^{1/m})\\ &\Longrightarrow \varphi\left(-z^2da(z^{1/m})/dz\right) = -m^{-1}t^{-(\frac{1}{m}+1)}a'(t^{-1/m}).\end{aligned} \tag{68}$$

The equations (67) and (68) imply (66) and therefore Lemma 3.4 implies the algebra homomorphism (13)

$$\begin{array}{lcl}\Phi : \mathbf{A}(\tau) = \mathbf{K}(\tau)[-z^2\partial;-z^2d/dz] & \to & \mathbf{B}(\tau) = \mathbf{C}(\tau)[\partial_t;d/dt]\\ f = \sum_{j\in\mathbb{N}} a_j(z^{1/m})(-z^2\partial)^j & \mapsto & f_t := \sum_{j\in\mathbb{N}} a_j(t^{-1/m})\partial_t^j.\end{array} \tag{69}$$

and the induced module action (14).

Lemma 3.5. *(cf. [17, Thm. 1.8.4, Ex. 1.8.6], [4, Cor. 363]) The algebra $\mathbf{A} = \mathbf{K}[\partial;d/dz]$ is simple, i.e., 0 and $\mathbf{A}$ are its unique two-sided ideals.*

Corollary 3.6. *([17, Cor. 5.7.3]) Any f.g. $\mathbf{A}$-torsion module is cyclic.*

3.3 A directed system category

We embed the behaviors $\mathrm{cl}\left((\mathcal{B}(R,\tau))_{\tau\geq\sigma(R)}\right)$ *from* (19) *into an abelian category* $\mathfrak{B}$ *with good properties.*

Consider the strictly ordered and therefore directed ordered set $[0,\infty)$ of nonnegative real numbers and consider the category $\mathfrak{A} :=_{\mathbb{C}} \mathbf{Mod}^{[0,\infty)}$ of directed systems of $\mathbb{C}$-vector spaces according to Remark 3.1. Recall the signal spaces $W(\tau) = \mathrm{C}^{\infty}(\tau,\infty)$ or $W(\tau) = \mathcal{D}'(\tau,\infty)$. These give rise to the directed system

$$\left(W(\tau_1), \mathrm{res} : W(\tau_1) \to W(\tau_2) : w \mapsto w|_{(\tau_2,\infty)}\right)_{\tau_2\geq\tau_1\geq 0} \tag{70}$$

where res is the standard restriction map. If the directed system is defined on $[\tau_0,\infty)$ only, i.e., if $(V_\tau, \varphi_{\tau_2\tau_1} : V_{\tau_1} \to V_{\tau_2})_{\tau_2\geq\tau_1\geq\tau_0}$ is given, we extend this to a directed system on $[0,\infty)$ by

$$(V_{\tau_1}, \varphi_{\tau_2\tau_1})_{\tau_2\geq\tau_1\geq 0} \in_{\mathbb{C}} \mathbf{Mod}^{[0,\infty)} \text{ with } V_{\tau_1} := 0,\ \varphi_{\tau_2\tau_1} := 0 \text{ for } \tau_1 < \tau_0. \tag{71}$$

Likewise morphisms are extended and furnish an exact embedding functor

$${}_{\mathbb{C}}\mathbf{Mod}^{[\tau_0,\infty)} \to_{\mathbb{C}} \mathbf{Mod}^{[0,\infty)}. \tag{72}$$

To interpret the equivalence class behaviors from (19) as objects of a category we consider the quotient category of $\mathfrak{A}$ modulo the following equivalence relation $\equiv$:

$$\begin{aligned} &V = (V_{\tau_1}, \varphi_{\tau_2\tau_1})_{\tau_2\geq\tau_1\geq 0} \equiv V' = \left(V'_{\tau_1}, \varphi'_{\tau_2\tau_1}\right)_{\tau_2\geq\tau_1\geq 0} \\ &:\Longleftrightarrow \exists\tau_0\forall\tau_2 \geq \tau_1 \geq \tau_0 :\ V_{\tau_1} = V'_{\tau_1},\ \varphi_{\tau_2\tau_1} = \varphi'_{\tau_2\tau_1}. \end{aligned} \tag{73}$$

The equivalence class is denoted by $\mathrm{cl}(V)$. These $\mathrm{cl}(V)$ are the objects of the new category $\mathfrak{B}$. With the embedding from (72) we obtain

$$\mathrm{cl}\left((V_{\tau_1}, \varphi_{\tau_2\tau_1})_{\tau_2\geq\tau_1\geq 0}\right) = \mathrm{cl}\left((V_{\tau_1}, \varphi_{\tau_2\tau_1})_{\tau_2\geq\tau_1\geq\tau_0}\right). \tag{74}$$

The study of $\mathrm{cl}\left((V_{\tau_1}, \varphi_{\tau_2\tau_1})_{\tau_2\geq\tau_1\geq 0}\right)$ signifies that of the V_τ for possibly large τ. If $V = (V_{\tau_1}, \varphi_{\tau_2\tau_1})_{\tau_2\geq\tau_1\geq\tau_0}$ and $V' = \left(V'_{\tau_1}, \varphi'_{\tau_2\tau_1}\right)_{\tau_2\geq\tau_1\geq\tau'_0}$ are directed systems consider, for $\min(\tau_\Phi, \tau_\Psi) \geq \max(\tau_0, \tau'_0)$, directed system morphisms

$$\begin{aligned} \Phi &= (\Phi_\tau)_{\tau\geq\tau_\Phi} : (V_{\tau_1}, \varphi_{\tau_2\tau_1})_{\tau_2\geq\tau_1\geq\tau_\Phi} \to \left(V'_{\tau_1}, \varphi'_{\tau_2\tau_1}\right)_{\tau_2\geq\tau_1\geq\tau_\Phi} \\ \Psi &= (\Psi_\tau)_{\tau\geq\tau_\Psi} : (V_{\tau_1}, \varphi_{\tau_2\tau_1})_{\tau_2\geq\tau_1\geq\tau_\psi} \to \left(V'_{\tau_1}, \varphi'_{\tau_2\tau_1}\right)_{\tau_2\geq\tau_1\geq\tau_\Psi} \end{aligned} \tag{75}$$

and define the equivalence relation

$$\Phi \equiv \Psi :\Longleftrightarrow \exists\tau''_0 \geq \max(\tau_\Phi, \tau_\Psi)\forall\tau \geq \tau''_0 :\ \Phi_\tau = \Psi_\tau. \tag{76}$$

The equivalence class is denoted by $\mathrm{cl}(\Phi)$. Then the set of morphisms in $\mathfrak{B}$ from $\mathrm{cl}(V)$ to $\mathrm{cl}(V')$ is defined as

$$\begin{gathered} \mathfrak{B}(\mathrm{cl}(V), \mathrm{cl}(V')) := \mathrm{Hom}(\mathrm{cl}(V), \mathrm{cl}(V') := \\ \left\{\mathrm{cl}(\Phi);\ \Phi = (\Phi_\tau)_{\tau\geq\tau_\Phi} : (V_{\tau_1}, \varphi_{\tau_2\tau_1})_{\tau_2\geq\tau_1\geq\tau_\Phi} \to \left(V'_{\tau_1}, \varphi'_{\tau_2\tau_1}\right)_{\tau_2\geq\tau_1\geq\tau_\Phi}\right\}. \end{gathered} \tag{77}$$

Again with the componentwise $\mathbb{C}$-linear structure and composition, for sufficiently large τ_Φ, we obtain the category $\mathfrak{B}$. It is also abelian, kernels, cokernels and finite products etc. being formed componentwise for $\tau \in [\tau_0'', \infty)$ and sufficiently large τ_0''. The directed system from (70) gives rise to the objects

$$\mathcal{W} := \mathrm{cl}\left((W(\tau), \mathrm{res})_{\tau \geq 0}\right) \in \mathfrak{B} \text{ and } \forall q \geq 0 : \ \mathcal{W}^q := \mathrm{cl}\left((W(\tau)^q, \mathrm{res})_{\tau \geq 0}\right) \in \mathfrak{B}. \tag{78}$$

The behaviors from (19) induce the subobjects

$$\mathrm{cl}\left((\mathcal{B}(R,\tau), \mathrm{res})_{\tau \geq \sigma(R)}\right) \subseteq \mathcal{W}^q. \tag{79}$$

3.4 The functor $\mathbf{Mod}_{\mathbf{A}}^{\mathrm{fg}} \to \mathfrak{B},\ \mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U)$

The following results are fully analogous to those in [5, §2.3] and therefore the proofs are omitted. Assume the data

$$\begin{aligned} &R = \sum_j R_j(-z^2\partial)^j \in \mathbf{A}^{p\times q},\ U = \mathbf{A}^{1\times p}R,\ M = \mathbf{A}^{1\times q}/U, \\ &\forall \tau \geq \sigma(R) :\ \mathcal{B}(R,\tau) := \left\{ w \in W(\tau)^q;\ R \circ w = \sum_j R_j(t) w^{(j)} = 0 \right\}. \end{aligned} \tag{80}$$

Recall that for $R_j = A_j(z^{1/m})$ with $A_j \in \mathbb{C} << z >>^{p\times q}$ the function $R_j(t)$ is computed as $R_j(t) = A_j(t^{-1/m}) \in \mathrm{C}^\infty(\sigma(R_j), \infty)^{p\times q}$.
The standard basis $\delta = (\delta_1, \cdots, \delta_q)^\top \in (\mathbf{A}^{1\times q})^q$ gives rise to the column

$$\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_q)^\top \in M^q,\ \mathbf{w}_i := \delta_i + U, \tag{81}$$

of generators of M. Conversely,

$$\varphi_{\mathbf{w}} : \mathbf{A}^{1\times q} \to M,\ \xi = \xi\delta \mapsto \xi\mathbf{w} = \sum_{i=1}^q \xi_i \mathbf{w}_i, \text{ with } \varphi_{\mathbf{w}}(\delta) = \mathbf{w},\ \ker(\varphi_{\mathbf{w}}) = U, \tag{82}$$

shows that the system of generators $\mathbf{w}$ of M determines both the dimension of $\mathbf{A}^{1\times q}$ and its submodule U. The objects of the category ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ are pairs $(M, \mathbf{w})$ of f.g. modules M with a given list $\mathbf{w}$ of generators or a given representation $M = \mathbf{A}^{1\times q}/U$. The morphisms in ${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}}$ are just the $\mathbf{A}$-linear maps.

Lemma 3.7. *Assume the data from* (80). *Then the object*

$$\mathcal{B}(U) \underset{(80)}{:=} \mathrm{cl}\left((\mathcal{B}(R,\tau), \mathrm{res})_{\tau \geq \sigma(R)}\right) \in \mathfrak{B} \tag{83}$$

depends on U only and not on the special choice of R, and hence (19) *is justified. Moreover $U_1 \subseteq U_2$ implies $\mathcal{B}(U_2) \subseteq \mathcal{B}(U_1) \subseteq \mathcal{W}^q$.*

For two f.g. modules $M_i = \mathbf{A}^{1\times q_i}/U_i,\ i = 1, 2$, the following lemma establishes the standard correspondence of $\mathbf{A}$-linear maps and matrices.

Lemma 3.8. *(cf. [7, Cor. 2.1],[5, Cor. 2.5]) (i) There is the canonical isomorphism*

$$\begin{array}{ccc} \{P \in \mathbf{A}^{q_1\times q_2};\ U_1 P \subseteq U_2\} / \left\{\widetilde{P} \in \mathbf{A}^{q_1\times q_2};\ \mathbf{A}^{1\times q_1}\widetilde{P} \subseteq U_2\right\} & \cong & \mathrm{Hom}_{\mathbf{A}}(M_1, M_2) \\ P + \left\{\widetilde{P} \in \mathbf{A}^{q_1\times q_2};\ \mathbf{A}^{1\times q_1}\widetilde{P} \subseteq U_2\right\} & \mapsto & (\circ P)_{\mathrm{ind}} \end{array} \tag{84}$$

where $(\circ P)_{\text{ind}}(\xi + U_1) := \xi P + U_2$ *for* $\xi \in \mathbf{A}^{1\times q_1}$.
(ii) The map $(\circ P)_{\text{ind}} : \mathbf{A}^{1\times q_1}/U_1 \to \mathbf{A}^{1\times q_2}/U_2$ *is an isomorphism if and only if it is bijective or* $(\circ P)_{\text{ind}}^{-1} = (\circ Q)_{\text{ind}} : \mathbf{A}^{1\times q_2}/U_2 \to \mathbf{A}^{1\times q_1}/U_1$ *exists. The necessary and sufficient conditions for* $Q \in \mathbf{A}^{q_2\times q_1}$ *to satisfy* $(\circ Q)_{\text{ind}} = (\circ P)_{\text{ind}}^{-1}$ *are*

$$U_2 Q \subseteq U_1,\ \mathbf{A}^{1\times q_1}(PQ - \text{id}_{q_1}) \subseteq U_1,\ \mathbf{A}^{1\times q_2}(QP - \text{id}_{q_2}) \subseteq U_2. \tag{85}$$

The preceding Lemma 3.8 enables to extend the assignment $\mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U)$ to a contravariant functor, again in complete analogy to the discrete case [5]. For the data from (84) we additionally assume that $U_i = \mathbf{A}^{1\times p_i} R_i$. The condition $\mathbf{A}^{1\times p_1} R_1 P = U_1 P \subseteq U_2 = \mathbf{A}^{1\times p_2} R_2$ implies the existence of $X \in \mathbf{A}^{p_1\times p_2}$ with $R_1 P = X R_2$. We choose

$$\begin{aligned}
&\tau_1 \geq \max(\sigma(R_1), \sigma(R_2), \sigma(P), \sigma(X)) \underset{(69)}{\Longrightarrow} \forall \tau \geq \tau_1 :\ R_1, R_2, P, X \in \mathbf{A}(\tau_1)^{\bullet\times\bullet}\\
&\Longrightarrow \forall \tau \geq \tau_1 \forall w \in \mathcal{B}(R_2, \tau) :\\
&\qquad\qquad R_1 \circ (P \circ w) = X R_2 \circ w = X \circ (R_2 \circ w) = X \circ 0 = 0\\
&\Longrightarrow P\circ : \mathcal{B}(R_2, \tau) := \{w \in W(\tau)^{q_2};\ R_2 \circ w = 0\} \to \mathcal{B}(R_1, \tau),\ w \mapsto P \circ w.
\end{aligned} \tag{86}$$

Corollary and Definition 3.9. *For an* $\mathbf{A}$*-linear map*

$$\begin{aligned}
&\varphi = (\circ P)_{\text{ind}} : \mathbf{A}^{1\times q_1}/U_1 \to \mathbf{A}^{1\times q_2}/U_2 \textit{ define}\\
&\mathcal{B}(\varphi) := P\circ \underset{(86)}{:=} \text{cl}\left((P\circ : \mathcal{B}(R_2, \tau) \to \mathcal{B}(R_1, \tau))_{\tau \geq \tau_1}\right) :\\
&\qquad\qquad \mathcal{B}(U_2) = \text{cl}\left((\mathcal{B}(R_2, \tau), \text{res})_{\tau \geq \sigma(R_1)}\right) \to \mathcal{B}(U_1).
\end{aligned} \tag{87}$$

The map $\mathcal{B}(\varphi)$ *is well-defined, i.e., independent of the choice of* P *with* $\varphi = (\circ P)_{\text{ind}}$*, and the assignment*

$$\mathbf{Mod}_{\mathbf{A}}^{\text{fg}} \to \mathfrak{B},\ \mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U),\ \varphi = (\circ P)_{\text{ind}} \mapsto \mathcal{B}(\varphi) = P\circ, \tag{88}$$

is a contravariant additive functor. By definition the image of this functor is the category $\mathcal{B}eh$ *of behaviors.*

Remark 3.10. Define the space

$$W_\infty = \text{colim}_{\tau \geq 0} W(\tau) = \{(\tau, w); \tau \geq 0,\ w \in W(\tau)\} / \equiv\ \ni \text{cl}(\tau, w) \tag{89}$$

of germs of signals at ∞ (cf. [4, §5.4.2.3]). This is canonically an $\mathbf{A}$-module via

$$f \circ \text{cl}(\tau, w) := \text{cl}\left(\tau_1, f \circ w\right),\ \tau_1 := \max(\tau, \sigma(f)) \tag{90}$$

and indeed a large injective $\mathbf{A}$-cogenerator [4, Thm. 838]. For M from (80) the associated behavior is

$$\mathcal{B}_{W_\infty}(U) := \{w_\infty \in W_\infty^q;\ R \circ w_\infty = 0\} \cong \text{Hom}_{\mathbf{A}}(M, W_\infty). \tag{91}$$

The injective cogenerator property implies that $\mathcal{B}_{W_\infty}(U)$ determines

$$U = \left\{\xi \in \mathbf{A}^{1\times q};\ \xi \circ \mathcal{B}_{W_\infty}(U) = 0\right\},\ M = \mathbf{A}^{1\times q}/U \text{ and } \mathcal{B}(U). \tag{92}$$

But in general there is no τ such that for all $t \in (\tau, \infty)$ and $w_\infty \in \mathcal{B}_{W_\infty}(U)$ the initial condition $w_\infty(t_0)$ is defined. So the definition of u.e.s. and w.e.s. according to Def. 2.4 for $\mathcal{B}_{W_\infty}(U)$ is impossible. This suggested the new definition of behaviors of $\mathcal{B}(U)$ from (83). This $\mathcal{B}(U)$ is *not* isomorphic to $\text{Hom}_{\mathbf{A}}(M, \mathcal{F})$ for any signal module $\mathcal{F}$ with engineering significance.

3.5 The exactness of $\mathbf{A}^{1\times k}/U \mapsto \mathcal{B}(U)$

The exactness of the functor $\mathbf{A}^{1\times k}/U \mapsto \mathcal{B}(U)$ is also derived in full analogy to the discrete case [5, §2.4] and therefore the detailed proofs are again omitted.
Consider f.g. modules $M_i := \mathbf{A}^{1\times q_i}/U_i \in \mathbf{Mod}_{\mathbf{A}}^{\mathrm{fg}},\ i = 1,2,3$, and a sequence of $\mathbf{A}$-linear maps

$$M_1 \overset{\varphi=(\circ P)_{\mathrm{ind}}}{\longrightarrow} M_2 \overset{\psi=(\circ Q)_{\mathrm{ind}}}{\longrightarrow} M_3,\ U_1P \subseteq U_2,\ U_2Q \subseteq U_3. \tag{93}$$

Application of the functor $\mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U)$ furnishes the sequence of behaviors

$$\mathcal{B}(U_1) \overset{\mathcal{B}(\varphi)=P\circ}{\longleftarrow} \mathcal{B}(U_2) \overset{\mathcal{B}(\psi)=Q\circ}{\longleftarrow} \mathcal{B}(U_3). \tag{94}$$

Theorem 3.11. *The functor*

$$\mathbf{Mod}_{\mathbf{A}}^{\mathrm{fg}} \to \mathfrak{B},\ M = \mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U),$$

is exact, i.e., the exactness of (93) *implies that of* (94).

Proof. As in the proof [5, Thm. 2.9] the proof is reduced to the case $q_1 = 0,\ q_2 = q_3 = 1$, $U_2 = U_3 = 0$ and $Q = \sum_{j=0}^{d} Q_j(-z^2\partial)^j\mathbf{A} = \mathbf{K}[-z^2\partial; -z^2d/dz],\ Q_d \neq 0$. Since $\mathbf{A}$ is a domain the nonzero map $\circ Q : \mathbf{A} \to \mathbf{A}$ is a monomorphism. It has to be shown that $Q\circ : \mathcal{W} = \mathcal{B}(0) \to \mathcal{B}(0)$ is an epimorphism in the category $\mathcal{B}$ of classes of directed systems. We apply Cor.3.3 and choose $\tau_0 \geq \sigma(Q)\ (\geq \sigma(Q_d))$ such that Q_d has no zero in (τ, ∞). For $\tau \geq \tau_0$ and $u \in W(\tau)$ the differential equation

$$\begin{gathered} Q \circ y = u \text{ or} \\ y^{(d)} + Q_d(t)^{-1}Q_{d-1}(t)y^{(d-1)} + \cdots + Q_d(t)^{-1}Q_0(t)y = Q_d(t)^{-1}u \end{gathered} \tag{95}$$

has a solution $y \in W(\tau)$. This signifies that for $\tau \geq \tau_0$ the maps $Q\circ : W(\tau) \to W(\tau)$ are surjective. These induce the epimorphism

$$Q\circ = \mathrm{cl}\left((Q\circ)_{\tau\geq\tau_0}\right) : \mathcal{B}(0) = \mathcal{W} = \mathrm{cl}\left((W(\tau))_{\tau\geq\tau_0}\right) \to \mathcal{W}. \tag{96}$$

□

3.6 Autonomous behaviors

In this section we prove the analogue of the cogenerator property of the standard signal modules for the behaviors of this paper and simultaneously characterize autonomous behaviors. Again the proofs are analogous to those in the discrete case [5, §2.5].
In this section we choose the signal spaces of smooth functions

$$W(\tau) = \mathrm{C}^\infty(\tau, \infty) = \mathcal{B}(0, \tau) \text{ and } \mathcal{W} = \mathrm{cl}\left((W(\tau), \mathrm{res})_{\tau\geq 0}\right) = \mathcal{B}(0). \tag{97}$$

unless explicitly stated otherwise. A finitely generated $\mathbf{A}$-module $M_1 = \mathbf{A}^{1\times q_1}/U_1 = \mathbf{A}^{1\times q_1}/\mathbf{A}^{1\times p_1}R_1$ is isomorphic to a direct sum of cyclic modules [17, Cor. 5.7.19]. Hence there is an isomorphism

$$\begin{gathered} M_1 = \mathbf{A}^{1\times q_1}/U_1 \cong \mathbf{A}/\mathbf{A}f_1 \times \cdots \mathbf{A}/\mathbf{A}f_r \times \mathbf{A}^{1\times(q_2-r)} = \mathbf{A}^{1\times q_2}/U_2 =: M_2 \\ \text{where } q_2 \geq r \geq 0,\ 0 \neq f_i \in \mathbf{A} \text{ and } \deg_\partial(f_i) > 0 \text{ or } 0 \subsetneq \mathbf{A}f_i \subsetneq \mathbf{A}, \\ U_2 = \mathbf{A}^{1\times q_2}R_2,\ R_2 := \mathrm{diag}(f_1, \cdots, f_r, 0, \cdots, 0) \in \mathbf{A}^{q_2\times q_2}. \end{gathered} \tag{98}$$

Since $\mathbf{A}$ is simple the number r can be chosen as 0 or 1, cf. Lemma 3.5, Cor. 3.6. The functor $\mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U)$ is applied to this and furnishes the isomorphism

$$\mathcal{B}(U_1) \cong \mathcal{B}(U_2) = \mathcal{B}(\mathbf{A}f_1) \times \cdots \times \mathcal{B}(\mathbf{A}f_r) \times \mathcal{B}(0)^{q_2-r}. \tag{99}$$

The systems $\mathcal{B}(\mathbf{A}f_j)$ are particularly simple: Consider, more generally, any

$$\begin{aligned} & g = \sum_{i=0}^{d} g_i(-z^2\partial)^i \in \mathbf{A} = \mathbf{K}[-z^2\partial; -z^2 d/dz],\ \deg_\partial(g) = d,\ \text{ i.e., } g_d \neq 0 \\ & \Longrightarrow \mathbf{K}^{1\times d} \underset{\mathbf{K}}{\cong} \mathbf{A}/\mathbf{A}g,\ (h_0, \cdots, h_{d-1}) \mapsto \sum_{i=0}^{d-1} h_i(-z^2\partial)^i + \mathbf{A}g,\ h_i \in \mathbf{K}, \\ & \Longrightarrow d = \dim_{\mathbf{K}}(\mathbf{A}/\mathbf{A}g) < \infty. \end{aligned} \tag{100}$$

The preceding isomorphism is the usual consequence of euclidean division with remainder. According to Cor. 3.3 choose $\tau_0 > \sigma(g)$ such that no $t \geq \tau_0$ is a zero of g_d. For all $t_0 > \tau \geq \tau_0$ we obtain the isomorphisms

$$\mathcal{B}(g,\tau) = \left\{ w \in W(\tau);\ \sum_{i=0}^{d} g_i(t) w^{(i)}(t) = 0 \right\} \cong \mathbb{C}^d,\ w \mapsto (\partial(d) \circ w)(t_0). \tag{101}$$

We conclude

$$\begin{aligned} & \forall \tau \geq \tau_0 :\ \dim_{\mathbb{C}}(\mathcal{B}(g,\tau)) = d = \deg_\partial(g) \\ & \qquad \Longrightarrow (\mathcal{B}(\mathbf{A}g) = \mathrm{cl}\left((\mathcal{B}(g,\tau), \mathrm{res})_{\tau \geq \tau_0}\right) = 0 \iff d = 0). \end{aligned} \tag{102}$$

Remark 3.12. The distributional solutions of $x' = F(t)x$ with $F \in \mathrm{C}^\infty(\tau,\infty)^{q\times q}$ are smooth. The same follows for the solutions of $g \circ w = 0$ on (τ, ∞) in (101).

Corollary 3.13. *If $M = \mathbf{A}^{1\times q_1}/U_1$ in* (98) *is nonzero then so is $\mathcal{B}(U_1)$.*

Proof. In (99) the behavior $\mathcal{W} = \mathcal{B}(0)$ is nonzero and so are the behaviors $\mathcal{B}(\mathbf{A}f_j)$ by (102) since $\deg_q(f_j) > 0$. In (98) M_1 is nonzero if and only if $q_2 > 0$. The isomorphism (99) then implies that $\mathcal{B}(U_1) \cong \mathcal{B}(U_2)$ is nonzero too. □

Corollary 3.14. *For $M_i = \mathbf{A}^{1\times q_i}/U_i,\ i = 1, 2$, the $\mathbb{C}$-linear map*

$$\mathrm{Hom}_{\mathbf{A}}(M_1, M_2) \to \mathfrak{B}(\mathcal{B}(U_2), \mathcal{B}(U_1)),\ \varphi = (\circ P)_{\mathrm{ind}} \mapsto \mathcal{B}(\varphi) = P\circ, \tag{103}$$

is injective. By definition (cf. Cor. and Def. 3.9) the isomorphic image of (103)*, i.e.,*

$$\mathrm{Hom}_{\mathcal{B}eh}(\mathcal{B}(U_2), \mathcal{B}(U_1)) := \{\mathcal{B}(\varphi);\ \varphi \in \mathrm{Hom}_{\mathbf{A}}(M_1, M_2)\} \tag{104}$$

is the $\mathbb{C}$-space of behavior morphisms. *The objects $\mathcal{B}(U)$ and morphisms $\mathcal{B}(\varphi)$ form the subcategory $\mathcal{B}eh \subset \mathfrak{B}$ of behaviors. The functor*

$${}_{\mathbf{A}}\mathbf{Mod}^{\mathrm{fg}} \to \mathcal{B}eh,\ \mathbf{A}^{1\times q}/U \mapsto \mathcal{B}(U),\ \varphi \mapsto \mathcal{B}(\varphi), \tag{105}$$

is exact and a duality (contravariant equivalence). The proof of Thm. 2.3 is thus complete.

Proof. The proof is the same as that of [5, Cor. 2.11]. □

Let $\mathrm{tor}(M)$ denote the torsion submodule of M. As in [5, (77)] the isomorphism (98) implies the isomorphisms

$$\begin{gathered}\mathrm{tor}(M_1) \underset{\mathbf{A}}{\cong} \mathrm{tor}(M_2) = \oplus_{j=1}^r \mathbf{A}/\mathbf{A}f_j \underset{\mathbf{K}}{\cong} \mathbf{K}^{1\times d},\ d := \sum_{j=1}^r \deg_q(f_j),\ \text{and} \\ M_1/\operatorname{tor}(M_1) \underset{\mathbf{A}}{\cong} M_2/\operatorname{tor}(M_2) \underset{\mathbf{A}}{\cong} \mathbf{A}^{q_2-r}.\end{gathered} \tag{106}$$

The ranks of $R \in \mathbf{A}^{p\times q}$ resp. of $M = \mathbf{A}^{1\times q}/\mathbf{A}^{1\times p}R$ (cf. [5, (78)-(81)]) are the $\mathbf{Q}$-dimensions of the column space $R\mathbf{Q}^q$ resp. of $\mathbf{Q}\otimes_{\mathbf{A}} M$ where $\mathbf{Q}$ is the quotient field of $\mathbf{A}$, and then $q = \mathrm{rank}(R) + \mathrm{rank}(M)$.

Lemma and Definition 3.15. *(cf. [5, Lemma 2.13]) The following properties are equivalent for the data from* (98) *and* (99)*:*
(i) $\mathrm{rank}(M_1) = 0$ *or* $\mathrm{rank}(R_1) = q_1$.
(ii) M_1 *is a torsion module, i.e.,* $M_1 \cong \prod_{j=1}^r \mathbf{A}/\mathbf{A}f_j$ *in* (98), (106).
(iii) $d := \dim_{\mathbf{K}}(M_1) < \infty$.
(iv) There are $\tau_0 \geq 0$ *and* $d \in \mathbb{N}$ *such that* $\forall \tau \geq \tau_0 :\ \dim_{\mathbb{C}}(\mathcal{B}(R_1,\tau)) = d$.
(v) There is a nonzero $f \in \mathbf{A}$ *such that* $M_1 \cong \mathbf{A}/\mathbf{A}f$*, cf. Cor. 3.6.*
If these conditions are satisfied the behavior $\mathcal{B}(U_1)$ *is called* autonomous.
For sufficiently large τ *all distributional trajectories in* $\mathcal{B}(R_1,\tau)$ *are smooth.*

According to Cor. 3.6 all f.g. torsion modules and especially $\mathrm{tor}(M_1)$ are cyclic, hence

$$\begin{gathered}\mathrm{tor}(M_1) \underset{\mathbf{A}}{\cong} \mathrm{tor}(M_2) = \oplus_{j=1}^r \mathbf{A}/\mathbf{A}f_j \cong \mathbf{A}/\mathbf{A}f,\ f \neq 0, \\ d := \dim_{\mathbf{K}}(\mathrm{tor}(M_1)) = \sum_{j=1}^r \deg_\partial(f_j) = \deg_\partial(f).\end{gathered} \tag{107}$$

Definition 3.16. Consider a f.g. module

$$\begin{gathered}R = \sum_{j=0}^k R_j(-z^2\partial)^j \in \mathbf{A}^{p\times q},\ U = \mathbf{A}^{1\times p}R,\ M = \mathbf{A}^{1\times q}/U, \\ \forall \tau \geq \sigma(R):\ \mathcal{B}(R,\tau) := \left\{ w \in \mathrm{C}^\infty(\tau,\infty)^q;\ \sum_{j=0}^k R_j(t)w^{(j)} = 0 \right\}, \\ \mathcal{B}(U) = \mathrm{cl}\left((\mathcal{B}(R,\tau), \mathrm{res})_{\tau\geq\sigma(R)} \right).\end{gathered} \tag{108}$$

The behavior $\mathcal{B}(U)$ is called *trajectory-autonomous* (t-autonomous) of *memory size d* if there are $\tau_0 \geq \sigma(R)$ and $d \in \mathbb{N}$ such that for all $t_0 > \tau \geq \tau_0$ the *initial value map*

$$\mathcal{B}(R,\tau) \to \mathbb{C}^{dq},\ w \mapsto (\partial(d)\circ w)(t_0), \tag{109}$$

is injective, but not necessarily bijective. This signifies that for sufficiently large τ all trajectories $w \in \mathcal{B}(R,\tau)$ are uniquely determined by the *initial data* $w^{(j)}(t_0),\ 0 \leq j \leq d-1$. The number d is obviously not unique.

Corollary 3.17. *The behaviors* $\mathcal{B}(\mathbf{A}g)$ *from* (101) *resp.* $\mathcal{B}(\mathbf{A}f_1)\times\cdots\times\mathcal{B}(\mathbf{A}f_r)$ *from* (99) *are obviously t-autonomous of memory sizes*

$$\deg_q(g)\ \textit{resp.}\ \max\left\{\deg_q(f_j);\ j = 1,\cdots,r\right\}. \tag{110}$$

Lemma 3.18. *Trajectory-autonomy is preserved by isomorphisms.*

Proof. Consider two isomorphic f.g. modules and their associated isomorphic behaviors:

$$M_i := \mathbf{A}^{1\times q_i}/U_i,\ U_i = \mathbf{A}^{1\times p_i}R_i,\ R_i \in \mathbf{A}^{p_i\times q_i},$$
$$\varphi = (\circ P)_{\mathrm{ind}} : M_1 \cong M_2,\ \mathcal{B}(\varphi) : \mathcal{B}(U_2) \cong \mathcal{B}(U_1),\ P = \sum_{j=0}^{d} P_j(-z^2\partial)^j. \tag{111}$$

Assume that $\mathcal{B}(U_1)$ is t-autonomous with memory size d_1. Then there is a $\tau_1 \geq \max(\sigma(R_1), \sigma(R_2), \sigma(P))$ such that for all $t_0 > \tau \geq \tau_1$ the maps

$$\mathcal{B}(R_1,\tau) \to \mathbb{C}^{d_1q_1},\ w_1 \mapsto (\partial(d_1)\circ w_1)(t_0) = (w_1(t_0), \cdots, w_1^{(d_1-1)}(t_0))^\top, \text{ resp.}$$
$$P\circ : \mathcal{B}(R_2,\tau) \cong \mathcal{B}(R_1,\tau),\ w_2 \mapsto w_1 := P\circ w_2 = \sum_{j=0}^{d} P_j(t)w_2^{(j)}, \tag{112}$$

are injective resp. bijective. Define $d_2 := d_1 + d$ and assume $w_2^{(m)}(t_0) = 0$ for $m \leq d_2 - 1$. Then

$$\forall \ell \text{ with } 0 \leq \ell \leq d_1 - 1:\ w_1^{(\ell)}(t) = \sum_{j=0}^{d}\sum_{i=0}^{\ell}\binom{\ell}{i}P_j^{(\ell-i)}(t)w_2^{(i+j)}(t)$$
$$\Longrightarrow w_1^{(\ell)}(t_0) = \sum_{j=0}^{d}\sum_{i=0}^{\ell}\binom{\ell}{i}P_j^{(\ell-i)}(t_0)w_2^{(i+j)}(t_0) = 0 \Longrightarrow (\partial(d_1)\circ w_1)(t_0) = 0$$
$$\underset{\text{memory size } d_1}{\Longrightarrow} w_1 = 0 \underset{P\circ \text{ bijective}}{\Longrightarrow} w_2 = 0. \tag{113}$$

This signifies that $\mathcal{B}(U_2)$ is t-autonomous with memory size d_2. □

Theorem 3.19. *A behavior $\mathcal{B}(U_1)$ is trajectory-autonomous if and only if it is autonomous.*

Proof. The proof is analogous to that of ([5, Thm. 2.16]) and follows from the isomorphism (99), Cor. 3.17, Lemma 3.18 and the obvious fact that $\mathcal{W} = \mathcal{B}(0)$ is not t-autonomous. □

4 Weakly exponentially stable (w.e.s.) behaviors

4.1 State space behaviors

Unless stated otherwise we use the signal spaces $W(\tau) = \mathrm{C}^\infty(\tau,\infty),\ \tau \geq 0$, in this section. Functions of at most polynomial growth (p.g.f.) appear at several places. It is obvious that the sum, product and maximum of p.g.f. are again p.g.f. and that polynomials are p.g.f.. From (30) we know that every $f = a(z^{1/m}) \in \mathbf{K}$ defines the p.g.f. $f(t) = a(t^{-1/m})$ on every closed interval $[\tau,\infty),\ \tau > \sigma(f)$. This implies that for every $A \in \mathbf{K}^{q\times q}$ the norm $\|A(t)\|$ is a p.g.f. on $[\tau,\infty),\ \tau > \sigma(A)$.

Lemma 4.1. *If $0 < \beta \leq \alpha$ and $0 < \nu \leq \mu$ then there is a constant $c \geq 1$, independent of t_0, such that*

$$\forall t \geq t_0 \geq 0:\ e^{-\alpha(t^\mu - t_0^\mu)} \leq ce^{-\beta(t^\nu - t_0^\nu)}. \tag{114}$$

Hence in the definition of w.e.s. in Def. 2.4 the factor $e^{-\alpha(t^\mu - t_0^\mu)}$ *can be replaced by* $e^{-\beta(t^\nu - t_0^\nu)}$ *if* $0 < \beta \le \alpha$ *and* $0 < \nu \le \mu$.

Proof. Of course $e^{-\alpha s} \le e^{-\beta s}$ for $s \ge 0$. Consider the function

$$f(t, t_0) = (t^\mu - t_0^\mu) - (t^\nu - t_0^\nu) = t^\mu - t^\nu - (t_0^\mu - t_0^\nu),$$
$$f(t_0, t_0) = 0,\ f'(t) := df(t, t_0)/dt = \mu t^{\mu-1} - \nu t^{\nu-1}.$$

If

$$\begin{gathered} t_0 \ge 1 \text{ then } \forall t \ge t_0 :\ f'(t) \ge 0 \Longrightarrow f(t, t_0) \ge 0 \Longrightarrow \\ e^{-\alpha f(t,t_0)} \le 1 \Longrightarrow e^{-\alpha(t^\mu - t_0^\mu)} \le e^{-\alpha(t^\nu - t_0^\nu)} \le e^{-\beta(t^\nu - t_0^\nu)}. \end{gathered} \tag{115}$$

If

$$\begin{gathered} t_0 < 1 \text{ then } f(1, t_0) = -(t_0^\mu - t_0^\nu) \ge 0,\ \forall t \ge 1 :\ f'(t) \ge 0 \Longrightarrow \\ \forall t \ge 1 :\ f(t, t_0) \ge 0 \Longrightarrow \forall t \ge 1 :\ e^{-\alpha(t^\mu - t_0^\mu)} \le e^{-\alpha(t^\nu - t_0^\nu)} \le e^{-\beta(t^\nu - t_0^\nu)}. \end{gathered} \tag{116}$$

Let

$$\begin{aligned} &c_1 = \max\left\{e^{-\beta f(t,t_0)}; t_0 \le t \le 1\right\} \Longrightarrow \forall t_0 \le t \le 1 :\ e^{-\beta(t^\mu - t_0^\mu)} \le c_1 e^{-\beta(t^\nu - t_0^\nu)} \\ &\underset{(116)}{\Longrightarrow} \forall t \ge t_0 : e^{-\alpha(t^\mu - t_0^\mu)} \le e^{-\beta(t^\mu - t_0^\mu)} \le c e^{-\beta(t^\nu - t_0^\nu)},\ c := \max(1, c_1). \end{aligned} \tag{117}$$

The inequalities (115) and (117) prove the assertion of the lemma. □

Lemma 4.2. *If* $0 < \beta < \alpha$, $\mu > 0$ *and* $\varphi > 0$ *is a p.g.f. on* $[\tau, \infty),\ \tau \ge 0$, *there is a p.g.f.* $\psi > 0$ *on* $[\tau, \infty)$ *such that*

$$\forall t \ge t_0 \ge \tau :\ \varphi(t) e^{-\alpha(t^\mu - t_0^\mu)} \le \psi(t_0) e^{-\beta(t^\mu - t_0^\mu)}. \tag{118}$$

Proof. (i) At first we consider the case $\mu = 1$: By definition there are $c_1 \ge 1$ and $p \in \mathbb{N}$ with $\varphi(t) \le c_1 t^p$ for $t \ge \tau$. There is a constant $c_2 \ge 1$ such that

$$\begin{aligned} &\forall i = 0, \cdots, p \forall t \ge 0 :\ t^i e^{-\alpha t} \le c_2 e^{-\beta t} \Longrightarrow \forall t \ge t_0 \ge \tau : \\ &\varphi(t) e^{-\alpha(t - t_0)} \le c_1 t^p e^{-\alpha(t - t_0)} = \sum_{i=0}^{p} c_1 \binom{p}{i} t_0^{p-i} (t - t_0)^i e^{-\alpha(t - t_0)} \\ &\le \sum_{i=0}^{p} c_1 \binom{p}{i} t_0^{p-i} c_2 e^{-\beta(t - t_0)} = \psi(t_0) e^{-\beta(t - t_0)},\ \psi(t) := \sum_{i=0}^{p} c_1 c_2 \binom{p}{i} t^{p-i}. \end{aligned} \tag{119}$$

The function ψ is a polynomial and hence a p.g.f..
(ii) For arbitrary $\mu > 0$ define $s_0 := t_0^\mu$ and $\varphi_1(s) := \varphi(s^{1/\mu})$. Then φ_1 is a p.g.f. of the variable s on $[\tau^\mu, \infty)$. According to (i) there is a p.g.f. ψ_1 on $[\tau^\mu, \infty)$ such that

$$\begin{aligned} &\forall s \ge s_0 = t_0^\mu \ge \tau^\mu :\ \varphi_1(s) e^{-\alpha(s - s_0)} \le \psi_1(s_0) e^{-\beta(s - s_0)} \underset{s := t^\mu}{\Longrightarrow} \\ &\forall t \ge t_0 \ge \tau :\ \varphi(t) e^{-\alpha(t^\mu - t_0^\mu)} = \varphi_1(t^\mu) e^{-\alpha(t^\mu - t_0^\mu)} \\ &\qquad \le \psi_1(t_0^\mu) e^{-\beta(t^\mu - t_0^\mu)} = \psi(t_0) e^{-\beta(t^\mu - t_0^\mu)},\ \psi(t_0) := \psi_1(t_0^\mu), \end{aligned} \tag{120}$$

where $\psi(t)$ is obviously also a p.g.f. on $[\tau, \infty)$. □

Lemma 4.3. *Consider the state space equation*

$$w'(t) = F(t)w(t) + u(t) \text{ with } F = A(z^{1/m}) \in \mathbf{K}(\tau_0)^{q\times q},\ A \in \mathbb{C} << z >>^{q\times q},$$
$$F(t) = A(t^{-1/m}) \in \mathrm{C}^\infty(\tau_0,\infty)^{q\times q},\ w,u \in W(\tau_0)^q. \tag{121}$$

If $\ell \in \mathbb{N}$ and $\tau_1 > \tau_0$ there is a p.g.f. $\varphi_\ell > 0$ on $[\tau_1,\infty)$ such that

$$\forall t \geq \tau_1:\ \|w^{(\ell)}(t)\| \leq \varphi_\ell(t)\max\left(\|w(t)\|, \|(\partial(\ell)\circ u)(t)\|\right). \tag{122}$$

Proof. All derivatives $d^jF(t)/dt^j =: F^{(j)}(t)$ have the form $F^{(j)}(t) = G_j(t)$, $G_j \in \mathbf{K}(\tau_0)^{q\times q}$. Hence the norms $\|F^{(j)}(t)\|$ are p.g.f. on each closed interval $[\tau,\infty),\ \tau > \tau_0$. Differentiation of $w'(t) = F(t)w(t) + u(t)$ and induction furnish equations

$$w^{(\ell)}(t) = F_\ell(t)w(t) + \sum_{i=0}^{\ell-1} G_{\ell,i}(t)u^{(i)}(t),\ F_0 = G_{1,0} = \mathrm{id}_q,\ F_1 = F,\ \ell \in \mathbb{N},$$
$$\text{where } F_\ell,\ G_{\ell,i} \in \mathbf{K}^{q\times q}, \|F_\ell(t)\|, \|G_{\ell,i}\| \text{ p.g.f. on } [\tau_1,\infty). \tag{123}$$

The equation (123) implies the assertion since

$$\|w^{(\ell)}(t)\| \leq \|F_\ell(t)\|\|w(t)\| + \sum_{i=0}^{\ell-1}\|G_{\ell,i}(t)\|\|u^{(i)}(t)\|$$
$$\leq \varphi_\ell(t)\max\left(\|w(t)\|, \|(\partial(\ell)\circ u)(t)\|\right) \tag{124}$$

where

$$\varphi_\ell(t) := \|F_\ell(t)\| + \sum_{i=0}^{\ell-1}\|G_{\ell,i}(t)\| \text{ and}$$
$$\|(\partial(\ell)\circ u)(t)\| = \max(\|u^{(i)}(t)\|;\ 0 \leq i \leq \ell-1). \tag{125}$$

□

We recall here and use several times that for $\tau_1 > \tau_0$ and $u \in W(\tau_0)^q$ a solution w of $w'(t) = F(t)w(t)$ or of $w'(t) = F(t)w(t) + u(t)$ on (τ_1,∞) can be uniquely extended to (τ_0,∞). Notice that this does not hold for implicit equations $R\circ w = 0$ in general.

Lemma 4.4. *As in Lemma 4.3 consider the operator $R := -z^2\partial - F \in \mathbf{A}(\tau_0)^{q\times q}$ and the behavior*

$$\mathcal{B}(R,\tau_0) = \left\{w \in \mathrm{C}^\infty(\tau_0,\infty)^q;\ w'(t) = F(t)w(t)\right\}. \tag{126}$$

If $\mathcal{B}\left(\mathbf{A}^{1\times q}(-z^2\partial - F)\right)$ is w.e.s. the following inequality is satisfied with the initial vector $w(t_0)$ instead of $(\partial(d)\circ w)(t_0)$ from (33)*:*

$$\exists\tau_1 > \tau_0\exists\alpha,\mu > 0\forall\ell \in \mathbb{N}\exists \text{ p.g.f. } \varphi_\ell \in \mathbb{C}^{[\tau_1,\infty)} \text{ with } \varphi_\ell > 0$$
$$\forall t \geq t_0 \geq \tau_1\forall w \in \mathcal{B}(R,\tau_0):$$
$$\|w^{(\ell)}(t)\| \leq \varphi_\ell(t_0)e^{-\alpha(t^\mu - t_0^\mu)}\|w(t_0)\|. \tag{127}$$

The same result holds with a different φ_ℓ if $w^{(\ell)}(t)$ is replaced by $(\partial(\ell)\circ w)(t)$.

Proof. By definition of w.e.s. the following inequality holds for some d, suitable constants and p.g.f. $\psi_\ell > 0$:

$$\|w^{(\ell)}(t)\| \leq \psi_\ell(t_0)e^{-\alpha(t^\mu - t_0^\mu)}\|(\partial(d)\circ w)(t_0)\|. \tag{128}$$

From (122) for $u = 0$ we infer an inequality $\|(\partial(d)\circ w)(t_0)\| \leq \psi(t_0)\|w(t_0)\|$ for some p.g.f. $\psi > 0$ on $[\tau_1, \infty)$ and conclude

$$\|w^{(\ell)}(t)\| \leq \varphi_\ell(t_0)e^{-\alpha(t^\mu - t_0^\mu)}\|w(t_0)\| \tag{129}$$

with the p.g.f. $\varphi_\ell := \psi_\ell\psi$. □

Lemma 4.5. *(i) In the situation of Lemma 4.3 let $\ell \in \mathbb{N}$ and $\tau_1 > \tau_0$. Assume $\alpha, \mu > 0$, a p.g.f. $\varphi > 0$ and a function $a \geq 0$ on $[\tau_1, \infty)$ such that for all solutions $w \in W(\tau_0)^q$ of $w'(t) = F(t)w(t)$ and all $t \geq t_0 \geq \tau_1$ the following inequalities hold:*

$$\|w(t)\| \leq \varphi(t_0)e^{-\alpha(t^\mu - t_0^\mu)}\|w(t_0)\|,\ \|(\partial(\ell)\circ u)(t)\| \leq \varphi(t_0)e^{-\alpha(t^\mu - t_0^\mu)}a(t_0). \tag{130}$$

Then for each β with $0 < \beta < \alpha$ there is a p.g.f. $\psi > 0$ on $[\tau_1, \infty)$ such that for all solutions $w \in W(\tau_0)^q$ of $w'(t) = F(t)w(t) + u(t)$ and all $t \geq t_0 \geq \tau_1$

$$\|w^{(\ell)}(t)\| \leq \psi(t_0)e^{-\beta(t^\mu - t_0^\mu)}\max(\|w(t_0)\|, a(t_0)). \tag{131}$$

(ii) In particular, if $u := 0$ and $a := 0$, the first inequality in (130) *implies the inequality* (131) *for all $\ell \in \mathbb{N}$ and therefore the w.e.s. of $\mathcal{B}(\mathbf{A}^{1\times q}(-z^2\partial\,\mathrm{id}_q - F))$.*

Proof. Wlog we assume $\varphi(t) = c_1t^p,\ c_1 \geq 1, p \in \mathbb{N}$. Let $\Phi(t, t_0) \in \mathrm{Gl}_q(\mathrm{C}^\infty(\tau_0, \infty))$ be the transition matrix of $w'(t) = F(t)w(t)$, i.e., $\Phi'(t, t_0) = F(t)\Phi(t, t_0)$ and $\Phi(t_0, t_0) = \mathrm{id}_q$. Then the solution of $w' = F(t)w + u$ is

$$\begin{aligned} w(t) &= \Phi(t, t_0)w(t_0) + \int_{t_0}^{t}\Phi(t, s)u(s)ds \Longrightarrow \\ \|w(t)\| &\leq \|\Phi(t, t_0)\|\|w(t_0)\| + \int_{t_0}^{t}\|\Phi(t, s)\|\|u(s)\|ds. \end{aligned} \tag{132}$$

Equation (130) implies for all $t \geq t_0 \geq \tau_1$:

$$\begin{aligned} &\|\Phi(t, t_0)\| \leq c_1t_0^pe^{-\alpha(t^\mu - t_0^\mu)},\ \text{hence } \|\Phi(t, s)\| \leq c_1s^pe^{-\alpha(t^\mu - s^\mu)} \text{ and} \\ &\qquad \|u(s)\| \leq c_1t_0^pe^{-\alpha(s^\mu - t_0^\mu)}a(t_0) \\ &\underset{(132)}{\Longrightarrow} \|w(t)\| \leq c_1t_0^pe^{-\alpha(t^\mu - t_0^\mu)}\|w(t_0)\| + c_1^2t_0^pa(t_0)e^{-\alpha(t^\mu - t_0^\mu)}\int_{t_0}^{t}s^pds \\ &\qquad \leq 2c_1(1 + c_1t^{2p+1})e^{-\alpha(t^\mu - t_0^\mu)}\max(\|w(t_0)\|, a(t_0)). \end{aligned} \tag{133}$$

From (122) we get a p.g.f. $\varphi_1 > 0$ on $[\tau_1, \infty)$ such that for all $t \geq t_0 \geq \tau_1$ and all solutions $w \in W(\tau_0)^q$ of $w'(t) = F(t)w(t) + u(t)$

$$\begin{aligned} &\|w^{(\ell)}(t)\| \leq \varphi_1(t)\max\left(\|w(t)\|, \|(\partial(\ell)\circ u)(t)\|\right) \underset{(133),(130)}{\leq} \\ &\varphi_1(t)\max\left(2c_1(1 + c_1t^{2p+1})e^{-\alpha(t^\mu - t_0^\mu)}\max(\|w(t_0)\|, a(t_0)), c_1t_0^pe^{-\alpha(t^\mu - t_0^\mu)}a(t_0)\right) \\ &= \varphi_1(t)2c_1(1 + c_1t^{2p+1})e^{-\alpha(t^\mu - t_0^\mu)}\max(\|w(t_0)\|, a(t_0)). \end{aligned} \tag{134}$$

Finally Lemma 4.2 furnishes a p.g.f. $\psi > 0$ on $[\tau_1, \infty)$ such that (131) holds, i.e.,

$$\forall t \geq t_0 \geq \tau_1 : \ \|w^{(\ell)}(t)\| \leq \psi(t_0) e^{-\beta(t^\mu - t_0^\mu)} \max(\|w(t_0)\|, a(t_0)).$$

□

Corollary 4.6. *Consider*

$$w'(t) = F(t)w(t), \ F \in \mathbf{K}(\tau_0)^{q\times q}, \ w \in W(\tau_0)^q. \tag{135}$$

If this state space equation is uniformly exponentially stable (u.e.s.) (cf. (35)*) then the behavior* $\mathcal{B}(\mathbf{A}^{1\times q}(-z^2\partial \operatorname{id}_q - F))$ *is w.e.s..*

Proof. The assumption signifies that there are $c \geq 1$, $\alpha > 0$ and $\tau_1 > \tau_0$ such that for all solutions $w \in W(\tau_0)^q$ of $w'(t) = F(t)w(t)$ and all $t \geq t_0 \geq \tau_1$ the inequality

$$\|w(t)\| \leq c e^{-\alpha(t-t_0)} \|w(t_0)\| \tag{136}$$

holds. Part (ii) of Lemma 4.5 implies the w.e.s. of $\mathcal{B}\left(\mathbf{A}^{1\times q}(-z^2\partial \operatorname{id}_q - F)\right)$. □

Corollary 4.7. *If* $F \in \mathbb{C}^{q\times q}$ *is a constant matrix then* $\mathcal{B}\left(\mathbf{A}^{1\times q}(-z^2\partial \operatorname{id}_q - F)\right)$ *is w.e.s. if and only if* F *is asymptotically stable, i.e., if* $\operatorname{spec}(F) \subset \mathbb{C}_-$.

Proof. It is a standard result that $\operatorname{spec}(A) \subset \mathbb{C}_-$ implies u.e.s. and thus w.e.s. by Cor. 4.6. Conversely,

$$\Phi(t, t_0) = e^{F(t-t_0)}, \ \|\Phi(t, t_0)\| \leq \varphi(t_0) e^{-\alpha(t^\mu - t_0^\mu)} \Longrightarrow$$
$$\lim_{t\to\infty} \Phi(t, t_0) = 0 \Longrightarrow \lim_{t\to\infty} e^{tF} = 0 \Longrightarrow \operatorname{spec}(F) \subset \mathbb{C}_-.$$

□

We apply Lemma 4.5 to any $f := \sum_{i=0}^d f_i(-z^2\partial)^i \mathbf{A}, \ \deg_\partial(f) = d$, and choose $\tau_0 > \sigma(f)$ such that f_d has no zeros on $[\tau_0, \infty)$ (cf. Cor. 3.3). Define

$$F := \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -f_d^{-1} f_0 & -f_d^{-1} f_1 & \cdots & \cdots & \cdots & \cdots & -f_d^{-1} f_{d-1} \end{pmatrix} \in \mathbf{K}^{d\times d}, \ R := -z^2\partial \operatorname{id}_d - F \in \mathbf{A}^{d\times d}. \tag{137}$$

With $\delta_{0,d} := (\overset{0}{1}, 0, \cdots, \overset{d-1}{0})$ these data imply the standard $\mathbf{A}$-linear isomorphism

$$\begin{aligned} &(\circ\delta_{0,d})_{\text{ind}} : \mathbf{A}/\mathbf{A}f \overset{\cong}{\longleftrightarrow} \mathbf{A}^{1\times d}/\mathbf{A}^{1\times d}R : (\circ\partial(d))_{\text{ind}}, \\ &\eta + \mathbf{A}f \to \eta\delta_{0,d} + \mathbf{A}^{1\times d}R = (\overset{0}{\eta}, 0, \cdots, \overset{d-1}{0}) + \mathbf{A}^{1\times d}R \\ &\xi\partial(d) + \mathbf{A}f = \sum_{i=0}^{d-1} \xi_i(-z^2\partial)^i + \mathbf{A}f \leftarrow \xi + \mathbf{A}^{1\times d}(-z^2\partial \operatorname{id}_d - F). \end{aligned} \tag{138}$$

For $t_0 > \tau_0$ this module isomorphism gives rise to the behavior isomorphism

$$\begin{aligned} &\mathcal{B}(f, \tau_0) = \left\{ w \in W(\tau_0); \ f_d(t) w^{(d)} + \cdots + f_0(t) w = 0 \right\} \\ &\quad \cong \mathcal{B}(-z^2\partial \operatorname{id}_d - F, \tau_0) = \left\{ x \in W(\tau_0)^d; \ x' = F(t)x \right\} \cong \mathbb{C}^d, \\ &w(t) = x_0(t) \longleftrightarrow x(t) = (x_0(t), \cdots, x_{d-1}(t))^\top = (\partial(d) \circ w)(t) \\ &\quad \longleftrightarrow x(t_0) = (\partial(d) \circ w)(t_0). \end{aligned} \tag{139}$$

Since every f.g. torsion module $M = \mathbf{A}^{1\times q}/\mathbf{A}^{1\times p}R, \ \operatorname{rank}(R) = q$, is cyclic and thus isomorphic to some $\mathbf{A}/\mathbf{A}f$ the isomorphisms in (138) and (139) also imply

Corollary 4.8. *If* $M = \mathbf{A}^{1\times q}/U$ *with* $U := \mathbf{A}^{1\times p}R$, $R \in \mathbf{A}^{p\times q}$, $\operatorname{rank}(R) = q$ *and* $\dim_{\mathbf{K}}(M) = d < \infty$ *is a f.g. torsion module then there are inverse module resp. behavior isomorphisms*

$$\begin{array}{llcll} (\circ P)_{\mathrm{ind}}: & \mathbf{A}^{1\times q}/\mathbf{A}^{1\times p}R & \stackrel{\cong}{\longleftrightarrow} & \mathbf{A}^{1\times d}/\mathbf{A}^{1\times d}(-z^2\partial\,\mathrm{id}_d - F): & (\circ Q)_{\mathrm{ind}} \\ Q\circ: & \mathcal{B}(U) & \stackrel{\cong}{\longleftrightarrow} & \mathcal{B}(\mathbf{A}^{1\times d}(\partial\,\mathrm{id}_d - F)) & : P\circ \end{array}$$
$$\text{where } d = \dim_{\mathbf{K}}(M),\ F \in \mathbf{K}^{d\times d},\ P \in \mathbf{A}^{q\times d},\ Q \in \mathbf{A}^{d\times q}. \tag{140}$$

The isomorphisms (138) and (139) and Lemma 4.5 imply

Corollary 4.9. *If* $f := \sum_{i=0}^{d} f_i(-z^i\partial)^i \in \mathbf{A}(\tau_0)$, *and if* f_d *has no zero in* (τ_0, ∞) *the behavior* $\mathcal{B}(\mathbf{A}f)$ *is w.e.s. if and only if there are* $\tau_1 > \tau_0$, $\alpha, \mu > 0$ *and a p.g.f.* $\varphi > 0$ *on* $[\tau_1, \infty)$ *such that*

$$\begin{aligned} &\forall t \geq t_0 \geq \tau_1 \forall w \in \mathcal{B}(f, \tau_0): \\ &\qquad \|(\partial(d) \circ w)(t)\| \leq \varphi(t_0) e^{-\alpha(t^\mu - t_0^\mu)} \|(\partial(d) \circ w)(t_0)\|. \end{aligned} \tag{141}$$

4.2 Preservation of w.e.s. under isomorphisms

Example 4.10. The following examples (i) resp. (ii) show that in contrast to w.e.s. (cf. the next Lemma 4.11) *stability* resp. *u.e.s.* are *not invariant under behavior isomorphisms* and therefore not discussed in this paper.
(i) Consider the isomorphic state space behaviors

$$\begin{aligned} &\mathcal{B}_1 := \{w_1 \in W(0);\ w_1' = 0\} = \mathbb{C} \cdot 1 \\ &\quad \cong \mathcal{B}_2 := z^{-1} \circ \mathcal{B}_1 = t\mathcal{B}_1 = \left\{w_2 \in W(0); w_2' = t^{-1}w_2\right\} = \mathbb{C} \cdot t. \end{aligned} \tag{142}$$

Obviously $\mathcal{B}_1$ is stable, but $\mathcal{B}_2$ is not.
(ii) Consider the uniformly exponentially stable LTI-behavior

$$\begin{aligned} &\mathcal{B}_1 := \left\{w = \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right) \in W(0)^2;\ w'(t) = F_1 w(t)\right\},\ F_1 := \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right),\ \lambda_1 < \lambda_2 < 0, \\ &P(t) := \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Gl}_2\left(\mathrm{C}^\infty(0,\infty)\right),\ \mathcal{B}_1 \cong \mathcal{B}_2 := P(z^{-1}) \circ \mathcal{B}_1 = P(t)\mathcal{B}_1,\ w \mapsto v := Pw, \\ &\mathcal{B}_2 := \left\{v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) \in W(0)^2;\ v'(t) = F_2(t)v(t)\right\} \\ &\qquad\qquad \text{where} F_2 = (P' + PF_1)P^{-1} = \left(\begin{smallmatrix} \lambda_1 & (\lambda_2 - \lambda_1)t + 1 \\ 0 & \lambda_2 \end{smallmatrix}\right) \\ &\Longrightarrow v(t) = \left(\begin{smallmatrix} e^{\lambda_1(t-t_0)} & te^{\lambda_2(t-t_0)} - t_0 e^{\lambda_1(t-t_0)} \\ 0 & e^{\lambda_2(t-t_0)} \end{smallmatrix}\right) v(t_0) \end{aligned} \tag{143}$$

With $v(t_0) := (0,1)^\top$, hence $\|v(t_0)\| = 1$, and $s := t - t_0 \geq 0$ we get

$$v_1(t) = te^{\lambda_2(t-t_0)} - t_0 e^{\lambda_1(t-t_0)} \geq t_0\left(e^{\lambda_2 s} - e^{\lambda_1 s}\right) \geq 0. \tag{144}$$

If $\mathcal{B}_2$ was u.e.s. there would exist $c, \alpha > 0$ such that

$$v_1(t) \leq ce^{-\alpha(t-t_0)} \leq c \Longrightarrow \forall t_0 > 0 \forall s \geq 0:\ t_0\left(e^{\lambda_2 s} - e^{\lambda_1 s}\right) \leq c. \tag{145}$$

For $s := 1$ and $t_0 > c\left(e^{\lambda_2} - e^{\lambda_1}\right)^{-1}$ this yields a contradiction. This is no contradiction to [21, Thm. 6.15] since $P(t)$ is not a *Lyapunov transformation*.

Recall $\mathbf{A} = \mathbf{K}[\partial; d/dz] = \mathbf{K}[-z^2\partial; -z^2 d/dz] \ni f = \sum_{j\in\mathbb{N}} f_j(-z^2\partial)^j,\ f_j \in \mathbf{K}$.

Lemma 4.11. *Weak exponential stability is preserved by isomorphisms, i.e., if* $\mathbf{A}^{1\times q_1}/U_1 \cong \mathbf{A}^{1\times q_2}/U_2$ *and if* $\mathcal{B}(U_1)$ *is w.e.s. then so is* $\mathcal{B}(U_2)$ $(\cong \mathcal{B}(U_1))$*. In other words: The w.e.s. of a behavior* $\mathcal{B}(U)$ *of a submodule* $U \subseteq \mathbf{A}^{1\times q}$ *depends on the factor module* $M = \mathbf{A}^{1\times q}/U$ *only, but not on the special presentation of* M*.*

Proof. Consider operators $P_1 \in \mathbf{A}^{q_1\times q_2}$ and $P_2 \in \mathbf{A}^{q_2\times q_1}$ that induce isomorphisms

$$(\circ P_1)_{\rm ind} : \mathbf{A}^{1\times q_1}/U_1 \xrightarrow{\cong} \mathbf{A}^{1\times q_2}/U_2,\ U_i = \mathbf{A}^{1\times p_i}R_i,$$
$$\text{and its inverse } (\circ P_1)_{\rm ind}^{-1} = (\circ P_2)_{\rm ind} \text{ where } P_i = \sum_{j=0}^{d_i} P_{ij}(-z^2\partial)^j. \tag{146}$$

Choose $\tau_0 \geq \max(\sigma(P_1), \sigma(P_2))$ sufficiently large such that for all $\tau \geq \tau_0$ the operators P_1 and P_2 induce inverse isomorphisms

$$P_2\circ : \mathcal{B}(R_1,\tau) \overset{\cong}{\longleftrightarrow} \mathcal{B}(R_2,\tau) : P_1\circ,\ w_1 = P_1\circ w_2 \longleftrightarrow w_2 = P_2 \circ w_1. \tag{147}$$

Assume that $\mathcal{B}(U_1) = \mathrm{cl}\left((\mathcal{B}(R_1,\tau), \mathrm{res})_{\tau\geq\tau_0}\right)$ is w.e.s. and therefore

$$\exists\tau_1 > \tau_0 \exists d \geq 0 \exists\alpha,\mu > 0 \forall m \in \mathbb{N} \exists \text{ p.g.f. } \varphi_m > 0 \forall t \geq t_0 > \tau \geq \tau_1$$
$$\forall w_1 \in \mathcal{B}(R_1,\tau) : \|(\partial(m)\circ w_1)(t)\| \leq \varphi_m(t_0)e^{-\alpha(t^\mu - t_0^\mu)}\|(\partial(d)\circ w_1)(t_0)\|. \tag{148}$$

Let $\tau \geq \tau_1$ and consider an arbitrary $w_2 \in \mathcal{B}(R_2,\tau)$ and $m \in \mathbb{N}$ and define

$$w_1 := P_1 \circ w_2 \Longrightarrow w_1(t) = \sum_{j=0}^{d_1} P_{1j}(t) w_2^{(j)}(t) \text{ and } w_2 := P_2 \circ w_1 \Longrightarrow$$
$$\forall k \in \mathbb{N} :\ w_1^{(k)} = \sum_{i=0}^{k}\sum_{j=0}^{d_1}\binom{k}{i} P_{1j}^{(k-i)} w_2^{(i+j)},\ P_{1j}^{(k-i)}(t) := d^{k-i}P_{1j}(t)/dt^{k-i}. \tag{149}$$

The norms $\|P_{1j}^{(k-i)}(t)\|$ are p.g.f. on $[\tau_1,\infty)$. Taking norms in (149) thus furnishes a p.g.f. $\psi_1 > 0$ on $[\tau_1,\infty)$ such that

$$\forall t > \tau :\ \|(\partial(d)\circ w_1)(t)\| \leq \psi_1(t)\|(\partial(d+d_1)\circ w_2)(t)\|. \tag{150}$$

Likewise $w_2 = P_2 \circ w_1$ and there is a p.g.f. $\psi_2 > 0$ on $[\tau_1,\infty)$ such that

$$\forall t > \tau :\ \|w_2^{(m)}(t)\| \leq \psi_2(t)\|(\partial(m+d_2)\circ w_1)(t)\|. \tag{151}$$

Together the preceding inequalities furnish, for $t \geq t_0 > \tau$,

$$\|w_2^{(m)}(t)\| \underset{(148),(151)}{\leq} \psi_2(t)\varphi_{m+d_2}(t_0)e^{-\alpha(t^\mu - t_0^\mu)}\|(\partial(d)\circ w_1)(t_0)\|$$
$$\underset{(150)}{\leq} \psi_2(t)e^{-\alpha(t^\mu - t_0^\mu)}\varphi_{m+d_2}(t_0)\psi_1(t_0)\|(\partial(d+d_1)\circ w_2)(t_0)\|. \tag{152}$$

We choose β with $0 < \beta < \alpha$. Lemma 4.2 furnishes a p.g.f. $\psi_3 > 0$ on $[\tau_1,\infty)$ such that for all $t \geq t_0 > \tau$

$$\psi_2(t)e^{-\alpha(t^\mu - t_0^\mu)} \leq \psi_3(t_0)e^{-\beta(t^\mu - t_0^\mu)} \underset{(152)}{\Longrightarrow}$$
$$\|w_2^{(m)}(t)\| \leq \left(\varphi_{m+d_2}(t_0)\psi_1(t_0)\psi_3(t_0)\right)e^{-\beta(t^\mu - t_0^\mu)}\|(\partial(d+d_1)\circ w_2)(t_0)\|. \tag{153}$$

Since $\varphi_{m+d_2}(t_0)\psi_1(t_0)\psi_3(t_0)$ is also a p.g.f. the preceding inequality implies the w.e.s. of $\mathcal{B}(U_2)$ with memory size $d + d_1$. ☐

The validity of the preceding lemma suggested the definition of w.e.s. in Def. 2.4. Recall that u.e.s. of state space behaviors is not preserved by behavior isomorphisms and necessitates the introduction of so-called Lyapunov transformations [21, Def. 6.14] that are not defined for arbitrary behaviors, but only for state space equations.

Corollary 4.12. *In the situation of Cor. 4.8 the behavior $\mathcal{B}(U)$ is w.e.s. if and only if $\mathcal{B}(\mathbf{A}^{1\times d}(-z^2\partial\,\mathrm{id}_d-F))$ is w.e.s., ie., if the first inequality of* (130) *is satisfied for $-z^2\partial\,\mathrm{id}_d-F$.*

Definition 4.13. A f.g. $\mathbf{A}$-module is called w.e.s. if and only if for one and thus for all representations $M=\mathbf{A}^{1\times q}/U$ the behavior $\mathcal{B}(U)$ is w.e.s..

Notice that in contrast to most situations in systems theory the analytic property w.e.s. of $\mathcal{B}(U)$ is the primary property whereas the algebraic property w.e.s. of M is the derived one.

4.3 The proof of Theorem 2.7

In this section we prove Thm. 2.7. For the needed algebraic preparations we refer to [5, §3.3]

We assume an exact sequence of f.g. $\mathbf{A}$-modules

$$0\to M_1\xrightarrow{\varphi}M_2\xrightarrow{\psi}M_3\to 0 \tag{154}$$

of f.g. $\mathbf{A}$-modules. After the choice of presentations $M_i=\mathbf{A}^{1\times\ell_i}/U_i$ the exact sequence (154) induces an exact behavior sequence

$$0\leftarrow\mathcal{B}_1\xleftarrow{\mathcal{B}(\varphi)}\mathcal{B}_2\xleftarrow{\mathcal{B}(\psi)}\mathcal{B}_3\leftarrow 0 \tag{155}$$

For Thm. 2.7 we are going to show that $\mathcal{B}(U_2)$ is e.s. if and only if $\mathcal{B}(U_1)$ and $\mathcal{B}(U_3)$ are. Due to Lemma 4.11 we may choose the representations arbitrarily. According to Cor. 4.8 and [5, Lemma 3.9] we may and do assume that the sequence (154) has the special form

$$\begin{gathered}0\to\mathbf{A}^{1\times q_1}/U_1\xrightarrow{(\circ(0,\mathrm{id}_{q_1}))_{\mathrm{ind}}}\mathbf{A}^{1\times q_2}/U_2\xrightarrow{\left(\circ\left(\begin{smallmatrix}\mathrm{id}_{q_3}\\0\end{smallmatrix}\right)\right)_{\mathrm{ind}}}\mathbf{A}^{1\times q_3}/U_3\to 0\\ \text{where } q_2=q_3+q_1,\ U_i=\mathbf{A}^{1\times q_i}R_i,\ R_i\in\mathbf{A}^{q_i\times q_i},\ i=1,2,3,\\ R_1=-z^2\partial\,\mathrm{id}_{q_1}-F_1,\ R_3=-z^2\partial\,\mathrm{id}_{q_3}-F_3,\ F_i\in\mathbf{K}^{q_i\times q_i},\\ R_2=\left(\begin{smallmatrix}R_3&R\\0&R_1\end{smallmatrix}\right),\ R\in\mathbf{A}^{q_3\times q_1}.\end{gathered} \tag{156}$$

For sufficiently large $\tau_0>\max\{\sigma(R_i);\ i=1,2,3\}$, $t_0>\tau\geq\tau_0$ and $W(\tau)=\mathrm{C}^\infty(\tau,\infty)$ there result the exact behavior sequences

$$\begin{array}{ccccccccc}0&\longleftarrow&W(\tau)^{q_1}&\xleftarrow{(0,\mathrm{id}_{\ell_1})\circ}&W(\tau)^{q_3+q_1}&\xleftarrow{\left(\begin{smallmatrix}\mathrm{id}_{\ell_3}\\0\end{smallmatrix}\right)\circ}&W(\tau)^{q_3}&\longleftarrow&0\\ &&\uparrow\subseteq&&\uparrow\subseteq&&\uparrow\subseteq&&\\ 0&\longleftarrow&\mathcal{B}(R_1,\tau)&\longleftarrow&\mathcal{B}(R_2,\tau)&\longleftarrow&\mathcal{B}(R_3,\tau)&\longleftarrow&0\\ &&w_1&\longleftarrow\!\shortmid&\left(\begin{smallmatrix}w_3\\w_1\end{smallmatrix}\right),\ \left(\begin{smallmatrix}w_3\\0\end{smallmatrix}\right)&\longleftarrow\!\shortmid&w_3&&\end{array} \tag{157}$$

and

$$\begin{aligned}
&\mathcal{B}(R_1,\tau)=\{w_1\in W(\tau)^{q_1};\ w_1'(t)=F_1(t)w_1(t)\}\cong\mathbb{C}^{q_1},\ w_1\mapsto w_1(t_0),\\
&\mathcal{B}(R_3,\tau)=\{w_3\in W(\tau)^{q_3};\ w_3'(t)=F_3(t)w_3(t)\}\cong\mathbb{C}^{q_3},\ w_3\mapsto w_3(t_0),\\
&\mathcal{B}(R_2,\tau)=\left\{\left(\begin{smallmatrix}w_3\\ w_1\end{smallmatrix}\right)\in W(\tau)^{q_3+q_1};\ (*)\right\}\\
&\ \text{where } (*):\ w_1'(t)=F_1(t)w_1(t),\ w_3'(t)=F_3(t)w_3(t)-(R\circ w_1)(t)
\end{aligned}\tag{158}$$

The next theorem coincides with Thm. 2.7.

Theorem 4.14. *Weakly exponentially stable behaviors form a Serre subcategory of the abelian category of all LTV behaviors. For the data from* (154) *and* (155) *or, without loss of generality, from from* (156) *and* (157)*, this signifies that $\mathcal{B}_2$ is w.e.s. if and only if $\mathcal{B}_1$ and $\mathcal{B}_3$ are.*

Proof. We assume (156) and (157). The time instant τ_0 in (157) is chosen sufficiently large.
1. If $\mathcal{B}_2$ is w.e.s then so are $\mathcal{B}_1$ and $\mathcal{B}_3$: The proof is fully analogous to the parts 1. and 2. of the proof of [5, Thm. 3.11] and therefore omitted.
2. Assume that $\mathcal{B}_1$ and $\mathcal{B}_3$ are w.e.s.: Choose $m_1\in\mathbb{N}$. By assumption and Lemma 4.4 there are $\tau_1>\tau_0$, $\mu,\alpha>0$ and a p.g.f. φ on $[\tau_1,\infty)$ such that

$$\begin{aligned}
&\forall t\geq t_0>\tau\geq\tau_1\forall w_i\in\mathcal{B}(-z^2\partial\,\mathrm{id}_{q_i}-F_i,\tau),\ i=1,3:\\
&\qquad\|(\partial(m_1)\circ w_i)(t)\|\leq\varphi(t_0)e^{-\alpha(t^\mu-t_0^\mu)}\|w_i(t_0)\|,\ i=1,3
\end{aligned}\tag{159}$$

W.l.o.g. due to Lemma 4.1 we here assume the same μ,α,φ for both w_3 and w_1. Notice that all three w_i can be uniquely extended from (τ,∞) to (τ_0,∞). Let $w_2=\left(\begin{smallmatrix}w_3\\ w_1\end{smallmatrix}\right)\in\mathcal{B}(R_2,\tau)$ be arbitrary . Then $w_i\in W(\tau)^{q_i},\ i=1,3,$ are the unique solutions of

$$\begin{aligned}
&w_1'(t)=F_1(t)w_1(t)\text{ with initial value }w_1(t_0)\\
&w_3'(t)=F_3(t)w_3(t)+u(t)\text{ with initial value }w_3(t_0)\text{ and }u(t):=-(R\circ w_1)(t).
\end{aligned}\tag{160}$$

Let $m\in\mathbb{N}$ and $d:=\deg_\partial(R)$ and define $m_1:=d+m$. Choose α_2 with $0<\alpha_2<\alpha$. According to Lemma (4.3) and (150) there are p.g.f. $\varphi_2,\varphi_4>0$ on $[\tau_1,\infty)$ such that

$$\begin{aligned}
\forall t\geq t_0:\ &\|(\partial(m)\circ u)(t)\|\leq\varphi_2(t)\|(\partial(m+d)\circ w_1)(t)\|\\
\underset{(159)}{\Longrightarrow}\ &\|(\partial(m)\circ u)(t)\|\leq\varphi_2(t)\varphi(t_0)e^{-\alpha(t^\mu-t_0^\mu)}\|w_1(t_0)\|\\
\underset{(114)}{\Longrightarrow}\ &\|(\partial(m)\circ u)(t)\|\leq\varphi_4(t_0))e^{-\alpha_2(t^\mu-t_0^\mu)}\|w_1(t_0)\|.
\end{aligned}\tag{161}$$

This shows that $u=-R\circ w_1$ satisfies the condition (130). The inequality (159) for w_3 implies that also $w'=F_3w$ satisfies the condition of (130). We now apply Lemma 4.5 to $w_3'=F_3w_3+u$ and obtain $0<\alpha_4<\alpha_2$ and a p.g.f. φ_5 on $[\tau_1,\infty)$ such that

$$\begin{aligned}
\forall t\geq t_0:\ &\|w_3^{(m)}(t)\|\leq\varphi_5(t_0)e^{-\alpha_4(t^\mu-t_0^\mu)}\|w_2(t_0)\|,\\
&\text{with }\|w_2(t_0)\|=\max\left(\|w_3(t_0)\|,\|w_1(t_0)\|\right).
\end{aligned}\tag{162}$$

The inequalities (159) for w_1 and (162) for w_3 finally imply for all $t\geq t_0$

$$\begin{aligned}
&\|w_2^{(m)}(t)\|=\max\left(\|w_3^{(m)}(t)\|,\|w_1^{(m)}(t)\|\right)\leq\psi(t_0)e^{-\alpha_4(t^\mu-t_0^\mu)}\|w_2(t_0)\|\\
&\qquad\text{where }\psi(t)=\max(\varphi(t),\varphi_5(t)).
\end{aligned}\tag{163}$$

□

Recall from Def. 4.13 that a f.g. $\mathbf{A}$-module is w.e.s. if for one or for all representations $M = \mathbf{A}^{1\times q}/U$ the behavior $\mathcal{B}(U)$ is w.e.s..

Corollary and Definition 4.15. *The f.g. weakly exponentially stable $\mathbf{A}$-modules form a Serre subcategory of ${}_{\mathbf{A}}\mathbf{Mod}$. Every f.g. $\mathbf{A}$-module M has a largest w.e.s. submodule $\mathrm{Ra}_{wes}(M)$ that is called the* w.e.s. radical *of M. Moreover $\mathrm{Ra}_{wes}\left(M/\,\mathrm{Ra}_{wes}(M)\right) = 0$.*

Corollary 4.16. *(i) Let $f = f_1 f_2$ be a nonzero product in $\mathbf{A}$, hence $0 \neq \mathbf{A}f \subseteq \mathbf{A}f_2$. Then f is weakly exponentially stable if and only if f_1 and f_2 are.*
(ii) If $M \cong \mathbf{A}/\mathbf{A}f$ is any f.g. torsion module and hence cyclic and if $f = f_1 \cdots f_r$ is a decomposition of f into irreducible factors f_i then M is w.e.s. if and only if all $\mathbf{A}/\mathbf{A}f_i$ are.

Proof. The application of Thm. 4.14 to the exact sequences

$$\begin{array}{c} 0 \to \mathbf{A}/\mathbf{A}f_1 \xrightarrow{(\circ f_2)_{\mathrm{ind}}} \mathbf{A}/\mathbf{A}f \xrightarrow{\mathrm{can}} \mathbf{A}/\mathbf{A}f_2 \to 0 \\ a + \mathbf{A}f_1 \longmapsto af_2 + \mathbf{A}f,\ b + \mathbf{A}f \longmapsto b + \mathbf{A}f_2 \\ \\ 0 \leftarrow \mathcal{B}(\mathbf{A}f_1) \xleftarrow{\mathcal{B}(f_2)} \mathcal{B}(\mathbf{A}f) \xleftarrow{\supseteq} \mathcal{B}(\mathbf{A}f_2) \leftarrow 0 \end{array} \tag{164}$$

furnishes the result.
(ii) follows from (i) by induction. □

Recall that $0 \neq g \in \mathbf{A}$ is irreducible if $\mathbf{A}g$ is a maximal left ideal or $\mathbf{A}/\mathbf{A}g$ is simple.

5 Algebraic characterization of exponential stability

5.1 Lattices

Weak exponential stability (w.e.s.) of an autonomous system $\mathcal{B} = \mathcal{B}(U)$ with $U = \mathbf{A}^{1\times p}R$, $R \in \mathbf{A}^{p\times q}$ and $\mathrm{rank}(R) = q$ or, equivalently, of its system module $M = \mathbf{A}^{1\times q}/U$ with $n := \dim_{\mathbf{K}}(M) < \infty$ is defined by analytic properties of the trajectories of $\mathcal{B}$. In Section 5 we characterize the w.e.s. of $\mathcal{B}$ by that of associated complex matrices for most autonomous systems.
Such U, M and $\mathcal{B}$ are given in the sequel. A f.g. $\mathbb{C} << z >> [\partial; d/dz]$-torsion module is a f.d. $\mathbb{C} << z >>$-vector space and is also called a *meromorphic connection* [16, Def. 4.3.1].
From the Sections 3.1 and 3.2 recall the valuation ring $\mathbf{L}$, its quotient field $\mathbf{K}$ with its valuation v and derivation d/dz and the algebra $\mathbf{A}$ of differential operators:

$$\begin{aligned} &\mathbf{L} = \bigcup_{m\geq 1} \mathbf{L}_m \subset \mathbf{K} = \bigcup_{m\geq 1} \mathbf{K}_m,\ \mathbf{L}_m = \mathbb{C} < z^{1/m} > \subset \mathbf{K}_m = \mathbb{C} << z^{1/m} >> \\ &\qquad \text{with } v\left(\sum_{i=k}^{\infty} a_i z^{i/m}\right) = k/m \text{ if } k \in \mathbb{Z},\ a_k \neq 0,\ v(0) = \infty \\ &\qquad \text{and } d/dz\left(\sum_{i=k}^{\infty} a_i z^{i/m}\right) = \sum_{i=k}^{\infty} a_i (i/m) z^{(i/m)-1} \\ &\mathbf{A} = \mathbf{K}[\partial; d/dz] = \mathbf{K}[b\partial; bd/dz] = \oplus_{j=0}^{\infty}\mathbf{K}(b\partial)^j,\ 0 \neq b \in \mathbf{K}. \end{aligned} \tag{165}$$

Since

$$v\left(d/dz(z^{1/m})\right) = v\left((1/m)z^{(1/m)-1}\right) = -(m-1)m^{-1} < 0 \text{ for } m > 1 \tag{166}$$

the algebra $\mathbf{L}$ is not invariant under d/dz and hence $\mathbf{L}[\partial; d/dz]$ does not exist. However, for $\delta := z\partial \in \mathbf{A}$, $a \in \mathbb{C} < z >$ and hence $v(a(z^{1/m})) \geq 0$ we get

$$\forall m \in \mathbb{N}: \ \delta a(z^{1/m}) = a(z^{1/m})\delta + m^{-1}z^{1/m}a'(z^{1/m}) \in \mathbf{L}[\delta] := \oplus_j \mathbf{L}\delta^j \tag{167}$$

and hence $\mathbf{L}[\delta; zd/dz] = \mathbf{L}[\delta]$ is a subalgebra of $\mathbf{A}$. A f.g. generated $\mathbf{L}$-submodule N of M is free since $\mathbf{L}$ is a Bézout domain and called a *lattice* of ${}_{\mathbf{K}}M$ if $\mathbf{K}N = M$ [3, §VII.4.1], [20, Def. 3.7]. An $\mathbf{L}$-basis of N is also a $\mathbf{K}$-basis of M. In general the lattice N is not invariant under δ.

5.2 Regular singular equations

The following derivations are a simple adaption of [16, Thm. 1.1.1, p. 45] to the situation here. The notations of Section 5.1 are in force.

Definition 5.1. The f.g. torsion module $M = \mathbf{A}^{1\times q}/U$ is called *regular singular* [20, Def. 3.9], if M contains a lattice N with $\delta N \subseteq N$ where $\delta := z\partial$.
A f.g. regular singular $\mathbb{C} << z >> [\partial; d/dz]$-torsion module is also called a *regular meromorphic connection* or *meromorphic connection with regular singularity* [16, Def. I.5.1.1, Cor. II.1.1.6].

Obviously regular singularity is preserved by isomorphisms and assumed in this section. Let $\mathbf{v} = (\mathbf{v}_1, \cdots, \mathbf{v}_n)^\top \in N^n$ be an $\mathbf{L}$-basis of N and $\mathbf{e} := (\mathbf{e}_1, \cdots, \mathbf{e}_n)^\top$ the standard basis of $\mathbb{C}^{1\times n} \subset \mathbf{A}^{1\times n}$. Since $\delta N \subseteq N$ and $\mathbf{L} = \bigcup_{m\geq 1} \mathbb{C} < z^{1/m} >$ there are $m \geq 1$ and $A \in \mathbb{C} < z >^{n\times n}$ such that

$$\delta\mathbf{v} = A(z^{1/m})\mathbf{v}. \tag{168}$$

The epimorphism $\mathbf{A}^{1\times n} \to M, \ \mathbf{e}_j \mapsto \mathbf{v}_j$, induces the isomorphism

$$\begin{array}{ccc} \mathbf{A}^{1\times n}/\mathbf{A}^{1\times n}(\delta\,\mathrm{id}_n - A(z^{1/m})) & \cong & M \\ \overline{\xi} := \xi + \mathbf{A}^{1\times n}(\delta\,\mathrm{id}_n - A(z^{1/m})) & \mapsto & \xi\mathbf{v} = \sum_{j=1}^n \xi_j \mathbf{v}_j \end{array}, \quad \text{hence } \mathcal{B}\left(\mathbf{A}^{1\times n}(\delta\,\mathrm{id}_n - A(z^{1/m}))\right) \cong \mathcal{B}(U). \tag{169}$$

Recall

$$\begin{aligned} \mathbf{K}_m = \mathbb{C} << z^{1/m} >> \subset \mathbf{A}_m := \mathbf{K}_m[\delta; zd/dz], \text{ especially} \\ \mathbf{K}_1 = \mathbb{C} << z >> \subset \mathbf{A}_1 := \mathbb{C} << z >> [\delta; zd/dz]. \end{aligned} \tag{170}$$

Define $V_m := \oplus_{j=1}^n \mathbf{K}_m \mathbf{v}_j \subset M$. The equation $\delta\mathbf{v} = A(z^{1/m})\mathbf{v}$ implies that V_m is an $\mathbf{A}_m$-submodule of M of dimension $\dim_{\mathbf{K}_m}(V_m) = n$ and that

$$\begin{aligned} \mathbf{A}_m^{1\times n}/\mathbf{A}_m^{1\times n}(\delta\,\mathrm{id}_n - A(z^{1/m})) \underset{\mathbf{A}_m}{\cong} V_m, \ \overline{\mathbf{e}_j} \mapsto \mathbf{v}_j, \Longrightarrow \\ M \underset{\mathbf{A}}{\cong} \mathbf{A}^{1\times n}/\mathbf{A}^{1\times n}(\delta\,\mathrm{id}_n - A(z^{1/m})) \underset{\mathbf{A}}{\cong} \mathbf{A} \otimes_{\mathbf{A}_m} V_m = \mathbf{K} \otimes_{\mathbf{K}_m} V_m. \end{aligned} \tag{171}$$

Define $V := \mathbf{A}_1^{1\times n}/\mathbf{A}_1^{1\times n}(\delta\,\mathrm{id}_n - mA(z))$. This is an $\mathbf{A}_1$-torsion module with the $\mathbf{K}_1$-basis $\overline{\mathbf{e}}$ and $\delta\overline{\mathbf{e}} = mA(z)\overline{\mathbf{e}}$. According to [20, Thm. 5.1], [16, Cor. 1.1.6,(3),(4), p.

47] there is a *constant* matrix $B_1 \in \mathbb{C}^{n\times n}$ and $P \in \mathrm{Gl}_n(\mathbb{C} < z >)$ with $P(0) = \mathrm{id}_n$ such that

$$\begin{gathered} zP' = (mA)P - PB_1 \Longrightarrow (\delta\,\mathrm{id}_n - mA)P = P(\delta\,\mathrm{id}_n - B_1) \Longrightarrow \\ (\circ P)_{\mathrm{ind}} : V = \mathbf{A}_1^{1\times n}/\mathbf{A}_1^{1\times n}\,(\delta\,\mathrm{id}_n - mA(z)) \underset{\mathbf{A}_1}{\cong} \mathbf{A}_1^{1\times n}/\mathbf{A}_1^{1\times n}\,(\delta\,\mathrm{id}_n - B_1)\,. \end{gathered} \tag{172}$$

The equations

$$\begin{aligned} &\delta a(z) = a(z)\delta + za'(z) \in \mathbf{A}_1 \text{ and} \\ &\delta a(z^{1/m}) = a(z^{1/m})\delta + m^{-1}z^{1/m}a'(z^{1/m}) \in \mathbf{A}_m \text{ imply} \\ &(m\delta)a(z^{1/m}) = a(z^{1/m})(m\delta) + z^{1/m}a'(z^{1/m}) \in \mathbf{A}_m \end{aligned} \tag{173}$$

and, via Lemma 3.4, the algebra isomorphism

$$\mathbf{A}_1 = \mathbf{K}_1[\delta; zd/dz] \cong \mathbf{A}_m = \mathbf{K}_m[\delta; zd/dz],\ \sum_j a_j(z)\delta^j \mapsto \sum_j a_j(z^{1/m})(m\delta)^j. \tag{174}$$

The isomorphisms (172) and (174) imply the isomorphism

$$\begin{aligned} V_m &\underset{\mathbf{A}_m}{\cong} \mathbf{A}_m^{1\times n}/\mathbf{A}_m^{1\times n}\left(\delta\,\mathrm{id}_n - A(z^{1/m})\right) = \mathbf{A}_m^{1\times n}/\mathbf{A}_m^{1\times n}\left((m\delta)\,\mathrm{id}_n - mA(z^{1/m})\right) \\ &\underset{\mathbf{A}_m}{\cong} \mathbf{A}_m^{1\times n}/\mathbf{A}_m^{1\times n}\left((m\delta)\,\mathrm{id}_n - B_1\right) = \mathbf{A}_m^{1\times n}/\mathbf{A}_m^{1\times n}\left(\delta\,\mathrm{id}_n - B\right),\ B := m^{-1}B_1. \end{aligned} \tag{175}$$

Together with the isomorphism (169), (171) we get the isomorphisms

$$\begin{aligned} M &\underset{\mathbf{A}}{\cong} \mathbf{A}^{1\times n}/\mathbf{A}^{1\times n}\left(\delta\,\mathrm{id}_n - A(z^{1/m})\right) \underset{\mathbf{A}}{\cong} \mathbf{A}^{1\times n}/\mathbf{A}^{1\times n}\left(\delta\,\mathrm{id}_n - B\right) \\ \mathcal{B}(U) &\cong \mathcal{B}\left(\mathbf{A}^{1\times n}(\delta\,\mathrm{id}_n - A(z^{1/m}))\right) \cong \mathcal{B}\left(\mathbf{A}^{1\times n}\left(\delta\,\mathrm{id}_n - B\right)\right). \end{aligned} \tag{176}$$

The equations $\delta = z\partial = -z^{-1}(-z^2\partial)$ and $(-z^2\partial)\circ v = v'$ for $v \in W(0,\infty)$ imply

$$\begin{gathered} \delta \circ v = -tv',\ \text{hence } \forall x \in W(0,\infty)^n :\ (\delta\,\mathrm{id}_n - B)\circ x = -tx' - Bx \Longrightarrow \\ \mathcal{B}(\delta\,\mathrm{id}_n - B, 0) = \left\{x \in W(0)^n;\ x' + t^{-1}Bx = 0\right\}. \end{gathered} \tag{177}$$

The transition matrix of the state space system

$$x' + t^{-1}Bx = 0 \text{ is } \Phi(t,t_0) = \exp\left(-(\ln(t) - \ln(t_0))B\right) \text{ for } t, t_0 > 0. \tag{178}$$

Corollary 5.2. *If the module is regular singular then it is weakly exponentially stable if and only if $M = 0$, hence Thm. 2.8,(i), holds.*

Proof. Since w.e.s. is preserved by isomorphism the isomorphism (176) and equation (177) show that this has to be shown for the state system $x' + t^{-1}Bx = 0$ only. Assume that $M \neq 0$ and thus $n > 0$ and that this equation is w.e.s. Then for sufficiently large τ there are constants $\alpha, \mu > 0$ and a p.g.f. $\varphi(t) > 0$ such that (cf. (127))

$$\forall t \geq t_0 > \tau :\ \|x(t)\| \leq \varphi(t_0)\exp\left(-\alpha(t^\mu - t_0^\mu)\right)\|x(t_0)\|,\ \alpha, \mu > 0. \tag{179}$$

Let $x(t_0)$ be a nonzero eigenvector of B for the eigenvalue λ. Then

$$\begin{aligned} x(t) &= \Phi(t,t_0)x(t_0) = \exp\left(-(\ln(t) - \ln(t_0))B\right)x(t_0) \\ &= \exp\left(-(\ln(t) - \ln(t_0))\lambda\right)x(t_0) \Longrightarrow \\ \|x(t)\| &= \exp\left(-(\ln(t) - \ln(t_0))\Re(\lambda)\right)\|x(t_0)\| = t^{-\Re(\lambda)}t_0^{\Re(\lambda)}\|x(t_0)\|. \end{aligned} \tag{180}$$

Even for $\Re(\lambda) > 0$ the power $t^{-\Re(\lambda)}$ decreases more slowly than $\exp(-\alpha t^{\mu})$ and hence the inequality (179) cannot hold. This implies $M = 0$ and the theorem.

□

5.3 Irregular singular equations

This Section is devoted to the proof of Thm. 2.8,(ii) and (iii),(a),(b), (c).

Definition 5.3. The torsion module $M = \mathbf{A}^{1\times q}/U$ is called *irregular singular* if it is not regular singular.

In this Section 5.3 we assume such an M. Again this property is preserved by isomorphisms. Thm. 5.4 below is an important standard result. We carry out its simple proof since those of [4, (6.37)] and [20, p. 68] are only indications and contain slight errors. Since every f.g. torsion module is cyclic we assume an isomorphism

$$M \cong \mathbf{A}/\mathbf{A}L, L = \sum_{i=0}^{d} a_i\delta^{d-i} \in \mathbf{A} = \mathbf{K}[\delta; zd/dz],\ \delta := z\partial,\ a_0 = 1, a_i \in \mathbf{K}. \tag{181}$$

This L is also called irregular singular.We write it in the form

$$L = (1, a_1, \cdots, a_d)\mathbf{D},\ \mathbf{D} := (\delta^d, \delta^{d-1}, \cdots .\delta, 1)^\top \in \mathbf{A}^{d+1}. \tag{182}$$

If $L \in \mathbf{L}[\delta; zd/dz]$ then

$$N := \mathbf{L}[\delta; zd/dz]/\mathbf{L}[\delta; zd/dz]L = \oplus_{i=0}^{d-1}\mathbf{L}\overline{\delta^i} \subset \mathbf{A}/\mathbf{A}L = \oplus_{i=0}^{d-1}\mathbf{K}\overline{\delta^i} \tag{183}$$

and N is a $\mathbf{L}[\delta; zd/dz]$-submodule of $\mathbf{A}/\mathbf{A}L$ and a lattice with $\delta N \subseteq N$. By definition this implies that $\mathbf{A}/\mathbf{A}L$ is regular singular. Since this is excluded in Section 5.3 we conclude that not all coefficients a_i of L belong to $\mathbf{L}$ and that hence $v(a_i) < 0$ for at least one $i = 1, \cdots, d$.

Theorem 5.4. *([20, p. 68], [4, modified (6.37)]) Assume that*

$$L = (z\partial)^d + a_1(z\partial)^{d-1} + \cdots + a_d \in \mathbf{A} = \mathbf{K}[z\partial; zd/dz] \tag{184}$$

is irregular singular, i.e., there is a coefficient $a_i \notin \mathbf{L}$ or with $v(a_i) < 0$. Define

$$\lambda := \min\{v(a_i)/i;\ i = 1, \cdots, d\} < 0,\ E := z^{-\lambda}(z\partial) = z^{1-\lambda}\partial. \tag{185}$$

Then

$$z^{-\lambda d}L = E^d + b_1E^{d-1} + \cdots + b_d \quad \text{where } b_i \in \mathbf{L} = \{a \in \mathbf{K};\ v(a) \geq 0\},\ \exists k \text{ with } v(b_k) = 0. \tag{186}$$

Proof. (i) We show inductively that

$$E^k = z^{-k\lambda}p_k(\delta),\ p_k \in \mathbb{C}[\delta],\ \deg_\delta(p_k) = k,\ p_k \text{ monic}. \tag{187}$$

For $k = 0, 1$ this is obviously true with $p_0 = 1, p_1 = \delta$. Inductively we then get

$$E^{k+1} = EE^k = z^{-\lambda}(\delta z^{-k\lambda})p_k(\delta) = z^{-\lambda}\left(z^{-k\lambda}\delta - k\lambda z^{-k\lambda}\right)p_k(\delta) = z^{-(k+1)\lambda}(\delta - k\lambda)p_k(\delta) = z^{-(k+1)\lambda}p_{k+1}(\delta),\ p_{k+1}(\delta) := (\delta - k\lambda)p_k(\delta). \tag{188}$$

(ii) Define

$$\mathbf{E} := (E^d, E^{d-1}, \cdots, E, 1)^\top \in \mathbf{A}^{d+1}. \tag{189}$$

Equation (186) furnishes an upper triangular matrix $U \in \mathrm{Gl}_{d+1}(\mathbb{C})$ with 1 in the main diagonal, $U_{ii} = 1$, and the diagonal matrix $D = \mathrm{diag}(z^{-d\lambda}, z^{-(d-1)\lambda}, \cdots, 1)$ such that

$$\begin{aligned}\mathbf{E} = DU\mathbf{D} \Longrightarrow \mathbf{D} = U^{-1}D^{-1}\mathbf{E} \Longrightarrow z^{-d\lambda}L = z^{-d\lambda}(1, a_1, \cdots, a_d)\mathbf{D}\\ = (1, a_1, \cdots, a_d)U^{-1}(z^{-d\lambda}D^{-1})\mathbf{E} =: (b_0, b_1, \cdots, b_d)\mathbf{E}\\ \Longrightarrow (b_0, b_1, \cdots, b_d) = (1, a_1, \cdots, a_d)U^{-1}D_1\\ \text{with } D_1 := z^{-d\lambda}D^{-1} = \mathrm{diag}(1, z^{-\lambda}, \cdots, z^{-d\lambda}).\end{aligned} \tag{190}$$

The matrix U^{-1} is also upper triangular with 1 in the main diagonal. The preceding equation thus implies

$$\begin{aligned}b_0 = 1,\ \forall j \geq 1:\ b_j &= \sum_{i=0}^{j} a_i(U^{-1})_{ij}z^{-j\lambda} = \sum_{i=0}^{j}(a_i z^{-j\lambda})(U^{-1})_{ij}\\ &= a_j z^{-j\lambda} + \sum_{i=0}^{j-1}(a_i z^{-j\lambda})(U^{-1})_{ij}.\end{aligned} \tag{191}$$

The definition of $\lambda < 0$ furnishes

$$\begin{aligned}&\forall j \geq i \geq 0:\ v(a_i z^{-j\lambda}) = \begin{cases} j\left((v(a_j)/j) - \lambda\right) \geq 0 & \text{if } i = j\\ i\left((v(a_i)/i) - \lambda\right) + (j-i)(-\lambda) > 0 & \text{if } i < j\end{cases} \Longrightarrow\\ &v(b_j) = \begin{cases} 0 & \text{if } v(a_j)/j = \lambda\\ > 0 & \text{if } v(a_j)/j > \lambda\end{cases} \Longrightarrow b_j \in \mathbf{L},\ \exists k \text{ with } v(b_k) = 0 \Longrightarrow\\ &z^{-d\lambda}L = E^d + b_1E^{d-1} + \cdots \in \mathbf{L}[E; z^{1-\lambda}d/dz],\ E := z^{-\lambda}\delta = z^{1-\lambda}\partial.\end{aligned} \tag{192}$$

□

Since $b_i \in \mathbf{L} = \bigcup_{m\geq 1}\mathbb{C} < z^{1/m} >$ there is an m such that

$$\forall i = 1, \cdots, d:\ b_i \in \mathbb{C} < z^{1/m} >,\ m \geq 1. \tag{193}$$

For the action on $y \in W(\tau)$ the operator E is written as

$$E = z^{-\lambda}z\partial = -z^{-1-\lambda}(-z^2\partial) \text{ and acts as } (E \circ y)(t) = -t^{1+\lambda}y'(t). \tag{194}$$

As usual we define

$$\begin{aligned}x = (x_0, \cdots, x_{d-1})^\top := (y, E\circ y, \cdots, E^{d-1}\circ y)^\top \text{ and}\\ A = A(z^{1/m}) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1\\ -b_d & -b_{d-1} & -b_{d-2} & \cdots & -b_1\end{pmatrix} \in \mathbb{C} < z^{1/m} >^{d\times d}\end{aligned} \tag{195}$$

and conclude

$$\begin{aligned}&L\circ y = 0 \iff z^{-d\lambda}L\circ y = 0 \iff E^d\circ y + \sum_{j=0}^{d-1} b_{d-j}E^j\circ y = 0\\ &\iff (E\,\mathrm{id}_d - A)\circ x = 0 \iff -t^{1+\lambda}x'(t) - A(t^{-1/m})x(t) = 0\\ &\iff x'(t) + t^{-1-\lambda}A(t^{-1/m})x(t) = 0.\end{aligned} \tag{196}$$

On the module level we obtain the inverse isomorphisms (cf. (138))

$$\begin{gathered} M=\mathbf{A}^{1\times q}/U \overset{(\circ P)_{\mathrm{ind}}}{\cong} \mathbf{A}/\mathbf{A}L\cong \mathbf{A}^{1\times d}/\mathbf{A}^{1\times d}(E\,\mathrm{id}_d-A),\ \overline{\eta}\leftrightarrow\overline{f}\leftrightarrow\overline{\xi}\\ \text{where } \overline{\eta}=\eta+U,\ \overline{f}:=f+\mathbf{A}L,\ \overline{\xi}:=\xi+\mathbf{A}^{1\times d}(E\,\mathrm{id}_d-A)\\ \eta=fQ,\ f=\eta P=\sum_{i=0}^{d-1}\xi_iE^i,\ \xi=(f,0,\cdots,0). \end{gathered} \tag{197}$$

where $P\in\mathbf{A}^{q\times 1}$ and $Q\in\mathbf{A}^{1\times q}$ define the inverse isomorphisms between M and $\mathbf{A}/\mathbf{A}L$. For the behaviors this implies

$$\begin{gathered} \mathcal{B}(U)\cong\mathcal{B}(\mathbf{A}L)\cong\mathcal{B}(\mathbf{A}^{1\times d}(E\,\mathrm{id}_d-A)),\ w\leftrightarrow y\leftrightarrow x,\\ w=P\circ y,\ y=Q\circ w=x_0,\ x=(y,E\circ y,\cdots,E^{d-1}\circ y)^\top. \end{gathered} \tag{198}$$

The matrix A is contained in $\mathbb{C}<z^{1/m}>^{d\times d}$ and we decompose it according to (56):

$$A(z^{1/m})=A_0+z^{1/m}A_1(z^{1/m}),\ A(t^{-1/m})=A_0+t^{-1/m}A_1(t^{-1/m}),\ A_0\in\mathbb{C}^{d\times d}. \tag{199}$$

Notice that $A_1(z^{1/m})$ is a power series in $z^{1/m}$ and not only a Laurent series and hence $A_1(t^{-1/m})$ is bounded on $[\tau,\infty)$ for all $\tau>\sigma(A_1)$. Summing up the preceding derivations we obtain

Theorem 5.5. *Assume that* $M=\mathbf{A}^{1\times q}/\mathbf{A}^{1\times p}R\cong\mathbf{A}/\mathbf{A}L\cong\mathbf{A}^{1\times d}/\mathbf{A}^{1\times d}(E\,\mathrm{id}_d-A)$ *is irregular singular and the derived data from above. There is a time*

$$\tau_0>\max(1,\sigma(A_1),\sigma(R),\sigma(P),\sigma(Q)) \tag{200}$$

such that the equation (196) *gets the form*

$$x'(t)+t^{-1-\lambda}\left(A_0+t^{-1/m}A_1(t^{-1/m})\right)x(t)=0,\ t>\tau_0, \tag{201}$$

where A resp. A_0,A_1 are defined in (195) *resp.* (199) *and where $A_1(t^{-1/m})$ is bounded on $[\tau_0,\infty)$. Moreover the behavior isomorphisms* (198) *imply the behavior isomorphisms*

$$\begin{gathered} \forall\tau\geq\tau_0:\mathcal{B}(R,\tau)=\{w\in W(\tau)^q;R\circ w=0\}\cong\mathcal{C}(\tau):=\\ \left\{x\in W(\tau)^d;\ x'(t)+t^{-1-\lambda}\left(A_0+t^{-1/m}A_1(t^{-1/m})\right)x(t)=0\right\},\ w\leftrightarrow x,\ \textit{where}\\ w=(w_1,\cdots,w_q)^\top=P\circ x_0,\ x=(x_0,\cdots,x_{d-1})^\top=(1,E,\cdots,E^{d-1})^\top Q\circ w. \end{gathered} \tag{202}$$

By the variable transformation $t\mapsto t^{-\lambda^{-1}},\ -\lambda^{-1}>0$, the preceding equations can be simplified as follows: Define

$$v(t):=x\left(t^{-\lambda^{-1}}\right),\ \text{hence } v\left(t^{-\lambda}\right)=x\left((t^{-\lambda})^{-\lambda^{-1}}\right)=x(t). \tag{203}$$

The differential equation (201) and (203) imply

$$\begin{aligned} &v'(t)=-\lambda^{-1}t^{-\lambda^{-1}-1}x'(t^{-\lambda^{-1}})\\ &=-\lambda^{-1}t^{-\lambda^{-1}-1}(-1)\left(t^{-\lambda^{-1}}\right)^{-1-\lambda}\left(A_0+\left(t^{-\lambda^{-1}}\right)^{-1/m}A_1\left(\left(t^{-\lambda^{-1}}\right)^{-1/m}\right)\right)v(t)\\ &=\lambda^{-1}\left(A_0+t^{1/(\lambda m)}A_1(t^{1/(\lambda m)})\right)v(t),\ 1/(\lambda m)<0,\ A_1(t^{1/(\lambda m)})\text{ bounded.} \end{aligned} \tag{204}$$

Result 5.6. *([21, Thm. 8.6]) Consider a state space system*

$$v'(t) = (B(t) + C(t))v(t),\ v(t) \in \mathrm{C}(\tau_0, \infty)^d,\ B, C \in \mathrm{C}^0(\tau_0, \infty)^{d\times d}, \tag{205}$$

where C is considered as a perturbation of B. Assume that $\|B(t)\| \le \beta$ for all $t > \tau_0$ and that the system $v' = Bv$ is u.e.s. Then there is a constant $\gamma > 0$ such that also $v' = (B + C)v$ is u.e.s. if $\|C(t)\| \le \gamma$ for all $t > \tau_0$.

Theorem 5.7. *Assume $R, U, M, \mathcal{B}$ as in the beginning of Section 5.1 with irregular singular M and the derived data from Thm. 5.5 and* (201)*. If the eigenvalues of A_0 have positive real parts then $\mathcal{B}$ is weakly exponentially stable. For sufficiently large $\tau_1 > \tau_0$ the trajectories $x \in W(\tau)^d$ satisfy an inequality*

$$\|x(t)\| \le c\exp\left(-\alpha(t^{-\lambda} - t_0^{-\lambda})\right)\|x(t_0)\|, \tag{206}$$

Proof. 1. We first consider the equation (204)

$$\begin{aligned} v'(t) &= \lambda^{-1}\left(A_0 + t^{1/(\lambda m)}A(t^{1/(\lambda m)})\right)v(t) \text{ and define} \\ B(t) &:= \lambda^{-1}A_0,\ \beta := \|\lambda^{-1}A_0\|,\ C(t) = \lambda^{-1}t^{1/(\lambda m)}A_1(t^{1/(\lambda m)}). \end{aligned} \tag{207}$$

Since $\lambda^{-1} < 0$ and the eigenvalues of A_0 have positive real parts those of $\lambda^{-1}A_0$ have negative real parts. Therefore $v' = Bv$ is u.e.s. Let $\gamma > 0$ be the number from Result 5.6. The function $A_1(t^{1/(\lambda m)}),\ 1/(\lambda m) < 0,$ is bounded for $t \to \infty$ and $\lim_{t\to\infty} t^{1/\lambda m} = 0$. We choose

$$\tau_1 > \tau_0 \text{ such that } \forall t \ge \tau_1 : \|C(t)\| = \|\lambda^{-1}t^{1/(\lambda m)}A_1(t^{1/(\lambda m)})\| \le \gamma. \tag{208}$$

According to Result 5.6 the equation $v' = (B + C)v$ is u.e.s. on (τ, ∞) for all $\tau \ge \tau_1$. Therefore there are $c, \lambda > 0$ such that

$$\forall t \ge t_0 > \tau \ge \tau_1 :\ \|v(t)\| \le c\exp(-\alpha(t - t_0))\|v(t_0)\|. \tag{209}$$

2. We use $x(t) = v(t^{-\lambda})$ from (203). The preceding inequality (209) implies:

$$\begin{aligned} &\forall t \ge t_0 > \tau \ge \tau_1^{-\lambda^{-1}}, \text{ hence } t^{-\lambda} \ge t_0^{-\lambda} > \tau^{-\lambda} \ge \tau_1 = \left(\tau_1^{-\lambda^{-1}}\right)^{-\lambda} : \\ &\|x(t)\| = \|v(t^{-\lambda})\| \le c\exp\left(-\alpha(t^{-\lambda} - t_0^{-\lambda})\right)\|x(t_0)\|. \end{aligned} \tag{210}$$

3. According to part 2. of the proof and Cor. 4.6 the state space behavior

$$\mathcal{B}(\tau) := \left\{x \in W(\tau)^d;\ x'(t) + t^{-\lambda-1}\left(A_0 + t^{-1/m}A_1(t^{-1/m}\right) = 0\right\},\ \tau \ge \tau_1^{-\lambda^{-1}},$$

is w.e.s.. Since w.e.s. is preserved by behavior isomorphisms also $\mathcal{B} = \mathcal{B}(U)$ is w.e.s.. □

The next theorem is an *instability* result. For $\xi \in \mathbb{C}^d$ we have to use the 2-norm $\|\xi\|_2 = \left(\sum_i |\xi_i|^2|\right)^{1/2}$ with $\|\xi\| = \max_i |\xi| \le \|\xi\|_2 \le d^{1/2}\|\xi\|$.

Theorem 5.8. *Assume the data from* (204) *with the abbreviations from* (207) *and assume that B has an eigenvalue with positive real part. Then there are $t_0 > \tau_0$, $c, \alpha > 0$ and* one *solution $v \in W(\tau_0)^d$ of $v'(t) = (B + C(t))v(t)$ such that*

$$\begin{aligned} \forall t \ge t_0 :\ \|v(t)\| &\ge c\exp\left(\alpha(t - t_0)\right)\|v(t_0)\|, \textit{ hence} \\ \|x(t)\| &\ge c\exp\left(\alpha(t^{-\lambda} - t_0^{-\lambda})\right)\|x(t_0)\|. \end{aligned} \tag{211}$$

In particular, $v(t)$ and $x(t)$ are not u.e.s. and $\mathcal{B}$ is not w.e.s..

Proof. 1. Let $I := \mathrm{id}_d$. By assumption the matrix B has an eigenvalue ν with positive real part. Let $B^* = \overline{B}^\top$ denote the adjoint matrix and $\mathcal{H} \subset \mathbb{C}^{d\times d}$ the space of Hermitean matrices H ($H = H^*$). The map $\mathcal{H} \to \mathcal{H},\ P \mapsto B^*P + PB$, is well-defined and an isomorphism if for two different eigenvalues $\lambda_1 \neq \lambda_2$ of B the inequality $\lambda_1 + \overline{\lambda_2} \neq 0$ holds. We choose ϵ_1 with $0 < \epsilon_1 < \Re(\nu)$ and such that the matrix $B - \epsilon_1 I$ satisfies this condition. So there is a unique Hermitean matrix $P = P^*$ such that

$$(B^* - \epsilon_1 I)P + P(B - \epsilon_1 I) = -I, \text{ hence } B^*P + PB - 2\epsilon_1 P = -I. \tag{212}$$

2. *The quadratic form $V(\xi) := \xi^* P\xi$ is not positive semi-definite*: Let η be a nonzero eigenvector of B with $B\eta = \nu\eta$. Then

$$\begin{aligned} &B\eta = \nu\eta,\ \eta^*B^* = \overline{\nu}\eta^*,\ \eta^*(B^*P + PB)\eta - 2\epsilon_1\eta^*P\eta = -\|\eta\|_2^2 \\ &\Longrightarrow 2\left(\Re(\nu) - \epsilon_1\right)V(\eta) = -\|\eta\|_2^2 < 0 \underset{\Re(\nu)>\epsilon_1}{\Longrightarrow} V(\eta) < 0. \end{aligned} \tag{213}$$

We infer that P has an eigenvalue $\mu < 0$ and eigenvector ξ such that

$$\mu < 0,\ \xi \neq 0,\ P\xi = \mu\xi,\ V(\xi) = \xi^*P\xi = \mu\|\xi\|_2^2 < 0. \tag{214}$$

3. Let $v(t) \in W(\tau)^d$ be a solution of $v'(t) = (B + C(t))v(t)$ and define $f(t) := V(v(t))$. Then

$$\begin{aligned} f = v^*Pv \Longrightarrow f' &= v'^*Pv + v^*Pv' = v^*(B^* + C^*)Pv + v^*P(B + C)v \\ &= v^*(B^*P + PB)v + v^*(C^*P + PC)v \\ &= v^*(-I + 2\epsilon_1 P)v + v^*Hv = -\|v\|_2^2 + 2\epsilon_1 V(v) + v^*Hv \\ &= 2\epsilon_1 f - \|v\|_2^2 + v^*Hv \end{aligned} \tag{215}$$

where $H(t) = H(t)^* := C(t)^*P + PC(t)$ is a Hermitean matrix.
4. The condition $\lim_{t\to\infty} C(t) = 0$ implies $\lim_{t\to\infty} H(t) = 0$. We choose ϵ_2 with $0 < \epsilon_2 < 1$ and $\tau_1 > \tau_0$ such that $|\eta^*H(t)\eta| \leq \epsilon_2\|\eta\|_2^2$ for $\eta \in \mathbb{C}^d$ and $t \geq \tau_1$. This choice implies

$$\begin{aligned} &f' \underset{(215)}{=} 2\epsilon_1 f - \|v\|_2^2 + v^*Hv \leq 2\epsilon_1 f - (1 - \epsilon_2)\|v\|_2^2 \Longrightarrow f' = 2\epsilon_1 f - h(t) \\ &\qquad \text{with } h(t) \geq 0 \Longrightarrow \forall t \geq t_0 \geq \tau_1 : \\ &f(t) = f(t_0)e^{2\epsilon_1(t-t_0)} - \int_{t_0}^t e^{2\epsilon_1(t-s)}h(s)ds \leq f(t_0)e^{2\epsilon_1(t-t_0)}. \end{aligned} \tag{216}$$

We apply the data from (214) and consider the solution

$$\begin{aligned} &v \in W(\tau_1)^d \text{ of } v'(t) = (B + C(t))v(t) \text{ with } v(t_0) := \xi \\ &\Longrightarrow f(t_0) = V(\xi) = \mu\|\xi^2\| < 0 \\ &\Longrightarrow |f(t)| = |V(v(t))| \underset{(216)}{\geq} (-\mu)\|v(t_0)\|_2^2 e^{2\epsilon_1(t-t_0)}. \end{aligned} \tag{217}$$

5. Let finally $\rho := \max\{|\beta|;\ \beta \in \mathrm{spec}(B)\} \geq -\mu$ be the spectral radius of P. Then $|V(\eta)| \leq \rho\|\eta\|_2^2$ for all $\eta \in \mathbb{C}^d$ and therefore especially

$$\forall t \geq t_0 \geq \tau_1 :\ \|v(t)\|_2^2 \geq \rho^{-1}|V(v(t))| \underset{(217)}{\geq} \rho^{-1}(-\mu)e^{2\epsilon_1(t-t_0)}\|v(t_0)\|_2^2. \tag{218}$$

Taking roots, replacing $\|v(t)\|_2$ by $\|v(t)\|$ and substituting $v(t) = x(t^{-\lambda})$ furnishes (211) and thus the theorem. The inequalities (216)-(218) also follow from *Chetaev's instability theorem* [10, Thm. 25.3]. $\square$

Example 5.9. If $L \in \mathbf{A}$ is monic of degree 2 and not regular singular with $\lambda < 0$ then we can write

$$\begin{aligned} &E := z^{-\lambda}(z\partial) = z^{1-\lambda}\partial \text{ and } z^{-2\lambda}L = E^2 + b_1E + b_2 \text{ with} \\ &b_1, b_2 \in \mathbb{C} < z^{1/m} >, \text{ hence } v(b_i) \geq 0, \text{ and } \min(v(b_1), v(b_2)) = 0. \end{aligned} \tag{219}$$

Conversely, any choice of $\lambda < 0$, $m \geq 1$, and b_i with the indicated properties gives rise to $E := z^{1-\lambda}\partial$ and $L := z^{2\lambda}(E^2 + b_1E + b_2)$ with the assumed properties of L. The corresponding differential equation is

$$\begin{aligned} &x'(t) + t^{-1-\lambda}A(t^{-1/m})x(t) = 0 \text{ with } b_1(0) \neq 0 \text{ or } b_2(0) \neq 0, \\ &A(z^{1/m}) := \begin{pmatrix} 0 & 1 \\ -b_2 & -b_1 \end{pmatrix} = A_0 + z^{1/m}A_1(z^{1/m}), \\ &A_0 := \begin{pmatrix} 0 & 1 \\ -b_2(0) & -b_1(0) \end{pmatrix}, \ A_1(z^{1/m}) = \begin{pmatrix} 0 & 0 \\ -z^{-1/m}(b_2 - b_2(0)) & -z^{-1/m}(b_1 - b_1(0)) \end{pmatrix} \end{aligned} \tag{220}$$

We make the simplest choice $\lambda := -1, \ m := 1$ and obtain the differential equation

$$\begin{aligned} &x'(t) + A(t^{-1})x(t) = 0 \text{ with } b_1(0) \neq 0 \text{ or } b_2(0) \neq 0, \\ &A(z) := \begin{pmatrix} 0 & 1 \\ -b_2 & -b_1 \end{pmatrix} = A_0 + zA_1(z), \\ &A_0 := \begin{pmatrix} 0 & 1 \\ -b_2(0) & -b_1(0) \end{pmatrix}, \ A_1(z) = \begin{pmatrix} 0 & 0 \\ -z^{-1}(b_2 - b_2(0)) & -z^{-1}(b_1 - b_1(0)) \end{pmatrix}. \end{aligned} \tag{221}$$

We choose $b_1(0) := -1$, $b_2(0) := 0$, hence $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ with the eigenvalues $0, 1$. Thm. 2.8 is not applicable. We choose $A_1(z) = \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}$ and obtain the differential equation

$$x'(t) + \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + t^{-1}\begin{pmatrix} 0 & 0 \\ 0 & t^{-1} \end{pmatrix}\right)x(t) = 0 \text{ or } x_1' + x_2 = 0, \ x_2' + (1 + t^{-2})x_2 = 0. \tag{222}$$

For $t \geq t_0 \geq 1$ we obtain the special solutions with $x_1(t_0) := 0, \ x_2(t_0) := 1$:

$$\begin{aligned} &x_2(t) = \exp\left(-(t - t^{-1}) + (t_0 - t_0^{-1})\right), \\ \Longrightarrow\ &x_1(t) = -\int_{t_0}^t x_2(s)ds = -\int_{t_0}^t \exp\left(-(s - s^{-1}) + (t_0 - t_0^{-1})\right)ds \\ \Longrightarrow\ &|x_1(t)| = \int_{t_0}^t \exp\left(-s + \left(s^{-1} + t_0 - t_0^{-1}\right)\right)ds \geq \int_{t_0}^t e^{-s}ds = e^{-t_0} - e^{-t}. \end{aligned} \tag{223}$$

Since $\lim_{t\to\infty} |x_1(t)| \geq e^{-t_0} > 0$ the trajectory $x(t)$ with $x(t_0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is not exponentially stable and hence the corresponding behavior $\mathcal{B}(U)$ is not w.e.s..

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