

## DUALITY FOR DIFFERENTIAL-DIFFERENCE SYSTEMS OVER LIE GROUPS\*

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**Abstract.** In modern mathematical systems theory, there exist two consistent ways of defining and describing a linear system: (i) in the *behavioral approach*, a linear system is the kernel  $\mathfrak{B}$  of a matrix-valued operator  $R$  in a power of a signal space  $W$ ; (ii) in the *module-theoretic setting*, a linear system is the cokernel  $M$  of the above matrix  $R$ . These two formulations have connections. The minimal conditions under which they are equivalent are investigated in this paper. The general theory is applied to differential-difference systems over Lie groups.

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**1. Introduction.** The notion of linear system can be defined in many ways (by a transfer function, a state-space representation, etc.). Two of them are of a particular interest since they are complete (especially, hidden modes are well represented) and intrinsic (i.e., independent of a choice made a priori, e.g., of the state and of the input- and output-variables). They can be characterized as follows, denoting by  $\mathbf{A}$  a ring of operators:

(i) A linear system is characterized by its *behavior*, i.e., the *kernel*  $\mathfrak{B}$  of a matrix-valued operator  $R \in \mathbf{A}^{q \times k}$  in a power of a solution space  $W$ , assumed to be an  $\mathbf{A}$ -module. This approach was initiated by Willems in the 80's [68] (see also [70]).

(ii) A linear system is associated with a *finitely presented*  $\mathbf{A}$ -module  $M$ , i.e., the *cokernel* of a matrix-valued operator  $R$  such as the one above. This formulation was initiated by Malgrange [45] and developed by experts in the theory of  $\mathcal{D}$ -modules (e.g., Palamodov, Björk, Kashiwara, and Schapira, to cite a few of them: see [42], [21], [3], [55] and the references therein) and was popularized in systems theory by Fliess [29].

Each of the two above approaches has its advantages. Approach (i) emphasizes the role of *solutions* and therefore could be more appealing for engineers. Approach (ii) emphasizes the role of *equations* which are the base of science. Given the ring  $\mathbf{A}$ , the behavior  $\mathfrak{B}$  can always be deduced from the module  $M$  and the signal space  $W$ , and thus it can be written  $\mathfrak{B}_W(M)$ ; for the converse to hold true, the space  $W$  must be “rich enough” (for example, if  $W = \{0\}$ , then  $\mathfrak{B}_W(M) = \{0\}$  whatever the module  $M$  is). On the other hand, the ring  $\mathbf{A}$  must be appropriate both to reflect the nature of the system and for the equations over  $\mathbf{A}$  to have solutions in the suitable space  $W$ ; thus there must be the *right interplay* between  $\mathbf{A}$  and  $W$ . If  $W$  is a *cogenerator*—a classical notion in homological algebra—then  $M$  is determined by  $\mathfrak{B}_W(M)$  since the map  $M \mapsto \mathfrak{B}_W(M)$  is injective. The second author extensively used the notion of cogenerator and showed its importance in systems theory (see [50] and related references).

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However, the condition that  $W$  be a cogenerator is somewhat restrictive (as already noticed by Wood [72]). The aim of this paper is to further investigate the *minimal conditions* under which approaches (i) and (ii) are equivalent. This leads us to enlarge the notion of cogenerator, usually defined in (and for) an abelian category  $\mathcal{C}$  (such as  $\mathbf{A}\mathbf{Mod}$ , the category of left  $\mathbf{A}$ -modules). Here we define a cogenerator in an additive category  $\mathcal{C}$  for a full subcategory  $\mathcal{D}$  of  $\mathcal{C}$ . A typical example is when  $\mathcal{C}$  is  $\mathbf{A}\mathbf{Mod}$  and  $\mathcal{D}$  is  $\mathbf{A}\mathbf{Mod}^{fp}$ , the category of *finitely presented* left  $\mathbf{A}$ -modules.

Then we apply the general theory to differential-difference systems over Lie groups. Such systems—and, more generally, convolution systems over Lie groups—have applications in various fields such as robotics, image processing, vision, and polymer science (see, e.g., [14], [75], and [16]).

The paper is organized as follows: the general definition of a cogenerator for a subcategory is given in section 2 after some preliminaries and illustrated by two examples: that already mentioned ( $\mathcal{C} = \mathbf{A}\mathbf{Mod}$ ,  $\mathcal{D} = \mathbf{A}\mathbf{Mod}^{fp}$ ) and another one, similar but with additional topological considerations. The application to differential systems with shifts over a Lie group is developed in sections 3 (case of lumped shifts) and 4 (case of distributed shifts). Section 5 is devoted to concluding remarks. The main results are stated in theorems and the other ones in lemmas and corollaries. Two appendices are proposed in section A for the reader's convenience: the first one on categories and functors and the second one on Lie groups, Lie algebras, and convolution algebras over Lie groups.

## 2. Cogenerators: A categorical point of view.

### 2.1. A general definition of a cogenerator.

**2.1.1.  $U$ -cokernels and behaviors.** The following notation is used throughout this paper.

**NOTATION 2.1.** *Let  $\mathbf{R}$  be a ring. (i) The morphisms of left (resp., right)  $\mathbf{R}$ -modules are left (resp., right) morphisms (see Appendix 1).*

*(ii) Let  $M$  be a left (resp., right)  $\mathbf{R}$ -module and consider a power and a copower of  $M$  with set of indices  $I$ ; the power is denoted by  $M^I$  (resp.,  ${}^I M$ ), the copower by  $M^{(I)}$  (resp.,  ${}^{(I)} M$ ), and both of them by  $M^n$  (resp.,  ${}^n M$ ) if  $I = \{1, \dots, n\}$ ; the elements of this power and of this copower are represented by rows (resp., columns), and the set of all  $n \times m$  matrices with entries in  $M$  is denoted by  ${}^n M^m$ .*

*(iii) The set of nonzero elements of  $\mathbf{R}$  is denoted by  $\mathbf{R}^\times$ .*

In what follows,  $\mathcal{C}$  is a preabelian category with arbitrary products (see Appendix 1), except when otherwise stated. Let  $U$  and  $W$  be objects of  $\mathcal{C}$  and  $\mathbf{A} \triangleq \text{Hom}_{\mathcal{C}}(U, U)$  and  $\mathbf{E} \triangleq \text{Hom}_{\mathcal{C}}(W, W)$  the rings of *left* endomorphisms of  $U$  and  $W$ , respectively (Appendix 1, section A.1.1). Then for any objects  $M, N$  of  $\mathcal{C}$ , the abelian groups  $\text{Hom}_{\mathcal{C}}(M, W)$  and  $\text{Hom}_{\mathcal{C}}(U, N)$  of all left morphisms  $\bullet f : M \rightarrow W$  and  $\bullet g : U \rightarrow N$  are, respectively, a right  $\mathbf{E}$ -module and a left  $\mathbf{A}$ -module via composition of morphisms. The abelian group  $\text{Hom}_{\mathcal{C}}(U, W)$  is an  $(\mathbf{A}, \mathbf{E})$ -bimodule [4, subsect. II.1.14]. The additive functor  $\mathfrak{B}_W = \text{Hom}_{\mathcal{C}}(\bullet, W)$  is contravariant and left exact from  $\mathcal{C}$  to the category  $\mathbf{Mod}_{\mathbf{E}}$  of right  $\mathbf{E}$ -modules.

Let  $I$  be a set of indices, arbitrary if  $\mathcal{C}$  has arbitrary coproducts and finite otherwise. Consider a subobject  $N$  of  $U^{(I)}$  (written  $N \subseteq U^{(I)}$ ),  $\iota : N \hookrightarrow U^{(I)}$  the inclusion, and the associated factor object  $M = \text{coker } \iota$  (also written  $M = U^{(I)}/N$ ) defined by the canonical exact sequence of left morphisms

$$(2.1) \quad N \xrightarrow{\bullet \iota} U^{(I)} \xrightarrow{\bullet \varphi} M \longrightarrow 0.$$

DEFINITION 2.2. (i) A factor of  $U^{(I)}$  is called  $U$ -generated (and  $U$ -finitely generated if  $I$  is finite).

(ii) The above object  $M = \text{coker } \iota$  is called a  $U$ -cokernel.

Remark 2.3. (i) All objects of  $\mathcal{C}$  are  $U$ -generated if and only if  $U$  is a generator. If the category  $\mathcal{C}$  is not abelian, a  $U$ -generated object need not be a  $U$ -cokernel since Noether’s first isomorphism theorem is not valid (see also Remark 2.22 below).

(ii) The above item holds because we allow the set  $I$  of indices to be arbitrary.

This is also necessary for the proof of Theorem 2.13(iii) and thus for that of Theorem 4.5(iv) below to be correct.

The right  $\mathbf{E}$ -modules  ${}^I \text{Hom}_{\mathcal{C}}(U, W)$  and  $\text{Hom}_{\mathcal{C}}(U^{(I)}, W)$  are identified (as in Appendix 1, section A.1.2). The subobject  $N$  gives rise to the  $W$ -behavior

$$(2.2) \quad N^\perp \triangleq \left\{ w \in \text{Hom}_{\mathcal{C}}(U^{(I)}, W) : \iota w = 0 \right\}$$

which is an object of  $\mathbf{Mod}_{\mathbf{E}}$ . The result below is a straightforward generalization of a classical result due to Malgrange [45, sect. 2, 2°], where multidimensional systems with constant coefficients are considered.

LEMMA 2.4. There is a canonical  $\mathbf{E}$ -linear isomorphism

$$\sigma : \text{Hom}_{\mathcal{C}}(M, W) \xrightarrow{\sim} N^\perp : \hat{w} \mapsto w$$

given by  $w = \varphi \hat{w}$ .

Proof. The kernel  $\ker \mathfrak{B}_W(\iota)$  is the largest subobject of  $\mathfrak{B}_W(U^{(I)}) = {}^I \text{Hom}_{\mathcal{C}}(U, W)$  annihilated by  $\mathfrak{B}_W(\iota) : w \mapsto \iota w$  (Appendix 1, section A.1.1); thus  $N^\perp$  is a representative of  $\ker \mathfrak{B}_W(\iota)$ . Applying  $\mathfrak{B}_W$  to (2.1) we obtain the exact sequence of right morphisms

$$(2.3) \quad 0 \longrightarrow \mathfrak{B}_W(M) \xrightarrow{\mathfrak{B}_W(\varphi)} \mathfrak{B}_W(U^{(I)}) \xrightarrow{\mathfrak{B}_W(\iota)} \mathfrak{B}_W(N);$$

thus  $\mathfrak{B}_W(M)$  is another representative of  $\ker \mathfrak{B}_W(\iota)$ . The isomorphism  $\sigma : \hat{w} \mapsto \varphi \hat{w}$  is canonical and  $\mathbf{E}$ -linear.  $\square$

DEFINITION 2.5. Let  $\mathcal{D}$  be a full subcategory of  $\mathcal{C}$ , all objects of which are  $U$ -cokernels. The class of all  $W$ -behaviors associated with  $\mathcal{D}$  is  $\mathfrak{B}_W(\mathcal{D})$ .

Conversely, let  $\mathfrak{B} \subseteq \text{Hom}_{\mathcal{C}}(U^{(I)}, W)$  and consider the object

$$(2.4) \quad \mathfrak{B}^\perp \triangleq \bigcap_{w \in \mathfrak{B}} \ker w = \ker \left( (w)_{w \in \mathfrak{B}} : U^{(I)} \longrightarrow W^{\mathfrak{B}} \right) \subseteq U^{(I)}.$$

For any  $U$ -cokernel  $M$  (i.e., for any  $M$  such as in (2.1)) and  $\mathfrak{B} = N^\perp$  defined by (2.2) we define the canonical (left) morphism

$$(2.5) \quad \bullet \phi_M \triangleq (\hat{w})_{w \in \mathfrak{B}} : M \longrightarrow W^{\mathfrak{B}};$$

we have by Lemma 2.4  $\varphi \phi_M = (w)_{w \in \mathfrak{B}}$ , and thus by (2.4)

$$(2.6) \quad \ker(\bullet \phi_M) = \mathfrak{B}^\perp.$$

LEMMA 2.6. The two maps  $(\bullet)^\perp : N \mapsto N^\perp$  and  $\mathfrak{B} \mapsto \mathfrak{B}^\perp$  constitute an order-reversing Galois connection (for the inclusion).

Proof. (i) Let  $N_1 \subseteq N_2$  and let  $\iota_1 : N_1 \hookrightarrow U^{(I)}$  be the inclusion. For any  $w \in N_2^\perp$ ,  $\iota_1 w = 0$ ; thus  $N_2^\perp \subseteq N_1^\perp$ . Similarly,  $\mathfrak{B}_2^\perp \subseteq \mathfrak{B}_1^\perp$  whenever  $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ . (ii) We have to prove that [19, sect. 7.6]

$$N \subseteq N^{\perp\perp}, \quad \mathfrak{B} \subseteq \mathfrak{B}^{\perp\perp}.$$

We have  $N \subseteq \ker \varphi \subseteq \ker \varphi \phi_M = N^{\perp\perp}$ , and the proof of the second inclusion is similar.  $\square$

**2.1.2. Cogenerators.** First let us establish the following.

LEMMA 2.7. *Let  $\mathcal{C}$  be a preabelian (or more generally an additive) category with arbitrary products,  $M$  and  $W$  objects of  $\mathcal{C}$ , and the canonical morphism*

$$\bullet\phi_M \triangleq (\hat{w})_{\hat{w} \in \text{Hom}_{\mathcal{C}}(M, W)} : M \rightarrow W^{\text{Hom}_{\mathcal{C}}(M, W)}.$$

(1) *The following conditions are equivalent:*

- (i)  $\phi_M$  is a monomorphism;
- (ii) there exists a monomorphism  $M \hookrightarrow W^J$  for some set of indices  $J$ ;
- (iii) for every object  $M_1$  of  $\mathcal{C}$ , the arrow map

$$\mathfrak{B}_W : \text{Hom}_{\mathcal{C}}(M_1, M) \mapsto \text{Hom}_{\mathbf{E}}(\mathfrak{B}_W(M), \mathfrak{B}_W(M_1))$$

*is injective.*

(2) *If  $U$  is a generator, the above conditions are equivalent to the following:*

- (iv) *For any nonzero morphism  $\bullet h : U \rightarrow M$ , there exists  $\bullet\beta : M \rightarrow W$  with  $\bullet h\beta \neq 0$ .*

*Proof.* (1) (i) $\Rightarrow$ (ii): This is clear (take  $J = \text{Hom}_{\mathcal{C}}(M, W)$ ). (ii) $\Rightarrow$ (iii): Let  $\bullet f : M_1 \rightarrow M$  and  $\bullet\eta = (\bullet\eta_j)_{j \in J} : M \hookrightarrow W^J$ . If  $\mathfrak{B}_W(f) = 0$ , then  $\mathfrak{B}_W(f)(\eta_j) = \bullet f\eta_j = 0$  for all  $j \in J$ ; thus  $\bullet f\eta = 0$ , and since  $\bullet\eta$  is a monomorphism, this implies  $f = 0$ . (iii) $\Rightarrow$ (i): By (2.5),  $\mathfrak{B}_W(f) = 0$  if and only if  $f\phi_M = 0$ ; thus, if  $\mathfrak{B}_W$  is injective, then  $\phi_M$  is a monomorphism.

(2) Assume that  $U$  is a generator. (ii) $\Rightarrow$ (iv): Let  $\bullet\eta = (\bullet\eta_j)_{j \in J} : M \hookrightarrow W^J$  and  $\bullet h : U \rightarrow M$ ,  $h \neq 0$ . Then  $\bullet h\eta \neq 0$ ; thus there exists  $j \in J$  such that  $h\eta_j \neq 0$ , and we have found  $\beta = \eta_j : M \rightarrow W$  such that  $\bullet h\beta \neq 0$ . (iv) $\Rightarrow$ (iii): Let  $\bullet f : M_1 \rightarrow M$  be such that  $\mathfrak{B}_W(f) = 0$ ; thus  $\mathfrak{B}_W(f)(\beta) = f\beta = 0$  for any  $\bullet\beta : M \rightarrow W$ . If  $f \neq 0$ , there exists  $\bullet h_1 : U \rightarrow M_1$  such that  $h_1 f \neq 0$ , whereas  $h_1 f\beta = 0$ , i.e.,  $h\beta = 0$  with  $h = h_1 f \neq 0$ , and (iv) does not hold.  $\square$

COROLLARY AND DEFINITION 2.8. (i) *Let  $\mathcal{C}$  be a preabelian (or more generally an additive) category with arbitrary products. An object  $W \in \mathcal{C}$  is called a cogenerator for a full subcategory  $\mathcal{D}$  of  $\mathcal{C}$  if the conditions of Lemma 2.7(1) are satisfied for all objects  $M \in \mathcal{D}$ .*

(ii) *Then  $\mathfrak{B}_W(\mathcal{D})$  is a subcategory of  $\mathbf{Mod}_{\mathbf{E}}$  and the functor  $\mathfrak{B}_W$  induces an isomorphism  $\mathfrak{B}_W : \mathcal{D}^{\text{op}} \xrightarrow{\sim} \mathfrak{B}_W(\mathcal{D})$ .*

See Appendix 1 for the proof of (ii). When taking  $\mathcal{D} = \mathcal{C}$ , one meets the classical definition of a cogenerator again, and the statement of Lemma 2.7 can be easily generalized to the case when  $\mathcal{C}$  is any category with arbitrary products (this generalization is left to the reader).

Recall that a left  $\mathbf{A}$ -module  $M$  is said to be *faithful* if for any  $0 \neq a \in \mathbf{A}$ ,  $aM \neq 0$ . We have the following.

LEMMA 2.9. *Let  $\mathcal{D}$  be a full subcategory of the preabelian (or more generally the additive) category  $\mathcal{C}$ , let  $W \in \mathcal{C}$  be a cogenerator for  $\mathcal{D}$ , let  $U \in \mathcal{D}$ , and let  $\mathbf{A} \triangleq \text{Hom}_{\mathcal{C}}(U, U)$ . Then  $\mathfrak{B}_W(U)$  is a faithful left  $\mathbf{A}$ -module.*

*Proof.* By Lemma 2.7(1)(iii), the map

$$\mathfrak{B}_W : \mathbf{A} \rightarrow \text{Hom}_{\mathbf{E}}(\mathfrak{B}_W(U), \mathfrak{B}_W(U)), \quad a \mapsto a\bullet$$

(where  $a\bullet$  is the left composition by  $a$ ) is injective; thus if  $a$  is nonzero, then  $a\bullet$  and  $a\mathfrak{B}_W(U)$  are nonzero too, which implies that the left  $\mathbf{A}$ -module  $\mathfrak{B}_W(U)$  is faithful.  $\square$

**THEOREM 2.10.** *Let  $\mathcal{C}$  be a semi-abelian category (Appendix 1, section A.1.1) and  $U, W \in \mathcal{C}$ .*

(1) *Let  $I$  be a set of indices, let  $N \subseteq U^{(I)}$ , let  $\iota : N \hookrightarrow U^{(I)}$  be the inclusion, and let  $M = U^{(I)}/N$  be a  $U$ -cokernel. The conditions in Lemma 2.7(1) are equivalent to the following:*

(v)  $\text{im } \iota = N^{\perp\perp}$ .

(2) *Let  $\mathcal{D}$  be a full subcategory of  $\mathcal{C}$  and  $\mathcal{S}_{\mathcal{D}}(U^{(I)})$  be the set of all  $N \subseteq U^{(I)}$  such that  $U^{(I)}/N \in \mathcal{D}$  and  $\iota : N \hookrightarrow U^{(I)}$  is strict; in addition, let  $\mathcal{S}_{\mathcal{D}}(U^{(I)})^{\perp}$  be the set of all  $N^{\perp}$ ,  $N \in \mathcal{S}_{\mathcal{D}}(U^{(I)})$ . If  $W \in \mathcal{C}$  is a cogenerator for  $\mathcal{D}$ , the Galois connection in Lemma 2.6 induces an order-reversing Galois connection*

$$\mathcal{S}_{\mathcal{D}}(U^{(I)}) \longleftrightarrow \mathcal{S}_{\mathcal{D}}(U^{(I)})^{\perp} : N \mapsto N^{\perp}, \quad \mathfrak{B} \mapsto \mathfrak{B}^{\perp}$$

which is bijective.

*Proof.* (1) (i) $\Rightarrow$ (v): By (2.6),  $\ker \bullet\varphi\phi_M = N^{\perp\perp}$ . If (i) holds, then  $N^{\perp\perp} = \ker \varphi = \ker \text{coker } \iota = \text{im } \iota$ . (v) $\Rightarrow$ (i): If (v) holds, then  $\ker \varphi = \ker(\varphi\phi_M)$  and hence also by (A.2), (A.3):

$$\text{coim}(\varphi\phi_M) = \text{coim } \varphi = \text{coker } \ker \text{coker } \iota = \text{coker } \iota = \varphi.$$

The canonical decomposition of  $\varphi\phi_M$  yields

$$\varphi\phi_M = \text{coim}(\varphi\phi_M)g = \varphi g,$$

and thus  $\phi_M = g$ , which is a monomorphism since  $\mathcal{C}$  is semi-abelian.

(2) If  $\iota : N \hookrightarrow U^{(I)}$  is strict, then  $\iota = \text{im } \iota$  and by (v),  $N = N^{\perp\perp}$  since  $W$  is a cogenerator for  $\mathcal{D}$ , and hence  $N = \mathfrak{B}^{\perp}$ , where  $\mathfrak{B} = N^{\perp}$ ; thus the map  $\mathcal{S}_{\mathcal{D}}(U^{(I)}) \ni N \mapsto N^{\perp} \in \mathcal{S}_{\mathcal{D}}(U^{(I)})^{\perp}$  is bijective with inverse  $\mathfrak{B} \mapsto \mathfrak{B}^{\perp}$ .  $\square$

Theorem 2.10(2) can be viewed as a generalization of [50, sect. 2, Corollary (47)] and [63, Corollary 1.1], where  $\mathcal{D} = \mathcal{C} = \mathbf{A}\mathbf{Mod}$ . In everything that follows, we make the following.

**Assumption 2.11.**  $\mathcal{C}$  is a semi-abelian category and  $\mathcal{D}$  is a full subcategory of  $\mathcal{C}$ , all objects of which are  $U$ -cokernels. The two  $\mathbf{E}$ -modules in Lemma 2.4 are identified under  $\sigma$ .

**2.1.3.  $U$ -finitely presented objects.** Let  $\bullet\iota_j^q : U \hookrightarrow U^q$  be the  $j$ th injection and  $\bullet\pi_i^k : U^k \twoheadrightarrow U$  the  $i$ th projection. There exists a canonical  $\mathbf{A}$ -linear isomorphism  $\text{Hom}_{\mathcal{C}}(U^q, U^k) \xrightarrow{\sim} {}^q\mathbf{A}^k$  given by

$$\bullet f \xrightarrow{\sim} (R_{ij})_{i,j}, \quad R_{ij} \triangleq \iota_i^q f \pi_j^k, \quad f = \sum_{i,j} \pi_i^q R_{ij} \iota_j^k.$$

The matrix  $R = (R_{ij})_{i,j} \in {}^q\mathbf{A}^k$  is written  $\text{Mat}(f)$ . Let  $\bullet f : U^q \rightarrow U^k$  and  $\bullet g : U^k \rightarrow U^l$ . Then,  $\text{Mat}(fg) = \text{Mat}(f)\text{Mat}(g)$ . Let  $M$  be an object of  $\mathcal{C}$  and let  $x \in \text{Hom}_{\mathcal{C}}(M, U^q) = \text{Hom}_{\mathcal{C}}(M, U)^q$ ; then  $x = (x_i)_{1 \leq i \leq q}$  is denoted by a row with entries  $x_i$  ( $1 \leq i \leq q$ ) and

$$xf = \left( \sum_{1 \leq i \leq q} x_i R_{ij} \right)_{1 \leq j \leq k} \in \text{Hom}_{\mathcal{C}}(M, U^k) = \text{Hom}_{\mathcal{C}}(M, U)^k$$

is the row  $xR$ , and thus the left morphism  $\bullet f$  is identified with the right multiplication by the matrix  $R = \text{Mat}(f)$ ; this right multiplication is written  $\bullet R$ .

LEMMA AND DEFINITION 2.12. (i) *An object  $M \in \mathcal{C}$  is called  $U$ -finitely presented if there exists an exact sequence*

$$(2.7) \quad U^q \xrightarrow{\bullet f} U^k \xrightarrow{\bullet \varphi} M \longrightarrow 0.$$

(ii) *In that case,  $M = \text{coker } \iota$ , where  $\iota \triangleq \text{im } f$ , and  $\iota = \text{im } \iota = \ker \varphi$ .*

For the proof of (ii), see section A.1.1. The morphism  $\bullet f$  can also be written  $\bullet R$ ,  $R = \text{Mat}(f)$ . The  $W$ -behavior  $\mathfrak{B} \triangleq N^\perp$ , identified with  $\mathfrak{B}_W(M)$ , is obtained by applying the functor  $\mathfrak{B}_W$  to (2.7) which yields the exact sequence of right morphisms

$$0 \longrightarrow \mathfrak{B}_W(M) \xrightarrow{\mathfrak{B}_W(\varphi)} \mathfrak{B}_W(U^k) \xrightarrow{\mathfrak{B}_W(\bullet R)} \mathfrak{B}_W(U^q)$$

with the canonical  $\mathbf{E}$ -linear isomorphisms  $\mathfrak{B}_W(U^k) \cong {}^k W$  and  $\mathfrak{B}_W(U^q) \cong {}^q W$ ;  $\mathfrak{B}_W(\bullet R)$  is identified with the left multiplication by  $R$ , written  $R\bullet$ , and  $\mathfrak{B}_W(\varphi)$  with the inclusion  $\mathfrak{B}_W(M) \hookrightarrow {}^k W$ ; thus setting  $\mathfrak{B}_W(M) = \mathfrak{B}$  we obtain (in accordance with (2.2))

$$(2.8) \quad \mathfrak{B} = \{w \in {}^k W : Rw = 0\} \triangleq \ker_W R\bullet.$$

The following generalizes [50, sect. 2, Theorem (61)].

THEOREM 2.13. *Let  $M_i = \text{coker } \bullet R_i$  be  $U$ -finitely presented objects of  $\mathcal{C}$  ( $R_i \in {}^{q_i} \mathbf{A}^k$ ) and  $N_i = \text{im } \bullet R_i$  ( $i = 1, 2$ ), and let  $\mathcal{D}$  be a full subcategory of  $\mathcal{C}$ , all objects of which are  $U$ -finitely presented.*

- (i) *If there exists a matrix  $X \in {}^{q_1} \mathbf{A}^{q_2}$  such that  $R_1 = X R_2$ , then  $\ker_W R_2\bullet \subseteq \ker_W R_1\bullet$ .*
- (ii) *The converse of (i) holds true if  $M_2 \in \mathcal{D}$ ,  $U$  is projective in  $\mathcal{C}$ , and  $W \in \mathcal{C}$  is a cogenerator for  $\mathcal{D}$ .*
- (iii) *If  $U$  is a generator and the converse of (i) holds true whenever  $M_2 \in \mathcal{D}$ , then  $W \in \mathcal{C}$  is a cogenerator for  $\mathcal{D}$ .*

*Proof.* (i) is obvious.

(ii) Since  $\mathcal{C}$  is semi-abelian, the morphism  $\bullet R_i$  can be factored as  $\bullet R_i = \beta_i \omega_i$ , where  $\bullet \beta_i : U^{q_i} \twoheadrightarrow N_i$  is an epimorphism and  $\omega_i : N_i \hookrightarrow U^k$  is the inclusion. We have  $N_2^\perp \subseteq N_1^\perp$ ; thus  $N_1 \subseteq N_1^{\perp\perp} \subseteq N_2^{\perp\perp}$  by Lemma 2.6, and  $N_2^{\perp\perp} = N_2$  by Theorem 2.10(1) and Lemma and Definition 2.12(ii). Since  $N_1 \subseteq N_2$ , let  $\iota : N_1 \hookrightarrow N_2$  be the inclusion. We have  $\bullet R_1 = \gamma \omega_2$  with  $\gamma = \beta_1 \iota : U^{q_1} \rightarrow N_2$ . Since  $U$  is projective, so is  $U^{q_1}$  and there exists  $\bullet \xi : U^{q_1} \rightarrow U^{q_2}$  such that  $\gamma = \xi \beta_2$ , and thus  $\gamma \omega_2 = \xi \beta_2 \omega_2$ , i.e.,  $\bullet R_1 = \xi \bullet R_2$ . Therefore,  $R_1 = X R_2$ , where  $X = \text{Mat}(\xi)$ .

(iii) Since  $U$  is a generator, there is an epimorphism  $f = (\bullet R_i)_{i \in I} : U^{(I)} \twoheadrightarrow N_2^{\perp\perp}$ , where

$$\bullet R_i : U \xrightarrow{\iota_i} U^{(I)} \xrightarrow{f} N_2^{\perp\perp} \hookrightarrow U^k,$$

i.e.,  $R_i \in \mathbf{A}^k$ ,  $\text{im}(\bullet R_i) \subseteq N_2^{\perp\perp}$ . Then,

$$\ker_W(R_2\bullet) = N_2^\perp = N_2^{\perp\perp\perp} \subseteq \text{im}(\bullet R_i)^\perp = \ker_W(R_i\bullet).$$

If the converse of (i) holds, there are rows  $X_i \in \mathbf{A}^{q_2}$  with  $R_i = X_i R_2$  and therefore  $\text{im}(\bullet R_i) \subseteq \text{im}(\bullet R_2) = N_2$ , and hence  $f = (\bullet R_i)_{i \in I} : U^{(I)} \twoheadrightarrow N_2$  and  $\text{im } f \subseteq N_2$ . Since  $N_2^{\perp\perp}$  is a kernel and  $f : U^{(I)} \twoheadrightarrow N_2^{\perp\perp}$ , we have  $N_2^{\perp\perp} = \text{im } f$  and thus also  $N_2^{\perp\perp} \subseteq N_2$ ; finally,  $N_2^{\perp\perp} = N_2$ , and thus  $W$  is a cogenerator for  $\mathcal{D}$  by Theorem 2.10(1).  $\square$

**2.2. Classical example.**

**2.2.1. The category  $\mathbf{A}\mathbf{Mod}$ .** Let  $\mathbf{R}$  be a ring and  $\mathcal{C}$  the category  $\mathbf{A}\mathbf{Mod}$  of left  $\mathbf{R}$ -modules with its left morphisms  $x \mapsto (x) f = x f$ . Then  $U \triangleq \mathbf{R}$  is a projective generator and the map

$$\mathbf{A} \triangleq \text{Hom}_{\mathbf{R}}(\mathbf{R}, \mathbf{R}) \rightarrow \mathbf{R}, \quad f \mapsto (1) f$$

is a ring-isomorphism with inverse  $r \mapsto \bullet r$ . We identify  $\mathbf{A}$  and  $\mathbf{R}$ , and with this identification,

$$\text{Hom}_{\mathbf{R}}(\mathbf{R}, M) \rightarrow M : g \mapsto (1) g$$

is an isomorphism of left  $\mathbf{A}$ -modules. The category  $\mathcal{C}$  is abelian and every object of  $\mathcal{C}$  is an  $\mathbf{A}$ -cokernel [5, subsect. I.2.8].

In the present context, Lemma 2.9 can be rephrased as in the corollary below which generalizes a classical result of homological algebra [39, Corollary (19.7)(1)], obtained when  $\mathcal{D} = \mathcal{C} = \mathbf{A}\mathbf{Mod}$ .

**COROLLARY 2.14.** *Let  $\mathcal{D}$  be a full subcategory of  $\mathcal{C}$  such that  $\mathbf{A} \in \mathcal{D}$  and let  $W \in \mathcal{C}$  be a cogenerator for  $\mathcal{D}$ . Then  $W$  is faithful.*

Let  $N \subseteq \mathbf{A}^{(I)}$ ,  $M = \mathbf{A}^{(I)}/N$ , and  $W \in \mathbf{A}\mathbf{Mod}$ . According to Pillai and Shankar [53] (see also [63], [72], [64]),  $N$  is called  $W$ -closed if  $N = N^{\perp\perp}$ . By Theorem 2.10, this means (using the terminology in Corollary and Definition 2.8) that  $W$  is a cogenerator for the subcategory of  $\mathcal{C}$  consisting of  $M$  alone.

**2.2.2. Controllability.** The full subcategory of  $\mathbf{A}\mathbf{Mod}$ , the objects of which are finitely presented, is denoted by  $\mathcal{D} = \mathbf{A}\mathbf{Mod}^{fp}$  in what follows. Let us recall that a module  $M \in \mathbf{A}\mathbf{Mod}$  is (i) *torsion-free* if whenever  $a \in \mathbf{A}$  is right-regular and  $am = 0$  ( $m \in M$ ), then  $m = 0$ ; (ii) *torsionless* if  $\mathbf{A}$  is a cogenerator for the subcategory of  $\mathbf{A}\mathbf{Mod}$  consisting of  $M$  alone [46, 3.4.2]. For further reference, the following gathers several results which can be found in the literature, except (ii) and (iv) which slightly generalize [50, sect. 2, Lemma and Definition (27)] and [76, Lemma 1], respectively.

**LEMMA 2.15.** *Let  $\mathbf{A}$  be a ring and consider the sequence (2.9) below, where  $R_1 \in {}^q\mathbf{A}^k$ ,  $R_2 \in {}^k\mathbf{A}^r$ ,*

$$(2.9) \quad \mathbf{A}^q \xrightarrow{\bullet R_1} \mathbf{A}^k \xrightarrow{\bullet R_2} \mathbf{A}^r \rightarrow 0,$$

and let  $M_1 = \text{coker } \bullet R_1$ .

- (i) *The following conditions are equivalent: (a) the sequence (2.9) is exact at  $\mathbf{A}^r$ ; i.e., the map  $\bullet R_2$  is surjective; (b)  $\bullet R_2$  is left-invertible; (c) the algebraic transpose  $R_2 \bullet$  of  $\bullet R_2$  is left-invertible; (d) the matrix  $R_2$  is left-invertible over  $\mathbf{A}$ .*
- (ii) *Given  $R_2 \in {}^k\mathbf{A}^r$ , if  $\mathbf{A}$  is a left coherent ring, there exists  $R_1 \in {}^q\mathbf{A}^k$  such that the sequence (2.9) is exact at  $\mathbf{A}^k$ .*
- (iii) *If the sequence (2.9) is exact at  $\mathbf{A}^k$ , then there exists an embedding  $M_1 \hookrightarrow \mathbf{A}^r$ ; in particular, in that case  $M_1$  is both torsion-free and torsionless.*
- (iv) *Conversely, given  $R_1 \in {}^q\mathbf{A}^k$ , if there exists an embedding  $M_1 \hookrightarrow \mathbf{A}^r$ , there exists a matrix  $R_2 \in {}^k\mathbf{A}^r$  such that the sequence (2.9) is exact at  $\mathbf{A}^k$ . Such an embedding exists if the ring  $\mathbf{A}$  is right coherent and  $M_1$  is torsionless, or if the ring  $\mathbf{A}$  is semiprime Goldie (in particular, if  $\mathbf{A}$  is an Ore domain<sup>1</sup>) and  $M_1$  is torsion-free.*

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<sup>1</sup>In this paper, the expressions *Ore*, *Noetherian*, *coherent*, etc., are used for *two-sided Ore*, *two-sided Noetherian*, *two-sided coherent*, etc.

(v) Given  $R_1 \in {}^q\mathbf{A}^k$ , there exist a positive integer  $r$  and a matrix  $R_2 \in {}^k\mathbf{A}^r$  such that the sequence (2.9) is exact if and only if  $M_1$  is free.

*Proof.* (i) Obviously, (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (a). If (a) holds,  $\mathbf{A}^k/\ker \bullet R_2 \cong \mathbf{A}^r$  by Noether’s first isomorphism theorem; thus  $\ker \bullet R_2$  is a direct summand of  $\mathbf{A}^k$ , and since  $\bullet R_2$  is surjective, it is left-invertible [4, subsect. II.1.9, Cor. 1 of Prop. 15].

(ii) If  $\mathbf{A}$  is left coherent, then  $\ker \bullet R_2 \subseteq \mathbf{A}^k$  is finitely generated [18, Theorem A.9]; a matrix  $R_1$ , the rows of which form a finite generator of  $\ker \bullet R_2$ , is universal with respect to  $R_1 R_2 = 0$  [50, sect. 2, Lemma and Definition (27)], and then the sequence (2.9) is exact at  $\mathbf{A}^k$ .

(iii) If the sequence (2.9) is exact at  $\mathbf{A}^k$ , there exists an induced map  $(\bullet R_2)_{ind} : M_1 \rightarrow \mathbf{A}^r$  which is an embedding, and then  $M_1$  is both torsion-free and torsionless since so is  $\mathbf{A}^r$ .

(iv) (a) Consider the exact sequence

$$\mathbf{A}^q \xrightarrow{\bullet R_1} \mathbf{A}^k \xrightarrow{\varphi_1} M_1 \rightarrow 0.$$

If there exists an embedding  $\bullet \iota : M_1 \hookrightarrow \mathbf{A}^r$ , set  $\bullet R_2 = \bullet \varphi_1 \iota : \mathbf{A}^k \rightarrow \mathbf{A}^r$ . Then the sequence (2.9) is exact at  $\mathbf{A}^k$ .

(b) If  $M_1$  is torsionless, there exists an embedding  $\varepsilon : M_1 \hookrightarrow \mathbf{A}^I$  of  $M_1$  into a power  $\mathbf{A}^I$  of  $\mathbf{A}$ . Set  $\bullet \tau = \bullet \varphi_1 \varepsilon : \mathbf{A}^k \rightarrow \mathbf{A}^I$ ; for any  $x \in \mathbf{A}^k$ ,  $x\tau$  can be written  $xT$ , where the matrix  $T \in {}^k\mathbf{A}^I$  has possibly infinitely many columns. The sequence

$$(2.10) \quad \mathbf{A}^q \xrightarrow{\bullet R_1} \mathbf{A}^k \xrightarrow{\bullet T} \mathbf{A}^I$$

is exact; thus  $R_1 T = 0$ . If  $\mathbf{A}$  is right coherent, there exists a natural integer  $r$  and matrices  $R_2 \in {}^k\mathbf{A}^r$ ,  $D \in {}^r\mathbf{A}^I$  such that  $R_1 R_2 = 0$ ,  $T = R_2 D$  [18, Theorem A.6(f)] and the exactness of the sequence (2.10) at  $\mathbf{A}^k$  implies that of the sequence (2.9) at  $\mathbf{A}^k$ ; thus there exists an embedding  $M_1 \hookrightarrow \mathbf{A}^r$  by (iii).

(c) The same holds if  $\mathbf{A}$  is semiprime Goldie and  $M$  is torsion-free [46, Prop. 3.4.7], and any Ore domain is a semiprime Goldie ring.

(v) (a) Assume  $M_1$  is free of rank  $r$ . The above monomorphism  $\iota : M_1 \hookrightarrow \mathbf{A}^r$  is then an isomorphism; thus  $\bullet R_2 = \bullet \varphi_1 \iota$  is an epimorphism and the sequence (2.9) is exact. (b) Conversely, if the sequence (2.9) is exact, (iii) holds and  $(\bullet R_2)_{ind}$  is an isomorphism; thus  $M_1 \cong \mathbf{A}^r$  is free.  $\square$

We consider now the exact sequence (2.7) with  $U = \mathbf{A}$  and the  $W$ -behavior  $\mathfrak{B}_W(M) = \mathfrak{B}$  defined by (2.8). As shown in [69], [54, sect. 6.6], [58], [73], [53], [74], [63], where various kinds of rings and signal spaces are considered, the “behavioral controllability” is linked to the existence of an “image representation” of  $\mathfrak{B}_W(M)$ , i.e., to the existence of matrix  $S \in {}^k\mathbf{A}^r$  such that

$$(2.11) \quad \mathfrak{B}_W(M) = \text{im}_W S \bullet \triangleq \{w \in {}^k W : \exists \xi \in {}^r W, w = S \xi\}.$$

When  $\mathfrak{B}_W(M)$  has an image representation of the form (2.11), where  $S \bullet : {}^r W \rightarrow \mathfrak{B}_W(M)$  is injective (thus bijective), this image representation is said to be “observable” since then the “latent variable”  $\xi$  is uniquely determined by the “manifest variable”  $w$  [54, sect. 5.3], and  $\mathfrak{B}_W(M)$  is said to be *strongly controllable* [59], [71].

On the other hand, according to Fliess and Mounier (see [29], [48], [49]), a linear system is a finitely presented left  $\mathbf{A}$ -module  $M$ . Such a system is said to be *free controllable* (resp., *torsion-free controllable*) if the module  $M$  is free (resp., torsion-free). In accordance with Zerz [76], a linear system  $M$  is said to be *torsionless controllable*



if the module  $M$  is torsionless. A free controllable linear system  $M$  is also said to be *flat*, and a basis of the free module  $M$  is called a *flat output* [30].

The consistency between the two above points of view has been widely studied when the  $\mathbf{A}$ -module  $W$  is injective (see, e.g., [53], [17], [71], [72]). It is clarified below without this assumption.

**THEOREM 2.16.** (1) *If  $\bullet S : \mathbf{A}^k \rightarrow \mathbf{A}^r$  is left-invertible, then the image representation (2.11) is observable, and the converse holds true if  $W$  is a cogenerator for  $\mathcal{D} = {}_{\mathbf{A}}\mathbf{Mod}^{fp}$ .*

(2) (i) *If  $M = \text{coker } \bullet R$  is a free module, then  $\mathfrak{B}_W(M)$  has an observable image representation. (ii) *Conversely, if  $W$  is cogenerator for  $\mathcal{D}$  in  ${}_{\mathbf{A}}\mathbf{Mod}$  and  $\mathfrak{B}_W(M)$  has an observable image representation, then  $M$  is free.**

(3) (i') *Assume that the ring  $\mathbf{A}$  is right coherent (resp., semiprime Goldie) and that the  $\mathbf{A}$ -module  $W$  is faithful. If  $M$  is torsionless (resp., torsion-free), then there exists a matrix  $S$  over  $\mathbf{A}$  such that  $\mathfrak{B}_W(M) = (\text{im}_W S\bullet)^{\perp\perp}$ . (ii') *Conversely, if  $W$  is a cogenerator for  $\mathcal{D}$ , then the equality  $\mathfrak{B}_W(M) = (\text{im}_W S\bullet)^{\perp\perp}$  implies that  $M$  is both torsionless and torsion-free.**

*Proof.* (1) If  $\bullet S : \mathbf{A}^k \rightarrow \mathbf{A}^r$  is left-invertible, then the image representation (2.11) is obviously observable. Conversely, consider the exact sequence

$$\mathbf{A}^k \xrightarrow{\bullet S} \mathbf{A}^r \xrightarrow{\psi} N \rightarrow 0$$

which yields (through the functor  $\mathfrak{B}_W(\bullet)$ ) the exact sequence

$$0 \rightarrow \mathfrak{B}_W(N) \rightarrow {}^rW \xrightarrow{S\bullet} {}^kW.$$

If  $S\bullet : {}^rW \rightarrow \mathfrak{B}_W(M)$  is injective, this implies  $\mathfrak{B}_W(N) = 0$ , and finally  $N = 0$  if in addition  $W$  is a cogenerator for  $\mathcal{D}$ . Therefore,  $\bullet S$  is surjective and thus left-invertible by Lemma 2.15(i).

(2) (i) is obvious.

(ii) Assuming that  $\mathfrak{B}_W(M)$  has an observable image representation, (2.11) holds with  $\bullet S$  left-invertible by (1); thus the sequence

$$(2.12) \quad 0 \rightarrow {}^rW \xrightarrow{S\bullet} {}^kW \xrightarrow{R\bullet} {}^qW$$

is exact in  $\mathbf{Mod}_{\mathbf{E}}$  and there exists a matrix  $T \in {}^r\mathbf{A}^k$  such that  $TS = I_r$ . This equality proves that  $\bullet S$  is surjective, and the exactness of (2.12) proves that  $S\bullet$  is injective and  $RS\bullet = 0$ ; thus  $RS = 0$  since  $W$  is a cogenerator for  $\mathcal{D}$ . Therefore,  $\mathbf{A}^q R \subseteq \ker \bullet S$  and there exists a map  $\bullet f$  such that the diagram below is commutative:

$$\begin{array}{ccc} \mathbf{A}^k & \xrightarrow{\bullet S} & \mathbf{A}^r \\ \downarrow \varphi & f \nearrow & \\ M & & \end{array}$$

where  $\bullet \varphi : \mathbf{A}^k \twoheadrightarrow M = \mathbf{A}^k / \mathbf{A}^q R \in {}_{\mathbf{A}}\mathbf{Mod}^{fp}$  is the canonical epimorphism and  $\bullet f : M \rightarrow \mathbf{A}^r$  is an epimorphism too since so is  $\bullet S$ . Thus  $\mathfrak{B}_W(M) = \ker_W (R\bullet) = \text{im}_W (S\bullet) \cong {}^rW$  (by exactness of (2.12)), and since the functor  $\mathfrak{B}_W$  is left exact, the surjectivity of  $\bullet f$  implies the exactness of

$$0 \rightarrow {}^rW \xrightarrow{\mathfrak{B}_W(\bullet f)} \mathfrak{B}_W(M);$$

thus  $\mathfrak{B}_W(\bullet f) : {}^rW \rightarrow \mathfrak{B}_W(M)$  is an embedding. Therefore,  $\mathfrak{B}_W(\bullet f)$  is an isomorphism  ${}^rW \xrightarrow{\sim} \mathfrak{B}_W(M)$ . Now we have

$$Id_{\mathbf{A}^r} = \bullet TS = \bullet T\varphi f = gf, \quad g \triangleq \bullet T\varphi : \mathbf{A}^r \rightarrow M.$$

Thus  $\bullet f = \bullet fgf$ , and hence  $\mathfrak{B}_W(\bullet f)\bullet = \mathfrak{B}_W(\bullet fg)\mathfrak{B}_W(\bullet f)\bullet$ , which implies  $\mathfrak{B}_W(\bullet fg) = Id_{rW}$  for  $\mathfrak{B}_W(\bullet f)$  is invertible. Since  $W$  is a cogenerator for  $\mathcal{D}$ , this implies  $\bullet fg = Id_{\mathbf{A}^r}$ , and thus  $\bullet f$  is injective; finally,  $\bullet f$  is an isomorphism  $M \xrightarrow{\sim} \mathbf{A}^r$ , and  $M$  is free.

(3) (i') By Lemma 2.15(iv), there exists a matrix  $S$  such that the sequence

$$\mathbf{A}^q \xrightarrow{\bullet R} \mathbf{A}^k \xrightarrow{\bullet S} \mathbf{A}^r \rightarrow 0$$

is exact at  $\mathbf{A}^k$ , i.e.,  $\text{im } \bullet R = \ker \bullet S$ , which implies  $(\text{im } \bullet R)^\perp = (\ker \bullet S)^\perp$ . We have  $(\text{im } \bullet R)^\perp = \ker_W R\bullet$ . Let  $r \in \ker \bullet S$ ; then  $rS = 0$ , which implies that  $rS\xi = 0$  for any  $\xi \in W^r$ , and thus  $\ker \bullet S \subseteq (\text{im}_W S\bullet)^\perp$ . If  $W$  is faithful, we conclude that

$$(2.13) \quad \ker \bullet S = (\text{im}_W S\bullet)^\perp$$

and thus  $\ker_W R\bullet = (\text{im}_W S\bullet)^{\perp\perp}$ .

(ii') Assuming that  $\ker_W R\bullet = (\text{im}_W S\bullet)^{\perp\perp}$  and  $W$  is a cogenerator for  $\mathbf{A}\mathbf{Mod}^{fp}$  in  $\mathbf{A}\mathbf{Mod}$ , we have  $(\ker_W R\bullet)^\perp = (\text{im}_W S\bullet)^\perp$ ; thus  $\text{im } \bullet R = (\text{im}_W S\bullet)^\perp = \ker \bullet S$  by Theorem 2.10, Corollary 2.14, and (2.13). The conclusion follows from Lemma 2.15(iii) with  $R_1 = R$ ,  $R_2 = S$ .  $\square$

*Remark 2.17.* (i) A nontrivial proof of statement (2)(ii) of Theorem 2.16 is needed because the  $\mathbf{A}$ -module  $W$  is not assumed to be injective; therefore the exactness of a sequence

$$(2.14) \quad \mathfrak{B}_W(M_3) \xrightarrow{R_2\bullet} \mathfrak{B}_W(M_2) \xrightarrow{R_1\bullet} \mathfrak{B}_W(M_1)$$

in  $\mathbf{Mod}_{\mathbf{E}}$  does not imply the exactness of

$$M_1 \xrightarrow{\bullet R_1} M_2 \xrightarrow{\bullet R_2} M_3$$

in  $\mathbf{A}\mathbf{Mod}$ , despite Corollary and Definition 2.8(ii); compare with [56, sect. 3] and [17, Theorem 3(6)]. The reason for this is that the sequence (2.14) is not necessarily exact in  $\mathfrak{B}_W(\mathcal{D})$ , for this category is not full in  $\mathbf{Mod}_{\mathbf{E}}$ . According to [50, sect. 2, Duality Theorem (56)],  $\mathfrak{B}_W(\mathcal{D})$  is full in  $\mathbf{Mod}_{\mathbf{E}}$  if  $W$  is a large injective cogenerator.

(ii) To specify the above item, recall that a cogenerator in  $\mathbf{A}\mathbf{Mod}$  need not be injective.

- (a) An example of a noninjective cogenerator is given in [39, sect. 19, Exercise 20].
- (b) Let us give a second example: let  $\mathbf{A}$  be a ring,  $\mathbf{A}W_1$  an injective cogenerator, and  $W_2 \neq 0$  a noninjective left  $\mathbf{A}$ -module, e.g.,  $\mathbf{A} = \mathbb{C}[d/dt]$ ,  $W_1 = \mathcal{E}(\mathbb{R})$ ,  $W_2 = \mathcal{D}(\mathbb{R})$ . Then  $W \triangleq W_1 \oplus W_2$  is a cogenerator which is a noninjective left  $\mathbf{A}$ -module by [39, Prop. 3.4].

### 2.3. Topological modules and behaviors.

**2.3.1. The category  $\mathbf{A}\mathbf{TMod}$ .** In everything that follows, we make the following.

*Assumption 2.18.*  $\mathbf{k}$  is the field of real or complex numbers, and all algebras are associative and unitary and defined over  $\mathbf{k}$ .

Let  $\mathbf{A}$  be an algebra, let  $W$  be a  $\mathbf{k}$ -vector space, and consider a nondegenerate bilinear form [7, subsect. II.6.1]

$$\langle -, - \rangle : \mathbf{A} \times W \rightarrow \mathbf{k}.$$

We also consider the nondegenerate bilinear form

$$(2.15) \quad \langle -, - \rangle^\sim : W \times \mathbf{A} \rightarrow \mathbf{k}$$

defined by  $\langle w, a \rangle^\sim = \langle a, w \rangle$  for any  $a \in \mathbf{A}$  and  $w \in W$ . The *weak topology*  $\sigma(\mathbf{A}, W)$  (resp.,  $\sigma(W, \mathbf{A})$ ) is the coarsest topology in  $\mathbf{A}$  (resp., in  $W$ ) which makes continuous all linear forms  $a \mapsto \langle a, w \rangle$ ,  $w \in W$  (resp.,  $w \mapsto \langle a, w \rangle$ ,  $a \in \mathbf{A}$ ). The following facts are classical [7, subsect. II.6.2]: the space  $W$  is canonically isomorphic to, and identified with, the topological dual  $\mathbf{A}'$  of  $(\mathbf{A}, \sigma(\mathbf{A}, W))$ , i.e., of  $\mathbf{A}$  endowed with the topology  $\sigma(\mathbf{A}, W)$ ; this statement is also valid with  $W$  and  $\mathbf{A}$  interchanged;  $(\mathbf{A}, \sigma(\mathbf{A}, W))$  and  $(W, \sigma(W, \mathbf{A}))$  (resp., written  $\mathbf{A}$  and  $W$  in what follows, for short) are *locally convex topological vector spaces* (LCTVSs) and *Hausdorff*.

A left  $\mathbf{A}$ -module  $M$  is called *topological* if it is endowed with a structure of LCTVS and if all maps  $\mathbf{A} \rightarrow M : a \mapsto am$ ,  $m \in M$ , are continuous; a topological right  $\mathbf{A}$ -module  $M$  is similarly defined. (Note that the definition of a topological module over a topological ring is different [6, subsect. III.6.6].) The algebra  $\mathbf{A}$  may itself be a topological left or right  $\mathbf{A}$ -module. The algebraic dual  $\mathbf{A}^* = \text{Hom}_{\mathbf{k}}(\mathbf{A}, \mathbf{k})$  is a left and right  $\mathbf{A}$ -module via

$$(x)\alpha a = (xa)\alpha \text{ and } (\alpha a)x = \alpha(ax), \quad \alpha \in \mathbf{A}^*, \ a, x \in \mathbf{A}$$

(using Notation 2.1(i)). In general,  $\mathbf{A}' = W$  is not an  $\mathbf{A}$ -submodule of  $\mathbf{A}^*$ , and when this happens it is characterized as below.

LEMMA 2.19. (1) *The following conditions are equivalent:*

- (i) *The left module  ${}_{\mathbf{A}}\mathbf{A}$  is topological; i.e., all maps  $a \mapsto ab$  ( $b \in \mathbf{A}$ ) are continuous.*
- (ii)  *$W$  is a left  $\mathbf{A}$ -module with  $\langle a, bw \rangle = \langle ab, w \rangle$  for all  $a, b \in \mathbf{A}$  and  $w \in W$ .*

*Then also all maps  $W \rightarrow W : w \mapsto bw$  ( $b \in \mathbf{A}$ ) are continuous.*

(2) *The following properties are equivalent:*

- (i') *The right module  $\mathbf{A}_{\mathbf{A}}$  is topological.*
- (ii')  *$W$  is a right  $\mathbf{A}$ -module with  $\langle wa, b \rangle^\sim = \langle w, ab \rangle^\sim$  for all  $a, b \in \mathbf{A}$ .*

*Then also all maps  $W \rightarrow W : w \mapsto wb$  ( $b \in \mathbf{A}$ ) are continuous.*

(3) *If the conditions of (1) hold, then those of (2) are equivalent to the following:*

- (iii') *The left module  ${}_{\mathbf{A}}W$  is topological.*

*Proof.* (1) (i) $\Rightarrow$ (ii): If  $a \mapsto ab$  is continuous, then so is  $b\alpha : a \mapsto a(b\alpha) = (ab)\alpha$  for any  $\alpha \in \mathbf{A}' = W$  with the identification  $W \ni w \mapsto \langle -, w \rangle \in \mathbf{A}'$ . Hence  $\mathbf{A}' = W$  is an  $\mathbf{A}$ -submodule of  $\mathbf{A}^*$ . (ii) $\Rightarrow$ (i): If (ii) holds, then  $bw \in W$  for any  $b \in \mathbf{A}$ ,  $w \in W$ ; therefore the map  $\mathbf{A} \ni a \mapsto \langle a, bw \rangle = \langle ab, w \rangle \in \mathbf{k}$  is continuous, which implies that so is the map  $a \mapsto ab$ . The maps  $w \mapsto \langle a, bw \rangle$  are continuous by the above equality. (2) is shown like (1).

(3) If (ii) holds, the following conditions are equivalent: all maps  $b \mapsto bw$  are continuous, i.e., (iii') holds; all maps  $b \mapsto \langle a, bw \rangle = \langle ab, w \rangle$  are continuous; all maps  $b \mapsto ab$  are continuous, i.e., (i') holds.  $\square$

DEFINITION 2.20. *Assuming that all conditions of Lemma 2.19 are satisfied, the category of topological left  $\mathbf{A}$ -modules and continuous linear maps is called the category of topological left  $\mathbf{A}$ -modules and written  ${}_{\mathbf{A}}\mathbf{TMod}$ .*

LEMMA 2.21.  *${}_{\mathbf{A}}\mathbf{TMod}$  is a semi-abelian category with arbitrary products and coproducts and with projective generator  $\mathbf{A}$ .*

*Proof.* (a) The category  $\mathcal{C} = {}_{\mathbf{A}}\mathbf{TMod}$  is a subcategory of  ${}_{\mathbf{A}}\mathbf{Mod}$ ;  $\mathcal{C}$  is obviously additive and has kernels and cokernels which are the algebraic ones endowed with the induced and coinduced topology, respectively; thus  $\mathcal{C}$  is preabelian. Let  $\bullet f : X \rightarrow Y$

be a morphism of  $\mathcal{C}$  and  $f_{ind}$  be its induced morphism in  $\mathbf{A}\mathbf{Mod}$ . Then  $f_{ind}$  is bijective (since  $\mathbf{A}\mathbf{Mod}$  is abelian) and continuous by definition of the topologies of  $\text{coim } f$  and of  $\text{im } f$ . Therefore,  $f_{ind}$  is the induced morphism of  $f$  in  $\mathcal{C}$ . Since  $f_{ind}$  is both a monomorphism and an epimorphism, the preabelian category  $\mathcal{C}$  is semi-abelian. Last,  $\mathcal{C}$  has arbitrary products and coproducts which are the algebraic ones endowed with the product topology and the locally convex direct sum topology, respectively [7, subsect. II.4.5, Definition 2].

(b) Let  $M \in \mathcal{C}$ . There exists an  $\mathbf{A}$ -linear surjection  $\bullet\varphi : \mathbf{A}^{(I)} \rightarrow M$ . Let  $\bullet\iota : \mathbf{A} \hookrightarrow \mathbf{A}^{(I)}$  be the canonical injection. Let  $\bullet f = \bullet\iota\varphi : \mathbf{A} \rightarrow M$ ; for any  $a \in \mathbf{A}$ ,  $(a) f = a m$ , where  $m = (1) f$ , and thus  $f$  is continuous since the module  $M$  is topological. Therefore,  $\varphi$  is continuous (if  $\mathbf{A}^{(I)}$  is endowed with the locally convex direct sum topology) according to [7, subsect. II.4.5, Prop. 6]. This proves that  $\mathbf{A}$  is a generator of  $\mathcal{C}$ .

(c) Let  $\bullet f : \mathbf{A} \rightarrow M$  and  $\bullet g : M_1 \rightarrow M$  be, respectively, a morphism and an epimorphism of  $\mathcal{C}$ . Since  $\bullet f$  and  $\bullet g$  are, respectively, a morphism and an epimorphism of  $\mathbf{A}\mathbf{Mod}$ , there exists a morphism  $\bullet h : \mathbf{A} \rightarrow M_1$  of  $\mathbf{A}\mathbf{Mod}$  such that  $\bullet f = \bullet h g$ . Since  $M_1$  is topological,  $\bullet h$  is continuous, i.e., is a morphism of  $\mathcal{C}$ , and thus  $\mathbf{A}$  is projective in  $\mathcal{C}$ .  $\square$

**2.3.2. Cogenerators for topological modules.** For  $X, Y \in \mathbf{A}\mathbf{TMod}$ , we denote by  $\text{Hom}_{\mathbf{A},c}(X, Y)$  the  $\mathbf{k}$ -vector space consisting of all continuous  $\mathbf{A}$ -linear maps  $X \rightarrow Y$ . Endowed with the topology of pointwise convergence,  $\text{Hom}_{\mathbf{A},c}(X, Y)$  is an LCTVS; in particular we have the additive left exact contravariant functor

$$\mathfrak{B}_W : \mathbf{A}\mathbf{TMod} \rightarrow \mathbf{Mod}_{\mathbf{E}} : M \mapsto \mathfrak{B}_W \triangleq \text{Hom}_{\mathbf{A},c}(M, W), \quad \mathbf{E} \triangleq \text{Hom}_{\mathbf{A},c}(W, W).$$

The image  $\mathfrak{B}_W(\mathbf{A}\mathbf{TMod})$  is called the class of all (topological)  $W$ -behaviors. All results in subsection 2.1 are valid, in particular, Theorems 2.10 and 2.13.

*Remark 2.22.* Let  $M \in \mathcal{C}$ . By Lemma 2.21, there exists a set of indices  $I$  such that  $M$  is a factor of  $\mathbf{A}^{(I)}$ ; i.e., there exists a continuous  $\mathbf{A}$ -linear surjection  $\varphi : \mathbf{A}^{(I)} \rightarrow M$ . There exists a continuous bijection  $\varphi_{ind} : \mathbf{A}^{(I)}/N \rightarrow M$ , where  $N = \ker \varphi$ . But  $\varphi_{ind}^{-1}$  is not continuous in general, i.e., is not an isomorphism of  $\mathcal{C}$ , and thus  $\mathbf{A}^{(I)}/N$  and  $M$  cannot be identified; then,  $M$  is not an  $\mathbf{A}$ -cokernel.

Let  $I$  be a set of indices. The nondegenerate form  $\langle -, - \rangle$  is extended to the nondegenerate form

$$\langle -, - \rangle : \mathbf{A}^{(I)} \times {}^I W : (x, w) \mapsto \sum_{i \in I} \langle x_i, w_i \rangle,$$

where all terms of the sum are zero except a finite number of them. The locally convex topologies induced by this form in  ${}^I W$  and  $\mathbf{A}^{(I)}$  are, respectively, the product topology  $\sigma({}^I W, \mathbf{A}^{(I)}) = \prod_{i \in I} \sigma(W, \mathbf{A})$  [7, subsect. II.6.6, Prop. 8] and the topology  $\sigma(\mathbf{A}^{(I)}, {}^I W)$ . The latter coincides with the locally convex direct sum topology  $\bigoplus_{i \in I} \sigma(\mathbf{A}, W)$  if and only if  $I$  is finite [61, Chap. IV, Exercise 8]. In what follows,  $\mathbf{A}^{(I)}$  is endowed with the topology  $\sigma(\mathbf{A}^{(I)}, {}^I W)$  and  $I$  may be assumed to be finite, to avoid confusion. If  $N \subseteq \mathbf{A}^{(I)}$  in  $\mathbf{A}\mathbf{TMod}$  (which means that  $N$  is a subspace of  $\mathbf{A}^{(I)}$ ), its polar  $N^0 \subseteq {}^I \mathbf{A}$  is defined as usual:

$$N^0 = \{w \in {}^I W : \langle N, w \rangle = 0\} = \bigcap_{x \in N} \ker \langle x, - \rangle$$

and  $N^0$  is closed in  ${}^I W$ . According to the bipolar theorem [7, subsect. II.6.3],  $N^{00} = \overline{N}$ , where  $\overline{N}$  denotes the closure of  $N$  in  $\mathbf{A}^{(I)}$ .

THEOREM 2.23. (i) Let  $N \subseteq \mathbf{A}^{(I)}$ ; then

$$N^\perp = N^0 \quad \text{and} \quad N^{\perp\perp} = N^{00} = \overline{N}.$$

- (ii) Let  $M = \mathbf{A}^{(I)}/N$  be an  $\mathbf{A}$ -cokernel in  $\mathcal{C} = \mathbf{A}\mathbf{TMod}$ ; then  $M$  is Hausdorff if and only if  $N = N^{\perp\perp}$ .
- (iii) Let  $\mathcal{D}$  be a full subcategory of  $\mathcal{C}$ , all objects of which are  $\mathbf{A}$ -cokernels. Then  $W \in \mathcal{C}$  is a cogenerator for  $\mathcal{D}$  if and only if all objects  $M = \mathbf{A}^{(I)}/N$  of  $\mathcal{D}$  are Hausdorff.

Proof. (i) (a) The identity

$$\langle ax, w \rangle = \langle a, xw \rangle, \quad a \in \mathbf{A}, \quad x \in \mathbf{A}^{(I)}, \quad w \in {}^I W,$$

holds. By definition,  $w \in N^\perp$  if and only if for any  $x = (x_i)_{i \in I} \in \mathbf{A}^{(I)}$ ,  $xw \triangleq \sum_{i \in I} x_i w_i = 0$ . Since  $\langle -, - \rangle$  is nondegenerate, the last equality is equivalent to  $0 = \langle a, xw \rangle = \langle ax, w \rangle$  for any  $a \in \mathbf{A}$  and thus to  $\langle x, w \rangle = 0$ . Thus  $N^\perp = N^0$ . (b) Since the module  $\mathbf{A}_\mathbf{A}$  is topological, the maps  $\mathbf{A} \rightarrow \mathbf{A} : x \mapsto ax$  and  $\mathbf{A}^{(I)} \rightarrow \mathbf{A}^{(I)} : x \mapsto ax$  ( $a \in \mathbf{A}$ ) are continuous [7, subsect. II.4.5]. This implies that  $a\overline{N} \subseteq \overline{aN} \subseteq \overline{N}$  and that  $\overline{N}$  is a left  $\mathbf{A}$ -submodule of  $\mathbf{A}^{(I)}$ . Now,  $x \in N^{\perp\perp}$  if and only if for any  $w \in N^\perp = N^0$ ,  $xw = 0$ . The last equality holds if and only if for all  $a \in \mathbf{A}$ ,  $0 = \langle a, xw \rangle = \langle ax, w \rangle$ , i.e.,  $ax \in N^{00} = \overline{N}$ . Since  $\overline{N}$  is a left  $\mathbf{A}$ -submodule of  $\mathbf{A}^{(I)}$ , this is equivalent to  $x \in \overline{N}$ .

(i) implies (ii), which implies (iii) by Lemma and Definition 2.12(ii) and Theorem 2.10.  $\square$

Remark 2.24. (i) Theorem 2.23 is a generalization of [28, Theorem 4.1] and of [33, Theorem 3.4], where  $W = \mathcal{E}(\mathbb{R})$ ,  $\mathbf{A}$  is the convolution algebra  $(W', \star)$ , and  $\mathbf{A}$  acts on  $W$  via the convolution product.

- (ii) Let  $W$  be an LCTVS and let  $\mathbf{A} \subseteq W'$ . Then, the restriction to  $\mathbf{A} \times W$  of the canonical bilinear form  $\langle -, - \rangle : W' \times W \rightarrow \mathbf{k}$  is possibly degenerate; the topology  $\sigma(\mathbf{A}, W)$  is Hausdorff (since it is induced from  $\sigma(W', W)$ ), but  $\sigma(W, \mathbf{A})$  is not in general [7, subsect. II.6.1, Prop. 1]. Therefore, the statements of Theorem 2.23 are no longer valid. This situation is further investigated in the next two sections, where convolution systems are considered.

### 3. Differential systems with lumped shifts over Lie groups.

3.1. Actions over a Lie group and convolution algebra. In everything that follows, we'll use the following.

NOTATION 3.1.  $\mathbf{G}$  is a Lie group with Lie algebra  $\mathfrak{g}$ ,  $W = \mathcal{E}(\mathbf{G})$  is the space of all  $\mathbf{k}$ -valued indefinitely differentiable functions on  $\mathbf{G}$ , and its topological dual  $W'$  is the space of all  $\mathbf{k}$ -valued distributions with compact support included in  $\mathbf{G}$  (Appendix 2, section A.2.2).

The group  $\mathbf{G}$  continuously acts on  $W$  via a left and a right action, both of them denoted by  $\circ$ , and defined by

- (i)  $(y)(a \circ \varphi) = (ya)\varphi$  (left action of  $\mathbf{G}$  on a left map, which is a right shift),
- (ii)  $(\varphi \circ a)(x) = \varphi(ax)$  (right action of  $\mathbf{G}$  on a right map, which is a left shift),

for any  $x, y, a \in \mathbf{G}$  and  $\varphi \in W$ . By (Appendix 2, (A.12)), this yields for any  $S, T \in W'$

$$\langle S, a \circ \varphi \rangle = \langle S \star \delta_a, \varphi \rangle, \quad \langle \varphi \circ a, T \rangle^\sim = \langle \varphi, \delta_a \star T \rangle^\sim$$

(see (A.9) for the latter notation which is consistent with (2.15)).

These actions induce, respectively, a right and a left action on  $W'$ , both of them also denoted by  $\circ$ , according to

$$(3.1) \quad \langle S \circ a, \varphi \rangle = \langle S, a \circ \varphi \rangle, \quad \langle \varphi, a \circ T \rangle^\sim = \langle \varphi \circ a, T \rangle^\sim$$

and from the above

$$S \circ a = S \star \delta_a, \quad a \circ T = \delta_a \star T.$$

We also define a left and a right action of  $W'$  on  $W$ , both of them denoted by  $\circ$ , respectively, by

$$(3.2) \quad S \circ W \rightarrow W : (S \circ \varphi)(x) = \langle S, \varphi \circ x \rangle,$$

$$(3.3) \quad \circ S : W \rightarrow W : (x)(\varphi \circ S) = \langle x \circ \varphi, S \rangle^{\sim}$$

for any  $\varphi \in W$  and  $S \in W'$ . As a consequence of (Appendix 2, (A.13)) one easily obtains

$$\varphi \circ S = \check{S} \star \varphi, \quad S \circ \varphi = \varphi \star \check{\Delta}^{-1} \check{S},$$

where  $\Delta$  is the modulus function,  $\check{\varphi}(y) = \varphi(y^{-1})$ , and  $\langle \check{S}, \varphi \rangle = \langle S, \check{\varphi} \rangle$  for any  $y \in \mathbf{G}$ ,  $\varphi \in W$ , and  $S \in W'$ . We have the following.

LEMMA 3.2. (i) *The equality*

$$\langle T, S \circ \varphi \rangle = \langle \varphi \circ T, S \rangle^{\sim}$$

holds for any  $S, T \in W'$  and  $\varphi \in W$ .

(ii) *Both actions  $\circ$  of  $W'$  on  $W$  induce one bilinear law of composition  $\circ : W' \times W' \rightarrow W'$  according to*

$$(3.4) \quad \langle T \circ S, \varphi \rangle = \langle T, S \circ \varphi \rangle, \quad \langle \varphi, T \circ S \rangle^{\sim} = \langle \varphi \circ T, S \rangle^{\sim},$$

*and this law of composition is the usual convolution product  $\star$ .*

(iii) *For any  $S, T \in W'$  and  $\varphi \in W$ ,*

$$(3.5) \quad (T \circ S) \circ \varphi = T \circ (S \circ \varphi), \quad \varphi \circ (T \circ S) = (\varphi \circ T) \circ S;$$

*i.e.,  $W$  is a left and right  $W'$ -module.*

*Proof.* (i) We have

$$\langle T, S \circ \varphi \rangle = \int_{\mathbf{G}} dT(x) \int_{\mathbf{G}} dS(y) (\varphi \circ x)(y) = \int_{\mathbf{G}} dT(x) \int_{\mathbf{G}} dS(y) \varphi(xy)$$

and thus

$$(3.6) \quad \langle T, S \circ \varphi \rangle = \langle T \star S, \varphi \rangle.$$

Similarly (denoting  $\varphi$  as a left map)

$$\langle \varphi \circ T, S \rangle^{\sim} = \int_{\mathbf{G}} dS(y) \int_{\mathbf{G}} dT(x) (x)(y \circ \varphi) = \int_{\mathbf{G}} dS(y) \int_{\mathbf{G}} dT(x) (xy) \varphi$$

and thus

$$(3.7) \quad \langle \varphi \circ T, S \rangle^{\sim} = \langle T \star S, \varphi \rangle.$$

(ii) This is an obvious consequence of (3.6) and (3.7).

(iii) By (3.2), (3.4) we get

$$\langle U, T \circ (S \circ \varphi) \rangle = \langle U \star T \star S, \varphi \rangle = \langle U, (T \star S) \circ \varphi \rangle = \langle U, (T \circ S) \circ \varphi \rangle,$$

which proves the first equality of (3.5). The second one is likewise deduced from (3.3) and (3.4).  $\square$

DEFINITION 3.3. *A convolution algebra is a subalgebra of  $(W', \circ)$ .*

**3.2. Fundamental principle and divisibility.** Let  $\mathbf{B}$  be a convolution algebra. The restriction to  $\mathbf{B} \times W$  of the action  $\circ : W' \times W \rightarrow W$  and that of the canonical bilinear form  $\langle -, - \rangle : W' \times W \rightarrow \mathbf{k}$  are still denoted by  $\circ$  and  $\langle -, - \rangle$ , respectively. In general the latter is degenerate; then the topology  $\sigma(\mathbf{B}, W)$  of  $\mathbf{B}$  is Hausdorff but not the topology  $\sigma(W, \mathbf{B})$  of  $W$  (see Remark 2.24(ii)). Therefore, the theory in subsection 2.3 cannot be directly applied.

*Example 3.4.* Consider the case when  $\mathbf{B} = U_{\mathbf{k}}(\mathfrak{g})$  (Appendix 2, section A.2.4); we have  $\mathbf{B} = \bigoplus_{\alpha \in \mathbb{N}^n} D_{\xi}^{\alpha} \delta_e$  (where  $\mathbb{N}$  is the set of nonnegative integers and the notation in section A.2.2 is used); the polar  $\mathbf{A}^{\circ}$  of  $\mathbf{A}$  is given by

$$\mathbf{A}^{\circ} = \{w \in W : \forall \alpha \in \mathbb{N}^n, D_{\xi}^{\alpha} w(e) = 0\}$$

and thus  $\mathbf{A}^{\circ} \neq 0$  (except in the trivial case  $\mathbf{G} = \{e\}$ ). We have

$$\mathbf{A}^{\perp} = \{w \in W : \forall \alpha \in \mathbb{N}^n, D_{\xi}^{\alpha} \delta_e \circ w = 0\} = 0,$$

where  $D_{\xi}^{\alpha} \delta_e \circ w = D_{\xi}^{\alpha} w$ ; thus  $\mathbf{A}^{\circ} \neq \mathbf{A}^{\perp}$ , and statement (ii) of Theorem 2.23 does not hold.

Let  $I$  be a set of indices; the canonical bilinear form  $\langle -, - \rangle : \mathbf{B} \times W \rightarrow \mathbf{k}$  and the action  $\circ : \mathbf{B} \times W \rightarrow W$  are extended to  $\mathbf{B}^{(I)} \times {}^I W$  according to

$$\begin{aligned} \langle -, - \rangle : \mathbf{B}^{(I)} \times {}^I W : (T, w) &\mapsto \sum_{i \in I} \langle T_i, w_i \rangle, \\ \circ : \mathbf{B}^{(I)} \times {}^I W : (T, w) &\mapsto \sum_{i \in I} T_i \circ w_i. \end{aligned}$$

As usual, the orthogonal space with respect to  $\langle -, - \rangle$ , i.e., the polar, is denoted by  $(-)^0$  and that with respect to  $\circ$  by  $(-)^{\perp}$ .

The notions of *crossed product* and of *polycyclic by finite group* are useful for what follows [46, 1.5.8 and 1.5.12]. Both of them are recalled below for the reader's convenience with a notation adapted to the present context.

**DEFINITION 3.5.** *Let  $\mathbf{A}$  be a ring and  $\mathbf{H}$  a group. Let  $\mathbf{B}$  be an over-ring of  $\mathbf{A}$  and a map  $\varepsilon : \mathbf{H} \rightarrow \mathbf{B}$  such that*

- (i)  $\varepsilon$  is injective,  $\varepsilon(e) = 1_{\mathbf{B}}$ , and all  $\varepsilon(h)$  ( $h \in \mathbf{H}$ ) are units of  $\mathbf{B}$ ;
- (ii) for all  $h, h_1, h_2 \in \mathbf{H}$ ,

$$\varepsilon(h) \mathbf{A} = \mathbf{A} \varepsilon(h), \varepsilon(h_1 h_2) \mathbf{A} = \varepsilon(h_1) \varepsilon(h_2) \mathbf{A} = \mathbf{A} \varepsilon(h_1 h_2) = \mathbf{A} \varepsilon(h_1) \varepsilon(h_2);$$

- (iii)  $\mathbf{B}$  is a free right (and, by (ii), left)  $\mathbf{A}$ -module with basis  $(\varepsilon(h))_{h \in \mathbf{H}}$ .

*Then  $\mathbf{B}$  is called a crossed product of  $\mathbf{A}$  by  $\mathbf{H}$ , written  $\mathbf{A} * \mathbf{H}$ .*

**DEFINITION 3.6.** *Let  $\mathbf{H}$  be a group with a finite chain*

$$(3.8) \quad \{e\} = \mathbf{H}_0 \triangleleft \mathbf{H}_1 \triangleleft \dots \triangleleft \mathbf{H}_n = \mathbf{H},$$

where  $\mathbf{H}_i \triangleleft \mathbf{H}_{i+1}$  means that  $\mathbf{H}_i$  is a normal subgroup of  $\mathbf{H}_{i+1}$ , and the factor  $\mathbf{H}_{i+1}/\mathbf{H}_i$  is either infinite cyclic or finite. Without loss of generality, the only finite factor is  $\mathbf{H}_n/\mathbf{H}_{n-1}$ , and  $\mathbf{H}$  along with the above chain is called a polycyclic by finite group (and a polycyclic group if  $\mathbf{H}_n/\mathbf{H}_{n-1} = \{e\}$ ).

The following definition will also be needed in what follows.

**DEFINITION 3.7.** *Let  $\mathbf{G}$  be a Lie group; then  $\mathfrak{P}^{cf}(\mathbf{G})$  is the set of all subgroups  $\mathbf{H}$  of  $\mathbf{G}$  such that all finitely generated subgroups of  $\mathbf{H}$  are polycyclic by finite.*

Note that if  $\mathbf{G}$  is commutative or, more generally, if  $\mathbf{G}$  is a Dedekind group, i.e., if every subgroup of  $\mathbf{G}$  is normal [36], then all subgroups of  $\mathbf{G}$  belong to  $\mathfrak{P}^{cf}(\mathbf{G})$ .

In everything that follows, we'll use the following.

NOTATION 3.8.  $\mathbf{H}$  is a subgroup of  $\mathbf{G}$ , and  $\mathbf{A}$  is a Noetherian subalgebra of  $U_{\mathbf{k}}(\mathfrak{g})$ .

The two main cases are  $\mathbf{A} = U_{\mathbf{k}}(\mathfrak{g})$ —the algebra of invariant differential operators—and  $\mathbf{A} = Z_{\mathbf{k}}(\mathfrak{g})$ —the algebra of bi-invariant differential operators—when the latter is Noetherian (see section A.2.4). Let

$$\mathbf{B} = \sum_{x \in \mathbf{H}} \delta_x \circ \mathbf{A}, \quad \mathbf{C} = \sum_{x \in \mathbf{G}} \delta_x \circ \mathbf{A}.$$

Setting  $\varepsilon(x)S \triangleq \delta_x \circ S$  ( $x \in \mathbf{G}$ ,  $S \in \mathbf{A}$ ), this yields by (A.14)  $\mathbf{B} = \mathbf{A} * \mathbf{H}$  and  $\mathbf{C} = \mathbf{A} * \mathbf{G}$ . We have

$$\mathbf{C} = \sum_{x \in \bar{\mathbf{G}}} \delta_x \circ \left( \sum_{x \in \mathbf{H}} \delta_x \circ \mathbf{A} \right) = \sum_{x \in \bar{\mathbf{G}}} \delta_x \circ \mathbf{B},$$

where  $\bar{\mathbf{G}} \subseteq \mathbf{G}$  is a set of representatives of the cosets of  $\mathbf{G}$  modulo  $\mathbf{H}$ . All distributions belonging to  $\mathbf{A}$  have their support included in  $\{e\}$ ; thus again by (A.14) all above sums are direct since  $\mathbf{G}$  is the disjoint union of its distinct cosets modulo  $\mathbf{H}$ . In particular,  $\mathbf{B}$  is a subalgebra of  $\mathbf{C}$  and

$$(3.9) \quad \mathbf{C} = \bigoplus_{x \in \bar{\mathbf{G}}} \delta_x \circ \mathbf{B} = \mathbf{C} \otimes_{\mathbf{B}} \mathbf{B}$$

is a free right  $\mathbf{B}$ -module.

When  $\mathbf{G} = \mathbb{R}^n$  and  $\mathbf{H}$  is a *finitely generated* subgroup of  $\mathbf{G}$ , then  $\mathbf{B}$  is a convolution algebra of differential-difference operators with lumped shifts, denoted by  $\mathbf{A}^{shift}$ .

LEMMA 3.9. *Let  $\mathbf{H} \in \mathfrak{P}^{cf}(\mathbf{G})$ ; then  $\mathbf{B} = \mathbf{A} * \mathbf{H}$  is coherent.*

*Proof.* Let  $\mathfrak{H}$  be the directed system of all finitely generated subgroups  $\mathbf{H}$  of  $\mathbf{G}$ . The ring  $\mathbf{A}$  is Noetherian; thus if  $\mathbf{H}_1 \in \mathfrak{H}$ , then the ring  $\mathbf{A} * \mathbf{H}_1$  is Noetherian again [46, Theorem 1.5.12], and thus it is coherent. Moreover, if  $\mathbf{H}_1 \subseteq \mathbf{H}_2$  are finitely generated subgroups of  $\mathbf{G}$ , then  $\mathbf{A} * \mathbf{H}_2$  is a free—hence flat—left and right  $\mathbf{A} * \mathbf{H}_1$ -module [46, Lemma 1.5.9(i)], and thus

$$\mathbf{B} = \bigcup_{\mathbf{I} \in \mathfrak{H}} \mathbf{A} * \mathbf{I} = \lim_{\substack{\longrightarrow \\ \mathbf{I} \in \mathfrak{H}}} \mathbf{A} * \mathbf{I}$$

is coherent according to [5, sect. I.2, Exercise 12(e)].  $\square$

Since the two rings  $\mathbf{B} \subseteq \mathbf{C}$  are subrings of  $W'$ ,  $W$  is both a left and right module over  $\mathbf{B}$  and over  $\mathbf{C}$  via the actions  $\circ$  induced by those defined in (3.2) and (3.3). The nondegenerate bilinear form  $\langle -, - \rangle : W' \times W \rightarrow \mathbf{k}$  induces a bilinear form  $\langle -, - \rangle : \mathbf{C} \times W \rightarrow \mathbf{k}$ . We have the following.

LEMMA 3.10. *The bilinear form  $\langle -, - \rangle : \mathbf{C} \times W \rightarrow \mathbf{k}$  is nondegenerate.*

*Proof.* The ring  $\mathbf{C}$  contains all distributions  $\delta_x$ ,  $x \in \mathbf{G}$ . If  $w \in W$  is such that  $\langle \mathbf{C}, w \rangle = 0$ , then  $\langle \delta_x, w \rangle = w(x) = 0$  for all  $x \in \mathbf{G}$ , and thus  $w = 0$ .  $\square$

Let  $R_2 \in {}^q\mathbf{B}^k$ ,  $N = \mathbf{B}^q \circ R_2$ , and  $M = \mathbf{B}^k / \mathbf{B}^q \circ R_2$ ; then the sequence

$$\mathbf{B}^q \xrightarrow{\circ R_2} \mathbf{B}^k \longrightarrow M \longrightarrow 0$$



is exact in  $\mathbf{BMod}$  and  $M$  is finitely presented in that category. The map  $\circ R_2$  is weakly\* continuous (i.e., is continuous when  $\mathbf{B}^q$  and  $\mathbf{B}^k$  are endowed with the topologies  $\sigma(\mathbf{B}^q, {}^qW)$  and  $\sigma(\mathbf{B}^k, {}^kW)$ , respectively) since the convolution product  $\star : W' \times W \rightarrow W$  is separately weakly continuous [11, Chap. I, Prop. 1;2]. It induces a weakly\* continuous map  $\circ R_2 : \mathbf{C}^q \rightarrow \mathbf{C}^k$ .

LEMMA 3.11. (i) Assume that  $\mathbf{H} \in \mathfrak{P}^{cf}(\mathbf{G})$ . Then there exist  $r \in \mathbb{N}$  and a matrix  $R_1 \in {}^r\mathbf{B}^q$  such that the sequence below is exact in  $\mathbf{BMod}$ :

$$(3.10) \quad \mathbf{B}^r \xrightarrow{\circ R_1} \mathbf{B}^q \xrightarrow{\circ R_2} \mathbf{B}^k.$$

(ii) The exactness of the sequence (3.10) in  $\mathbf{BMod}$  is equivalent to the exactness of the sequence below in  $\mathbf{CMod}$ :

$$(3.11) \quad \mathbf{C}^r \xrightarrow{\circ R_1} \mathbf{C}^q \xrightarrow{\circ R_2} \mathbf{C}^k.$$

*Proof.* (i) The ring  $\mathbf{B}$  is coherent by Lemma 3.9; thus there exists a matrix  $R_1$  such that the sequence (3.10) is exact in  $\mathbf{BMod}$  by Lemma 2.15(ii).

(ii) As seen above, the right  $\mathbf{B}$ -module  $\mathbf{C} = \mathbf{C} \otimes_{\mathbf{B}} \mathbf{B}$  is free and thus faithfully flat [39, Example (4.72)(3)]; thus the exactness of the sequence (3.10) is equivalent to that of (3.11) [39, Theorem (4.70)].  $\square$

The following expresses when a *fundamental principle* for  $W$ -behaviors is valid (compare with the Ehrenpreis–Malgrange–Palamodov fundamental principle [27, sect. IV.2], [44], [51, sect. IV.5]) using the above notation.

COROLLARY 3.12. Let  $R_2 \in {}^q\mathbf{B}^k$  be given and  $R_1 \in {}^r\mathbf{B}^q$  be as in Lemma 3.11(i) if it exists. The following properties are equivalent:

- (a) The fundamental principle holds, i.e.,  $R_2 \circ {}^kW = \{u \in {}^rW : R_1 \circ u = 0\}$  ( $R_2 \in {}^q\mathbf{B}^k, R_1 \in {}^r\mathbf{B}^q$ ); i.e., the equation  $R_2 \circ w = u$  ( $u \in {}^rW$ ) has a solution  $w \in {}^kW$  if and only if  $u$  satisfies the compatibility (or integrability) condition  $R_1 \circ u = 0$ .
- (b) The map  $\circ R_2 : \mathbf{C}^q \rightarrow \mathbf{C}^k$  is strict, i.e.,  $\text{coim}_{\mathbf{C}} \circ R_2 \cong \text{im}_{\mathbf{C}} \circ R_2 = \mathbf{C}^q \circ R_2$ .

*Proof.* By Lemma 3.10 and (3.4), the transpose of  $R_2 \circ : {}^kW \rightarrow {}^qW$  is  $\circ R_2 : \mathbf{C}^q \rightarrow \mathbf{C}^k$  and that of  $R_1 \circ : {}^qW \rightarrow {}^rW$  is  $\circ R_1 : \mathbf{C}^r \rightarrow \mathbf{C}^q$ . The fundamental principle holds if and only if the sequence

$${}^kW \xrightarrow{R_2 \circ} {}^qW \xrightarrow{R_1 \circ} {}^rW$$

is exact in the category  $\mathbf{Vect}_{\mathbf{k}}$  of right  $\mathbf{k}$ -vector spaces. By [7, subsect. II.6.5, Remark 1 after Cor. 4 of Prop. 7], this holds true if and only if the map  $\circ R_2 : \mathbf{C}^q \rightarrow \mathbf{C}^k$  is strict.  $\square$

Remark 3.13. (i) If  $\mathbf{B} = \mathbf{U}_{\mathbf{k}}(\mathfrak{g})$ , then  $R_1$  can be constructed via the Gröbner basis algorithm which is applicable to universal enveloping algebras of finite-dimensional Lie groups [50, sect. 5, Definition and Corollary (37)].

- (ii) Let  $0 \neq \mu \in \mathcal{D}(\mathbb{R}^n) \subseteq W'$ , where  $\mathcal{D}(\mathbb{R}^n)$  is the usual space of indefinitely differentiable  $\mathbf{k}$ -valued functions with compact support in  $\mathbb{R}^n$ ; then  $\mu \star W \subsetneq W$  [43, subsect. II.3.1, Corollary 1], and thus the module  ${}_{W'}W$  is not divisible and a fortiori not injective.
- (iii) If  $\mathbf{G} = \mathbb{R}^n$  and  $\mathbf{B} = \mathbf{A}$ , the fundamental principle holds [44, Theorem 3.2]; thus condition (b) in Corollary 3.12 is always satisfied.
- (iv) If  $\mathbf{G}$  is solvable simply connected and  $\mathbf{A} = \mathbf{Z}_{\mathbf{k}}(\mathfrak{g})$ , then  ${}_{\mathbf{A}}W$  is divisible [24, Prop. 2]. We conjecture that  ${}_{\mathbf{A}}W$  is injective. Since  $\mathbb{R}^n$  is such a group, this implies the Ehrenpreis–Malgrange–Palamodov fundamental principle; therefore, if this conjecture is correct, its proof is probably difficult.

- (v) If  $\mathbf{A} = \mathbf{U}_{\mathbf{k}}(\mathfrak{g})$ , the ring of all invariant differential operators over  $\mathbf{G}$ , and  ${}_{\mathbf{A}}W$  is divisible, then either  $\mathbf{G}$  is abelian or it has an abelian normal subgroup of codimension 1 [15, Prop. 2].
- (vi) If  $\mathbf{G}$  is the compact abelian torus  $\mathbb{R}/\mathbb{Z}$  and  $\mathbf{A} = \mathbf{U}_{\mathbf{k}}(\mathfrak{g}) = \mathbf{Z}_{\mathbf{k}}(\mathfrak{g}) = \mathbf{k}[d/dt]$ , then  ${}_{\mathbf{A}}W$  is not divisible [62, p. 225] and hence not injective.
- (vii) If  $\mathbf{G} = \mathbb{R}^n$  and  $\mathbf{B} = \mathbf{A}^{shift}$ , then  ${}_{\mathbf{B}}W$  is divisible [25].

Items (iv) and (v) of the above remark suggest preferring the commutative algebra  $\mathbf{Z}_{\mathbf{k}}(\mathfrak{g})$  of bi-invariant differential operators over the algebra  $\mathbf{U}_{\mathbf{k}}(\mathfrak{g})$  of invariant differential operators regarding the existence of global  $C^\infty$  solutions of nonhomogeneous differential equations over Lie groups.

**COROLLARY AND DEFINITION 3.14.** (1) *Let  $\mu$  be a left-regular element of  $\mathbf{B}$ . The following conditions are equivalent:*

- (i)  $\mu \circ : W \rightarrow W$  is an epimorphism;
  - (ii)  $\circ \mu : \mathbf{C} \rightarrow \mathbf{C}$  is a strict monomorphism, i.e.,  $\mathbf{C} \circ \mu \cong \mathbf{C}$ ;
  - (iii)  $\circ \mu : W' \rightarrow W'$  is a strict monomorphism for the weak\* topology, i.e.,  $W' \circ \mu \cong W'$ ;
  - (iv) the principal ideal  $W' \circ \mu$  is weakly\* closed in  $W'$ ;
  - (v) the principal ideal  $W' \circ \mu$  is strongly closed in  $W'$ .
- (2) *A distribution  $\mu \in \mathbf{B}$  is called invertible<sup>2</sup> if it satisfies the conditions in (1).*
- (3) *Assuming that all nonzero elements of  $\mathbf{B}$  are left-regular, the left  $\mathbf{B}$ -module  $W$  is divisible if and only if all nonzero elements of  $\mathbf{B}$  are invertible distributions, and then the  $\mathbf{B}$ -module  $W$  is faithful.*

*Proof.* (1) Let  $\mu$  be left-regular in  $\mathbf{B}$ ; i.e.,  $\nu \circ \mu = 0$  and  $\nu \in \mathbf{B}$  imply  $\nu = 0$  [20, sect. 7.1]. Then the sequence (3.10) is exact with  $r = 0$ ,  $q = k = 1$ ,  $R_1 = 0$ , and  $R_2 = \mu$ ; therefore, by Lemma 3.11(ii),  $\circ \mu : \mathbf{C} \rightarrow \mathbf{C}$  is a monomorphism, i.e.,  $\text{coim}_{\mathbf{C}} \circ \mu = \mathbf{C} / \ker \circ \mu = \mathbf{C} / \{0\} = \mathbf{C}$ . Thus, by Corollary 3.12, (i) holds if and only if  $\circ \mu : \mathbf{C} \rightarrow \mathbf{C}$  is strict, i.e.,  $\mathbf{C} = \text{coim}_{\mathbf{C}} \circ \mu \cong \text{im}_{\mathbf{C}} \circ \mu$ . As a result, (i)  $\Leftrightarrow$  (ii).

(i)  $\Leftrightarrow$  (iii) by [7, subsect. IV.4.2, Corollary 2 of Theorem 1] since  $W$  is a Fréchet space, and (i)  $\Leftrightarrow$  (iv) by [7, subsect. IV.4.2, Theorem 1].

Since  $W$  is a Montel space [13, sect. 4, Prop. 6], it is reflexive [7, subsect. IV.2.5] and the weak\* closure of  $W' \circ \mu$  is equal to its weak closure and therefore also to its strong closure. Therefore, (iv)  $\Leftrightarrow$  (v).

(3) If all nonzero elements of  $\mathbf{B}$  are left-regular, the divisibility of  $W$  over  $\mathbf{B}$  is equivalent to condition (1)(i) for all  $0 \neq \mu \in \mathbf{B}$  [39, Definition (3.16)]. This condition implies the faithfulness of  $W$  over  $\mathbf{B}$ ; i.e.,  $\mu \circ W = 0$  implies  $\mu = 0$ .  $\square$

**3.3. A cogenerator property.** First let us establish a general result.

**LEMMA 3.15.** *Let  $\mathbf{B} = \mathbf{A} * \mathbf{H}$  be a crossed product of a Noetherian domain  $\mathbf{A}$  of characteristic zero by a polycyclic by finite group  $\mathbf{H}$ .*

- (1)  *$\mathbf{B}$  is a semiprime Noetherian ring, and a Noetherian domain if  $\mathbf{H}$  is polycyclic.*
- (2) *For a finitely generated left  $\mathbf{B}$ -module  $M$ , the following conditions are equivalent:*
  - (i)  *$M$  is torsionless;*
  - (ii) *there exists an embedding  $M \hookrightarrow \mathbf{B}^r$  for some  $r \in \mathbb{N}$ ;*
  - (iii)  *$M$  is torsion-free.*

<sup>2</sup>This denomination is in accordance with that of Ehrenpreis [26] and Hörmander [37, Definition 16.3.12] but in a more general context. In Schwartz's terminology [62, sect. VI.10], such a distribution  $\mu$  is called *completely invertible*.

The same result holds for right modules.

*Proof.* (1) Since the ring  $\mathbf{A}$  is Noetherian, so is again the ring  $\mathbf{B}$  [46, Theorem 1.5.12]. In (3.8), assume without loss of generality that  $\mathbf{H}_i/\mathbf{H}_{i-1}$  is infinite cyclic (thus isomorphic to  $\mathbb{Z}$ ) for  $i \leq n - 1$  and finite for  $i = n$ . (a) The ring  $\mathbf{A} * \mathbb{Z}$  is isomorphic to a skew Laurent polynomial ring  $\mathbf{A} [X, X^{-1}; \alpha]$ , where  $\alpha$  is an automorphism of  $\mathbf{A}$  [46, Proposition 1.5.11]; thus it is an integral domain of characteristic zero. Thus for  $i \leq n - 1$ ,  $\mathbf{A} * \mathbf{H}_i \cong (\mathbf{A} * \mathbf{H}_{i-1}) * \mathbb{Z}$  is an integral domain of characteristic zero by induction. (b) We have  $\mathbf{B} \cong (\mathbf{A} * \mathbf{H}_{n-1}) * (\mathbf{H}_n/\mathbf{H}_{n-1})$ , where  $\mathbf{H}_n/\mathbf{H}_{n-1}$  is finite; thus  $\mathbf{B}$  is semiprime by [46, Theorem 10.5.11].

(2) Since (1) holds true, conditions (i), (ii), and (iii) are equivalent by [46, Props. 3.4.3 and 3.4.7].  $\square$

Let  $\mathbf{G}$  be a Lie group and consider  $W$ -behaviors ( $W = \mathcal{E}(\mathbf{G})$ ) over the ring  $\mathbf{B} = \mathbf{A} * \mathbf{H}$ , where  $\mathbf{H}$  is a polycyclic by finite subgroup of  $\mathbf{G}$ . Thus let  $\mathbf{C} = \mathbf{A} * \mathbf{G}$ ,  $R_1 \in {}^q\mathbf{B}^k$ ,  $N = \mathbf{B}^q \circ R_1 \subseteq \mathbf{B}^k$ ,  $M = \mathbf{B}^k/N$ ,  $\mathfrak{B} = \mathfrak{B}_W(M) = N^\perp$ , and  $\mathfrak{B}^\perp = N^{\perp\perp} = \{T \in \mathbf{B}^k : T \circ \mathfrak{B}\} = 0$ .

**THEOREM 3.16.** (1) *If the finitely presented module  $M$  satisfies the equivalent conditions of part (2) of Lemma 3.15, then the following equivalent properties hold:*

- (i) *the canonical map  $\phi_M : M \rightarrow W^{\mathfrak{B}} : m \mapsto (m \circ w)_{w \in \mathfrak{B}}$  is injective;*
  - (ii)  *$N = N^{\perp\perp}$ .*
- (2)  *$W$  is a cogenerator for the full subcategory  $\mathcal{D}$  of  ${}_{\mathbf{B}}\mathbf{Mod}^{fp}$ , all objects of which are torsionless (or equivalently torsion-free).*

*Proof.* (1) By Lemma 2.15(iv) and Lemma 3.11(ii), there exists a matrix  $R_2 \in {}^k\mathbf{B}^r$  such that the sequences (3.10), (3.11) are exact. In particular, with  $N = \mathbf{B}^q \circ R_1$ , the module  $\mathbf{C} \otimes_{\mathbf{B}} N = \mathbf{C}^q \circ R_1 = \ker_{\mathbf{C}} \circ R_2$  is closed; hence by Lemma 3.10 and Theorem 2.23(i),  $\mathbf{C} \otimes_{\mathbf{B}} N = (\mathbf{C} \otimes_{\mathbf{B}} N)^{\perp\perp}$  and the canonical map

$$\tilde{\phi}_{\tilde{M}} : \tilde{M} \rightarrow W^{\mathfrak{B}} : \tilde{m} \mapsto (\tilde{m} \circ w)_{w \in \mathfrak{B}}$$

( $\tilde{M} = \mathbf{C} \otimes_{\mathbf{B}} M$ ,  $\tilde{m} = 1 \otimes m$ ) is injective. By (3.9), the map

$$\Psi : M \rightarrow \tilde{M} : m \mapsto 1 \otimes m = \delta_e \otimes m$$

is injective too. In addition,  $\bullet\phi_M = \bullet\Psi \tilde{\phi}_{\tilde{M}}$  since  $m \circ w = (1 \otimes m) \circ w$  for the given identification; hence  $\bullet\phi_M$  is injective.

(2) This is a straightforward consequence of (1)(i) and of Corollary and Definition 2.8.  $\square$

*Remark 3.17.* (i) If  $M \in {}_{\mathbf{B}}\mathbf{Mod}^{fp}$  is not torsion-free,  $W$  is not a cogenerator of the category consisting of  $M$  alone, as shown by [31, Example 2.3].

(ii) All  $\mathbf{B}$ -modules  $M \in \mathcal{D}$  are Hausdorff by Lemma 3.15(2)(ii) since the map  $\circ R_2$  in (3.10) induces a continuous injection  $M = \mathbf{B}^k/\mathbf{B}^q R_1 \hookrightarrow \mathbf{B}^r$ , where  $\mathbf{B}^r \subseteq (W')^r$  is Hausdorff.

(iii) Theorem 3.16 is new, even in the standard case of delay-differential systems with  $\mathbf{G} = \mathbb{R}$  and  $\mathbf{H} = \bigoplus_{1 \leq i \leq m} \mathbb{Z}\tau_i$  with delays  $\tau_1, \dots, \tau_m$  (thus  $\mathbf{B} = \mathbf{k}[d/dt, \sigma, \sigma^{-1}]$ ,  $\sigma = (\sigma_1, \dots, \sigma_m)$ ,  $\sigma^{-1} \triangleq (\sigma_1^{-1}, \dots, \sigma_m^{-1})$ ,  $\sigma_i : w(t) \mapsto w(t + \tau_i)$ ). Nevertheless, Theorem 3.16 has connections with the results in [72, Lemmas 3.5 and 3.7] which are slightly generalized below.

**LEMMA 3.18** (compare with [72, Lemmas 3.5 and 3.7]). *Let  $\mathbf{B}$  be a ring and  $W$  a left  $\mathbf{B}$ -module with a free element  $w$ , i.e., such that the map  $\varphi_w : \mathbf{B} \rightarrow W : b \mapsto bw$  is injective. Then  $W$  is a cogenerator for the full subcategory  $\mathcal{D}$  of  ${}_{\mathbf{B}}\mathbf{Mod}$ , all objects of which are torsionless  $\mathbf{B}$ -modules. The category  $\mathcal{D}$  contains all modules  $M$  for which*

there exists an embedding  $M \hookrightarrow \mathbf{B}^r$ ,  $r \geq 0$ , and such modules are described in Lemma 2.15(iv).

If  $\mathbf{B}$  is a domain, the assumption means that the module  $W$  is not a torsion module.

*Proof.* If  $M$  is torsionless, there exists by definition an embedding  $\psi : M \hookrightarrow \mathbf{B}^I$  for some set of indices  $I$  (see section 2.2.2). The assumption implies that

$$\varphi_w^I : \mathbf{B}^I \rightarrow W^I : (b_i)_{i \in I} \mapsto (b_i w)_{i \in I}$$

is again a monomorphism, and so is the composition

$$\phi_M : M \xrightarrow{\psi} \mathbf{B}^I \xrightarrow{\varphi_w^I} W^I;$$

therefore  $W$  has the desired cogenerator property according to Corollary and Definition 2.8.  $\square$

When the Lie group  $\mathbf{G}$  is solvable, Lemma 3.18 implies Theorem 3.16, as shown by the following.

LEMMA 3.19. *Assuming that the Lie group  $\mathbf{G}$  is solvable, the left  $U_{\mathbf{k}}(\mathfrak{g})$ -module  $W$  is not torsion, and neither is the  $\mathbf{A}$ -module  $W$ .*

*Proof.* (a) Considering the sequence (A.15), let  $\mathfrak{a}$  be a one-codimensional ideal of  $\mathfrak{g}$  and  $\mathbf{k}X = \mathfrak{g}/\mathfrak{a}$ , and thus  $\mathfrak{g} = \mathfrak{a} \oplus \mathbf{k}X$ . The restriction  $\delta$  to  $U_{\mathbf{k}}(\mathfrak{a})$  of the adjoint map  $\text{ad}(X)$ , i.e.,  $\delta : U_{\mathbf{k}}(\mathfrak{a}) \rightarrow U_{\mathbf{k}}(\mathfrak{a}) : r \mapsto rX - Xr$ , is an inner derivation. By the Poincaré–Birkhoff–Witt theorem (Appendix 2, section A.2.4),  $U_{\mathbf{k}}(\mathfrak{g}) = \bigoplus_{n \geq 0} U_{\mathbf{k}}(\mathfrak{a}) X^n$ ; thus  $U_{\mathbf{k}}(\mathfrak{g}) = U_{\mathbf{k}}(\mathfrak{a})[X; \delta]$  is a skew polynomial ring with indeterminate  $X$  over  $U_{\mathbf{k}}(\mathfrak{a})$ .

(b) Let  $\mathbf{R}$  be a  $\mathbf{k}$ -algebra, let  $\delta : \mathbf{R} \rightarrow \mathbf{R}$  be a  $\mathbf{k}$ -derivation, let  $\mathbf{S} = \mathbf{R}[X; \delta]$ , and let  $\mathbf{R}^* = \text{Hom}_{\mathbf{k}}(\mathbf{R}, \mathbf{k})$ ,  $\mathbf{S}^* = \text{Hom}_{\mathbf{k}}(\mathbf{S}, \mathbf{k})$  be the duals of  $\mathbf{R}$  and  $\mathbf{S}$ , respectively. Generalizing [50, sect. 3, Lemma (29)], we obtain the following.

If the (left or right)  $\mathbf{R}$ -module  $\mathbf{R}^*$  is not a torsion module, then neither is the (left or right)  $\mathbf{R}$ -module  $\mathbf{S}^*$ .

This is obvious when  $\delta = 0$  since then

$$\mathbf{S}^* = \text{Hom}_{\mathbf{k}}(\mathbf{S}, \mathbf{k}) \cong \text{Hom}_{\mathbf{k}}(\mathbf{R}^{(\mathbb{N})}, \mathbf{k}) \cong {}^{\mathbb{N}}\text{Hom}_{\mathbf{k}}(\mathbf{R}, \mathbf{k}) = {}^{\mathbb{N}}\mathbf{R}^*$$

(Appendix 1, section A.1.2), and if the  $\mathbf{R}$ -module  ${}^{\mathbb{N}}\mathbf{R}^*$  is torsion, then  $\mathbf{R}^*$  is torsion too; this rationale can be extended to the general case, but we skip the technical details due to space limitation.

(c) By induction, we obtain from the above that  $U_{\mathbf{k}}(\mathfrak{g})^*$  is not a torsion (left or right)  $U_{\mathbf{k}}(\mathfrak{g})$ -module, and the same holds for  $W$  due to the Borel epimorphism (Appendix 2, section A.2.4).

(d) Since  $\mathbf{A} \subseteq U_{\mathbf{k}}(\mathfrak{g})$ ,  $W$  is not a torsion  $\mathbf{A}$ -module.  $\square$

**3.4. Dense image representation.** Let us study torsion-free and torsionless controllability as in subsection 2.2.

THEOREM 3.20. *Let  $(\mathbf{B}, \circ)$  be a convolution algebra, and let  $R_1 \in {}^a\mathbf{B}^k$ ,  $N = \mathbf{B}^q \circ R_1$ ,  $M = \mathbf{B}^k/N$ , and  $\mathfrak{B} = \mathfrak{B}_W(M) = \{w \in {}^k W : R_1 \circ w = 0\}$ .*

(1) *Assume that  $\mathbf{G}$  is a Lie group and that  $\mathbf{H} \in \mathfrak{P}^{cf}(\mathbf{G})$ , so that  $\mathbf{B} = \mathbf{A} * \mathbf{H}$  is coherent according to Lemma 3.9. If  $M$  is torsionless, then it is Hausdorff and  $\mathfrak{B}$  admits a dense image representation; i.e., the following property (P)*

holds:

(P): There are  $r \in \mathbb{N}$  and  $R_2 \in {}^k\mathbf{B}^r$  such that  $\mathfrak{B} = \overline{R_2 \circ ({}^rW)}$  (where the closure is taken in the usual Fréchet topology of compact convergence in  ${}^rW$ ).

In addition, if  $\mathbf{H}$  is polycyclic by finite, then  $M$  is torsionless if and only if it is torsion-free by Lemma 3.15.

(2) Conversely, if  $\mathbf{B}$  is dense in  $W'$ ,  $M$  is Hausdorff, and  $\mathfrak{B}$  admits a dense image representation, then  $M$  is both torsion-free and torsionless.

(3) If  $\mathbf{B}$  is an Ore domain (e.g., when  $\mathbf{G}$  is a Lie group and  $\mathbf{B} = \mathbf{A} * \mathbf{H}$ , where  $\mathbf{H}$  is polycyclic by Lemma 3.15(1)) and  $\langle -, - \rangle : \mathbf{B} \times W \rightarrow \mathbf{k}$  is nondegenerate, or in other words if  $\mathbf{B}$  is weakly\* dense in  $W'$ , then  $M$  is torsion-free if and only if  $M$  is Hausdorff (i.e.,  $N$  is closed in  $\mathbf{B}^k$ ) and  $\mathfrak{B}$  admits a dense image representation.

*Proof.* (1) By Lemma 2.15(iv) there exists a matrix  $R_2 \in {}^k\mathbf{B}^r$  such that both sequences (3.10), (3.11) are exact. The exactness of (3.10) implies that  $N = \text{im}_{\mathbf{B}}(\circ R_1) = \text{ker}_{\mathbf{B}}(\circ R_2)$ . Since  $\circ R_2 : \mathbf{B}^k \rightarrow \mathbf{B}^r$  is continuous and  $\mathbf{B}$  is Hausdorff, the kernel  $N$  of this map is closed and the factor  $M = \mathbf{B}^k/N$  is Hausdorff. The exactness of (3.11) implies that  $\text{im}_{\mathbf{C}}(\circ R_1) = \text{ker}_{\mathbf{C}}(\circ R_2)$ . Therefore, by [7, subsect. II.6.3, Cor. 3 of Prop. 4] and [7, subsect. II.6.4, Cor. 4 of Prop. 6],

$$\overline{R_2 \circ {}^rW} = (\text{ker}_{\mathbf{C}} \circ R_2)^0 = (\text{im}_{\mathbf{C}} \circ R_1)^0 = \text{ker}_W (R_1 \circ) = \mathfrak{B},$$

where the closure is taken in the topology  $\sigma({}^rW, \mathbf{C}^r)$  or equivalently in the initial Fréchet topology of  ${}^rW$ .

(2) Since  $M = \mathbf{B}^k/N$  is Hausdorff,  $N = \mathbf{B}^q \circ R_1$  is closed in  $\mathbf{B}^k$ . In addition, the bilinear form  $\langle -, - \rangle : \mathbf{B} \times W \rightarrow \mathbf{k}$  is nondegenerate since  $\mathbf{B}$  is dense—thus weakly\* dense—in  $W'$  [7, subsect. II.6.2, Cor. 4 of Prop. 3]. Therefore,  $N = N^{\perp\perp} = \mathfrak{B}^{\perp}$  by Theorem 2.23(i); thus  $N = \overline{(R_2 \circ {}^rW)}^{\perp} = (R_2 \circ {}^rW)^{\perp} = \text{ker}_{\mathbf{B}} \circ R_2$ . Thus the sequence (3.10) is exact and  $M$  is both torsion-free and torsionless by Lemma 2.15(iii).

(3) (a) Necessary condition: Since  $\mathbf{B}$  is an Ore domain, the finitely generated torsion-free module  $M$  admits by Lemma 2.15(iv) a continuous embedding in some  $\mathbf{B}^r$  and there exists an exact sequence (3.10). Since the form  $\langle -, - \rangle : \mathbf{B} \times W \rightarrow \mathbf{k}$  is nondegenerate, the property  $\overline{R_2 \circ {}^rW} = \mathfrak{B}$  is shown as in (1).

(b) The proof of the sufficient condition is the same as that of (2).  $\square$

**COROLLARY 3.21.** Assume  $\mathbf{G} = \mathbb{R}^n$  with the data of Theorem 3.20. Then  $W'$  is a commutative domain and so is any convolution algebra  $\mathbf{B}$ . A module  $M = \mathbf{B}^k/\mathbf{B}^q \circ R_1$  is torsion-free if and only if it admits an embedding in some  $\mathbf{B}^r$ . Therefore, Theorem 3.20 gets the following form:

- (1) If  $\mathbf{H}$  is any subgroup of  $\mathbb{R}^n$ ,  $\mathbf{B} = \mathbf{A} * \mathbf{H}$ , and  $M$  is torsion-free, then  $N = \mathbf{B}^q \circ R_1$  is closed in  $\mathbf{B}^k$ , and  $\mathfrak{B}$  admits a dense image representation.
- (2) If  $\langle -, - \rangle : \mathbf{B} \times W \rightarrow \mathbf{k}$  is nondegenerate, i.e., if  $\mathbf{B}$  is weakly\* dense in  $W'$ , then  $M$  is torsion-free if and only if  $M$  is Hausdorff and  $\mathfrak{B}$  admits a dense image representation.

*Remark 3.22.* (i) It can be deduced from [67, Theorems 11 and 14] that when  $\mathbf{G} = \mathbb{R}$ ,  $\mathbf{B} = W'$ , and  $M$  is Hausdorff, then  $\mathfrak{B}$  admits a dense image representation if and only if  $M$  is torsion-free. Corollary 3.21 is both more precise and more general.

(ii) The matrix  $R_2$  in Theorem 3.20(1) can be constructed using the Gröbner basis algorithm.

#### 4. Differential systems with distributed shifts over Lie groups.

**4.1. The ring  $\mathcal{H}(\mathbf{B})$  and its basic properties.** Recall that  $\mathbf{A}$  is a Noetherian subalgebra of  $U_{\mathbf{k}}(\mathfrak{g})$  (Notations 3.1 and 3.8). In addition, in what follows we make the following.

*Assumption 4.1.*  $\mathbf{H}$  is a polycyclic subgroup of  $\mathbf{G}$  such that all nonzero elements of  $\mathbf{B} = \mathbf{A} * \mathbf{H}$  are invertible distributions (Corollary and Definition 3.14).

*Example 4.2.* (i) If  $\mathbf{G} = \mathbb{R}^n$ , Assumption 4.1 is satisfied if  $\mathbf{A} \triangleq U_{\mathbf{k}}(\mathfrak{g}) = \mathbf{k} \left[ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right]$  and  $\mathbf{H}$  is any finitely generated subgroup of  $\mathbf{G}$  by Remark 3.13(vi).

(ii) Let  $\mathbf{G}$  be a solvable simply connected Lie group such that  $\mathbf{A} \triangleq Z_{\mathbf{k}}(\mathfrak{g})$  is Noetherian (not all solvable Lie groups have this property: see section A.2.4).

Then Assumption 4.1 is satisfied with  $\mathbf{H} = \{e\}$  by Remark 3.13(iv).

The ring  $\mathcal{H}(\mathbf{B})$  studied below is a generalization of the ring  $\mathcal{H}$  introduced, in the case  $\mathbf{G} = \mathbb{R}$ , by Glüsing-Lüerssen [31] when the delays are commensurable (see also [32]) and by Habets [35] without this assumption. By Lemma 3.15(1),  $\mathbf{B}$  is a Noetherian domain, and thus an Ore domain, and it admits a quotient division ring  $Q(\mathbf{B})$  of *left fractions*  $(a)^{\circ-1} \circ b$  and of *right fractions*  $b \circ (a)^{\circ-1}$ , where  $c = (a)^{\circ-1} \circ b$  (resp.,  $c = b \circ (a)^{\circ-1}$ ) means by definition that  $a \circ c = b$  (resp.,  $c \circ a = b$ ).

Recall the following: (a)  $\mu \circ \nu = \mu * \nu$  for any  $\mu, \nu \in W'$  by Lemma 3.2(ii). (b) A ring  $\mathbf{R}$  is called *weakly finite* if for any integer  $n \geq 1$ , every generating set of  $n$  elements in  $\mathbf{R}^n$  is free; over for such a ring, a square matrix is left- (or right-) invertible if and only if it is invertible [18, sect. 0.2].

**THEOREM AND DEFINITION 4.3.** (1) *Let*

$$\mathcal{H}(\mathbf{B}) \triangleq \left\{ b \circ (a)^{\circ-1} \in Q(\mathbf{B}) : \exists \mu \in W', b = \mu \circ a \right\}.$$

*Then  $\mathcal{H}(\mathbf{B})$  is a subring of both  $Q(\mathbf{B})$  and  $W'$ , and  $\mathcal{H}(\mathbf{B}) = Q(\mathbf{B}) \cap W'$ .*

(2) *The following conditions are equivalent for a right fraction and  $b \circ (a)^{\circ-1} \in Q(\mathbf{B})$ :*

(i)  $b \circ (a)^{\circ-1} \in \mathcal{H}(\mathbf{B})$ ;

(ii)  $\ker_W a \circ \subseteq \ker_W b \circ$ .

(3)  *$\mathcal{H}(\mathbf{B})$  is an Ore domain, and thus is weakly finite, and its quotient division ring is  $Q(\mathbf{B})$ .*

(4) *When  $\mathbf{G} = \mathbb{R}$  and  $\mathbf{H} \neq \{e\}$  is cyclic,  $\mathcal{H}(\mathbf{B})$  is not Noetherian.*

*Proof.* (1) is clear. (2) By definition,  $b \circ (a)^{\circ-1} \in Q(\mathbf{B})$  belongs to  $\mathcal{H}(\mathbf{B})$  if and only if there exists  $\mu \in W'$  such that  $b = \mu \circ a$ . Let  $w \in \ker_W a \circ$ ; then  $a \circ w = 0$  and  $b \circ w = (\mu \circ a) \circ w = \mu \circ (a \circ w) = 0$  by Lemma 3.2(iii), which implies that  $\ker_W a \circ \subseteq \ker_W b \circ$ .

Conversely, let  $a, b \in \mathbf{B}$ ,  $a \neq 0$ , such that  $\ker_W a \circ \subseteq \ker_W b \circ$ .

For any  $\mu \in \mathbf{B}$ , let  $\mathfrak{B}_\mu = (W' \circ \mu)^\perp$ . We have  $\mathfrak{B}_\mu = \ker_W \mu \circ$  and  $\overline{W' \circ \mu} = \mathfrak{B}_\mu^\perp$  by Theorem 2.23(i) since  $\langle -, - \rangle : W' \times W \rightarrow \mathbf{k}$  is nondegenerate. By Corollary and Definition 3.14(iii),  $\overline{W' \circ \mu} = W' \circ \mu$ .

Thus the assumption  $\ker_W a \circ \subseteq \ker_W b \circ$  implies  $W' \circ b \subseteq W' \circ a$ ; hence  $b \in W' \circ a$  and  $b \circ (a)^{\circ-1} \in \mathcal{H}(\mathbf{B})$ .

(3)  $\mathcal{H}(\mathbf{B})$  is an integral domain since  $\mathcal{H}(\mathbf{B}) \subseteq Q(\mathbf{B})$ .

Let us show that  $\mathcal{H}(\mathbf{B})$  is right Ore, i.e., whenever  $f_1, f_2 \in \mathcal{H}(\mathbf{B})^\times$ ,  $f_1 \circ \mathcal{H}(\mathbf{B}) \cap f_2 \circ \mathcal{H}(\mathbf{B}) \neq 0$ ; i.e., there exist  $g_1, g_2 \in \mathcal{H}(\mathbf{B})^\times$  such that  $f_1 \circ g_1 = f_2 \circ g_2$ . This happens if and only if  $g_2 = (f_2^{\circ-1} \circ f_1) \circ g_1$ , where  $f_2^{\circ-1} \circ f_1 \in Q(\mathbf{B})$ . There exist  $b, a \in \mathbf{B}^\times \subseteq W'^\times$  such that  $f_2^{\circ-1} \circ f_1 = b \circ (a)^{\circ-1}$ . Take  $g_1 = a$ ,  $g_2 = b$ . Then  $g_1, g_2 \in \mathcal{H}(\mathbf{B})^\times$  and  $f_1 \circ g_1 = f_2 \circ g_2$  as desired.

The definition of  $\mathcal{H}(\mathbf{B})$  is left/right symmetric; thus  $\mathcal{H}(\mathbf{B})$  is a left Ore domain too.

An Ore domain is weakly finite according to [18, sect. 0.8, Corollary 8.7].

We have  $\mathbf{B} \subseteq \mathcal{H}(\mathbf{B}) \subseteq \mathbf{Q}(\mathbf{B})$ ; thus the quotient division ring of  $\mathcal{H}(\mathbf{B})$  is  $\mathbf{Q}(\mathbf{B})$ .

(4) When  $\mathbf{G} = \mathbb{R}$  and  $\mathbf{H} \neq \{e\}$  is cyclic, then  $\mathcal{H}(\mathbf{B})$  is not Noetherian by [31, Prop. 3.1(c)].  $\square$

*Remark 4.4.* (a) Part (1) of Theorem and Definition 4.3 generalizes [32, section 3.5], where it is assumed that  $\mathbf{G} = \mathbb{R}$  and  $\mathbf{H} \neq \{e\}$  is cyclic (see also [10]).

(b) In addition, Corollary and Definition 3.14(3) implies that Assumptions 2.1 and 4.1 of Habets [35] are satisfied, and thus  $\mathcal{H}(\mathbf{B})$  is the over-ring of  $\mathbf{B}$  denoted by  $\mathbf{B}_W$  in [35, Definition 3.1]; therefore, extending Habets' terminology to the noncommutative case,  $\mathcal{H}(\mathbf{B})$  is the ring of all *admissible right fractions*  $b \circ (a)^{\circ-1} \in \mathbf{Q}(\mathbf{B})$  with respect to  $W$ .

**4.2. Cogenerator properties.** Since  $\mathcal{H}(\mathbf{B}) \subseteq W'$ ,  $W$  has a canonical structure of  $\mathcal{H}(\mathbf{B})$ -module, by restriction of the ring of scalars.

**THEOREM 4.5.** *For  $i = 1, 2$ , let  $R_i \in {}^{q_i}\mathcal{H}(\mathbf{B})^k$ ,  $U_i = W'^{q_i} \circ R_i$ , and  $\mathfrak{B}_i = U_i^\perp = \ker_W(R_i \circ)$ . Assume that  $\text{rk } R_2 = q_2$ . Then the following hold:*

- (i) *The map  $R_2 \circ : {}^k W \rightarrow {}^{q_2} W$  is surjective; in particular all nonzero distributions belonging to  $\mathcal{H}(\mathbf{B})$  are invertible and the  $\mathcal{H}(\mathbf{B})$ -module  $W$  is both divisible and faithful.*
- (ii)  *$\mathfrak{B}_2^\perp = U_2^{\perp\perp} = U_2$ , where  $\mathfrak{B}_2^\perp = \mathfrak{B}_2^{\perp(W')}$   $\triangleq \{w' \in {}^k W' : w' \circ \mathfrak{B}_2\} = 0$ .*
- (iii)  *$\ker_W(R_2 \circ) \subseteq \ker_W(R_1 \circ)$  if and only if there exists  $X \in {}^{q_1}\mathcal{H}(\mathbf{B})^{q_2}$  such that  $R_1 = X \circ R_2$ .*
- (iv) *The  $\mathcal{H}(\mathbf{B})$ -module  $W$  is a cogenerator for the full subcategory  $\mathcal{D}$  of  ${}_{\mathcal{H}(\mathbf{B})}\mathbf{Mod}$ , the objects  $M$  of which are finitely presented by a short exact sequence*

$$0 \longrightarrow \mathcal{H}(\mathbf{B})^q \xrightarrow{\circ R} \mathcal{H}(\mathbf{B})^k \longrightarrow M \longrightarrow 0.$$

*Proof.* (i) The assumption means that  $R_2$  is right-invertible over  $\mathbf{Q}(\mathbf{B})$  and implies that  $R_2$  is right-invertible over  $\mathbf{Q}(\mathbf{B})$ ; i.e., there exists a matrix  $X_2 \in {}^k \mathbf{Q}(\mathbf{B})^{q_2}$  such that  $R_2 \circ X_2 = I_{q_2}$ . Since  $\mathbf{B}$  is an Ore domain, for each  $j \in \{1, \dots, q_2\}$ , all entries of the  $j$ th column of  $X_2$  have a common right denominator  $\mu_j \in \mathbf{B}^\times$  [46, Prop. 2.1.16(i)]; i.e., there exist elements  $(Y_2)_{ij} \in \mathbf{B}$  ( $1 \leq i \leq k$ ) such that  $(X_2)_{ij} = (Y_2)_{ij} \circ \mu_j^{\circ-1}$ ; then  $(Y_2)_{ij} = (X_2)_{ij} \circ \mu_j \in \mathbf{B}$  and

$$R_2 \circ Y_2 = R_2 \circ \left( X_2 \circ \text{diag}(\mu_j)_{1 \leq j \leq q_2} \right) = \text{diag}(\mu_j)_{1 \leq j \leq q_2}.$$

Therefore, for any  $w \in {}^{q_2} W$ ,  $R_2 \circ (Y_2 \circ w) = \text{diag}(\mu_j)_{1 \leq j \leq q_2} \circ w = (\mu_j \circ w_j)_{1 \leq j \leq q_2}$ . By hypothesis, each distribution  $\mu_j$  is invertible and  $\mu_j \circ : W \rightarrow W$  is surjective. Therefore,  $R_2 \circ : {}^k W \rightarrow {}^{q_2} W$  is surjective too.

(ii) The transpose of  $R_2 \circ : {}^k W \rightarrow {}^{q_2} W$  is  $\circ R_2 : W'^{q_2} \rightarrow W'^k$  and the spaces  ${}^k W, {}^{q_2} W$  are Fréchet spaces. Therefore, by (i) and [37, Lemma 16.5.8],  $\circ R_2 : W'^{q_2} \rightarrow W'^k$  is injective and  $W'^{q_2} \circ R_2$  is weakly\* closed in  $W'^k$ . Therefore,  $\mathfrak{B}_2^\perp = U_2^{\perp\perp} = U_2$  by Theorem 2.23(i) since  $\langle -, - \rangle : {}^k W' \times W'^k \rightarrow \mathbf{k}$  is nondegenerate.

(iii) The sufficient condition is obvious. Let us prove the necessary condition. If  $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$ , then  $W'^{q_1} \circ R_1 \subseteq \mathfrak{B}_1^\perp \subseteq \mathfrak{B}_2^\perp$  and  $\mathfrak{B}_2^\perp = W'^{q_2} \circ R_2$  by (ii). Thus  $W'^{q_1} \circ R_1 \subseteq W'^{q_2} \circ R_2$ , and hence every row of  $R_1$  and therefore  $R_1$  itself are left-multiples of  $R_2$ ; i.e., there exists  $X \in {}^{q_2} W'^{q_1}$  such that  $R_1 = X \circ R_2$ . Since  $\text{rk } R_2 = q_2$ , after a possible permutation of columns we obtain

$$R_i = \begin{bmatrix} R'_i & R''_i \end{bmatrix} \in {}^{q_i}\mathcal{H}(\mathbf{B})^{q_i+(k-q_i)} \quad (i = 1, 2),$$

where  $R'_2$  is invertible over  $\mathbf{Q}(\mathbf{B})$ . Then  $R'_1 = X \circ R'_2$  and  $X = R'_1 \circ R'^{-1}_2$ . As in (i) we obtain  $Y \in {}^q\mathbf{B}^{q_2}$  and  $\nu_i \in \mathbf{B}^\times$  ( $1 \leq i \leq q_2$ ) with  $R'_2 \circ Y = \text{diag}(\nu_i)_{1 \leq i \leq q_2} = \Delta$ , and hence

$$X = R'_1 \circ R'^{-1}_2 = R'_1 \circ Y \circ \Delta^{-1} \in {}^{q_1}(\mathbf{Q}(\mathbf{B}) \cap W')^{q_2} = {}^{q_1}\mathcal{H}(\mathbf{B})^{q_2}.$$

(iv) is a consequence of (iii) and of Theorem 2.13(iii).  $\square$

**COROLLARY 4.6.** *Let  $R \in {}^q\mathcal{H}(\mathbf{B})^k$ ,  $N = \mathcal{H}(\mathbf{B})^q \circ R$ ,  $M = \mathcal{H}(\mathbf{B})^k / N$ ,  $\mathfrak{B} = N^\perp = \ker_W(R \circ)$ .*

- (i)  $\text{rk } R = q$  if and only if  $R \circ {}^k W = {}^q W$ .
- (ii) If  $R$  has a left-inverse  $X$  with coefficients in  $W'$ , i.e.,  $X \in {}^k W'^q$ ,  $X \circ R = \delta_e I_k$ , then  $\mathfrak{B} = 0$ .
- (iii) Assuming that  $\text{rk } R = q$ ,  $\ker_W(R \circ) = 0$  if and only if  $R \in \text{GL}_q(\mathcal{H}(\mathbf{B}))$ .
- (iv) In particular, for  $R \in \mathcal{H}(\mathbf{B})$ ,  $\ker_W(R \circ) = 0$  if and only if  $R$  is a unit in  $\mathcal{H}(\mathbf{B})$ .

*Proof.* (i) The necessary condition follows from Theorem 4.5(i).

Conversely, assume  $\text{rk } R < q$ . Since  $\text{rk } R$  is (by definition) the dimension of the left  $\mathbf{Q}(\mathbf{B})$ -vector space generated by the rows of  $R$ , there exists a nonzero row  $y \in \mathbf{Q}(\mathbf{B})^q$  such that  $y \circ R = 0$ . Let  $d$  be a common left denominator of all elements of  $y$  and  $x = d \circ y \in \mathcal{H}(\mathbf{B})^q$ ; then  $x \circ R = 0$  and  $x \neq 0$ . Thus  $R \circ ({}^k W) \subseteq \ker_W(x \circ) \subseteq {}^q W$ . Assuming that  $\ker_W(x \circ) = {}^q W$ , then  $x \circ {}^q W = 0$  and  $x_i \circ W = 0$  for all  $i \in \{1, \dots, q\}$ , a contradiction since the  $\mathcal{H}(\mathbf{B})$ -module  $W$  is faithful by Theorem 4.5(i). Therefore  $R \circ ({}^k W) \subsetneq {}^q W$ .

(iii) Assume that  $\text{rk } R = q$ . Clearly,  $\ker_W(R \circ) = 0$  if and only if  $\ker_W(R \circ) \subseteq \ker_W(I_k \delta_e \circ)$ , and by Theorem 4.5(iii) this is equivalent to the existence of a matrix  $X \in {}^k \mathcal{H}(\mathbf{B})^{q_2}$  such that  $I_k \delta_e = X \circ R$ ; thus  $R$  is left-invertible over  $\mathcal{H}(\mathbf{B})$  and  $\text{rk } R = k$ . Therefore,  $R$  is square, and since  $\mathcal{H}(\mathbf{B})$  is weakly finite by Theorem and Definition 4.3(2), the left-invertibility of the square matrix  $R$  over  $\mathcal{H}(\mathbf{B})$  is equivalent to its invertibility over that ring.

(ii) is obvious and (iv) is a particular case of (iii).  $\square$

**Remark 4.7.** Consider the data of Theorem 4.5 and Corollary 4.6 for the special case  $\mathbf{G} = \mathbb{R}$ .

- (a) Statement (iii) of Theorem 4.5 is proved in [33, Theorem 5.3] without any assumption on  $\text{rk } R_2$  when  $\mathbf{H} \neq 0$  is cyclic. In that case, indeed,  $\mathcal{H}(\mathbf{B})$  is a Bézout domain; thus  $R_2$  is left-equivalent to a matrix  $\begin{bmatrix} Q \\ 0 \end{bmatrix}$ , where  $Q$  is full row rank [18, sect. 1.1, Theorem 1.1(g)], and thus the assumption on  $\text{rk } R_2$  becomes superfluous. Except in that case, the statement of Theorem 4.5 is not valid when the matrix  $R_2$  is not assumed to be full row rank, as shown by [66, Example after the proof of Theorem 3].
- (b) Statement (iii) of Theorem 4.5 can be viewed as an extension to the noncommutative case of [35, Theorem 4.9] (see Remark 4.4(b)).
- (c) The units in  $\mathcal{H}(\mathbf{B})$  are explicitly described in Corollary 4.12 below when  $\mathbf{G} = \mathbb{R}^n$ .

**4.3. Differential systems with distributed shifts over  $\mathbb{R}^n$ .** In what follows,  $\mathbf{G} = \mathbb{R}^n$  and  $\mathbf{H}$  is a finitely generated subgroup of  $\mathbf{G}$ . The ring of all entire complex-valued functions over  $\mathbb{C}^n$  is denoted by  $\mathcal{O}(\mathbb{C}^n)$ . For  $s, z \in \mathbb{C}^n$ , we set

$$s \bullet z \triangleq \sum_{1 \leq i \leq n} s_i z_i, \quad |s| = \left( \sum_{1 \leq i \leq n} |s_i|^2 \right)^{1/2}.$$



Let  $\mathbf{A} \triangleq \mathbf{U}_{\mathbf{k}}(\mathfrak{g}) = \mathbf{k} \left[ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right]$  and  $\mathbf{B} = \mathbf{A} * \mathbf{H}$ . The convolution algebra  $\mathbf{B}$  is now a commutative domain; thus the left fractions  $a^{\circ-1} \circ b \in \mathbf{Q}(\mathbf{B})$  coincide with the right fractions  $b \circ a^{\circ-1} \in \mathbf{Q}(\mathbf{B})$  and are denoted by  $b/^{\circ}a$  in what follows. As is well known, the Laplace transform

$$\mathcal{L} : W' \ni \mu \mapsto \hat{\mu}(s) \triangleq \int_{\mathbb{R}^n} e^{-s \bullet x} d\mu(x), \quad s \in \mathbb{C}^n,$$

yields the following isomorphisms:

$$\mathbf{A} \cong \mathbb{C}[s], \quad \mathbf{B} \cong \mathcal{PE}(\mathbf{H}) \triangleq \sum_{\tau \in \mathbf{H}} \mathbb{C}[s] e^{-\tau \bullet s}, \quad W' \cong PW(\mathbb{C}^n),$$

where  $PW(\mathbb{C}^n)$  is the set of all functions  $f \in \mathcal{O}(\mathbb{C}^n)$  for which there exist  $c > 0$ ,  $a > 0$ , and  $N \in \mathbb{N}$  such that

$$|f(s)| \leq c(1 + |s|)^N e^{a|\operatorname{Re}(s)|}.$$

The elements of  $\mathcal{PE}(\mathbf{B})$  are called *complex exponential-polynomial functions* [2], and the third isomorphism is given by the Paley–Wiener–Schwartz theorem [22, (22.18.7)].

The following results complement Corollary and Definition 3.14 and are due to Ehrenpreis and Malgrange; they are nicely expounded by Hörmander [37] and are gathered below for convenience.

LEMMA 4.8 (see [37, Theorems 16.3.10, 16.5.7, and 16.5.19]; [1, Appendix]). *For  $\mu \in W'$ , consider the convolution equation*

$$(4.1) \quad \mu \star u = f,$$

where the distribution  $\mu$  is given and the distribution  $u$  is looked for. The following conditions are equivalent:

- (i) if  $f \in W'$  and  $\hat{f}(s)/\hat{\mu}(s) \in \mathcal{O}(\mathbb{C}^n)$ , then there exists a (unique)  $u \in W'$  such that  $\hat{f}(s)/\hat{\mu}(s) = \hat{u}(s)$  (which implies (4.1));
- (ii) (4.1) has a solution  $u \in W$  for any  $f \in W$ ; i.e.,  $\mu$  is invertible;
- (iii) (4.1) has a solution  $u \in \mathcal{D}'(\mathbb{R}^n)$  for any  $f \in \mathcal{D}'(\mathbb{R}^n)$ ;
- (iv)  $\hat{\mu}$  is slowly decreasing (see [37, Theorem 16.3.10 and Definition 16.3.12]; [1]).

Remark 4.9. Malgrange established the equivalence between Corollary and Definition 3.14(1)(v) and Lemma 4.8(i) for  $0 \neq \mu \in \mathcal{E}'(\mathbb{R}^n)$  [43, Corollary on p. 310]. Ehrenpreis [26] introduced the notion of slowly decreasing  $\hat{\mu}$  and proved the equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) of Lemma 4.8. The elements of  $\mathcal{PE}(\mathbf{H})$  are easily seen to be slowly decreasing, and this proves Ehrenpreis' result in Remark 3.13(vi).

The following was proved and partially stated by Berenstein and Dostal [2] when  $n \geq 1$  and is explicit in [35] when  $n = 1$ .

LEMMA 4.10. *Let  $F, G \in \mathcal{PE}(\mathbf{H})$ ,  $G \neq 0$ , be such that  $F/G \in \mathcal{O}(\mathbb{C}^n)$ . There exist  $H \in \mathcal{PE}(\mathbf{H})$  and  $0 \neq P \in \mathbb{C}[s]$  such that  $F/G = H/P$ .*

*Proof.* (a) By [2, Main Theorem], there exist  $H \in \mathcal{PE}(\mathbb{R}^n)$  and  $0 \neq P \in \mathbb{C}[s]$  such that  $F/G = H/P$ .

(b) For any  $0 \neq L(s) \in \mathcal{PE}(\mathbb{R}^n)$ , write  $L(s) = \sum_{\tau \in \mathbf{T}} l_{\tau}(s) e^{-\tau \bullet s}$ , where  $\mathbf{T}$  is a finite subset of  $\mathbb{R}^n$  and  $0 \neq l_{\tau}(s) \in \mathbb{C}[s]$  for any  $\tau \in \mathbf{T}$ . The elements  $\tau \in \mathbf{T}$  are the exponents [35] (or, more loosely speaking, the “frequencies” [2]) of  $L(s)$ . The above element  $H \in \mathcal{PE}(\mathbb{R}^n)$  is constructed by induction in [2, Proof of Theorem 1]. At each step, the exponents involved are  $\mathbb{Z}$ -linear combinations of those of  $F$  and  $G$ ,

i.e., belong to  $\mathbf{H}$ ; see [2, (2.20)–(2.26)] for more details. Since the procedure stops in a finite number of steps,  $H \in \mathcal{PE}(\mathbf{H})$ . (When  $n = 1$ , the fact that the exponents of  $H$  are  $\mathbb{Z}$ -linear combinations of those of  $F$  and  $G$  is used in the proof of [35, Theorem 5.9].)  $\square$

The following is a generalization of [35, Lemma 3.2 and Theorems 5.4, 5.7, and 5.9] to the case  $n \geq 1$ .

**THEOREM 4.11.** *The following rings coincide with the ring  $\mathcal{H}(\mathbf{B})$ :*

$$\begin{aligned} \mathcal{H}_1(\mathbf{B}) &= \left\{ b/^{\circ}a \in \mathbf{Q}(\mathbf{B}) : \hat{b}(s)/\hat{a}(s) \in PW(\mathbb{C}^n) \right\}, \\ \mathcal{H}_2(\mathbf{B}) &= \left\{ b/^{\circ}a \in \mathbf{Q}(\mathbf{B}) : \hat{b}(s)/\hat{a}(s) \in \mathcal{O}(\mathbb{C}^n) \right\}, \\ \mathcal{H}_3(\mathbf{B}) &= \left\{ b/^{\circ}a \in \mathbf{Q}(\mathbf{B}) \cap W' : b \in \mathbf{B}, a \in \mathbf{A}^{\times} \right\}. \end{aligned}$$

*Proof.* (a) By definition, if  $u = b/^{\circ}a \in \mathcal{H}(\mathbf{B})$ , then  $\hat{u}(s) \in PW(\mathbb{C}^n) \subseteq \mathcal{O}(\mathbb{C}^n)$ , which proves that  $\mathcal{H}(\mathbf{B}) \subseteq \mathcal{H}_1(\mathbf{B})$ . The inclusion  $\mathcal{H}_1(\mathbf{B}) \subseteq \mathcal{H}_2(\mathbf{B})$  is obvious.

(b) Let  $b \in \mathbf{B}$ ,  $a \in \mathbf{B}^{\times}$  be such that  $\hat{b}(s)/\hat{a}(s) \in \mathcal{O}(\mathbb{C}^n)$ . By Lemma 4.8, the distribution  $a$  is invertible and there exists  $u \in W'$  such that  $a \circ u = b$ . Therefore,  $b/^{\circ}a \in \mathcal{H}(\mathbf{B})$ , and hence  $\mathcal{H}_2(\mathbf{B}) \subseteq \mathcal{H}(\mathbf{B})$ .

(c) Last,  $\mathcal{H}_3(\mathbf{B}) = \mathcal{H}_2(\mathbf{B})$  by Lemma 4.10.  $\square$

**COROLLARY 4.12.** *An element  $T \in \mathcal{H}(\mathbf{B})$  is a unit if and only if it has the form  $T = \alpha\delta_{\tau}$ ,  $\tau \in \mathbf{H}$ ,  $\alpha \in \mathbf{k}^{\times}$ , i.e., is also a unit in  $\mathbf{B}$ .*

*Proof.* The indicated elements are obviously units. The ring  $\mathbf{B}$  can be written  $\mathbf{A}[\sigma, \sigma^{-1}]$  with the notation in Remark 3.17(iii) and  $\mathbf{A} = \mathbf{k}[D]$ ,  $D = (\partial/\partial x_i)_{1 \leq i \leq n}$ . Assume that  $T_1 \in \mathcal{H}(\mathbf{B})$  is invertible and that

$$\delta_0 = T_1 \circ T_2, \quad T_i = b_i/^{\circ}a_i, \quad b_i \in \mathbf{B}, \quad a_i \in \mathbf{A}^{\times} \quad (i = 1, 2);$$

therefore  $b_1 \circ b_2 = a_1 \circ a_2$  in  $\mathbf{B} = \mathbf{A}[\sigma, \sigma^{-1}]$ , where we have used  $\mathcal{H}(\mathbf{B}) = \mathcal{H}_3(\mathbf{B})$ . We infer  $b_1 = a'_1 \circ \delta_{\tau_1}$  for some  $a'_1 \in \mathbf{A}$  and  $\tau_1 \in \mathbf{H}$ ; thus  $T_1 = b_1/^{\circ}a_1 = (a'_1/^{\circ}a_1) \circ \delta_{\tau_1}$ . Since  $\hat{T}_1(s)$  is entire, so is the rational function  $\hat{a}'_1(s)/\hat{a}_1(s) = \hat{a}''_1(s)$ . Thus  $\hat{a}''_1(s)$  is a polynomial and  $T_1 = a''_1 \circ \delta_{\tau_1}$  with  $a''_1 \in \mathbf{A}$ . The same argument applies to  $T_2$  and furnishes

$$\delta_0 = T_1 \circ T_2 = (a''_1 \circ a''_2) \circ \delta_{\tau_1 + \tau_2},$$

and thus  $a''_1 \circ a''_2 = \delta_0$  and  $\tau_1 + \tau_2 = 0$ . Since  $\mathbf{A}$  is a polynomial algebra,  $a''_1 = \alpha \in \mathbf{k}^{\times}$ , and hence  $T = \alpha\delta_{\tau}$  as asserted.  $\square$

**Remark 4.13.** (a) We consider the space  $\mathcal{PE} \triangleq \mathcal{PE}(\mathbb{R}^n)$  of real polynomial-exponential functions ( $\mathcal{PE} \subseteq \mathcal{E}(\mathbb{R}^n)$ ). As proved by Gurevič [34, Theorem 3.2], there exists a column  $R \in {}^k\mathcal{E}'(\mathbb{R}^n)$ ,  $n \geq 2$ ,  $k = 6$ , such that

$$0 = \ker_{\mathcal{PE}}(R \circ) = {}^k(\mathcal{PE}) \cap \ker_W(R \circ), \quad \ker_W(R \circ) \neq 0;$$

i.e.,  $R \circ w = 0$  has nonzero solutions in  $W$  but no polynomial-exponential ones. Thus  $\ker_{\mathcal{PE}}(R \circ) = \ker_{\mathcal{PE}}(1 \circ) = 0$ , and according to Malgrange's theorem [43, Preliminaries, sect. 4, Theorem 3], there exists a row  $X \in \mathcal{O}(\mathbb{C}^n)^k$  such that  $X \hat{R} = 1$ . However,  $\ker_{PW(\mathbb{C}^n)}(\hat{R} \bullet) \not\subseteq \ker_{PW(\mathbb{C}^n)}(1 \bullet) = 0$ , so that [66, Theorem 3] does not hold true for  $n > 1$ . This is because this result is based on the Schwartz–Kelleher–Taylor theorem on mean periodic *vector-valued* functions [38, Theorem 7.3] (see also [52]), which implies that for any natural integer  $k$ , every shift-invariant closed subspace of  ${}^k W$  is spanned by the  ${}^k\mathbb{C}$ -valued exponential-polynomial functions it contains when  $n = 1$ . Gurevič's counterexample shows that the last result does not hold for  $n > 1$ .

- (b) With the notation from Remark 3.17(iii), the ring  $\mathcal{H}(\mathbf{B}) = \mathcal{H}_3(\mathbf{B})$  is isomorphic to  $\mathbf{D} \cap \mathcal{O}(\mathbb{C}^n)$ , where  $\mathbf{D} = \mathbf{k}(s)[\sigma]$ ,  $s = (s_1, \dots, s_n)$ ,  $\sigma = (\sigma_1, \dots, \sigma_m)$ . The ring  $\mathbf{D}$  is projective-free [40, sect. IV.4, Cor. 4.10]; thus a finitely presented  $\mathbf{D}$ -module  $M$  has a full row rank matrix of definition if and only if  $M$  has projective dimension  $\leq 1$ . Assume  $m > 1$ . Since  $\mathbf{D}$  has global dimension  $m$  [46, 7.5.3(iv)], such a module  $M$  is very peculiar, and so are probably also the finitely presented  $\mathcal{H}(\mathbf{B})$ -modules in Theorem 4.5(iv), for we conjecture that  $\mathcal{H}(\mathbf{B})$  has global dimension  $\geq m$ .

**5. Concluding remarks.** Using the definition of a cogenerator for a subcategory, many results of the literature have been generalized: see, e.g., Theorems 2.10, 2.13, 2.23, 3.16, and 4.5. Surprisingly, the statement of several results is purely algebraic, whereas their proof is based on the theory of locally convex topological spaces in duality: see, e.g., Theorems 3.16 and 4.5. Section 3 can be extended to systems over *general locally compact groups* using Bruhat’s theory [13], with the same proofs. A natural extension of this paper is the study of the fundamental principle and of the cogenerator properties for *analytic* differential-difference systems; this is the subject of a forthcoming paper [9].

**Appendix A. Appendices.**

**A.1. Appendix 1: Preabelian and semi-abelian categories.** We refer the reader to MacLane’s classical book [41] for a general presentation of categories and functors. A preabelian category is an additive category in which every morphism has a kernel and a cokernel and thus an image and a coimage [57, sect. 2.1]. All categories considered in this paper are preabelian, except when otherwise stated.

**A.1.1. Objects and morphisms.** Consider a category  $\mathcal{C}$ , its class of objects  $\text{Ob}(\mathcal{C})$ , and its class of morphisms  $\text{Mor}(\mathcal{C})$ . We write  $X \in \mathcal{C}$  (instead of  $X \in \text{Ob}(\mathcal{C})$ ) to mean that  $X$  is an object of  $\mathcal{C}$ . The opposite of  $\mathcal{C}$ , denoted by  $\mathcal{C}^{\text{op}}$ , is deduced from  $\mathcal{C}$  by “reversing the arrows.” This leads to defining *left and right morphisms*. To clarify this, consider the category **Set** such that  $\text{Ob}(\mathbf{Set})$  is the class of all sets and  $\text{Hom}_{\mathbf{Set}}(X, Y)$  is the set of all maps  $f : X \rightarrow Y$ . If  $f$  is written  $x \mapsto xf$  (resp.,  $x \mapsto fx$ ), we call it a left (resp., right) map since  $f$  acts on the left (resp., on the right), and to avoid confusions  $f$  can be written  $\bullet f$  (resp.,  $f\bullet$ ). Consider the composition

$$(A.1) \quad X \xrightarrow{f} Y \xrightarrow{g} Z.$$

If  $f$  and  $g$  are left (resp., right) maps, this composition is  $fg$  (resp.,  $gf$ ). Now, let  $\mathcal{C}$  be any category and consider the composition (A.1), where  $f, g \in \text{Mor}(\mathcal{C})$ . Generalizing the above, we’ll say that  $f$  and  $g$  are *left morphisms* (written  $\bullet f$  and  $\bullet g$  to avoid confusions) if this composition is denoted as  $fg$  and *right morphisms* (written  $f\bullet$  and  $g\bullet$ ) if this composition is denoted as  $gf$ . For example, in [41], all morphisms are right ones, whereas in [20, Chap. 2], all morphisms are left ones. In the rest of this appendix, all morphisms are right ones.

All facts below are classical. An isomorphism is a bimorphism, i.e., both a monomorphism and an epimorphism, but the converse does not hold in general. Let  $X$  be an object of  $\mathcal{C}$ . The set  $P_X$  of all morphisms with target  $X$  has preorder  $\leq$  with  $u\bullet \leq v\bullet$  to mean that  $u$  factors through  $v$  (i.e.,  $u = vu'$  for some morphism  $u'\bullet$ ). Two morphisms  $u, v \in P_X$  are called *equivalent* if both  $u\bullet \leq v\bullet$  and  $v\bullet \leq u\bullet$ . Dually, the set  $Q^X$  of all morphisms with source  $X$  has preorder  $\geq$  with  $f\bullet \geq g\bullet$  to mean that  $g$  factors through  $f$  (i.e.,  $g = g'f$  for some  $g'\bullet$ ) and  $f, g$  are called *equivalent* if both

$f\bullet \geq g\bullet$  and  $g\bullet \geq f\bullet$ . A *subobject* of  $X$  is an equivalence class of monomorphisms belonging to  $P_X$ . Thus a representative of a subobject of  $X$  is a pair  $(Y, \iota)$  such that  $\iota : Y \rightarrow X$  is a monomorphism, called the *canonical injection* (or the *inclusion*)  $Y \hookrightarrow X$ . Dually, a *factor object* of  $X$  is an equivalence class of epimorphisms belonging to  $Q^X$ . A representative of a factor of  $X$  is a pair  $(Z, \varphi)$  such that  $\varphi : X \rightarrow Z$  is an epimorphism, called the *canonical epimorphism*  $X \twoheadrightarrow Z$ . Abusing the language,  $(Y, \iota)$  is denoted by  $\iota$  or  $Y$ , and  $(Z, \varphi)$  by  $\varphi$  or  $Z$ , when there is no risk of confusion. The above applies to kernels, cokernels, images, and coimages. Let  $f\bullet : X \rightarrow Y$ ; then it is convenient to think of  $\ker f$  as the largest subobject  $(K, \kappa\bullet)$  of  $X$  annihilated by  $f$  (i.e., such that  $f\kappa = 0$ ) and of  $\operatorname{coker} f$  as the largest factor  $(C, \gamma\bullet)$  of  $Y$  annihilating  $f$  (i.e., such that  $\gamma f = 0$ ). The *image* and the *coimage* of  $f$ , respectively, written  $\operatorname{im} f$  and  $\operatorname{coim} f$ , are defined as

$$(A.2) \quad \operatorname{im} f = \ker \operatorname{coker} f, \quad \operatorname{coim} f = \operatorname{coker} \ker f;$$

they are, respectively, a subobject of  $Y$  and a factor of  $X$ . Let  $u \in P_X$  and  $g \in Q^X$ . Then  $u \leq \ker g \Leftrightarrow gu = 0 \Leftrightarrow \operatorname{coker} u \geq g$ , which means that the two functions  $\ker : Q^X \rightarrow P_X$ ,  $\operatorname{coker} : P_X \rightarrow Q^X$  constitute an order-reversing Galois connection. In particular,

$$(A.3) \quad \ker g = \ker \operatorname{coker} \ker g = \ker \operatorname{coim} g = \operatorname{im} \ker g,$$

$$(A.4) \quad \operatorname{coker} u = \operatorname{coker} \ker \operatorname{coker} u = \operatorname{coker} \operatorname{im} u = \operatorname{coim} \operatorname{coker} u.$$

Therefore (i)  $\iota$  is a kernel if and only if  $\iota = \operatorname{im} \iota$ , (ii)  $\beta$  is a cokernel if and only if  $\beta = \operatorname{coim} \beta$ , and (iii) if  $\iota = \operatorname{im} f$ , then  $\operatorname{coker} \iota = \operatorname{coker} f$  and  $\operatorname{im} \iota = \operatorname{im} f = \iota$ .

Let  $f : X \rightarrow Y$ . There exists a unique morphism  $f_{\operatorname{ind}}$ , called the *induced morphism*, such that  $f$  admits the factorization

$$(A.5) \quad f : X \xrightarrow{\alpha} \operatorname{coim} f \xrightarrow{f_{\operatorname{ind}}} \operatorname{im} f \xrightarrow{\omega} Y,$$

where  $\alpha$  is the canonical epimorphism and  $\omega$  is the inclusion. The morphism  $f$  is called *strict* if  $f_{\operatorname{ind}}$  is an isomorphism. A preabelian category  $\mathcal{C}$  is called *semi-abelian* if for any morphism  $f$ , the induced morphism  $f_{\operatorname{ind}}$  is a bimorphism [60, p. 168]; if for each morphism  $f$ ,  $f_{\operatorname{ind}}$  is an isomorphism, then  $\mathcal{C}$  is called *abelian*.

**A.1.2. Functors.** All categories are semi-abelian in what follows, and all functors are additive in this paper. Recall that a functor  $\mathfrak{F} : \mathcal{C} \rightarrow \mathcal{D}$  is said to be (i) *full* (resp., *faithful*, *fully faithful*) if its arrow map is surjective (resp., injective, bijective), (ii) *injective* (resp., *surjective*, *bijective*) if its object map has this property, and (iii) an *isomorphism* if it is fully faithful and bijective. The *image* of  $\mathfrak{F}$ , written  $\mathfrak{F}(\mathcal{C})$ , is a class consisting of two subclasses: that of all objects  $\mathfrak{F}(X)$ ,  $X \in \mathcal{C}$ , and that of all morphisms  $\mathfrak{F}(f)$ ,  $f \in \operatorname{Mor}(\mathcal{C})$ . The image  $\mathfrak{F}(\mathcal{C})$  is a subcategory of  $\mathcal{D}$  if  $\mathfrak{F}$  is injective [47, sect. II.10]. A sequence

$$(A.6) \quad X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \longrightarrow 0$$

is called *exact* if  $f_2$  (or, more specifically,  $(f_2, X_3)$ ) is a cokernel of  $f_1$ . Dually, a sequence

$$(A.7) \quad X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \longleftarrow 0$$

is called exact if  $f_2$  (or, more specifically,  $(f_2, X_3)$ ) is a kernel of  $f_1$ . A covariant functor  $\mathfrak{F} : \mathcal{C} \rightarrow \mathcal{D}$  is *left exact* if exactness of (A.7) implies exactness of

$$(A.8) \quad \mathfrak{F}(X_1) \xleftarrow{\mathfrak{F}(f_1)} \mathfrak{F}(X_2) \xleftarrow{\mathfrak{F}(f_2)} \mathfrak{F}(X_3) \longleftarrow 0.$$

Dually, a contravariant functor  $\mathfrak{F} : \mathcal{C} \rightarrow \mathcal{D}$  is left exact if it is a left exact covariant functor  $\mathcal{C}^{op} \rightarrow \mathcal{D}$ . The bifunctor  $\text{Hom}_{\mathcal{C}}(\bullet, \bullet) : \mathcal{C} \times \mathcal{C} \rightarrow \mathbf{Ab}$  (where  $\mathbf{Ab}$  denotes the category of abelian groups and  $\mathbb{Z}$ -linear maps) is left exact in each argument, covariant in the first one and contravariant in the second one. An object  $X$  is called *projective* in  $\mathcal{C}$  if  $\text{Hom}_{\mathcal{C}}(X, \bullet)$  transforms epimorphisms into surjections and *injective* in  $\mathcal{C}$  if it is projective in  $\mathcal{C}^{op}$ . Assuming that  $\mathcal{C}$  has products and coproducts indexed by the set  $I$  (possibly filtering),

$$\text{Hom}_{\mathcal{C}}\left(\prod_{i \in I} X_i, Y\right) \cong \prod_{i \in I} \text{Hom}_{\mathcal{C}}(X_i, Y), \quad \text{Hom}_{\mathcal{C}}\left(X, \prod_{i \in I} Y_i\right) \cong \prod_{i \in I} \text{Hom}_{\mathcal{C}}(X, Y_i),$$

where the isomorphisms are canonical and thus are identifications. A coproduct in  $\mathcal{C}$  is a product in  $\mathcal{C}^{op}$ . We define the power (resp., the copower)  $U^I = \prod_{i \in I} U_i$  (resp.,  $U^{(I)} = \prod_{i \in I} U_i$ ),  $U_i = U$ . An object  $U$  of category  $\mathcal{C}$  with arbitrary coproducts is a *generator* in  $\mathcal{C}$  if the following equivalent conditions are satisfied: (i) every object of  $\mathcal{C}$  is isomorphic to a factor of a copower  $U^{(I)}$ ; (ii) for any nonzero right morphism  $f \bullet : X \rightarrow Y$ , there exists  $\alpha \bullet : U \rightarrow X$  such that  $f \alpha \neq 0$ ; (iii) the functor  $\text{Hom}_{\mathcal{C}}(U, \bullet)$  is faithful. An object  $W$  of a category  $\mathcal{C}$  with arbitrary products is called a *cogenerator* in  $\mathcal{C}$  if it is a generator in  $\mathcal{C}^{op}$ .

**A.2. Appendix 2: Lie groups and Lie algebras.** We refer to Bourbaki’s and Dieudonné’s treatises [8], [22] for an exhaustive presentation of Lie groups and Lie algebras; the convolution of distributions and the differential operators over Lie groups are expounded in the latter.

**A.2.1. Lie groups.** All Lie groups are real and locally compact in this paper. A Lie group  $\mathbf{G}$  is a real analytic manifold which is a group (multiplicatively noted in general, with neutral element  $e$ , and additively noted if  $\mathbf{G}$  is abelian, with neutral element 0) such that the map  $\mathbf{G} \times \mathbf{G} \ni (x, y) \mapsto x^{-1}y \in \mathbf{G}$  is analytic. Let  $a \in \mathbf{G}$ ; there exists a chart  $(V, \xi)$  such that  $V$  is an open connected neighborhood of  $a$ ,  $\xi : V \rightarrow \Omega$  is a diffeomorphism onto an open connected neighborhood  $\Omega$  of 0 in  $\mathbb{R}^n$  (where  $n$  is the dimension of the manifold  $\mathbf{G}$ ), and  $\xi(a) = 0$ ; such a chart is said to be centered at  $a$ . Let  $f : V \rightarrow \mathbf{k}$  (where  $\mathbf{k}$  is the field of real or complex numbers) be a  $C^\infty$  function. There exists a uniquely defined  $C^\infty$  function  $\phi : \Omega \rightarrow \mathbf{k}$  such that  $f(x) = \phi(\xi_1(x), \dots, \xi_n(x))$ , where the analytic functions  $\xi_i$  are the local coordinates (which can also be considered as variables). For any multi-index  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  and any  $x \in V$ , we set [12, Part I, sect. 2.2]

$$|\alpha| = \sum_{1 \leq i \leq n} \alpha_i, \quad \alpha! = \prod_{1 \leq i \leq n} \alpha_i!, \quad \frac{\partial^\alpha}{\partial \xi^\alpha} = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \dots \partial \xi_n^{\alpha_n}}, \quad D_\xi^\alpha f(x) = \frac{\partial^\alpha \phi}{\partial \xi^\alpha}(\xi(x)).$$

**A.2.2. Distributions with compact support.** Let  $\mathcal{E}(\mathbf{G})$  be the space of all  $C^\infty$  functions  $\mathbf{G} \rightarrow \mathbf{k}$ , endowed with its usual Fréchet topology of compact convergence [22, (17.3.1)]. The space  $\mathcal{E}'(\mathbf{G})$  of all continuous linear forms  $\mathcal{E}(\mathbf{G}) \rightarrow \mathbf{k}$  is the space of distributions with compact support over  $\mathbf{G}$ . For  $T \in \mathcal{E}'(\mathbf{G})$  and  $\varphi \in \mathcal{E}(\mathbf{G})$ ,  $T(\varphi)$  is written

$$(A.9) \quad \langle T, \varphi \rangle = \langle \varphi, T \rangle^{\sim}$$

or, with a slight abuse of notation [22, (17.3.8)],

$$(A.10) \quad \int_{\mathbf{G}} dT(x) \varphi(x) = \int_{\mathbf{G}} \varphi(x) dT(x).$$

Let  $(V, \xi)$  be a chart centered at  $a \in \mathbf{G}$ ; the restriction  $T|_V$  is the linear form  $\varphi \mapsto T(\varphi)$ ,  $\varphi \in \mathcal{E}(V)$ , and its derivative with multi-index  $\alpha$  is expressed in function of  $(V, \xi)$  by  $\langle D_\xi^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D_\xi^\alpha \varphi \rangle$ .

Let  $T, S \in \mathcal{E}'(\mathbf{G})$ ; their convolution product  $T \star S \in \mathcal{E}'(\mathbf{G})$  is the linear form  $\mathcal{E}(\mathbf{G}) \ni \varphi \mapsto \langle T \otimes S, \varphi m \rangle \in \mathbf{k}$ , where  $m$  is the map  $(x, y) \mapsto xy$ . By (A.10), this can also be written [22, (14.5.4) and (17.10.3)]

$$\int_{\mathbf{G}} \varphi d(T \star S) = \int_{\mathbf{G}} \left( \int_{\mathbf{G}} \varphi(xy) dT(x) \right) dS(y) = \int_{\mathbf{G}} dS(y) \int_{\mathbf{G}} dT(x) \varphi(xy).$$

Let  $a \in \mathbf{G}$  and  $\delta_a : \varphi \mapsto \varphi(a)$  be the Dirac distribution at point  $a$ . From the above,

$$(A.11) \quad D_\xi^\alpha \varphi(a) = (-1)^{|\alpha|} \langle D_\xi^\alpha \delta_a, \varphi \rangle,$$

$$(A.12) \quad \langle \delta_a \star S, \varphi \rangle = \langle S, \varphi(a \bullet) \rangle, \quad \langle T \star \delta_a, \varphi \rangle = \langle T, \varphi(\bullet a) \rangle,$$

where  $\varphi(a \bullet)$  and  $\varphi(\bullet a)$ , respectively, denote the functions  $x \mapsto \varphi(ax)$  and  $x \mapsto \varphi(xa)$ .

The convolution algebra  $(\mathcal{E}'(\mathbf{G}), \star)$  is associative with unit element  $\delta_e$  and is commutative if and only if  $\mathbf{G}$  is commutative too. This ring is an integral domain in the classical case  $\mathbf{G} = \mathbb{R}^n$  according to the Paley–Wiener–Schwartz theorem [22, (22.18.7)] but not in general. For example, let  $\mathbb{T}$  be the torus and  $C$  a nonzero constant; then  $0 \neq C \in \mathcal{E}'(\mathbb{T})$ ,  $\delta_0 \neq 0$ , and  $\delta_0 \star C = 0$ .

Let  $dx$  be a left Haar measure and for any function  $f \in \mathcal{E}(\mathbf{G})$  consider the distribution  $T = f dx$ . The support of  $T$  is not compact in general, but for any  $S \in \mathcal{E}'(\mathbf{G})$ , the convolution products  $T \star S$  and  $S \star T$  exist and are absolutely continuous with respect to  $dx$ , with density function  $g$  (resp.,  $h$ ) belonging to  $\mathcal{E}(\mathbf{G})$ . Identifying  $f \star S$  with  $g$  and  $S \star f$  with  $h$  one obtains [22, (14.8.2) and (14.8.4)]

$$(A.13) \quad (S \star f)(a) = \int_{\mathbf{G}} f(s^{-1}a) dS(s), \quad (f \star S)(a) = \int_{\mathbf{G}} f(as^{-1}) \Delta(s^{-1}) dS(s)$$

and in particular

$$\delta_a \star f = f(a^{-1} \bullet), \quad f \star \delta_a = \Delta(a^{-1}) f(\bullet a^{-1}),$$

where  $\Delta$  is the *modular function* which is positive [22, sect. XIV.3] and analytic [8, subsect. III.3.16, Prop. 56]. The operator  $f(\bullet) \mapsto f(a \bullet)$  (resp.,  $f(\bullet) \mapsto f(\bullet a)$ ) is the *left* (resp., *right*) *shift* denoted by  $\gamma(a^{-1})$  (resp.,  $\delta(a)$ ) [22, (14.1.1)]. Let  $a \in \mathbf{G}$ , let  $(V, \xi)$  be a chart centered at  $e$ , and let  $f \in \mathcal{E}(\mathbf{G})$ ; as easily shown, for any  $x \in aV$ ,

$$D_\xi^\alpha \delta_a \star f(x) = D_\xi^\alpha f(a^{-1}x).$$

Denoting by  $\mathcal{T}_x^\infty(\mathbf{G})$  the  $\mathbf{k}$ -vector space consisting of all  $\mathbf{k}$ -valued distributions with support included in  $\{x\}$  ( $x \in \mathbf{G}$ ) and by  $\mathcal{T}^\infty(\mathbf{G})$  the  $\mathbf{k}$ -vector space consisting of all punctual  $\mathbf{k}$ -valued distributions (i.e., all  $\mathbf{k}$ -valued distributions with finite support),  $\mathcal{T}_e^\infty(\mathbf{G})$  and  $\mathcal{T}^\infty(\mathbf{G})$  are subalgebras of  $(\mathcal{E}'(\mathbf{G}), \star)$ , and  $\mathcal{T}^\infty(\mathbf{G})$  is commutative if and only if  $\mathbf{G}$  is commutative too. In addition,

$$(A.14) \quad \mathcal{T}^\infty(\mathbf{G}) = \bigoplus_{x \in \mathbf{G}} \mathcal{T}_x^\infty(\mathbf{G})$$

and  $\mathcal{T}_x^\infty(\mathbf{G}) = \delta_x \star \mathcal{T}_e^\infty(\mathbf{G}) = \mathcal{T}_e^\infty(\mathbf{G}) \star \delta_x$  [8, subsect. III.3.1].

**A.2.3. Lie algebra of a Lie group.** Let  $\mathfrak{g} \triangleq T_e(\mathbf{G})$  be the tangent space of  $\mathbf{G}$  at  $e$ . Consider the  $n$  first order elementary differential operators

$$X_i = \partial_i|_e : f \rightarrow D_\xi^\alpha f(e), \quad \alpha_j = \delta_{ij}, \quad i, j \in \{1, \dots, n\}$$

(where  $\delta_{ij}$  is the Kronecker symbol and  $(V, \xi)$  is a chart centered at  $e$ ) and set  $X = (X_1, \dots, X_n)$ ;  $X$  is a  $\mathbf{k}$ -basis of  $\mathfrak{g}$ . By (A.11), for any  $i \in \{1, \dots, n\}$   $X_i$  can be identified with the  $\mathbf{k}$ -valued distribution  $(-1)^{|\alpha|} D_\xi^\alpha \delta_e$ , where  $\alpha \in \mathbb{N}^n$  is the above multi-index; then  $\mathfrak{g}$  is identified with the  $\mathbf{k}$ -vector space consisting of all  $\mathbf{k}$ -valued distributions of order 1 with support  $\{e\}$ . By [22, (17.14.2)], for any  $Y, Z \in \mathfrak{g}$ ,

$$[Y, Z] \triangleq Y \star Z - Z \star Y \in \mathfrak{g}$$

and the bilinear map  $[\bullet, \bullet] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  is a Lie bracket;  $\mathfrak{g}$ , endowed with this bracket, is called the Lie algebra of  $\mathbf{G}$  [22, (19.3.3)]. An ideal of  $\mathfrak{g}$  is a  $\mathbf{k}$ -vector space  $\mathfrak{a} \subseteq \mathfrak{g}$  such that  $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$ . The Lie algebra  $\mathfrak{g}$  is called *simple* if it is noncommutative and if its only proper ideal is  $\{0\}$ , *semisimple* if its only abelian ideal is  $\{0\}$ , *nilpotent* if there exists a finite decreasing sequence of ideals  $(\mathfrak{g}_i)_{0 \leq i \leq p}$  such that  $\mathfrak{g}_0 = \mathfrak{g}$ ,  $\mathfrak{g}_p = \{0\}$  with  $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+1}$  ( $0 \leq i \leq p - 1$ ), and *solvable* if there exists a sequence

$$(A.15) \quad \mathfrak{g}_0 = \mathfrak{g} \supsetneq \mathfrak{g}_1 \supsetneq \dots \supsetneq \mathfrak{g}_n = \{0\}$$

of Lie subalgebras such that  $\mathfrak{g}_i$  is a one-codimensional ideal in  $\mathfrak{g}_{i-1}$ ,  $1 \leq i \leq n$  [8, subsect. I.5.1, Prop. 2]. All nilpotent Lie algebras are solvable.

Assuming that the Lie group  $\mathbf{G}$  is connected, it is called semisimple (resp., nilpotent, solvable) if its Lie algebra is semisimple (resp., nilpotent, solvable).

**A.2.4. The universal enveloping algebra and its center.** More generally, given any associative  $\mathbf{k}$ -algebra  $A$ , one can endow  $A$  with a  $\mathbf{k}$ -Lie algebra structure by defining the Lie bracket  $[a, b] = ab - ba$  ( $a, b \in A$ ). A *representation* of  $\mathfrak{g}$  is defined to be a  $\mathbf{k}$ -Lie algebra morphism from  $\mathfrak{g}$  to such  $A$ . There exists an associative  $\mathbf{k}$ -algebra  $U_{\mathbf{k}}(\mathfrak{g})$  (denoted by  $U(\mathfrak{g})$  if  $\mathbf{k} = \mathbb{R}$  and  $U_{\mathbb{C}}(\mathfrak{g})$  if  $\mathbf{k} = \mathbb{C}$ ) together with a representation  $\theta : \mathfrak{g} \rightarrow U_{\mathbf{k}}(\mathfrak{g})$  with the following universal property: for any associative  $\mathbf{k}$ -algebra representation  $\varphi : \mathfrak{g} \rightarrow A$ , there exists a unique  $\mathbf{k}$ -algebra morphism  $\psi : U_{\mathbf{k}}(\mathfrak{g}) \rightarrow A$  such that  $\psi\theta = \varphi$ . Thus  $U_{\mathbf{k}}(\mathfrak{g})$  is unique up to isomorphism and is called the *universal enveloping algebra* of  $\mathfrak{g}$ . Consider the *standard monomials*

$$\{X_{i_1} X_{i_2} \dots X_{i_n} : i_j \in \{1, \dots, n\}, \quad i_1 \leq \dots \leq i_n\}.$$

According to the Poincaré–Birkhoff–Witt theorem, these standard monomials form a  $\mathbf{k}$ -basis of  $U_{\mathbf{k}}(\mathfrak{g})$ ;  $U_{\mathbf{k}}(\mathfrak{g})$  is a skew polynomial ring and a noncommutative left and right Noetherian integral domain [46, section 1.7].

If  $\mathfrak{g}$  is the Lie algebra of a Lie group  $\mathbf{G}$ , then  $U_{\mathbf{k}}(\mathfrak{g})$  is isomorphic to, and identified with,  $\mathcal{T}_e^\infty(\mathbf{G})$ . The dual  $U_{\mathbf{k}}^*(\mathfrak{g}) \triangleq \text{Hom}_{\mathbf{k}}(U_{\mathbf{k}}(\mathfrak{g}), \mathbf{k})$  is isomorphic to the power series ring  $\mathbf{k}[[x]]$  (where  $x = (x_i)_{1 \leq i \leq n}$  is a sequence of  $n$  commuting indeterminates) and, according to E. Borel’s theorem [65, Theorem 38.1], there exists a canonical epimorphism  $\mathcal{E}(\mathbf{G}) \twoheadrightarrow U_{\mathbf{k}}^*(\mathfrak{g}) : f \mapsto \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} (X^\alpha f) x^\alpha$ , called the *Borel epimorphism*.

Let  $P : \mathcal{E}(\mathbf{G}) \rightarrow \mathcal{E}(\mathbf{G})$  be a differential operator [22, sect. XVII.13]. Such an operator is called *left-* (resp., *right-*) *invariant* if for any  $a \in \mathbf{G}$  and  $\varphi \in \mathcal{E}(\mathbf{G})$ ,

$$P\varphi = \gamma(a)(P\gamma(a^{-1})\varphi) \quad (\text{resp., } P\varphi = \delta(a)(P\delta(a^{-1})\varphi)).$$

For short, a differential operator is called *invariant* when it is left-invariant and *bi-invariant* when it is both left- and right-invariant. The algebra of all invariant (resp., bi-invariant) differential operators is identified with  $U_{\mathbf{k}}(\mathfrak{g})$  (resp.,  $Z_{\mathbf{k}}(\mathfrak{g})$ , i.e., the center of  $U_{\mathbf{k}}(\mathfrak{g})$ , which is a commutative domain) [22, (19.3.1) and (23.36.6)].

If  $\mathfrak{g}$  is semisimple, then  $Z_{\mathbf{k}}(\mathfrak{g})$  is the polynomial ring  $\mathbf{k}[y_1, \dots, y_l]$ , where the natural integer  $l$  is the “rank” of  $\mathfrak{g}$  [23, sections 1.9.8 and 7.3.8]; thus  $Z_{\mathbf{k}}(\mathfrak{g})$  is Noetherian. The case of solvable Lie algebras is quite different since there exists a nilpotent Lie algebra  $\mathfrak{g}$  of dimension 45 such that  $Z_{\mathbf{k}}(\mathfrak{g})$  is not Noetherian [23, section 4.9.20].

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