The significance of Gabriel localization for stability and stabilization of multidimensional input/output behaviors

To Peter Gabriel on his 77th birthday

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***Abstract*— Serre (1953) was the first who considered categories of groups up to negligible ones, negligibility being determined by the considered context. Gabriel in his thesis (1962) developed this idea into a complete theory of quotient categories, rings and modules which is called Gabriel localization. This theory was nicely exposed by Stenström (1975). The author has recently observed that this theory is a valuable tool for stability and stabilization of multidimensional behaviors where a finitely generated multivariate polynomial torsion module is considered negligible if its characteristic variety has points in a preselected stability region only or, equivalently, if its associated autonomous behavior is stable, i.e., has polynomial-exponential solutions with frequencies in this stability region only. Via the Integral Representation Formula of Ehrenpreis/Palamodov corresponding properties hold for all trajectories of the behavior. In the one-dimensional standard cases stability signifies asymptotic stability. In our approach to output feedback stabilization of multidimensional systems almost direct decompositions, i.e., direct sum decompositions up to negligible modules, are essential. Quadrat had observed the significance of direct sum decompositions in stabilization theory. These decompositions are usually hidden in coprime factorizations of transfer matrices which, however, do not always exist. Bisiacco, Valcher and Napp Avelli studied almost direct decompositions for two-dimensional polynomial modules and behaviors, but without using localization theory. This paper explains Gabriel's localization and quotient modules and their use in multidimensional stabilization theory. It also contains a new algorithm for the computation of the (Gabriel) quotient module of a finitely generated torsionfree module over a multivariate polynomial ring. This algorithm can also be used for the computation of Willems closures of such modules and thus generalizes work of Shankar, Sasane, Napp Avelli, van der Put et al.. It is also useful for the computation of the purity filtration of a finitely generated polynomial torsion module which is the subject of Barakat's talk at this conference where also the history of this filtration is discussed. The author gratefully acknowledges financial support of the Austrian FWF.** [1]

I. Introduction

J.P. Serre [18] was the first who considered categories of groups *up to negligible ones*, negligibility being determined by the considered context. P. Gabriel in his thesis [7] developed this idea into a complete theory of quotient categories, rings and modules which is called *Gabriel localization*. This theory was nicely exposed by B. Stenström [22]. The author has recently observed [13] that this theory is a valuable tool for *stability and stabilization of multidimensional behaviors* where a finitely generated multivariate polynomial torsion module M is considered negligible if its characteristic variety has points in a preselected stability region only or, equivalently, if its associated autonomous behavior $\mathcal{B}$ is *stable*, i.e., has polynomial-exponential solutions with frequencies in this stability region only. Via the *Integral Representation Formula* of Ehrenpreis/Palamodov corresponding properties hold for all trajectories of $\mathcal{B}$. In the one-dimensional standard cases stability of $\mathcal{B}$ signifies asymptotic stability. Direct decompositions of modules up to negligible ones, also called almost direct decompositions, are central to our approach to stabilization. The connection of output feedback stabilization with direct decompositions has been observed and investigated by A. Quadrat [16], [17]. These decompositions are usually hidden in coprime factorizations of rational transfer matrices which, however, do not exist in general. In systems theory almost direct decompositions were considered by M. Bisiacco, M.E. Valcher [2], [3] and by D. Napp Avelli [9] for two-dimensional behaviors.

The present article gives a survey of our approach to multidimensional stabilization and some new results, also on the computation of arbitrary Willems closures. Matlis' theory on injective modules [8, pp.145-150] is an essential ingredient of our proofs. The survey [15] contains a comprehensive list of researchers and references on multidimensional stabilization.

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II. Data

We present Gabriel localization and its systems theoretic application in a continuous standard case. Let $A := \mathbb{C}[s] = \mathbb{C}[s_1, \ldots, s_n]$ be the complex polynomial algebra in $n \geq 1$ indeterminates with the category Mod_A of A-modules and the sets $\mathrm{Max}(A) \subseteq \mathrm{Spec}(A)$ of maximal resp. prime ideals. We consider the standard injective cogenerator signal A-module $\mathcal{F} := \mathrm{C}^\infty(\mathbb{R}^n, \mathbb{C})$ of smooth functions or $\mathcal{F} :=$

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$\mathcal{D}'_{\mathbb{C}}(\mathbb{R}^n)$ of distributions with the action $s_i \circ y := \partial y/\partial x_i$ where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. If $\mathfrak{a} \subseteq A$ is an ideal we define

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(A);\ \mathfrak{a} \subseteq \mathfrak{p}\},$$
$$V_{\mathbb{C}}(\mathfrak{a}) := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n;\ \mathfrak{a}(\lambda) = 0\}.$$

For $\lambda \in \mathbb{C}^n$ let

$$\mathfrak{m}(\lambda) := \{f \in A;\ f(\lambda) = 0\} = \sum_{i=1}^{n} A(s_i - \lambda_i) \in \mathrm{Max}(A).$$

Every $\mathfrak{p} \in \mathrm{Spec}(A)$ and A-module M give rise to the local ring

$$A_{\mathfrak{p}} := \left\{at^{-1} \in \mathbb{C}(s);\ a \in A,\ t \notin \mathfrak{p}\right\} \text{ and}$$
$$_{A_{\mathfrak{p}}}M_{\mathfrak{p}} := \left\{xt^{-1};\ x \in M,\ t \notin \mathfrak{p}\right\}.$$

The *associator* of M is defined as

$$\mathrm{ass}(M) := \{\mathfrak{p} \in \mathrm{Spec}(A);\ A/\mathfrak{p} \subseteq M \text{ (up to isom.)}\}.$$

Consider a matrix $R \in A^{k\times\ell}$ and its associated finitely generated modules

$$U := A^{1\times k}R,\ M := A^{1\times\ell}/U \tag{1}$$

and associated $\mathcal{F}$-behavior

$$\mathcal{B} := U^{\perp} := \left\{w \in \mathcal{F}^{\ell};\ U \circ w = 0\right\} = \left\{w \in \mathcal{F}^{\ell};\ R \circ w = 0\right\} \text{ with } U = \mathcal{B}^{\perp}. \tag{2}$$

The module is torsion or $\mathcal{B}$ is *autonomous* if and only if $\mathrm{rank}(R) = \ell$. Then the *characteristic variety* $\mathrm{char}(M) = \mathrm{char}(\mathcal{B})$ of M or $\mathcal{B}$ is [5, Sect. 8.1.7]

$$\mathrm{char}(M) := \left\{\lambda \in \mathbb{C}^n;\ M_{\mathfrak{m}(\lambda)} \neq 0\right\} = \{\lambda \in \mathbb{C}^n;\ \mathrm{rank}(R(\lambda)) < \mathrm{rank}(R) = \ell\} = \bigcup_{\mathfrak{p}\in\mathrm{ass}(M)} V_{\mathbb{C}}(\mathfrak{p}). \tag{3}$$

The *Integral Representation Formula* of Ehrenpreis/Palamodov [5, Th. 8.1.3] says that every smooth trajectory $y \in \mathcal{B}$ has a representation

$$y(x) = \sum_{i=1}^{T} \int_{\mathbb{C}^n} a_i(\lambda, x) e^{\lambda \bullet x} d\mu_i(\lambda) \tag{4}$$

where $\lambda \bullet x := \lambda_1 x_1 + \ldots + \lambda_n x_n$ with the following specifications: The μ_i are measures on $\mathbb{C}^n$ with support $\mathrm{supp}(\mu_i) \subseteq \mathrm{char}(\mathcal{B})$. The $a_i(\lambda, x) \in \mathbb{C}[\lambda, x]^{\ell}$ are polynomial vectors with the property that for all $\lambda \in \mathrm{supp}(\mu_i)$ the polynomial-exponential trajectory $a_i(\lambda, x)e^{\lambda\bullet x}$ belongs to $\mathcal{B}$. The a_i can be computed [12]. The convergence conditions are omitted here.

For *stability* and *stabilization* we assume an arbitrary *stability decomposition*

$$\mathbb{C}^n := \Lambda_1 \uplus \Lambda_2,\ \Lambda_2 \neq \emptyset, \quad \Lambda_1 =: \text{stability region},\ \Lambda_2 =: \text{instability region}, \tag{5}$$

a standard case being $\Lambda_2 := \{z \in \mathbb{C};\ \Re(z) \geq 0\}^n$. In the one-dimensional standard case $\Lambda_1 = \mathbb{C}^-$ is the left open half-plane.

III. Gabriel localization

The results of this section except for those using (5) *are originally due to Gabriel [7] and are exposed in Stenström's book [22].*

The decomposition (5) induces the decomposition

$$\mathrm{Spec}(A) = \mathcal{P}_1 \uplus \mathcal{P}_2 \text{ where } \mathcal{P}_1 := \{\mathfrak{p} \in \mathrm{Spec}(A);\ V_{\mathbb{C}}(\mathfrak{p}) \subseteq \Lambda_1\}. \tag{6}$$

The sets $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy the conditions

$$\mathcal{P}_1 \ni \mathfrak{p} \subseteq \mathfrak{q} \implies \mathfrak{q} \in \mathcal{P}_1 \text{ and}$$
$$\mathcal{P}_2 \ni \mathfrak{p} \supseteq \mathfrak{q} \implies \mathfrak{q} \in \mathcal{P}_2$$

and give rise to the following full subcategory of Mod_A:

$$\mathfrak{C} := \left\{C \in \mathrm{Mod}_A;\ \forall \lambda \in \Lambda_2:\ C_{\mathfrak{m}(\lambda)} = 0\right\} = \{C \in \mathrm{Mod}_A;\ \mathrm{ass}(C) \subseteq \mathcal{P}_1\}.$$

This category is closed under isomorphisms, subobjects, factor objects, extensions and arbitrary direct sums. Such full subcategories are called *localizing* and give rise to the ideal set

$$\mathcal{T} := \{\mathfrak{a} \subseteq A;\ A/\mathfrak{a} \in \mathfrak{C}\}$$

with the following properties:

1) If $\mathcal{T} \ni \mathfrak{a} \subseteq \mathfrak{b}$ then $\mathfrak{b} \in \mathcal{T}$.
2) If $\mathfrak{a}, \mathfrak{b} \in \mathcal{T}$ then $\mathfrak{a} \bigcap \mathfrak{b} \in \mathcal{T}$.
3) If $\mathcal{T} \ni \mathfrak{a} \supseteq \mathfrak{b}$ and $(\mathfrak{b} : a) \in \mathcal{T}$ holds for all $a \in \mathfrak{a}$ then $\mathfrak{b} \in \mathcal{T}$.

Such a $\mathcal{T}$ is the basis of neighborhoods of 0 of a unique linear topology on A and is called a *Gabriel topology*.

Lemma and Definition III.1. 1) *There are bijective correspondences between the data* $\mathcal{P}_1, \mathcal{P}_2, \mathfrak{C}, \mathcal{T}$ *with the indicated properties given by*

$$\mathrm{Spec}(A) = \mathcal{P}_1 \uplus \mathcal{P}_2,$$
$$\mathcal{P}_1 = \mathcal{T} \bigcap \mathrm{Spec}(A) = \{\mathfrak{p} \in \mathrm{Spec}(A);\ A/\mathfrak{p} \in \mathfrak{C}\},$$
$$\mathcal{T} = \{\mathfrak{a} \subseteq A;\ V(\mathfrak{a}) \subseteq \mathcal{P}_1\} = \{\mathfrak{a} \subseteq A;\ A/\mathfrak{a} \in \mathfrak{C}\},$$
$$\mathfrak{C} = \{C \in \mathrm{Mod}_A;\ \mathrm{ass}(C) \subseteq \mathcal{P}_1\} = \{C \in \mathrm{Mod}_A;\ \forall \mathfrak{p}_2 \in \mathcal{P}_2:\ C_{\mathfrak{p}_2} = 0\} = \{C \in \mathrm{Mod}_A;\ \forall x \in C \exists \mathfrak{a} \in \mathcal{T} \text{ with } \mathfrak{a}x = 0\}.$$

The modules in $\mathfrak{C}$ *are called* $\mathfrak{C}$*- or* $\mathcal{T}$-small *or* negligible.

2) *For* $\mathcal{P}_1 \subsetneq \mathrm{Spec}(A)$ *or* $\{0\} \in \mathcal{P}_2$ *(what we always assume) each module in* $\mathfrak{C}$ *is a torsion module.*
3) *Not all* $\mathcal{P}_1$ *come from a decomposition* (5) *via* (6)*, for instance [14, p.478]*

$$\mathcal{P}_2(c) := \{\mathfrak{p} \in \mathrm{Spec}(A);\ \dim(A_{\mathfrak{p}}) < c\}$$
$$\text{where } 1 \leq c \leq n = \dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}),$$
$$\mathcal{P}_1(c) := \{\mathfrak{p} \in \mathrm{Spec}(A);\ \dim(A/\mathfrak{p}) \leq n - c\}$$

where dim *denotes the Krull dimension. For* $c = 2$ *the associated negligible modules are called* pseudo-zero *in [6, Section VII.4.4].*

4) *The largest submodule in* $\mathfrak{C}$ *of an* A*-module* M *exists, is called its* $\mathfrak{C}$-radical *or* $\mathcal{T}$-torsion module *and is denoted by*

$$\mathrm{Ra}_{\mathfrak{C}}(M) = \mathrm{tor}_{\mathcal{T}}(M) = \{x \in M;\ Ax \in \mathfrak{C}\}.$$

Then $\mathrm{Ra}_{\mathfrak{C}}(M/\mathrm{Ra}_{\mathfrak{C}}(M)) = 0$ *and*

$$\mathrm{Ra}_{\mathfrak{C}}(M) \neq 0 \iff \mathrm{ass}(M) \bigcap \mathcal{P}_1 \neq \emptyset.$$

If $\mathcal{P}_1 = \{\mathfrak{p} \in \mathrm{Spec}(A);\ V_{\mathbb{C}}(\mathfrak{p}) \subseteq \Lambda_1\}$ *according to* (6) *then*

$$\mathrm{Ra}_{\mathfrak{C}}(M) \neq 0 \Leftrightarrow \exists \mathfrak{p} \in \mathrm{ass}(M) \text{ with } V_{\mathbb{C}}(\mathfrak{p}) \subseteq \Lambda_1$$
$$\mathrm{Ra}_{\mathfrak{C}}(M) = 0 \Leftrightarrow \forall \mathfrak{p} \in \mathrm{ass}(M):\ V_{\mathbb{C}}(\mathfrak{p}) \bigcap \Lambda_2 \neq \emptyset.$$

In the sequel we assume that the data from Lemma and Definition III.1 are given.

Example III.2. Let $S \subseteq A$ be any multiplicatively closed set and $A_S \subseteq \mathbb{C}(s)$ resp. M_S the corresponding quotient ring resp. module. Then $\mathfrak{C}_S := \{C \in \mathrm{Mod}_A;\ C_S = 0\}$ is a localizing subcategory with Gabriel topology

$$\mathcal{T}_S = \left\{\mathfrak{a} \subseteq A;\ \mathfrak{a} \bigcap S \neq \emptyset\right\} \text{ and}$$
$$\mathrm{tor}_{\mathcal{T}_S}(M) = \ker(\mathrm{can}: M \to M_S) =$$
$$\{x \in M;\ \exists s \in S \text{ with } sx = 0\}.$$

Example III.3. Assume $1 \le c \le n$ and consider

$$\mathcal{P}_1(c) = \{\mathfrak{p} \in \mathrm{Spec}(A);\ \dim(A/\mathfrak{p}) \le n - c\}$$
$$\mathcal{P}_2(c) = \{\mathfrak{p} \in \mathrm{Spec}(A);\ \dim(A_{\mathfrak{p}}) < c\}$$

from Lemma III.1,(3), and the associated localizing subcategory

$$\mathfrak{C}(c) := \{C \in \mathrm{Mod}_A;\ \mathrm{ass}(C) \subset \mathcal{P}_1(c)\}.$$

If M is an A-module its (Krull) dimension resp. codimension is given resp. defined as

$$\dim(M) := \max\{\dim(A/\mathfrak{p});\ \mathfrak{p} \in \mathrm{ass}(M)\}$$
$$\mathrm{codim}(M) := n - \dim(M).$$

There results the equivalence

$$C \in \mathfrak{C}(c) \iff \dim(C) \le n - c \iff \mathrm{codim}(C) \ge c.$$

Hence $\mathrm{tor}_c(M) := \mathrm{Ra}_{\mathfrak{C}(c)}(M)$ is the largest submodule of M of codimension greater or equal to c or of dimension at most $n-c$. If M is finitely generated these torsion modules can be computed by means of Theorem IV.1,(2), below. Obviously the submodules decrease with increasing c, hence

$$M \supseteq \mathrm{tor}(M) = \mathrm{tor}_1(M) \supseteq \cdots \supseteq \mathrm{tor}_{n-1}(M) \supseteq \mathrm{tor}_n(M).$$

The equation

$$\mathrm{tor}_{c+1}(\mathrm{tor}_c(M)/\mathrm{tor}_{c+1}(M)) \subseteq$$
$$\mathrm{tor}_{c+1}(M/\mathrm{tor}_{c+1}(M)) = 0$$

implies that all $\mathfrak{p} \in \mathrm{ass}(\mathrm{tor}_c(M)/\mathrm{tor}_{c+1}(M))$ have dimension $n-c$, i.e., that $\mathrm{tor}_c(M)/\mathrm{tor}_{c+1}(M)$ is *pure* $n-c$*-dimensional*. Therefore this filtration is called the *purity filtration* of M in [1] where also the history of this filtration and a computation by means of spectral sequences are discussed. If M is finitely generated the module $\mathrm{tor}_n(M)$ is the largest $\mathbb{C}$-finite dimensional submodule of M whose algorithmic computation was described in [14].

The data from Lemma and Definition III.1,(1), also induce the saturated multiplicatively closed set

$$T := \bigcap_{\mathfrak{p}_2 \in \mathcal{P}_2} (A \setminus \mathfrak{p}_2) =: \{\text{stable polynomials}\} \tag{7}$$

and its quotient ring

$$A_T = \left\{at^{-1};\ a \in A, t \in T\right\} \subseteq \mathbb{C}(s)$$

of *stable rational functions*. Then

$$\forall \mathfrak{p}_2 \in \mathcal{P}_2: M_{\mathfrak{p}_2} = (M_T)_{\mathfrak{p}_2} \text{ and thus}$$
$$\mathfrak{C}_T \subseteq \mathfrak{C} \text{ and } \mathcal{T}_T \subseteq \mathcal{T}.$$

If $\mathcal{P}_1 = \{\mathfrak{p} \in \mathrm{Spec}(A);\ V_{\mathbb{C}}(\mathfrak{p}) \subseteq \Lambda_1\}$ (see (6)) then

$$\begin{aligned} T &:= \{t \in A;\ V_{\mathbb{C}}(t) \subseteq \Lambda_1\} \text{ and} \\ A_T &= \left\{at^{-1} \in \mathbb{C}(s);\ V_{\mathbb{C}}(t) \subseteq \Lambda_1\right\}. \end{aligned} \tag{8}$$

The equality $\mathcal{T}_T = \mathcal{T}$ holds only in case Λ_2 is *ideal-convex* [20, Prop. 3.1.20], i.e., $\mathrm{Max}(A_T) = \{\mathfrak{m}(\lambda)_T;\ \lambda \in \Lambda_2\}$ or, equivalently, $\mathfrak{C}$ is *perfect* [22, Prop. XI.3.4].

Corollary and Definition III.4. 1) *An autonomous behavior*

$$\mathcal{B} := \left\{w \in \mathcal{F}^{\ell};\ R \circ w = 0\right\},\ R \in A^{k \times \ell},$$

where $\mathrm{rank}(R) = \ell$ *is called* $\mathcal{T}$-small, negligible or stable*, if its module* $M := A^{1\times\ell}/A^{1\times k}R$ *is* $\mathcal{T}$*-small, ie., belongs to* $\mathfrak{C}$.

2) *Assume* $\mathcal{P}_1 = \{\mathfrak{p} \in \mathrm{Spec}(A);\ V_{\mathbb{C}}(\mathfrak{p}) \subseteq \Lambda_1\}$*. Then the behavior from item 1. is* $\mathcal{T}$*-stable if and only if its characteristic variety* $\mathrm{char}(\mathcal{B}) = \mathrm{char}(M)$ *is contained in* Λ_1*. In this case only polynomial-exponential solutions with frequencies* $\lambda \in \Lambda_1$ *contribute to the solution integral of* (4).

3) *In the standard one-dimensional case with* $\Lambda_1 = \mathbb{C}^-$ *the equality* $\mathcal{T} = \mathcal{T}_T$ *holds and* $\mathcal{B}$ *is* $\mathcal{T}$*-stable if and only if it is* asymptotically stable.

4) *In the standard cases of multidimensional systems theory the inclusion* $\mathfrak{a} \in \mathcal{T}$ *or* $V_{\mathbb{C}}(\mathfrak{a}) \subseteq \Lambda_1$ *can be decided via constructive real algebraic geometry.*

Open problem III.5. For the standard systems theoretic $\mathcal{P}_1$ and Λ_1 and $\mathfrak{a} \in \mathcal{T}$ decide $\mathfrak{a} \in \mathcal{T}_T$, i.e., $\mathfrak{a} \bigcap T \neq \emptyset$, algorithmically and eventually construct $t \in \mathfrak{a} \bigcap T$. This open problem was already stated in [23].

An A-module N is called $\mathfrak{C}$- or $\mathcal{T}$*-closed* if for each $\mathfrak{a} \in \mathcal{T}$ the canonical map

$$\mathrm{can}: N \to \mathrm{Hom}_A(\mathfrak{a}, N),\ x \mapsto (a \mapsto ax),\ \mathfrak{a} \in \mathcal{T},$$

is bijective. Let $\mathrm{Mod}_{A,\mathcal{T}}$ be the full subcategory of all $\mathcal{T}$-closed modules. In the situation of Example III.2 the category $\mathrm{Mod}_{A,\mathcal{T}_S}$ coincides with Mod_{A_S}. The following theorem constructs the analogue of the quotient module M_S.

Theorem and Definition III.6. (Quotient functor) *Data from Lemma and Definition III.1.*

1) *The injection* $\mathrm{inj}: \mathrm{Mod}_{A,\mathcal{T}} \to \mathrm{Mod}_A$ *has a left adjoint* $Q: \mathrm{Mod}_A \to \mathrm{Mod}_{A,\mathcal{T}}$ *with its associated functorial morphism* $\eta: \mathrm{id}_{\mathrm{Mod}_A} \to \mathrm{inj}\, Q$ *such that*
$$\mathrm{Hom}_A(Q(M), N) \cong \mathrm{Hom}_A(M, N),\ g \mapsto g\eta_M,$$
$$\text{for } M \in \mathrm{Mod}_A,\ N \in \mathrm{Mod}_{A,\mathcal{T}}\text{ . Then}$$
$$\mathrm{Ra}_{\mathfrak{C}}(M) = \ker(\eta_M) \text{ and } \mathrm{cok}(\eta_M) \in \mathfrak{C}.$$
2) *The category* $\mathrm{Mod}_{A,\mathcal{T}}$ *is abelian, the functor* Q *is exact and* inj *is left exact, but not exact in general.*
3) $\mathfrak{C} = \{C \in \mathrm{Mod}_A;\ Q(C) = 0\}$ *and* $\mathrm{Mod}_{A,\mathcal{T}} = \{N \in \mathrm{Mod}_A;\ N \cong Q(N)\}$.
4) *The category* $\mathrm{Mod}_{A,\mathcal{T}}$ *is a full subcategory of* Mod_{A_T}.

The use of Q *or of closed modules thus signifies to consider* A*-modules up to* $\mathcal{T}$*-negligible ones.*

Example III.7. In the standard case $\mathfrak{C} = \mathfrak{C}_S$ of Example III.2 the identity $Q(M) = M_S \in \mathrm{Mod}_{A,\mathcal{T}_S} = \mathrm{Mod}_{A_S}$ holds and inj is exact.

IV. Quotient computation

The following theorem was left as an open problem in [15, p.168],[14, p.479].

Theorem IV.1. *Data from Lemma and Definition III.1. Let* ${}_AU$ *be a finitely generated torsionfree module or* $U \subseteq A^{1\times\ell}$ *without loss of generality.*

1) $Q(A) = A_T$, $U_T \subseteq Q(U) \subseteq A_T^{1\times\ell}$ *and*
$$Q(U)/U_T = \mathrm{Ra}_{\mathfrak{C}}\left(A^{1\times\ell}/U\right)_T.$$
2) *If* $\mathfrak{p} \in \mathcal{P}_1$ *can be decided constructively there is an algorithm for the computation of* $\mathrm{Ra}_{\mathfrak{C}}\left(A^{1\times\ell}/U\right)$ *via constructive primary decompositions and therefore also one which computes* $Q(U)$ *in the form* $Q(U) = A_T^{1\times k'}R',\ R' \in A^{k'\times\ell}$. *All steps use the Gröbner basis algorithm.*
3) *If* $\mathcal{P}_1 = \{\mathfrak{p} \in \mathrm{Spec}(A);\ V_{\mathbb{C}}(\mathfrak{p}) \subseteq \Lambda_1\}$ *and* Λ_1 *is* semi-algebraic *in* $\mathbb{C}^n = \mathbb{R}^{2n}$ *then* $\mathfrak{p} \in \mathcal{P}_1$ *can be decided by constructive real algebraic geometry.*

For arbitrary finitely generated ${}_AM$ the module ${}_{A_T}Q(M)$ may not be finitely generated and there is presently no algorithm to compute it.

V. Large injective cogenerators

The injective cogenerators $\mathcal{F}$ from above and many others of systems theoretic significance are *large* [11, Th. 4.54]. This signifies that every finitely generated module can be embedded into a finite power of $\mathcal{F}$ or that $\mathrm{ass}(\mathcal{F}) = \mathrm{Spec}(A)$.

Theorem and Definition V.1. *Data from Lemma and Definition III.1.*

1) *The radical* $\mathrm{Ra}_{\mathfrak{C}}(\mathcal{F})$ *is injective and hence there is a (non-constructive) direct complement* $\mathcal{F}_2$ *with*
$$\mathcal{F} = \mathrm{Ra}_{\mathfrak{C}}(\mathcal{F}) \oplus \mathcal{F}_2 = \{\textit{negligible signals}\} \oplus \{\textit{essential signals}\}. \tag{9}$$
Any such $\mathcal{F}_2$ *satisfies*
$$\mathcal{F}_2 \cong Q(\mathcal{F}) \cong \mathcal{F}/\mathrm{Ra}_{\mathfrak{C}}(\mathcal{F})$$
and is an injective cogenerator in $\mathrm{Mod}_{A,\mathcal{T}}$.
2) *For the modules and behavior from* (1) *and* (2) *the decomposition* (9) *induces the decomposition* $\mathcal{B} = \mathrm{Ra}_{\mathfrak{C}}(\mathcal{B}) \oplus \left(\mathcal{B} \bigcap \mathcal{F}_2^\ell\right)$ *where*
$$\mathcal{B} \bigcap \mathcal{F}_2^\ell \cong \mathrm{Hom}_A(M, \mathcal{F}_2) \cong \mathrm{Hom}_A(Q(M), \mathcal{F}_2).$$
These isomorphisms, in turn, imply the equivalences
$$M \in \mathfrak{C} \iff Q(M) = 0 \iff$$
$$\mathrm{Hom}_A(M, \mathcal{F}_2) = 0 \iff \mathcal{B} = \mathrm{Ra}_{\mathfrak{C}}(\mathcal{B}).$$
3) Willems closures*: If* $M = A^{1\times\ell}/U$ *and* $\mathrm{Ra}_{\mathfrak{C}}(M) = \widetilde{U}/U$ *then*
$$\mathrm{Ra}_{\mathfrak{C}}(M) = \widetilde{U}/U =$$
$$\ker\left(M \to \mathcal{F}_2^{\mathrm{Hom}_A(M,\mathcal{F}_2)},\ x \mapsto (\varphi(x))_\varphi\right) \text{ or}$$
$$\widetilde{U} = \left\{\xi \in A^{1\times\ell};\ \xi \circ \left(\mathcal{B} \bigcap \mathcal{F}_2^\ell\right) = 0\right\} \text{ and hence}$$
$$\widetilde{U} = U \iff \mathrm{ass}(M) \subset \mathcal{P}_2.$$
$$\text{If } \mathcal{P}_1 = \{\mathfrak{p} \in \mathrm{Spec}(A);\ V_{\mathbb{C}}(\mathfrak{p}) \subseteq \Lambda_1\} \text{ then}$$
$$\widetilde{U} = U \iff \forall \mathfrak{p} \in \mathrm{ass}(M): V_{\mathbb{C}}(\mathfrak{p}) \bigcap \Lambda_2 \neq \emptyset.$$
The submodule $\widetilde{U}$ *is called the* Willems closure *of* U *with respect to* $\mathcal{F}_2$ *and can be calculated via Theorem IV.1. If* $U = \widetilde{U}$ *then* U *is called* Willems closed.
4) *For every injective module* I *there is a suitable Gabriel localization with its associated* $\mathcal{F}_2$ *such that the Willems closures with respect to* I *and* $\mathcal{F}_2$ *coincide.*

Item 3. generalizes computations of Shankar [21], [10], Sasane [19] et al.

VI. Stabilization

We assume the data from Lemma and Definition III.1 and from (7). Consider two input/output (IO)-subbehaviors $\mathcal{B}_i,\ i = 1, 2,$ of $\mathcal{F}^{p+m}$:
$$\mathcal{B}_1 := \left\{\left(\begin{smallmatrix} y_1 \\ u_1 \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P_1 \circ y_1 = Q_1 \circ u_1\right\},$$
$$P_1 \in A^{k_1\times p},\ \mathrm{rank}(P_1) = \mathrm{rank}(P_1, -Q_1) = p,$$
$$U_1 := A^{1\times k_1}(P_1, -Q_1) \subseteq A^{1\times(p+m)},$$
$$\mathcal{B}_2 := \left\{\left(\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P_2 \circ y_2 = Q_2 \circ u_2\right\},$$
$$P_2 \in A^{k_2\times m},\ \mathrm{rank}(P_2) = \mathrm{rank}(-Q_2, P_2) = m,$$
$$U_2 := A^{1\times k_2}(-Q_2, P_2) \subseteq A^{1\times(p+m)}.$$
Let $l := p + m$. Their output feedback behavior (compare Fig. 1) is
$$\mathcal{B} := \left\{\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{2l};\quad P \circ y = Q \circ u\right\} \text{ where}$$
$$y := \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) \in \mathcal{F}^{p+m},\ u := \left(\begin{smallmatrix} u_2 \\ u_1 \end{smallmatrix}\right) \in \mathcal{F}^{p+m},$$
$$P := \left(\begin{smallmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{smallmatrix}\right),\ Q := \left(\begin{smallmatrix} 0 & Q_1 \\ Q_2 & 0 \end{smallmatrix}\right),$$
$$U^0 := A^{1\times(k_1+k_2)}P = U_1 + U_2.$$

The feedback behavior $\mathcal{B}$ is *well-posed* if it is an IO-behavior with input u or, equivalently,

$$\operatorname{rank}(P) = p + m \text{ or } U^0 = U_1 \oplus U_2.$$

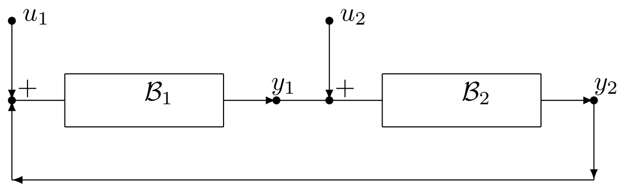

Fig. 1. The output feedback interconnection of two IO behaviors

Lemma and Definition VI.1. *The feedback behavior is called $\mathcal{T}$-stable if it is well-posed and if its autonomous part $\mathcal{B}^0 := \{y \in \mathcal{F}^\ell;\ P \circ y = 0\}$ is $\mathcal{T}$-stable (see Corollary and Definition III.4). Then $\mathcal{B}_2$ is called a $\mathcal{T}$-stabilizing compensator of $\mathcal{B}_1$, $\mathcal{B}_1$ is called $\mathcal{T}$-stabilizable and $Q(U^0) = Q(U_1) \oplus Q(U_2) = A_T^{1\times\ell}$.*

Proof. $\mathcal{B}^0$ is $\mathcal{T}$-stable if and only if

$$M^0 := A^{1\times\ell}/U^0 \in \mathfrak{C} \iff Q(M^0) = 0 \iff Q(U^0) = Q(A^{1\times\ell}) = A_T^{1\times\ell}.$$

□

Theorem VI.2. (compare [13], [15, Th. 8.1, Alg. 8.2]) *The behavior $\mathcal{B}_1$ is $\mathcal{T}$-stabilizable if and only if $Q(U_1)$ is an A_T-direct summand of $A_T^{1\times\ell}$. The module $Q(U_1)$ can be computed by means of Th. IV.1. There is an algorithm to decide $\mathcal{T}$-stabilizability if open problem III.5 can be solved. Then all stabilizing compensators can be constructed* (parametrization). *The algorithms use the Gröbner basis algorithm.*

The construction of a direct complement of $Q(U_1)$ in $A_T^{1\times\ell}$ is related to the algorithm of [24].

References

[1] M. Barakat, *Purity Filtration and the Fine Structure of Uncontrollability*, Proc. MTNS 2010
[2] M. Bisiacco, M.E. Valcher, *A note on the direct decomposition of two-dimensional behaviors*, IEEE Trans. Circuits Systems I: Fund. Theory Appl. 48(2001), 490-494
[3] M. Bisiacco, M.E. Valcher, *Two-dimensional behavior decompositions with finite-dimensional intersection: A complete characterization*, Multidimensional Syst. Signal Process. 16(2005), 335-354
[4] V.D. Blondel, A. Megretski (Editors), *Unsolved Problems in Mathematical Systems and Control Theory*, Princeton University Press, 2004
[5] J.-E. Björk, *Rings of Differential Operators*, North-Holland, 1979
[6] N. Bourbaki, *Commutative Algebra*, Hermann, Paris, 1972
[7] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France 90(1962), 323-448
[8] H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, 1986
[9] D. Napp Avelli, *Almost direct sum decomposition and implementation of 2D behaviors*, Math. Control Signals Systems 21(2009), 1-19
[10] D. Napp Avelli, M. van der Put, S. Shankar, *Periodic Behaviors*, submitted to SIAM J. Contr. Optim.
[11] U. Oberst, *Multidimensional Constant Linear Systems*, Acta Appl. Math. 20(1990), 1 - 175
[12] U. Oberst, *The Construction of Noetherian Operators*, J. Algebra 222(1999), 595-620
[13] U. Oberst, *Stability and Stabilization of Multidimensional Input/Output Behaviors*, SIAM J. Control Optim. 45(2006), 1467-1507
[14] U. Oberst, *Regular almost interconnection of multidimensional behaviors*, SIAM J. Control Optim. 48(2009), 463-480
[15] U. Oberst, M. Scheicher, *A survey of (BIBO) stability and (proper) stabilization of multidimensional input/output systems*, Radon Series Comp. Appl. Math. 3, 151-190, de Gruyter, 2007
[16] A. Quadrat, *A lattice approach to analysis and synthesis problems*, Mathematics of Control, Signals, and Systems 18(2006), 147-186
[17] A. Quadrat, *On a generalization of the Youla-Kučera parametrization. Part II: The lattice approach to MIMO systems*, Mathematics of Control, Signals, and Systems 18(2006), 199-235
[18] J.P. Serre, *Groupes de homotopie et classes de groupes abéliennes*, Ann. Math. 58(1953), 258-294
[19] A. Sasane, *On the Willems closure with respect to W_s*, IMA J. Mathematical Control and Information 20(2003), 217-232
[20] S. Shankar, V.R. Sule, *Algebraic geometric aspects of feedback stabilization*, SIAM J. Control Optim. 30(1992), 11-30
[21] S. Shankar, *The Nullstellensatz for systems of PDE*, Adv. in Appl. Math. 23(1999), 360-374
[22] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, 1975
[23] L. Xu, Z. Lin, J.-Q. Ying, O. Saito, Y. Anazawa, *Open problems in control of linear discrete multidimensional systems*, in [4, pp.221-228]
[24] E. Zerz, V. Lomadze, *A Constructive Solution to Interconnection and Decomposition Problems with Multidimensional Behaviors*, SIAM J. Control Optim. 40(2001), 1072-1086