The significance of Gabriel localization for stability and stabilization of multidimensional input/output behaviors

To Peter Gabriel on his 77th birthday

Ulrich Oberst
Institut für Mathematik, Universität Innsbruck,
Technikerstraße 13, A - 6020 Innsbruck, Austria
e-mail: Ulrich.Oberst@uibk.ac.at
Tel. **43-512-507-6073, Fax **43-512-507-2920

Abstract—Serre (1953) was the first who considered categories of groups up to negligible ones, negligibility being determined by the considered context. Gabriel in his thesis (1962) developed this idea into a complete theory of quotient categories, rings and modules which is called Gabriel localization. This theory was nicely exposed by Stenström (1975). The author has recently observed that this theory is a valuable tool for stability and stabilization of multidimensional behaviors where a finitely generated multivariate polynomial torsion module is considered negligible if its characteristic variety has points in a preselected stability region only or, equivalently, if its associated autonomous behavior is stable, i.e., has polynomial-exponential solutions with frequencies in this stability region only. Via the Integral Representation Formula of Ehrenpreis/Palamodov corresponding properties hold for all trajectories of B. In the one-dimensional standard cases stability of B signifies asymptotic stability. Direct decompositions of modules up to negligible ones, also called almost direct decompositions, are central to our approach to stabilization. The connection of output feedback stabilization with direct decompositions has been observed and investigated by A. Quadrat [16], [17]. These decompositions are usually hidden in coprime factorizations of rational transfer matrices which, however, do not exist in general. In systems theory almost direct decompositions were considered by M. Bisiacco, M.E. Valcher [2], [3] and by D. Napp Avelli [9] for two-dimensional behaviors.

The present article gives a survey of our approach to multidimensional stabilization and some new results, also on the computation of arbitrary Willems closures. Matlis' theory on injective modules [8, pp.145-150] is an essential ingredient of our proofs. The survey [15] contains a comprehensive list of researchers and references on multidimensional stabilization.

AMS-classification: 93D15, 93B25, 93C20, 13P25

Key words: Gabriel localization, multidimensional behavior, stability, stabilization, integral representation formula, Willems closure

I. INTRODUCTION

J.P. Serre [18] was the first who considered categories of groups up to negligible ones, negligibility being determined by the considered context. P. Gabriel in his thesis [7] developed this idea into a complete theory of quotient categories, rings and modules which is called Gabriel localization. This theory was nicely exposed by B. Stenström [22]. The author has recently observed [13] that this theory is a valuable tool for stability and stabilization of multidimensional behaviors where a finitely generated multivariate polynomial torsion module M is considered negligible if its characteristic variety has points in a preselected stability region only or, equivalently, if its associated autonomous behavior is stable, i.e., has polynomial-exponential solutions with frequencies in this stability region only. Via the Integral Representation Formula of Ehrenpreis/Palamodov corresponding properties hold for all trajectories of B. In the one-dimensional standard cases stability of B signifies asymptotic stability. Direct decompositions of modules up to negligible ones, also called almost direct decompositions, are central to our approach to stabilization. The connection of output feedback stabilization with direct decompositions has been observed and investigated by A. Quadrat [16], [17]. These decompositions are usually hidden in coprime factorizations of rational transfer matrices which, however, do not exist in general. In systems theory almost direct decompositions were considered by M. Bisiacco, M.E. Valcher [2], [3] and by D. Napp Avelli [9] for two-dimensional behaviors.

The present article gives a survey of our approach to multidimensional stabilization and some new results, also on the computation of arbitrary Willems closures. Matlis' theory on injective modules [8, pp.145-150] is an essential ingredient of our proofs. The survey [15] contains a comprehensive list of researchers and references on multidimensional stabilization.

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II. DATA

We present Gabriel localization and its systems theoretic application in a continuous standard case. Let $A := \mathbb{C}[s] = \mathbb{C}[s_1, \ldots, s_n]$ be the complex polynomial algebra in $n \geq 1$ indeterminates with the category Mod$_A$ of $A$-modules and the sets Max($A$) $\subseteq$ Spec($A$) of maximal resp. prime ideals. We consider the standard injective cogenerator signal $A$-module $F := C^\infty(\mathbb{R}^n, \mathbb{C})$ of smooth functions or $F :=$
$D_C(R^n)$ of distributions with the action $s_i \circ y := \partial y/\partial x_i$ where $x = (x_1, \ldots, x_n) \in R^n$. If $a \subseteq A$ is an ideal we define

$$V(a) := \{p \in \text{Spec}(A); \ a \subseteq p\},$$
$$V_C(a) := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in C^n; \ a(\lambda) = 0\}.$$ For $\lambda \in C^n$ let

$$m(\lambda) := \{f \in A; \ f(\lambda) = 0\} = \sum_{i=1}^{n} A(s_i - \lambda_i) \in \text{Max}(A).$$

Every $p \in \text{Spec}(A)$ and $A$-module $M$ give rise to the local ring

$$A_p := \{at^{-1} \in C(s) ; \ a \in A, \ t \notin p\} \quad \text{and} \quad A_p M_p := \{xt^{-1}; \ x \in M, \ t \notin p\}.$$ The **associator** of $M$ is defined as

$$\text{ass}(M) := \{p \in \text{Spec}(A); \ A/p \subseteq M \ (\text{up to isom})\}.$$ Consider a matrix $R \in A^{k \times \ell}$ and its associated finitely generated modules

$$U := A^{1 \times k} R, \ M := A^{1 \times \ell} / U$$

and associated $F$-behavior

$$B := U^\perp := \{w \in F^\ell; \ R \circ w = 0\} = \{w \in F^\ell; \ R \circ w = 0\} \quad \text{with} \quad U = B^\perp.$$ The module is torsion or $B$ is autonomous if and only if $\text{rank}(R) = \ell$. Then the **characteristic variety** $\text{char}(M) = \text{char}(B)$ of $M$ or $B$ is [5, Sect. 8.1.7]

$$\text{char}(M) := \{\lambda \in C^n; \ M_{(\lambda)} \neq 0\} = \{\lambda \in C^n; \ \text{rank}(R(\lambda)) < \text{rank}(R) = \ell\} = \bigcup_{p \in \text{ass}(M)} V_C(p).$$

**The Integral Representation Formula** of Ehrenpreis/Palamodov [5, Th. 8.1.3] says that every smooth trajectory $y \in B$ has a representation

$$y(x) = \sum_{i=1}^{T} \int_{C^n} a_i(\lambda, x)e^{\lambda \bullet x}d\mu_i(\lambda)$$

where $\lambda \bullet x := \lambda_1x_1 + \ldots + \lambda_nx_n$ with the following specifications: The $\mu_i$ are measures on $C^n$ with support $\text{supp}(\mu_i) \subseteq \text{char}(B)$. The $a_i(\lambda, x) \in C[\lambda, x]^\ell$ are polynomial vectors with the property that for all $\lambda \in \text{supp}(\mu_i)$ the polynomial-exponential trajectory $a_i(\lambda, x)e^{\lambda \bullet x}$ belongs to $B$. The $a_i$ can be computed [12]. The convergence conditions are omitted here. For **stability and stabilization** we assume an arbitrary stability decomposition

$$C^n := \Lambda_1 \cup \Lambda_2, \ \Lambda_2 \neq \emptyset,$$

$$\Lambda_1 =: \text{stability region}, \ \Lambda_2 =: \text{instability region},$$
a standard case being $\Lambda_2 := \{z \in C; \ Re(z) \geq 0\}^n$. In the one-dimensional standard case $\Lambda_1 = C^-$ is the left open half-plane.

**III. Gabriel localization**

The results of this section except for those using (5) are originally due to Gabriel [7] and are exposed in Stenström’s book [22].

The decomposition (5) induces the decomposition

$$\text{Spec}(A) = \mathcal{P}_1 \cup \mathcal{P}_2$$

where

$$\mathcal{P}_1 := \{p \in \text{Spec}(A); \ V_C(p) \subseteq \Lambda_1\}.$$ The sets $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy the conditions

$$\mathcal{P}_1 \ni p \subseteq q \implies q \in \mathcal{P}_1 \quad \text{and} \quad \mathcal{P}_2 \ni p \supseteq q \implies q \in \mathcal{P}_2$$

and give rise to the following full subcategory of $\text{Mod}_A$:

$$\mathcal{C} := \{C \in \text{Mod}_A; \ \forall \lambda \in \Lambda_2; \ C_{(\lambda)} = 0\} = \{C \in \text{Mod}_A; \ \text{ass}(C) \subseteq \mathcal{P}_1\}.$$ This category is closed under isomorphisms, subobjects, factor objects, extensions and arbitrary direct sums. Such full subcategories are called **localizing** and give rise to the ideal set

$$T := \{a \subseteq A; \ A/a \in \mathcal{C}\}$$

with the following properties:

1. If $T \ni a \subseteq b$ then $b \in T$.
2. If $a, b \in T$ then $a \circ b \in T$.
3. If $T \ni a \supseteq b$ and $(b : a) \in T$ holds for all $a \in T$.

Such a $T$ is the basis of neighborhoods of 0 of a unique linear topology on $A$ and is called a **Gabriel topology**.

**Lemma and Definition III.1.** 1) There are bijective correspondences between the data $\mathcal{P}_1, \mathcal{P}_2, \mathcal{C}, \ T$ with the indicated properties given by

$$\text{Spec}(A) = \mathcal{P}_1 \cup \mathcal{P}_2,$$

$$\mathcal{P}_1 = T \bigcap \text{Spec}(A) = \{p \in \text{Spec}(A); \ A/p \in \mathcal{C}\},$$

$$T = \{a \subseteq A; \ V(a) \subseteq \mathcal{P}_1\} = \{a \subseteq A; \ A/a \in \mathcal{C}\},$$

$$\mathcal{C} = \{C \in \text{Mod}_A; \ \text{ass}(C) \subseteq \mathcal{P}_1\} = \{C \in \text{Mod}_A; \ \forall x \in T \ni C_{(x)} = 0\}.$$ The modules in $\mathcal{C}$ are called $\mathcal{C}$- or $T$-small or negligible.

2) For $\mathcal{P}_1 \subsetneq \text{Spec}(A)$ or $\{0\} \in \mathcal{P}_2$ (what we always assume) each module in $\mathcal{C}$ is a torsion module.

3) Not all $\mathcal{P}_1$ come from a decomposition (5) via (6), for instance [14, p.478]

$$\mathcal{P}_2(c) := \{p \in \text{Spec}(A); \ \text{dim}(A_p) < c\}$$

where $1 \leq c \leq n = \text{dim}(A_p) + \text{dim}(A/p)$, and

$$\mathcal{P}_1(c) := \{p \in \text{Spec}(A); \ \text{dim}(A/p) \leq n - c\}$$

where $\text{dim}$ denotes the Krull dimension. For $c = 2$ the associated negligible modules are called pseudo-zero in [6, Section VII.4.4].
4) The largest submodule in $\mathcal{C}$ of an $A$-module $M$ exists, is called its $\mathcal{C}$-radical or $T$-torsion module and is denoted by

$$\text{Ra}_T(M) = \{ x \in M; Ax \in \mathcal{C} \}.$$

Then $\text{Ra}_T(M/\text{Ra}_T(M)) = 0$ and

$$\text{Ra}_T(M) \neq 0 \iff \text{ass}(M) \cap P_1 \neq \emptyset.$$

If $P_1 = \{ p \in \text{Spec}(A); V_C(p) \subseteq \Lambda_1 \}$ according to (6) then

$$\text{Ra}_T(M) \neq 0 \iff \exists p \in \text{ass}(M) \text{ with } V_C(p) \subseteq \Lambda_1$$

In the sequel we assume that the data from Lemma and Definition III.1 are given.

**Example III.2.** Let $S \subseteq A$ be any multiplicatively closed set and $A_S \subseteq \mathcal{C}$ the corresponding quotient ring resp. module. Then $\mathcal{C}_S := \{ C \in \text{Mod}_A; C_S = 0 \}$ is a localizing subcategory with Gabriel topology

$$T^*_S \leftarrow \{ a \subseteq A; a \cap S \neq \emptyset \}$$

and

$$\text{Tor}_{T^*_S}(M) = \ker \text{(can} : M \to M_{S}) = \{ x \in M; \exists s \in S \text{ with } sx = 0 \}.$$

**Example III.3.** Assume $1 \leq c \leq n$ and consider

$$P_1(c) = \{ p \in \text{Spec}(A); \text{dim}(A/p) \leq n - c \}$$

from Lemma III.1,(3), and the associated localizing subcategory

$$\mathcal{C}(c) := \{ C \in \text{Mod}_A; \text{ass}(C) \subseteq P_1(c) \}.$$ If $M$ is an $A$-module its (Krull) dimension resp. codimension is given resp. defined as

$$\text{dim}(M) := \max \{ \text{dim}(A/p); p \in \text{ass}(M) \}$$

and

$$\text{codim}(M) := n - \text{dim}(M).$$

There results the equivalence

$$C \in \mathcal{C}(c) \iff \text{dim}(C) \leq n - c \iff \text{codim}(C) \geq c.$$ Hence $\text{tor}_c(M) := \text{Ra}_{\mathcal{C}(c)}(M)$ is the largest submodule of $M$ of codimension greater or equal to $c$ or of dimension at most $n - c$. If $M$ is finitely generated these torsion modules can be computed by means of Theorem IV.1,(2), below. Obviously the submodules decrease with increasing $c$, hence

$$M \supseteq \text{tor}_c(M) \supseteq \cdots \supseteq \text{tor}_{n-1}(M) \supseteq \text{tor}_n(M).$$

The equation

$$\text{tor}_{c+1}(M/\text{tor}_{c+1}(M)) \subseteq \text{tor}_{c+1}(M/\text{tor}_{c+1}(M)) = 0$$

implies that all $p \in \text{ass}(\text{tor}_c(M)/\text{tor}_{c+1}(M))$ have dimension $n - c$, i.e., that $\text{tor}_c(M)/\text{tor}_{c+1}(M)$ is pure $n - c$-dimensional. Therefore this filtration is called the purity filtration of $M$ in [1] where also the history of this filtration and a computation by means of spectral sequences are discussed. If $M$ is finitely generated the module $\text{tor}_c(M)$ is the largest $\mathcal{C}$-finite dimensional submodule of $M$ whose algorithmic computation was described in [14].

The data from Lemma and Definition III.1,(1), also induce the saturated multiplicatively closed set

$$T := \bigcap_{p_2 \in P_2} (A \setminus p_2) = \{ \text{stable polynomials} \}$$

and its quotient ring

$$A_T = \{ at^{-1}; a \in A, t \in T \} \subseteq \mathcal{C}(s)$$

of stable rational functions. Then

$$\forall p_2 \in P_2 : M_{p_2} = (M_T)_{p_2}$$

and thus

$$\mathcal{C}_T \subseteq \mathcal{C} \text{ and } T_T \subseteq T.$$ If $P_1 = \{ p \in \text{Spec}(A); V_C(p) \subseteq \Lambda_1 \}$ (see (6)) then

$$T := \{ t \in A; V_C(t) \subseteq \Lambda_1 \}$$

and

$$A_T = \{ at^{-1} \in \mathcal{C}(s); V_C(t) \subseteq \Lambda_1 \}.$$ The equality $T_T = T$ holds only in case $\Lambda_2$ is ideal-convex [20, Prop. 3.1.20], i.e., $\text{Max}(A_T) = \{ \text{max}(\lambda)_T; \lambda \in \Lambda_2 \}$ or, equivalently, $\mathcal{C}$ is perfect [22, Prop. XI.3.4].

**Corollary and Definition III.4.**

1) An autonomous behavior

$$\mathcal{B} := \{ w \in \mathcal{F}; \ R \circ w = 0 \}, \ R \in A^{k \times t},$$

where $\text{rank}(R) = \ell$ is called $T$-small, negligible or stable, if its module $M := A^{1 \times t}/A^{1 \times k}R$ is $T$-small, i.e., belongs to $\mathcal{C}$.

2) Assume $P_1 = \{ p \in \text{Spec}(A); V_C(p) \subseteq \Lambda_1 \}$. Then the behavior from item 1 is $T$-stable if and only if its characteristic variety $\text{char}(\mathcal{B}) = \text{char}(M)$ is contained in $\Lambda_1$. In this case only polynomial-exponential solutions with frequencies $\lambda \in \Lambda_1$ contribute to the solution integral of (4).

3) In the standard one-dimensional case with $\Lambda_1 = \mathcal{C}$ the equality $T_T = T_T$ holds and $\mathcal{B}$ is $T$-stable if and only if it is asymptotically stable.

4) In the standard cases of multidimensional systems theory the inclusion $a \in T$ or $V_C(a) \subseteq \Lambda_1$ can be decided via constructive real algebraic geometry.

**Open problem III.5.** For the standard systems theoretic $P_1$ and $\Lambda_1$ and $a \in T$ decide $a \in T_T$, i.e., $a \cap T \neq \emptyset$, algorithmically and eventually construct $t \in a \cap T$. This open problem was already stated in [23].

An $A$-module $N$ is called $\mathcal{C}$- or $T$-closed if for each $a \in T$ the canonical map

$$\text{can} : N \to \text{Hom}_A(a, N), \ x \mapsto (a \mapsto ax), \ a \in T,$$

is bijective. Let $\text{Mod}_A, T$ be the full subcategory of all $T$-closed modules. In the situation of Example III.2 the category $\text{Mod}_A, T_S$ coincides with $\text{Mod}_A, S$. The following theorem constructs the analog of the quotient module $M_S$. 
Theorem and Definition III.6. (Quotient functor) Data from Lemma and Definition III.1.

1) The injection \( \text{inj} : \text{Mod}_{A,T} \rightarrow \text{Mod}_A \) has a left adjoint \( Q : \text{Mod}_A \rightarrow \text{Mod}_{A,T} \) with its associated functorial morphism \( \eta : \text{id}_{\text{Mod}_A} \rightarrow \text{inj} \) such that
\[
\text{Hom}_A(Q(M), N) \cong \text{Hom}_A(M, N),
\]
for \( M \in \text{Mod}_A, N \in \text{Mod}_{A,T} \). Then
\[
\text{Ra}_A(M) = \ker(\eta_M) \text{ and } \text{cok}(\eta_M) \in \mathcal{C}.
\]
2) The category \( \text{Mod}_{A,T} \) is abelian, the functor \( Q \) is exact and \( \text{inj} \) is left exact, but not exact in general.
3) \( \mathcal{C} = \{ C \in \text{Mod}_A ; Q(C) = 0 \} \) and \( \text{Mod}_{A,T} = \{ N \in \text{Mod}_A ; N \cong Q(N) \} \).
4) The category \( \text{Mod}_{A,T} \) is a full subcategory of \( \text{Mod}_A \).

The use of \( Q \) or of closed modules thus signifies to consider \( A \)-modules up to \( T \)-negligible ones.

Example III.7. In the standard case \( \mathcal{C} = \mathcal{C}_S \) of Example III.2 the identity \( Q(M) = M_S \in \text{Mod}_{A,T_S} = \text{Mod}_A \) holds and \( \text{inj} \) is exact.

IV. QUOTIENT COMPUTATION

The following theorem was left as an open problem in [15, p.168],[14, p.479].

Theorem IV.1. Data from Lemma and Definition III.1. Let \( A U \) be a finitely generated torsionfree module or \( U \subseteq A^{1 \times t} \) without loss of generality.

1) \( Q(A) = A_T, U_T \subseteq Q(U) \subseteq A_T^{1 \times t} \text{ and } Q(U)/U_T = \text{Ra}_A(A^{1 \times t}/U_T) \).

2) If \( p \in \mathcal{P}_1 \) can be decided constructively there is an algorithm for the computation of \( \text{Ra}_A(A^{1 \times t}/U) \) via constructive primary decompositions and therefore also one which computes \( Q(U) \) in the form \( Q(U) = A_T^{1 \times t} R_T, R_T \in A^{1 \times t} \). All steps use the Gröbner basis algorithm.

3) If \( \mathcal{P}_1 = \{ p \in \text{Spec}(A) ; \text{V}_C(p) \subseteq \mathcal{A}_1 \} \) and \( \mathcal{A}_1 \) is semi-algebraic in \( C^n = \mathbb{R}^{2n} \), then \( p \in \mathcal{P}_1 \) can be decided by constructive real algebraic geometry.

For arbitrary finitely generated \( A M \) the module \( A, Q(M) \) may not be finitely generated and there is presently no algorithm to compute it.

V. LARGE INJECTIVE COGENERATORS

The injective cogenerators \( F \) from above and many others of systems theoretic significance are large [11, Th. 4.54]. This signifies that every finitely generated module can be embedded into a finite power of \( F \) or that \( \text{ass}(F) = \text{Spec}(A) \).

Theorem and Definition V.1. Data from Lemma and Definition III.1.

1) The radical \( \text{Ra}_A(F) \) is injective and hence there is a (non-constructive) direct complement \( F_2 \) with
\[
F = \text{Ra}_A(F) \oplus F_2 = \{ \text{negligible signals} \} \oplus \{ \text{essential signals} \}.
\]

Any such \( F_2 \) satisfies
\[
F_2 \cong Q(F) \cong F/\text{Ra}_A(F)
\]
and is an injective cogenerator in \( \text{Mod}_{A,T} \).

2) For the modules and behavior from (1) and (2) the decomposition (9) induces the decomposition \( B = \text{Ra}_A(B) \oplus (B \cap F_2) \) where
\[
B \cap F_2 \cong Q(A(M), F_2) \cong Q(A(M), F_2).
\]

These isomorphisms, in turn, imply the equivalences
\[
M \in \mathcal{C} \iff Q(M) = 0 \iff \text{Hom}_A(M, F_2) = 0 \iff B = \text{Ra}_A(B).
\]

3) Willems closures: If \( M = A^{1 \times t}/U \) and \( \text{Ra}_A(M) = \bar{U}/U \) then
\[
\text{Ra}_A(M) = \bar{U}/U = \ker(M \rightarrow F_2 \text{Hom}_A(M, F_2), \ x \mapsto (\varphi(x))_p) \text{ or }
\bar{U} = \{ \xi \in A^{1 \times t} ; \xi \in (B \cap F_2) = 0 \} \text{ and hence }
\bar{U} = U \iff \text{ass}(M) \subseteq \mathcal{P}_2.
\]

If \( \mathcal{P}_1 = \{ p \in \text{Spec}(A) ; \text{V}_C(p) \subseteq \mathcal{A}_1 \} \) then
\[
\bar{U} = U \iff \forall p \in \text{ass}(M) ; \text{V}_C(p) \cap \mathcal{A}_2 \neq \emptyset.
\]

The submodule \( \bar{U} \) is called the Willems closure of \( U \) with respect to \( F_2 \) and can be calculated via Theorem IV.1. If \( U = \bar{U} \) then \( U \) is called Willems closed.

4) For every injective module \( I \) there is a suitable Gabriel localization with its associated \( F_2 \) such that the Willems closures with respect to \( I \) and \( F_2 \) coincide.

Item 3. generalizes computations of Shankar [21], [10], Sasane [19] et al.

VI. STABILIZATION

We assume the data from Lemma and Definition III.1 and from (7). Consider two input/output (IO)-subbehaviors \( B_i, i = 1, 2 \), of \( F^{p+m} \):
\[
B_1 := \{( u^{y_1}_1 ) \in F^{p+m} ; P_1 \circ y_1 = Q_1 \circ u_1 \},
\]
\[
P_1 \in A^{k_1 \times p}, \text{rank}(P_1) = \text{rank}(P_1, -Q_1) = p,
\]
\[
U_1 := A^{1 \times k_1}(P_1, -Q_1) \subseteq A^{1 \times (p+m)},
\]
\[
B_2 := \{( u^{y_2}_2 ) \in F^{p+m} ; P_2 \circ y_2 = Q_2 \circ u_2 \},
\]
\[
P_2 \in A^{k_2 \times m}, \text{rank}(P_2) = \text{rank}(P_2, -Q_2) = m,
\]
\[
U_2 := A^{1 \times k_2}(Q_2) \subseteq A^{1 \times (p+m)}.
\]

Let \( l := p + m \). Their output feedback behavior (compare Fig. 1) is
\[
B := \{( y u ) \in F^l ; P \circ y = Q \circ u \} \text{ where }
\]
\[
y := ( u^{y_1}_1 ) \in F^{p+m}, \ u := ( u^{y_2}_2 ) \in F^{p+m},
\]
\[
P := \left( \begin{array}{c}
P_1 & -Q_1 \\
-Q_2 & P_2
\end{array} \right), \ Q := \left( \begin{array}{c}
0 & Q_1 \\
Q_2 & 0
\end{array} \right),
\]
\[
U^0 := A^{1 \times (k_1 + k_2)}P = U_1 + U_2.
\]
The feedback behavior $B$ is well-posed if it is an IO-behavior with input $u$, or, equivalently,
\[
\text{rank}(P) = p + m \text{ or } U^0 = U_1 \oplus U_2.
\]

\[u_1 \quad + \quad B_1 \quad \quad y_1 \quad + \quad B_2 \quad y_2\]

Fig. 1. The output feedback interconnection of two IO behaviors

**Lemma and Definition VI.1.** The feedback behavior is called $T$-stable if it is well-posed and if its autonomous part $B^0 := \{y \in \mathcal{F}; P_y = 0\}$ is $T$-stable (see Corollary and Definition III.4). Then $B_2$ is called a $T$-stabilizing compensator of $B_1$. $B_1$ is called $T$-stabilizable and $Q(U^0) = Q(U_1) \oplus Q(U_2) = A_T^{1 \times \ell}$.

**Proof.** $B^0$ is $T$-stable if and only if
\[
M^0 := A^{1 \times \ell} / U^0 \in \mathcal{C} \iff Q(M^0) = 0 \iff Q(U^0) = Q(A^{1 \times \ell}) = A_T^{1 \times \ell}.
\]

**Theorem VI.2.** (compare [13], [15, Th. 8.1, Alg. 8.2]) The behavior $B_1$ is $T$-stabilizable if and only if $Q(U_1)$ is an $A_T$-direct summand of $A_T^{1 \times \ell}$. The module $Q(U_1)$ can be computed by means of Th. IV.1. There is an algorithm to decide $T$-stabilizability if open problem III.5 can be solved. Then all stabilizing compensators can be constructed (parametrization). The algorithms use the Gröbner basis algorithm.

The construction of a direct complement of $Q(U_1)$ in $A_T^{1 \times \ell}$ is related to the algorithm of [24].

**References**

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