Design, parametrization, and pole placement of stabilizing output feedback compensators via injective cogenerator quotient signal modules

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Abstract

Control design belongs to the most important and difficult tasks of control engineering and has therefore been treated by many prominent researchers and in many textbooks, the systems being generally described by their transfer matrices or by Rosenbrock equations and more recently also as behaviors. Our approach to controller design uses, in addition to the ideas of our predecessors on coprime factorizations of transfer matrices and on the parametrization of stabilizing compensators, a new mathematical technique which enables simpler design and also new theorems in spite of the many outstanding results of the literature: (1) We use an injective cogenerator signal module $\mathscr{F}$ over the polynomial algebra $\mathscr{D} = F[s]$ (F an infinite field), a saturated multiplicatively closed set T of stable polynomials and its quotient ring $\mathscr{D}_T$ of stable rational functions. This enables the simultaneous treatment of continuous and discrete systems and of all notions of stability, called T-stability. We investigate stabilizing control design by output feedback of input/output (IO) behaviors and study the full feedback IO behavior, especially its autonomous part and not only its transfer matrix. (2) The new technique is characterized by the permanent application of the injective cogenerator quotient signal module ${}_{\mathscr{D}_T}\mathscr{F}_T$ and of quotient behaviors $\mathscr{B}_T$ of ${}_{\mathscr{D}}\mathscr{F}$-behaviors $\mathscr{B}$. (3) For the control tasks of tracking, disturbance rejection, model matching and decoupling and not necessarily proper plants we derive necessary and sufficient conditions for the existence of proper stabilizing compensators with proper and stable closed loop behaviors, parametrize all such compensators as IO behaviors and not only their transfer matrices and give new algorithms for their construction. Moreover we solve the problem of pole placement or spectral assignability for the complete feedback behavior. The properness of the full feedback behavior ensures the absence of impulsive solutions in the continuous case, and that of the compensator enables its realization by Kalman state space equations or elementary building blocks. We note that every behavior admits an IO decomposition with proper transfer matrix, but that most of these decompositions do not have this property, and therefore we

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do not assume the properness of the plant. (4) The new technique can also be applied to more general control interconnections according to Willems, in particular to two-parameter feedback compensators and to the recent tracking framework of Fiaz/Takaba/Trentelman. In contrast to these authors, however, we pay special attention to the properness of all constructed transfer matrices which requires more subtle algorithms.

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1 Introduction

The present paper is an elaboration of the MTNS 2010 paper [6].

Problems of control design have always been of central interest in systems theory and have been investigated by many prominent researchers, among them Antsaklis and Michel [1, Ch.7, Part 2, pp.589-634], Bengtsson, Blomberg and Ylinen [3], Bourlès [7], Callier and Desoer [8, Chs.7,9, pp.196-242], Chen [9, Ch.9, pp.458-534], Falb, Feintuch and Saeks [10], Francis, Kailath [13, Sec.7.5, pp.532-538], Khargonekar, Kučera [14], Murray, Pearson, Pernebo [19], Schneider, Vardulakis [26, Ch.7, pp.335-354], Vidyasagar [27, Sec.5.7, Sec.7.5, pp.294-317], Wolovich [29, Ch.8, pp.269-323], Wonham [30], Youla, Zames, their coauthors and many other contributors. We refer to the quoted books for history, origin and development of the decisive ideas of control design which is generally described in difficult advanced chapters of these books. Due to the large number of researchers and original papers on control design we only refer to the books where these papers are quoted, used, and elaborated and to some newer papers on behavioral stabilization. *We present a new technique for controller design which enables both simpler proofs and new theorems in spite of the many outstanding results of the literature*, but we also use the ideas of our predecessors on coprime factorizations of transfer matrices and parametrization of stabilizing compensators. For observer constructions the corresponding work was done in [4] after Fuhrmann's authoritative survey article [12]. Our approach to the problems of the title is distinguished by the following *original features*:

1. We use an injective cogenerator signal module $\mathscr{F}$ over a polynomial algebra $\mathscr{D} = F[s]$ (F an infinite field) of differential or difference operators with the action $d \circ y$, $d \in \mathscr{D}$, $y \in \mathscr{F}$, and define T-stability and T-stabilization with respect to a *saturated multiplicatively closed subset or submonoid $T \subseteq \mathscr{D} \setminus \{0\}$ of stable polynomials*. This enables the simultaneous discussion of discrete and continuous systems and of different stability notions, in particular of all those discussed in [12]. An input/output (IO) behavior is T-stable if its autonomous part and its transfer matrix have this property. We investigate stabilization by output feedback and control design for ${}_{\mathscr{D}}\mathscr{F}$-IO *behaviors* instead of Rosenbrock systems or transfer matrices which are mostly used in the literature (see item 6) and pay special attention to the autonomous part of the IO feedback behavior and not only to its transfer matrix. We note that an injective and faithful ($d \circ \mathscr{F} = 0 \Longrightarrow d = 0$) signal module $\mathscr{F}$ is called *regular* in [3, Def.3 on p.81]. The signal module ${}_{F[s]}F(s)$ is regular, but not a cogenerator. The duality between equation modules and behaviors is valid for injective cogenerators, but not for regular signal modules.

2. The signal module ${}_{\mathscr{D}}\mathscr{F}$ gives rise to its quotient module $\mathscr{F}_T := \left\{\frac{y}{t};\, y \in \mathscr{F}, t \in T\right\}$ over the quotient ring $\mathscr{D}_T := \left\{\frac{d}{t};\, d \in \mathscr{D}, t \in T\right\} (\subseteq F(s))$ of *stable rational functions*

and to the direct sum decomposition $\mathscr{F} \cong \mathscr{F}_T \oplus \mathrm{t}_T(\mathscr{F})$ where ${}_{\mathscr{D}_T}\mathscr{F}_T$ is again an *injective cogenerator* with its own behavioral systems theory and where $\mathrm{t}_T(\mathscr{F})$ is the T-torsion submodule of T-*small* or T-*negligible* signals [18], [4]. Every behavior $\mathscr{B} \subseteq \mathscr{F}^q$ admits a corresponding direct sum decomposition $\mathscr{B} \cong \mathscr{B}_T \oplus \mathrm{t}_T(\mathscr{B})$ into the quotient ${}_{\mathscr{D}_T}\mathscr{F}_T$-behavior $\mathscr{B}_T$ and its T-small (T-negligible, T-autonomous) part $\mathrm{t}_T(\mathscr{B})$. The consideration of the ${}_{\mathscr{D}_T}\mathscr{F}_T$-behaviors $\mathscr{B}_T$ signifies to study ${}_{\mathscr{D}}\mathscr{F}$-behaviors *up to T-negligible ones*. A transfer matrix $H \in \mathscr{D}_T^{p\times m}$ of T-stable rational functions gives rise to the IO *operator* $H\circ : \mathscr{F}_T^m \to \mathscr{F}_T^p$, $u \mapsto y := H \circ u$, which plays an essential part in our derivations. We note that the widely used subring $\mathscr{S} \subset \mathscr{D}_T$ of *proper and T-stable* rational functions also acts on $\mathscr{F}_T$, but *not on $\mathscr{F}$. The use of quotient modules and especially of the injective cogenerator quotient signal module ${}_{\mathscr{D}_T}\mathscr{F}_T$ and the quotient behaviors $\mathscr{B}_T$ enables relatively short and conceptual proofs of all results on control design.*
3. Like all IO behaviors every considered plant $\mathscr{B}_1$ has a rational transfer matrix H_1. We do *not* assume that $\mathscr{B}_1$, i.e., H_1, is proper and can therefore admit arbitrary decompositions of the variables of $\mathscr{B}_1$ into input and output components. *In contrast we only consider proper IO compensators $\mathscr{B}_2$ such that the output feedback IO behavior* $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ *is proper and T-stable*. The properness of $\mathscr{B}_2$ enables its realization by Kalman equations or elementary building blocks while that of $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ ensures the absence of impulsive solutions in the continuous case. *For the control tasks of tracking, disturbance rejection, model matching and decoupling and not necessarily proper plants we derive necessary and sufficient conditions for the existence of proper stabilizing compensators with proper closed loop behaviors, parametrize all such compensators as IO behaviors and not only their transfer matrices and give new algorithms for their construction*. For a plant $\mathscr{B}_1$ in state space form we also obtain all possible T-stabilizing compensators and their feedback behavior in the same form. The parametrization of all not necessarily proper controllers, but with stable and proper feedback behavior is *considerably simpler* and derived in Thm. 3.12.
4. *The generality of the monoid T also permits to solve the problem of spectral assignability or pole placement for the considered control tasks constructively*: Under a necessary and sufficient condition on the plant $\mathscr{B}_1$ and the other data a least monoid $T_{\min}$ can be constructed for which a $T_{\min}$-stabilizing compensator $\mathscr{B}_2$ for the intended control task exists. This $T_{\min}$ is finitely generated up to units. The finitely many roots of the polynomials in $T_{\min}$ are then unavoidable as possible poles of the closed loop behavior. Any finite or infinite set of complex numbers which contains these unavoidable poles can be prescribed for the location of the closed loop poles.
5. New algorithms for the construction of all proper compensators $\mathscr{B}_2$ as described above are presented and exhibited in an example.
6. Comparison with the *behavioral* control interconnection literature: More general regular interconnections of plant and controller have been discussed by several authors from the behavioral point of view, for instance in [28],[2],[24],[21]. The latter paper [21], for instance, parametrizes the set of all regularly implementing, partially interconnected controllers for which the manifest controlled behavior is autonomous and stable. Since an autonomous behavior has no transfer matrix such matrices, their properness and use in control design as in [9], [8], [27] and in the present paper are, of course, not discussed in [21]. While our full feedback behavior is proper and stable as IO system which is necessary for the proper functioning of any machine realization the stability of the full interconnected behavior is not a subject of [21]. The newest paper [11] also treats control tasks in this framework. In Blumthaler's forthcoming thesis

our new technique is also applied to other control configurations like those in [3, pp. 187-189], [27, §6.7] (two-parameter compensators), [20, §10.8], [21], [11]. In contrast to the quoted references for the behavioral framework, appropriate transfer matrices and their properness and stability still play an important part in these considerations. Multidimensional proper stabilization was already treated in [18] and [25].
One reviewer has pointed out the importance of robustness and in particular the internal model principle as discussed, for instance, in [30, Ch.8], [9], [27, §7.5], [7, §9.3]. We agree, but have presently only limited insight into this problem and therefore postpone its study to the future. This has to start with the definition of a metric in the set of IO behaviors and especially in the set of compensators which realize different control tasks.
The plan of the paper is the following: In Section 2 we introduce the main data and explain the connection of the standard coprime factorizations and Bezout equations with the also standard split module sequences according to [16]. In systems theory this simple connection was observed by A. Quadrat [22], [23], for instance. Section 3 treats stabilization by output feedback with proper compensator and proper feedback behavior, but not necessarily proper plant and develops the new technique of injective cogenerator quotient signal module as far as needed later on. The construction of *all proper* compensators and the spectral assignability problem require extensive considerations. The main results of this paper on tracking, disturbance rejection, model matching and decoupling are contained in Sections 4 - 6. Section 7 contains the algorithms that make the results constructive. The paper concludes with a worked-out example in Section 8.

2 Preliminaries

The general situation which we consider is the same as in [5] and [4], and so are the mathematical techniques we apply.

Let $\mathscr{D}$ denote the polynomial ring $F[s]$ over some infinite field F, $\mathscr{K} := \mathrm{quot}(\mathscr{D}) = F(s)$ its quotient field, and let $\mathscr{F}$ be an injective cogenerator over $\mathscr{D}$. Later $\mathscr{D}$ will be the ring of operators (differential or difference operators in the standard cases), and $\mathscr{F}$ the signal module. The standard choices are the following: $F = \mathbb{R}, \mathbb{C}$, $\mathscr{F} = \mathscr{C}^\infty(\mathbb{R}, F)$ or $\mathscr{F} = \mathscr{D}'(\mathbb{R}, F)$ (continuous standard cases) or $\mathscr{F} = F^{\mathbb{N}}$ (discrete standard case). The action of the indeterminate s on a signal in $\mathscr{F}$ is defined as differentiation in the continuous cases and as left shift in the discrete case.

Furthermore, let T be a multiplicatively closed subset or submonoid of $\mathscr{D} \setminus \{0\}$ which we always assume saturated. The elements of T are called *T-stable polynomials*. As usual $\mathscr{D}_T$ denotes the quotient ring of $\mathscr{D}$ w.r.t. T (also referred to as the localization of $\mathscr{D}$ w.r.t. T or as the ring of *T-stable rational functions*), i.e.,

$$\mathscr{D}_T \quad = \quad \left\{ \frac{d}{t} \in F(s); d \in \mathscr{D}, t \in T \right\} \subseteq F(s). \tag{1}$$

More generally, for any $\mathscr{D}$-module M we consider the quotient module $M_T = \left\{ \frac{x}{t}; x \in M, t \in T \right\}$ which is a $\mathscr{D}_T$-module in the natural fashion, compare [15, Sec.II.3], [5, p.2424]. In particular we will need quotient modules U_T of row modules $U \subseteq \mathscr{D}^{1\times\ell}$, the quotient module $\mathscr{F}_T$ of the signal module $\mathscr{F}$ (which is an injective cogenerator over $\mathscr{D}_T$) [4, Sec.1], and quotient modules $\mathscr{B}_T$ of $\mathscr{F}$-behaviors $\mathscr{B}$ [4, Thm.1.8, Cor.1.9]. We will subsequently use the properties of $\mathscr{F}_T$ and $\mathscr{B}_T$ derived in [4, Sec.1]. We briefly repeat the terms T-autonomy and T-stability introduced in [5, Thm. and Def.2.15]:

Definition 2.1 (T-autonomy, T-small signals, T-stability)**.** 1. A behavior

$$\mathscr{B} = \left\{ w \in \mathscr{F}^{\ell};\, R \circ w = 0 \right\}$$

where $R \in \mathscr{D}^{k\times\ell}$ is called T*-autonomous* if there exists $t \in T$ such that $t \circ \mathscr{B} = 0$. This is equivalent to $\mathscr{B}_T = 0$ or to the existence of a left inverse matrix of R in $\mathscr{D}_T^{\ell\times k}$ (cf. [4, Thm.1.9.3]). Signals $w \in \mathscr{F}^{\ell}$ which are annihilated by some $t \in T$ are called T*-small*.

2. An input/output (IO) behavior [20, Sec.3.3], [17, Thm.2.69, p.37]

$$\mathscr{B} = \left\{ \binom{y}{u} \in \mathscr{F}^{p+m};\, P \circ y = Q \circ u \right\},$$

$(P, -Q) \in \mathscr{D}^{p\times(p+m)}$, $\det(P) \neq 0$, is called T*-stable* if its autonomous part $\mathscr{B}^0 := \{y \in \mathscr{F}^p;\, P \circ y = 0\}$ is T-autonomous.

Example 2.2. 1. Assume that $F = \mathbb{R}$, $\Lambda \subseteq \mathbb{C}$ such that Λ is equal to its complex conjugate $\overline{\Lambda}$, and $T := \{t \in \mathbb{R}[s] \setminus \{0\};\, \mathrm{V}_{\mathbb{C}}(t) \subseteq \Lambda\}$ where $\mathrm{V}_{\mathbb{C}}(t) := \{\lambda \in \mathbb{C};\, t(\lambda) = 0\}$ denotes the vanishing set of t in $\mathbb{C}$.

2. In particular, if we choose $\Lambda := \{\lambda \in \mathbb{C};\, \Re(\lambda) < 0\}$ in the continuous standard case resp. $\Lambda := \{\lambda \in \mathbb{C};\, |\lambda| < 1\}$ in the discrete standard case then a signal is T-small if and only if it is polynomial-exponential and asymptotically zero for $t \to \infty$. For other examples compare, e.g., [5, Ex.2.16].

In the subsequent sections the following two lemmas will be basic tools:

Lemma 2.3. *Let R be a commutative ring and $A_1 \in R^{p\times\ell}$, $B_1 \in R^{\ell\times m}$ such that*

$$0 \longrightarrow R^{1\times p} \xrightarrow{\circ A_1} R^{1\times\ell} \xrightarrow{\circ B_1} R^{1\times m} \longrightarrow 0 \tag{2}$$

is exact (especially $\ell = p + m$). Then the following assertions hold:

1. *There are a left inverse $A_2^0 \in R^{m\times\ell}$ of B_1, $A_2^0 B_1 = \mathrm{id}_m$, and a right inverse $B_2^0 \in R^{\ell\times p}$ of A_1, $A_1 B_2^0 = \mathrm{id}_p$, such that*

$$0 \longleftarrow R^{1\times p} \xleftarrow{\circ B_2^0} R^{1\times\ell} \xleftarrow{\circ A_2^0} R^{1\times m} \longleftarrow 0 \tag{3}$$

is exact too. Then $\circ A_2^0$ resp. $\circ B_2^0$ is called a section of $\circ B_1$ resp. a retraction of $\circ A_1$, and both sequences (2) *and* (3) *are split exact.*

2. *There are canonical bijections*

$$\begin{array}{cc}
\left\{U_2 \subseteq R^{1\times\ell};\, R^{1\times p}A_1 \oplus U_2 = R^{1\times\ell}\right\} & U_2 \\
\cong & \updownarrow \\
\left\{B_2 \in R^{\ell\times p};\, A_1 B_2 = \mathrm{id}_p\right\} & B_2 \\
\cong & \updownarrow \\
\left\{A_2 \in R^{m\times\ell};\, A_2 B_1 = \mathrm{id}_m\right\} & A_2 \\
\cong & \updownarrow \\
R^{m\times p} & X
\end{array}$$

with $U_2 = \ker(\circ B_2) = R^{1\times m}A_2$, $B_2 = B_2^0 - B_1 X$, $A_2 = A_2^0 + X A_1$. Then

$$0 \longleftarrow R^{1\times p} \xleftarrow{\circ B_2} R^{1\times\ell} \xleftarrow{\circ A_2} R^{1\times m} \longleftarrow 0 \tag{4}$$

is (split) exact too, and

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} (B_2, B_1) = \begin{pmatrix} \mathrm{id}_p & 0 \\ 0 & \mathrm{id}_m \end{pmatrix} = \mathrm{id}_{p+m}\,.$$

Proof. Assertion 1 and the first two bijections of assertion 2 follow from [16, Prop.I.4.1-I.4.3]. The last bijection in 2 follows from the equivalences

$$\begin{aligned} & A_2B_1 = \mathrm{id}_m = A_2^0 B_1 \quad \Leftrightarrow \quad (A_2 - A_2^0)B_1 = 0 \quad \Leftrightarrow \\ \Leftrightarrow \quad & R^{1\times m}(A_2 - A_2^0) \subseteq \ker(\circ B_1) = \mathrm{im}(\circ A_1) = R^{1\times p}A_1 \quad \Leftrightarrow \\ \Leftrightarrow \quad & \exists X \in R^{m\times p} : A_2 - A_2^0 = XA_1, \quad \text{i.e.,} \quad A_2 = A_2^0 + XA_1. \end{aligned}$$

□

The parameter X in the preceding lemma furnishes the parametrization of stabilizing compensators according to Kučera and Youla et al. The direct sum decompositions were introduced by Quadrat [22], [23] in this context, but were also considered by Rocha and Wood [24] in context with regular interconnections (according to Willems) and set-controllability. Behavioral direct sum decompositions were also discussed by Bisiacco, Bourlès, Fliess, Lomadze, Valcher, Zerz et al.

Lemma 2.4 (Coprime factorizations, controllable realizations)**.** *Let R denote a principal ideal domain with quotient field $K := \mathrm{quot}(R)$. Assume a matrix $H \in K^{p\times m}$.*

1. *There exists an essentially unique (i.e., unique up to row equivalence over R) matrix $(P, -Q) \in R^{p\times(p+m)}$ which satisfies the following equivalent conditions with $U := R^{1\times p}(P, -Q)$:*

   (a) *The sequence*

   $$0 \longrightarrow R^{1\times p} \xrightarrow{\circ(P,-Q)} R^{1\times(p+m)} \xrightarrow{\circ\left(\begin{smallmatrix} H \\ \mathrm{id}_m \end{smallmatrix}\right)} K^{1\times m}$$

   is exact.

   (b) i. *$PH = Q$, i.e., $(P, -Q)\left(\begin{smallmatrix} H \\ \mathrm{id}_m \end{smallmatrix}\right) = 0$, and*

   ii. *$(P, -Q)$ has a right inverse in $R^{(p+m)\times p}$, i.e.,*
   $\mathrm{rank}(P, -Q) = \dim_K(KU) = p$ *and U is a direct summand of $R^{1\times(p+m)}$ or*
   $\dim_K(KU) = p$ *and the elementary divisors of U (or $(P, -Q)$) are units in R.*

   In this case $R^{1\times p}P = \left\{\xi \in R^{1\times p};\ \xi H \in R^{1\times m}\right\}$, $\det(P) \neq 0$, and $H = P^{-1}Q$. The representation $H = P^{-1}Q$ is called a left coprime factorization (l.c.f.) *and $(P, -Q)$ the* controllable realization *of H over R.*

2. *Likewise, there is an essentially unique (i.e., unique up to column equivalence) matrix $\left(\begin{smallmatrix} N \\ D \end{smallmatrix}\right) \in R^{(p+m)\times m}$ such that $HD = N$ and $\left(\begin{smallmatrix} N \\ D \end{smallmatrix}\right)$ has a left inverse in $R^{m\times(p+m)}$, i.e.,*

   $$0 \longrightarrow R^m \xrightarrow{\left(\begin{smallmatrix} N \\ D \end{smallmatrix}\right)\circ} R^{p+m} \xrightarrow{(\mathrm{id}_p, -H)\circ} K^p$$

   is exact. Then $\det(D) \neq 0$ and $H = ND^{-1}$ is called a right coprime factorization (r.c.f.) *of H over R.*

3. *Let* $(P, -Q) \in R^{p\times(p+m)}$, $\det(P) \neq 0$, $\binom{N}{D} \in R^{(p+m)\times m}$, $\det(D) \neq 0$, *such that* $H = P^{-1}Q = ND^{-1}$. *Then*

$$0 \longrightarrow R^{1\times p} \xrightarrow{\circ(P,-Q)} R^{1\times(p+m)} \xrightarrow{\circ\binom{N}{D}} R^{1\times m} \longrightarrow 0$$

is exact (and thus Lemma 2.3 is applicable to it) if and only if $H = P^{-1}Q$ *is a left coprime factorization and* $H = ND^{-1}$ *is a right coprime factorization of* H *over* R.

4. *If* $(P, -Q)$ *resp.* $\binom{N}{D}$ *satisfies the conditions in 1 resp. 2 for the ring* R *this is also the case for any overring* R', $R \subseteq R' \subseteq K$.

3 Feedback systems and stabilizing compensators

We consider two input/output (IO) behaviors [20, Sec.3.3], [17, Thm.2.69, p.37], [5, p.2419]

$$\begin{aligned} \mathscr{B}_1 &= \left\{ \begin{pmatrix} y_1 \\ u_1 \end{pmatrix} \in \mathscr{F}^{p+m}; P_1 \circ y_1 = Q_1 \circ u_1 \right\}, \\ \mathscr{B}_2 &= \left\{ \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \in \mathscr{F}^{p+m}; P_2 \circ y_2 = Q_2 \circ u_2 \right\} \end{aligned}$$

where $(P_1, -Q_1) \in \mathscr{D}^{p\times(p+m)}$, $\det(P_1) \neq 0$, $(-Q_2, P_2) \in \mathscr{D}^{m\times(p+m)}$, $\det(P_2) \neq 0$, with associated modules of equations

$$U_1 = \mathscr{D}^{1\times p}(P_1, -Q_1), \quad U_2 = \mathscr{D}^{1\times m}(-Q_2, P_2).$$

Recall that (Kalman) state space equations give rise to IO behaviors by elimination of the state [20, Ch.6], [17, p.27].

Definition 3.1 (Feedback behavior). The *feedback behavior* (compare Figure 1) is defined as

$$\mathscr{B} := \mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2) := \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}^{(p+m)+(p+m)}; P \circ y = Q \circ u \right\} \quad \text{where}$$
$$y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ u := \begin{pmatrix} u_2 \\ u_1 \end{pmatrix},\ P := \begin{pmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{pmatrix},\ Q := \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix} \in \mathscr{D}^{(p+m)\times(p+m)}$$

with $\mathscr{B}^0 := \{y \in \mathscr{F}^p; P \circ y = 0\}$ and modules of equations $U = \mathscr{D}^{1\times(p+m)}(P, -Q)$ and $U^0 = \mathscr{D}^{1\times(p+m)}P = U_1 + U_2$. The feedback system is *well-posed* if $\mathscr{B}$ is an input/output behavior with input u and output y, i.e., if $\mathscr{B}^0$ is autonomous or

$$\mathrm{rank}(P) = p + m = \mathrm{rank}(P_1, -Q_1) + \mathrm{rank}(-Q_2, P_2) \quad \text{or} \quad U^0 = U_1 \oplus U_2.$$

Theorem 3.2 (Characterization of T-stable feedback behaviors). *For* $\mathscr{B} = \mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ *the following conditions are equivalent:*

1. $\mathscr{B}$ *is well-posed and* T*-stable or* $\mathscr{B}^0$ *is* T*-autonomous, i.e.,* $\mathscr{B}^0_T = 0$.

2. P *is invertible in* $\mathscr{D}_T$*, i.e.,* $\det(P) \in T$.

3. (a) $\mathscr{B}_T$ *is controllable and*

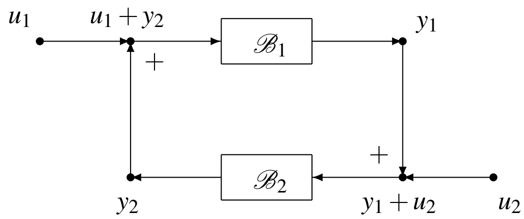

Figure 1: The feedback behavior $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$.

(b) $\mathscr{B}$ is well-posed and $H := P^{-1}Q \in \mathscr{D}_T^{(p+m)\times(p+m)}$.

4. $U_{1,T} \oplus U_{2,T} = \mathscr{D}_T^{1\times(p+m)}$.

Note that condition 4 implies that $M_{1,T} := \mathscr{D}_T^{1\times(p+m)}/U_{1,T} \cong U_{2,T}$ and in particular that $M_{1,T}$ is free since $U_{2,T}$ is so. This is equivalent to right invertibility of $(P_1, -Q_1)$ over $\mathscr{D}_T$ or to controllability of $\mathscr{B}_{1,T}$, compare [20, Thm.5.2.10], [17, Thms.7.21, 7.52, 7.53, p.141f, p.150ff].

Proof. The equivalence of 1, 2, and 3 has already been shown in [5, Thm. and Def.2.15]. The sum in 4. is direct since the feedback behavior is well-posed. Moreover, since localization preserves exactness,

$$U_{1,T} \oplus U_{2,T} = (U_1 \oplus U_2)_T = U_T^0 = (\mathscr{D}^{1\times(p+m)}P)_T = \mathscr{D}_T^{1\times(p+m)}P.$$

This $\mathscr{D}_T$-module is equal to $\mathscr{D}_T^{1\times(p+m)}$ if and only if P is invertible in $\mathscr{D}_T$, i.e., if condition 2. is satisfied. $\square$

We will primarily use the direct sum characterization from item 4, having in mind the parametrization of direct summands from Lemma 2.3.

Definition 3.3 (T-stabilizing compensators, T-stabilizable IO behaviors)**.** If the equivalent conditions of Thm. 3.2 are satisfied then $\mathscr{B}_2$ is called a *T-stabilizing compensator* of $\mathscr{B}_1$. If $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ is in addition proper we call $\mathscr{B}_2$ a *properly T-stabilizing compensator* of $\mathscr{B}_1$. The behavior $\mathscr{B}_1$ is said to be *T-stabilizable* if there exists a T-stabilizing compensator.

Remark 3.4. Assume that $\mathscr{B}_2$ is a T-stabilizing compensator of $\mathscr{B}_1$. Interconnection of $\mathscr{B}_1$ and $\mathscr{B}_2$ via $u_1 := y_2$ and $u_2 := y_1$ furnishes

$$\mathscr{B}_1 \cap \mathscr{B}_2 = \mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)^0 \subseteq \mathrm{t}_T(\mathscr{F})^{p+m}$$

where $\mathrm{t}_T(\mathscr{F})$ denotes the set of all T-small signals in $\mathscr{F}$. In Willems' language a T-small behavior, viz. $\mathscr{B}_1 \cap \mathscr{B}_2$, can be achieved from $\mathscr{B}_1$ by regular interconnection, compare [24]. Notice, however, that in contrast to [24] we do not specify the intersection $\mathscr{B}_1 \cap \mathscr{B}_2$, but only its T-smallness, and that $\mathrm{t}_T(\mathscr{F})^{p+m}$ is not a subbehavior of $\mathscr{F}^{p+m}$.

In the following we will first construct all T-stabilizing compensators with proper feedback behavior $\mathrm{fb}(\mathscr{B}_1,\mathscr{B}_2)$ and then, from Lemma 3.17 to Remark 3.28, those which are additionally themselves proper.

In order to study problems related to properness, we introduce the usual rings

$$F(s)_{\mathrm{pr}} := \left\{ \tfrac{f}{g} \in F(s);\, \deg\left(\tfrac{f}{g}\right) := \deg(f) - \deg(g) \le 0 \right\} \quad \text{resp.} \quad \mathscr{S} := \mathscr{D}_T \cap F(s)_{\mathrm{pr}}$$

of proper resp. of proper and T-stable rational functions, compare [8, p.169], [26, Ch.5], [27, Ch.2]. We will always assume that the set T contains an element $(s-\alpha)$ where $\alpha \in F$. Otherwise (in the case $F = \mathbb{C}$) the saturation of T would imply $T = F \setminus \{0\}$ and $\mathscr{S} = \mathbb{C}$.

According to [5, Def. and Lem.2.14, Lem.3.11]) we obtain

$$\begin{aligned} &\sigma := \frac{1}{s-\alpha},\ \widehat{\mathscr{D}} := F[\sigma],\ \mathscr{S} = \widehat{\mathscr{D}}_{\widehat{T}} \text{ with } \widehat{T} := \left\{ \tfrac{t}{(s-\alpha)^{\deg(t)}};\, t \in T \right\} \text{ and} \\ &\mathscr{D}_T = \mathscr{S}_\sigma := \left\{ \tfrac{\xi}{\sigma^j};\, \xi \in \mathscr{S},\, j \in \mathbb{N} \right\} = \mathscr{S}_{\{\beta\sigma^j;\, \beta \in F\setminus\{0\},\, j \ge 0\}}. \end{aligned} \tag{5}$$

The introduction of α and $\sigma = (s-\alpha)^{-1}$ is due to Pernebo [19]. All these rings are principal ideal domains with the following inclusions:

$$\begin{array}{ccccc} \widehat{\mathscr{D}} = F[\sigma] & \subseteq & \widehat{\mathscr{D}}_{\widehat{T}} = \mathscr{S} & \subseteq & F(s)_{\mathrm{pr}} \\ & & \cap\mid & & \cap\mid \\ \mathscr{D} = F[s] & \subseteq & \mathscr{D}_T = \mathscr{S}_\sigma & \subseteq & F(s) = F(\sigma) = \mathscr{K}. \end{array}$$

Remark 3.5 (Computation of the Smith form w.r.t. $\mathscr{S}$)**.** Note that, if $R \in \mathscr{K}^{k\times\ell}$ is a rational matrix, then its Smith form w.r.t. $\widehat{\mathscr{D}}$ is also the Smith form with respect to $\mathscr{S} = \widehat{\mathscr{D}}_{\widehat{T}}$, w.r.t. $\mathscr{D}_T = \mathscr{S}_\sigma$, w.r.t. $F(s)_{\mathrm{pr}}$, and w.r.t. $F(s)$.

In the following we will use the inclusions $\widehat{\mathscr{D}} \subseteq \mathscr{S} \subseteq \mathscr{D}_T$ and that these rings are quotient rings of $\widehat{\mathscr{D}}$. We will replace the defining matrices of the behaviors by matrices with entries in $\widehat{\mathscr{D}}$ or $\mathscr{S}$ which are row equivalent over $\mathscr{D}_T$ to the original matrices. Recall that T-stability depends on modules (or behaviors) *over* $\mathscr{D}_T$ only.

Assumption 3.6. In the sequel we assume that $\mathscr{B}_{1,T}$ is controllable, i.e., that $H_1 = P_1^{-1}Q_1$ is a left coprime factorization of H_1 over $\mathscr{D}_T$ (compare Lemma 2.4). According to Theorem 3.2 this is a necessary condition for T-stabilizability of $\mathscr{B}_1$. Let

$$H_1 = \widehat{P}_1^{-1}\widehat{Q}_1 = \widehat{N}_1\widehat{D}_1^{-1}, \quad \widehat{R}_1 := \left(\widehat{P}_1, -\widehat{Q}_1\right) \in \widehat{\mathscr{D}}^{p\times(p+m)}, \quad \begin{pmatrix} \widehat{N}_1 \\ \widehat{D}_1 \end{pmatrix} \in \widehat{\mathscr{D}}^{(p+m)\times m} \tag{6}$$

denote a left resp. right coprime factorization of H_1 over $\widehat{\mathscr{D}}$. This implies that

$$0 \longrightarrow \widehat{\mathscr{D}}^{1\times p} \xrightarrow{\circ\left(\widehat{P}_1, -\widehat{Q}_1\right)} \widehat{\mathscr{D}}^{1\times(p+m)} \xrightarrow{\circ\begin{pmatrix} \widehat{N}_1 \\ \widehat{D}_1 \end{pmatrix}} \widehat{\mathscr{D}}^{1\times m} \longrightarrow 0 \tag{7}$$

is exact. According to Lemma 2.3 let $\widehat{R}_2^0 = \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) \in \widehat{\mathscr{D}}^{m\times(p+m)}$ be a left inverse of $\begin{pmatrix} \widehat{N}_1 \\ \widehat{D}_1 \end{pmatrix}$ and $\begin{pmatrix} \widehat{D}_2^0 \\ \widehat{N}_2^0 \end{pmatrix} \in \widehat{\mathscr{D}}^{(p+m)\times p}$ a right inverse of $\left(\widehat{P}_1, -\widehat{Q}_1\right)$ such that

$$0 \longleftarrow \widehat{\mathscr{D}}^{1\times p} \xleftarrow{\circ\begin{pmatrix} \widehat{D}_2^0 \\ \widehat{N}_2^0 \end{pmatrix}} \widehat{\mathscr{D}}^{1\times(p+m)} \xleftarrow{\circ\left(-\widehat{Q}_2^0, \widehat{P}_2^0\right)} \widehat{\mathscr{D}}^{1\times m} \longleftarrow 0 \tag{8}$$

is also exact.

Corollary 3.7. *Assumption 3.6 is in force. Then*

$$U_{1,T} = \mathscr{D}_T^{1\times p} R_1 = \mathscr{D}_T^{1\times p}\widehat{R}_1, \quad \textit{hence}$$
$$\mathscr{B}_{1,T} = \left\{ \begin{pmatrix} y_1 \\ u_1 \end{pmatrix} \in \mathscr{F}_T^{p+m};\, P_1 \circ y_1 = Q_1 \circ u_1 \right\} = \left\{ \begin{pmatrix} y_1 \\ u_1 \end{pmatrix} \in \mathscr{F}_T^{p+m};\, \widehat{P}_1 \circ y_1 = \widehat{Q}_1 \circ u_1 \right\}$$

since $\mathscr{B}_{1,T} = U_{1,T}^{\perp}$. Recall that $\mathscr{F}_T$ is an injective $\mathscr{D}_T$-cogenerator and in particular a $\mathscr{D}_T$-module.

Proof. By assumption $H_1 = P_1^{-1}Q_1$ is a left coprime factorization of H_1 over $\mathscr{D}_T$. $H_1 = \widehat{P}_1^{-1}\widehat{Q}_1$ has this property over $\widehat{\mathscr{D}}$ and hence also over $\mathscr{D}_T \supseteq \mathscr{S} \supseteq \widehat{\mathscr{D}}$ (compare Lemma 2.4.4). The essential uniqueness of these factorizations implies that $\mathscr{D}_T^{1\times p}R_1 = \mathscr{D}_T^{1\times p}\widehat{R}_1$. □

Now assume that $\mathscr{B}_2 = \{\binom{u_2}{y_2} \in \mathscr{F}^{p+m};\, P_2 \circ y_2 = Q_2 \circ u_2\}$ is a T-stabilizing compensator of $\mathscr{B}_1$ where $R_2 := (-Q_2, P_2) \in \mathscr{D}^{m\times(p+m)}$, $\det(P_2) \neq 0$, $H_2 := P_2^{-1}Q_2$, and $U_2 := \mathscr{D}^{1\times m}R_2$. Hence, $U_{1,T} \oplus U_{2,T} = \mathscr{D}_T^{1\times(p+m)}$ and $H_2 = P_2^{-1}Q_2$ is a left coprime factorization of H_2 over $\mathscr{D}_T$ since $U_{2,T} = \mathscr{D}_T^{1\times m}R_2$ is a direct summand. Let $H_2 = \widehat{P}_2^{-1}\widehat{Q}_2$, $\widehat{R}_2 := \left(-\widehat{Q}_2, \widehat{P}_2\right) \in \widehat{\mathscr{D}}^{m\times(p+m)}$, be a left coprime factorization of H_2 over $\widehat{\mathscr{D}}$. As in Corollary 3.7 we conclude that $U_{2,T} = \mathscr{D}_T^{1\times m}R_2 = \mathscr{D}_T^{1\times m}\widehat{R}_2$ and

$$\mathscr{B}_{2,T} = \left\{ \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \in \mathscr{F}_T^{p+m};\, P_2 \circ y_2 = Q_2 \circ u_2 \right\} = \left\{ \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \in \mathscr{F}_T^{p+m};\, \widehat{P}_2 \circ y_2 = \widehat{Q}_2 \circ u_2 \right\}.$$

With the notation $\ell := p+m$, we define the following matrices:

$$\begin{aligned} P &:= \begin{pmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{pmatrix} \in \mathscr{D}^{\ell\times\ell}, \quad Q := \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix} \in \mathscr{D}^{\ell\times\ell}, \\ \widehat{P} &:= \begin{pmatrix} \widehat{P}_1 & -\widehat{Q}_1 \\ -\widehat{Q}_2 & \widehat{P}_2 \end{pmatrix} \in \widehat{\mathscr{D}}^{\ell\times\ell}, \quad \widehat{Q} := \begin{pmatrix} 0 & \widehat{Q}_1 \\ \widehat{Q}_2 & 0 \end{pmatrix} \in \widehat{\mathscr{D}}^{\ell\times\ell}. \end{aligned} \tag{9}$$

Hence, with $y := \binom{y_1}{y_2}$ and $u := \binom{u_2}{u_1}$ we get $\mathrm{fb}(\mathscr{B}_1,\mathscr{B}_2) = \left\{\binom{y}{u} \in \mathscr{F}^{\ell+\ell};\, P\circ y = Q\circ u\right\}$ and $\mathrm{fb}(\mathscr{B}_1,\mathscr{B}_2)^0 = \left\{y \in \mathscr{F}^{\ell};\, P\circ y = 0\right\}$.

Corollary 3.8. *Assume the data from* (9). *Then $\mathscr{D}_T^{1\times\ell}(P,-Q) = \mathscr{D}_T^{1\times\ell}\left(\widehat{P},-\widehat{Q}\right)$. For the quotient behaviors this implies that*

$$\begin{aligned} \mathrm{fb}(\mathscr{B}_1,\mathscr{B}_2)_T = \mathrm{fb}(\mathscr{B}_{1,T},\mathscr{B}_{2,T}) &= \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}_T^{\ell+\ell};\, P\circ y = Q\circ u \right\} \\ &= \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}_T^{\ell+\ell};\, \widehat{P}\circ y = \widehat{Q}\circ u \right\}. \end{aligned}$$

The assumption that the behavior $\mathscr{B}_2$ is a T-stabilizing compensator of $\mathscr{B}_1$ is equivalent to $\mathrm{fb}(\mathscr{B}_1,\mathscr{B}_2)_T^0 = 0$, i.e., $P \in \mathrm{Gl}_\ell(\mathscr{D}_T)$ or $\widehat{P} \in \mathrm{Gl}_\ell(\mathscr{D}_T)$. The transfer matrix of the feedback behavior is $H = P^{-1}Q = \widehat{P}^{-1}\widehat{Q}$.

Proof. By Corollary 3.7, there are (unique) matrices $A_1 \in \mathrm{Gl}_p(\mathscr{D}_T)$ and $A_2 \in \mathrm{Gl}_m(\mathscr{D}_T)$ such that $A_1R_1 = \widehat{R}_1$ and $A_2R_2 = \widehat{R}_2$, hence $A := \left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right) \in \mathrm{Gl}_\ell(\mathscr{D}_T)$ and $AP = \widehat{P}$, $AQ = \widehat{Q}$. We deduce that $\mathscr{D}_T^{1\times\ell}(P,-Q) = \mathscr{D}_T^{1\times\ell}(\widehat{P},-\widehat{Q})$. □

Theorem 3.9 (Characterization of properly T-stabilizing compensators)**.** *For the IO behavior $\mathscr{B}_1$ and its T-stabilizing compensator $\mathscr{B}_2$ and the data from above the following conditions are equivalent:*

1. $H \in \mathscr{S}^{\ell\times\ell}$*, i.e., H is also proper (recall $\mathscr{S} := \mathscr{D}_T \cap F(s)_{\mathrm{pr}}$).*
2. $\mathscr{S}^{1\times\ell}\widehat{P} = \mathscr{S}^{1\times p}\widehat{R}_1 \oplus \mathscr{S}^{1\times m}\widehat{R}_2 = \mathscr{S}^{1\times\ell}$.
3. $\widehat{P} \in \mathrm{Gl}_\ell(\mathscr{S})$.

Under these conditions $\mathscr{B}_2$ is a properly T-stabilizing compensator of $\mathscr{B}_1$ according to Definition 3.3.

Proof. The equivalence of 2 and 3 is obvious. Remember that the sum in 2 is direct since the feedback behavior is assumed to be well-posed. Condition 3 trivially implies 1 since $H = \widehat{P}^{-1}\widehat{Q}$. Now assume 1 and define $M := \begin{pmatrix} \mathrm{id}_p & 0 & 0 & 0 \\ 0 & \mathrm{id}_m & 0 & \mathrm{id}_m \\ \mathrm{id}_p & 0 & \mathrm{id}_p & 0 \\ 0 & \mathrm{id}_m & 0 & 0 \end{pmatrix} \in \mathrm{Gl}_{p+m+p+m}(F)$. Then $\begin{pmatrix} \widehat{P}_1 & -\widehat{Q}_1 & 0 & 0 \\ 0 & 0 & -\widehat{Q}_2 & \widehat{P}_2 \end{pmatrix} M = \begin{pmatrix} \widehat{P}_1 & -\widehat{Q}_1 & 0 & -\widehat{Q}_1 \\ -\widehat{Q}_2 & \widehat{P}_2 & -\widehat{Q}_2 & 0 \end{pmatrix} = (\widehat{P}, -\widehat{Q})$, whence the isomorphism

$$\mathscr{S}^{1\times\ell}/\mathscr{S}^{1\times p}\left(\widehat{P}_1, -\widehat{Q}_1\right) \times \mathscr{S}^{1\times\ell}/\mathscr{S}^{1\times m}\left(-\widehat{Q}_2, \widehat{P}_2\right) \cong \mathscr{S}^{1\times 2\ell}/\mathscr{S}^{1\times\ell}(\widehat{P}, -\widehat{Q}) \quad (10)$$

where $\left(\overline{\xi_1}, \overline{\xi_2}\right)$ is mapped to $\overline{(\xi_1, \xi_2)M}$. But $\widehat{R}_1 = \left(\widehat{P}_1, -\widehat{Q}_1\right)$ and $\widehat{R}_2 = \left(-\widehat{Q}_2, \widehat{P}_2\right)$ are right invertible over $\widehat{\mathscr{D}}$ and thus over $\mathscr{S} \supseteq \widehat{\mathscr{D}}$ by construction and hence $\mathscr{S}^{1\times\ell}/\mathscr{S}^{1\times p}\widehat{R}_1$ and $\mathscr{S}^{1\times\ell}/\mathscr{S}^{1\times m}\widehat{R}_2$ are free. The preceding isomorphism implies the same property for $\mathscr{S}^{1\times 2\ell}/\mathscr{S}^{1\times\ell}(\widehat{P}, -\widehat{Q})$ and thus the existence of $Z \in \mathscr{S}^{2\ell\times\ell}$ with $\mathrm{id}_\ell = (\widehat{P}, -\widehat{Q})Z = \widehat{P}((\mathrm{id}_\ell, -H)Z)$. Since $H \in \mathscr{S}^{\ell\times\ell}$ by 1, the matrix $(\mathrm{id}_\ell, -H)Z$ is an inverse of $\widehat{P}$ in $\mathscr{S}^{\ell\times\ell}$ and hence $\widehat{P} \in \mathrm{Gl}_\ell(\mathscr{S})$. □

We next *construct* all properly T-stabilizing compensators of $\mathscr{B}_1$ under the (necessary) condition of controllability of $\mathscr{B}_{1,T}$. From the preceding theorem we infer that direct summands of $\mathscr{S}^{1\times p}\widehat{R}_1$ in $\mathscr{S}^{1\times\ell}$ play a part. These have been classified in Lemma 2.3.

Lemma 3.10. *We use the data from above, in particular from Assumption 3.6 and equations* (6)-(8)*.*

1. *There are bijections*

$$\begin{array}{cc}
\left\{V \subseteq \mathscr{S}^{1\times\ell};\ \mathscr{S}^{1\times p}\widehat{R}_1 \oplus V = \mathscr{S}^{1\times\ell}\right\} & V \\
\cong & \updownarrow \\
\left\{\begin{pmatrix}\widehat{D}_2 \\ \widehat{N}_2\end{pmatrix} \in \mathscr{S}^{(p+m)\times p};\ \widehat{R}_1\begin{pmatrix}\widehat{D}_2 \\ \widehat{N}_2\end{pmatrix} = \mathrm{id}_p\right\} & \begin{pmatrix}\widehat{D}_2 \\ \widehat{N}_2\end{pmatrix} \\
\cong & \updownarrow \\
\left\{\widehat{R}_2 = \left(-\widehat{Q}_2, \widehat{P}_2\right) \in \mathscr{S}^{m\times(p+m)};\ \widehat{R}_2\begin{pmatrix}\widehat{N}_1 \\ \widehat{D}_1\end{pmatrix} = \mathrm{id}_m\right\} & \widehat{R}_2 = \left(-\widehat{Q}_2, \widehat{P}_2\right) \\
\cong & \updownarrow \\
\mathscr{S}^{m\times p} & X
\end{array}$$

where $V = \ker\left(\circ\begin{pmatrix}\widehat{D}_2\\ \widehat{N}_2\end{pmatrix}\right) = \mathscr{S}^{1\times m}\widehat{R}_2$, $\begin{pmatrix}\widehat{D}_2\\ \widehat{N}_2\end{pmatrix} = \begin{pmatrix}\widehat{D}_2^0\\ \widehat{N}_2^0\end{pmatrix} - \begin{pmatrix}\widehat{N}_1\\ \widehat{D}_1\end{pmatrix}X$,
$\widehat{R}_2 = \left(-\widehat{Q}_2, \widehat{P}_2\right) = \widehat{R}_2^0 + X\widehat{R}_1$, $\widehat{P}_2 = \widehat{P}_2^0 - X\widehat{Q}_1$, $\widehat{Q}_2 = \widehat{Q}_2^0 - X\widehat{P}_1$.
Moreover,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{S}^{1\times p} & \xrightarrow{\circ(\widehat{P}_1,-\widehat{Q}_1)} & \mathscr{S}^{1\times \ell} & \xrightarrow{\circ\begin{pmatrix}\widehat{N}_1\\ \widehat{D}_1\end{pmatrix}} & \mathscr{S}^{1\times m} & \longrightarrow & 0 \quad \textit{and} \\ 0 & \longleftarrow & \mathscr{S}^{1\times p} & \xleftarrow{\circ\begin{pmatrix}\widehat{D}_2\\ \widehat{N}_2\end{pmatrix}} & \mathscr{S}^{1\times \ell} & \xleftarrow{\circ(-\widehat{Q}_2,\widehat{P}_2)} & \mathscr{S}^{1\times m} & \longleftarrow & 0 \end{array} \tag{11}$$

are split exact sequences and

$$\begin{pmatrix}\widehat{P}_1 & -\widehat{Q}_1\\ -\widehat{Q}_2 & \widehat{P}_2\end{pmatrix}\begin{pmatrix}\widehat{D}_2 & \widehat{N}_1\\ \widehat{N}_2 & \widehat{D}_1\end{pmatrix} = \mathrm{id}_{p+m}.$$

2. *Almost all $\widehat{P}_2$ from 1 have non-zero determinant (in the sense specified in the proof and the next remark).*

Proof. 1. The sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{S}^{1\times p} & \xrightarrow{\circ(\widehat{P}_1,-\widehat{Q}_1)} & \mathscr{S}^{1\times \ell} & \xrightarrow{\circ\begin{pmatrix}\widehat{N}_1\\ \widehat{D}_1\end{pmatrix}} & \mathscr{S}^{1\times m} & \longrightarrow & 0 \quad \text{and} \\ 0 & \longleftarrow & \mathscr{S}^{1\times p} & \xleftarrow{\circ\begin{pmatrix}\widehat{D}_2^0\\ \widehat{N}_2^0\end{pmatrix}} & \mathscr{S}^{1\times \ell} & \xleftarrow{\circ(-\widehat{Q}_2^0,\widehat{P}_2^0)} & \mathscr{S}^{1\times m} & \longleftarrow & 0 \end{array}$$

with the retraction $\circ\begin{pmatrix}\widehat{D}_2^0\\ \widehat{N}_2^0\end{pmatrix}$ and the section $\circ\left(-\widehat{Q}_2^0, \widehat{P}_2^0\right)$ are exact since they can be obtained from (7) and (8) by applying $(-)_{\widehat{T}}$ and localization preserves exactness. Remember that $\mathscr{S} = \widehat{\mathscr{D}}_{\widehat{T}}$. Application of Lemma 2.3 to these exact sequences yields the assertion.

2. Let $\Xi = (\Xi_{ij})_{1\le i\le m, 1\le j\le p}$ be a matrix of indeterminates and consider the polynomial

$$g(\Xi) := \det\left(\widehat{P}_2^0 - \Xi\widehat{Q}_1\right).$$

Then, for $X \in \mathscr{S}^{m\times p}$, $g(X) = \det\left(\widehat{P}_2^0 - X\widehat{Q}_1\right) = \det\left(\widehat{P}_2\right)$. We have to show $g \neq 0$. Since $\mathscr{S} = F[\sigma]_{\widehat{T}}$ is an infinite integral domain this implies that also the polynomial function $(g|\mathscr{S}^{m\times p} : \mathscr{S}^{m\times p} \longrightarrow \mathscr{S})$ is non-zero, indeed $\{X \in \mathscr{S}^{m\times p}; g(X) \neq 0\}$ is an open dense subset of $\mathscr{S}^{m\times p}$ (in the Zariski topology). The matrix $\left(0, \widehat{D}_1^{-1}\right)$ is obviously a left inverse of $\begin{pmatrix}\widehat{N}_1\\ \widehat{D}_1\end{pmatrix}$ in $\mathscr{K}^{m\times(p+m)}$. Part 1 of the present lemma applied to $\mathscr{K} = F(s) = \mathrm{quot}(\mathscr{S})$ instead of $\mathscr{S}$ yields the existence of $X \in \mathscr{K}^{m\times p}$ such that $\left(0, \widehat{D}_1^{-1}\right) = \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) + X\left(\widehat{P}_1, -\widehat{Q}_1\right)$. Hence $g(X) = \det\left(\widehat{P}_2^0 - X\widehat{Q}_1\right) = \det(\widehat{D}_1)^{-1} \neq 0$.

□

Remark 3.11. If k is an infinite field a property of vectors $X \in k^N$ is called *generically* or *almost always* true if it holds on a non-empty Zariski open set or, equivalently, on a special open set $\left\{X \in k^N; g(X) \neq 0\right\}$ where g is a *non-zero* polynomial. In the preceding lemma this language is extended to the infinite integral domain $\mathscr{S} = F[\sigma]_{\widehat{T}}$.

Theorem 3.12 (Constructive parametrization of properly T-stabilizing compensators)**.**

1. *Assume that $\mathscr{B}_{1,T}$ is controllable or, equivalently, that R_1 is right invertible over $\mathscr{D}_T$ and the ensuing data from* (6) - (8).

   (a) *Choose $X \in \mathscr{S}^{m\times p}$ such that $\widehat{R}_2 := \left(-\widehat{Q}_2, \widehat{P}_2\right) := \widehat{R}_2^0 + X\widehat{R}_1$ from Lemma 3.10 satisfies* $\det(\widehat{P}_2) \neq 0$, *and define $H_2 := \widehat{P}_2^{-1}\widehat{Q}_2$. Let $R_{2,\mathrm{cont}} = (-Q_{2,\mathrm{cont}}, P_{2,\mathrm{cont}})$ be the controllable realization of H_2 over $\mathscr{D}$, i.e., let $H_2 = (P_{2,\mathrm{cont}})^{-1}Q_{2,\mathrm{cont}}$ be a left coprime factorization of H_2 over $\mathscr{D}$. Furthermore choose an arbitrary $A \in \mathscr{D}^{m\times m}$ with* $\det(A) \in T$ *and define $R_2 := (-Q_2, P_2) := AR_{2,\mathrm{cont}}$.*
   Then $\mathscr{B}_2 := \{\binom{u_2}{y_2} \in \mathscr{F}^{p+m}; P_2 \circ y_2 = Q_2 \circ u_2\}$ is a properly T-stabilizing compensator of $\mathscr{B}_1$, and all such compensators arise in this fashion.

   (b) *The pairs $(X, A) \in \mathscr{S}^{m\times p} \times \mathscr{D}^{m\times m}$ with* $\det(\widehat{P}_2^0 - X\widehat{Q}_1) \neq 0$ *and* $\det(A) \in T$ *parametrize the set of all properly T-stabilizing compensators $\mathscr{B}_2$ of $\mathscr{B}_1$ where $\mathscr{B}_2$ is constructed from (X, A) according to 1a. Two pairs (X, A) and (X', A') give rise to the same compensator $\mathscr{B}_2$ if and only if $X = X'$ and A is row equivalent to A' over $\mathscr{D}$.*

2. *The following conditions are equivalent for an IO behavior $\mathscr{B}_1$:*

   (a) *$\mathscr{B}_1$ is T-stabilizable, i.e., there exists a T-stabilizing compensator $\mathscr{B}_2$ of $\mathscr{B}_1$.*

   (b) *There exists a* properly *T-stabilizing compensator $\mathscr{B}_2$ of $\mathscr{B}_1$.*

   (c) *$\mathscr{B}_{1,T}$ is controllable.*

Proof. 1. (a) i. By construction

$$\mathscr{S}^{1\times p}\widehat{R}_1 \oplus \mathscr{S}^{1\times m}\widehat{R}_2 = \mathscr{S}^{1\times \ell}, \quad \text{hence also}$$
$$\mathscr{D}_T^{1\times p}R_1 \oplus \mathscr{D}_T^{1\times m}\widehat{R}_2 = \mathscr{D}_T^{1\times p}\widehat{R}_1 \oplus \mathscr{D}_T^{1\times m}\widehat{R}_2 = \mathscr{D}_T^{1\times \ell}$$

because $\mathscr{D}_T = \mathscr{S}_\sigma$. Since $\mathscr{S}^{1\times m}\widehat{R}_2 = \mathscr{S}^{1\times m}\left(-\widehat{Q}_2, \widehat{P}_2\right)$ is a direct summand of $\mathscr{S}^{1\times \ell}$, the factorization $H_2 = \widehat{P}_2^{-1}\widehat{Q}_2$ is left coprime over $\mathscr{S}$ and thus over $\mathscr{D}_T$. Also $H_2 = (P_{2,\mathrm{cont}})^{-1}Q_{2,\mathrm{cont}}$ is left coprime over $\mathscr{D}$ and hence over $\mathscr{D}_T$. The essential uniqueness of these factorizations implies $\mathscr{D}_T^{1\times m}R_{2,\mathrm{cont}} = \mathscr{D}_T^{1\times m}\widehat{R}_2$. By assumption $\det(A) \in T$, hence $A \in \mathrm{Gl}_m(\mathscr{D}_T)$ which implies

$$\mathscr{D}_T^{1\times m}R_2 = \mathscr{D}_T^{1\times m}R_{2,\mathrm{cont}} = \mathscr{D}_T^{1\times m}\widehat{R}_2 \quad \text{and} \quad \mathscr{D}_T^{1\times \ell} = \mathscr{D}_T^{1\times p}R_1 \oplus \mathscr{D}_T^{1\times m}R_2.$$

According to Theorem 3.2 $\mathscr{B}_2$ is a T-stabilizing compensator of $\mathscr{B}_1$. Now let $\widetilde{R}_2 := (-\widetilde{Q}_2, \widetilde{P}_2) \in \widehat{\mathscr{D}}^{m\times(p+m)}$ be the controllable realization of H_2 over $\widehat{\mathscr{D}}$ and hence also over $\mathscr{S}$. Therefore $H_2 = \widetilde{P}_2^{-1}\widetilde{Q}_2 = \widehat{P}_2^{-1}\widehat{Q}_2$ are two left coprime factorizations of H_2 over $\mathscr{S}$ which implies row equivalence of $\widetilde{R}_2$ and $\widehat{R}_2$ over $\mathscr{S}$, i.e., $\mathscr{S}^{1\times m}\widetilde{R}_2 = \mathscr{S}^{1\times m}\widehat{R}_2$, and hence $\mathscr{S}^{1\times p}\widehat{R}_1 \oplus \mathscr{S}^{1\times m}\widetilde{R}_2 = \mathscr{S}^{1\times \ell}$. According to Theorem 3.9 $\mathscr{B}_2$ is indeed a properly T-stabilizing compensator of $\mathscr{B}_1$. Notice that the matrix $\widehat{R}_2$ from Theorem 3.9 is denoted by $\widetilde{R}_2$ here. The matrices $\widetilde{R}_2 \in \widehat{\mathscr{D}}^{m\times(p+m)}$ and $\widehat{R}_2 \in \mathscr{S}^{m\times(p+m)}$ of the present proof are row equivalent over $\mathscr{S}$, but not identical.
ii. Let, conversely,

$$\mathscr{B}_2 = \left\{\begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \in \mathscr{F}^{p+m}; P_2 \circ y_2 = Q_2 \circ u_2\right\}, \ R_2 := (-Q_2, P_2) \in \mathscr{D}^{m\times(p+m)}, \ \det(P_2) \neq 0$$

be any properly T-stabilizing compensator of $\mathscr{B}_1$ with transfer matrix $H_2 := P_2^{-1}Q_2$. Then $\mathscr{D}_T^{1\times m}R_2$ is a direct summand by Theorem 3.2, and consequently $H_2 = P_2^{-1}Q_2$ is a left coprime factorization of H_2 over $\mathscr{D}_T$, compare Lemma 2.4. Let $R_{2,\text{cont}} := (-Q_{2,\text{cont}}, P_{2,\text{cont}}) \in \mathscr{D}^{m\times(p+m)}$ be the controllable realization of H_2 over $\mathscr{D}$. This implies a factorization $R_2 = AR_{2,\text{cont}}$ for some $A \in \mathscr{D}^{m\times m}$ with $\det(A) \neq 0$. Note that $H_2 = P_{2,\text{cont}}^{-1}Q_{2,\text{cont}}$ is a left coprime factorization of H_2 over $\mathscr{D}$ and hence also over $\mathscr{D}_T$. Since the left coprime factorization is unique up to row equivalence we deduce that $\mathscr{D}_T^{1\times m}R_2 = \mathscr{D}_T^{1\times m}R_{2,\text{cont}}$ and consequently that $A \in \mathrm{Gl}_m(\mathscr{D}_T)$, i.e., $\det(A) \in T$. Define $\widetilde{R}_2 := (-\widetilde{Q}_2, \widetilde{P}_2) \in \widehat{\mathscr{D}}^{m\times(p+m)}$ as in 1(a)i, i.e., $H_2 = \widetilde{P}_2^{-1}\widetilde{Q}_2$ is a left coprime factorization of H_2 over $\widehat{\mathscr{D}}$ and hence also over $\mathscr{D}_T \supseteq \widehat{\mathscr{D}}$. This implies $\mathscr{D}_T^{1\times m}\widetilde{R}_2 = \mathscr{D}_T^{1\times m}R_2$. Theorem 3.9 furnishes $\mathscr{S}^{1\times p}\widehat{R}_1 \oplus \mathscr{S}^{1\times m}\widetilde{R}_2 = \mathscr{S}^{1\times \ell}$. From Lemma 3.10 we obtain a unique $\widehat{R}_2 = \left(-\widehat{Q}_2, \widehat{P}_2\right) = \widehat{R}_2^0 + X\widehat{R}_1 \in \mathscr{S}^{m\times(p+m)}$ with $\widehat{R}_2\begin{pmatrix}\widehat{N}_1\\ \widehat{D}_1\end{pmatrix} = \mathrm{id}_m$ and $\mathscr{S}^{1\times m}\widetilde{R}_2 = \mathscr{S}^{1\times m}\widehat{R}_2$, hence also $\mathscr{D}_T^{1\times m}\widetilde{R}_2 = \mathscr{D}_T^{1\times m}\widehat{R}_2$ and $H_2 = \widehat{P}_2^{-1}\widehat{Q}_2$.

(b) From 1a we conclude that all properly T-stabilizing compensators of $\mathscr{B}_1$ are obtained from parameters (X,A) with the asserted properties. Assume that (X,A) and (X',A') give rise to the same compensator $\mathscr{B}_2$ with transfer matrix H_2. For the corresponding $\widehat{R}_2 = \left(-\widehat{Q}_2, \widehat{P}_2\right) = \widehat{R}_2^0 + X\widehat{R}_1$ and $\widehat{R}'_2 = (-\widehat{Q}'_2, \widehat{P}'_2) = \widehat{R}_2^0 + X'\widehat{R}_1$, the left coprime factorizations $H_2 = (\widehat{P}_2)^{-1}(\widehat{Q}_2) = (\widehat{P}'_2)^{-1}(\widehat{Q}'_2)$ of H_2 over $\mathscr{S}$ imply the existence of $B \in \mathrm{Gl}_m(\mathscr{S})$ with $\widehat{R}'_2 = B\widehat{R}_2$, hence $B = B\,\mathrm{id}_m = B\widehat{R}_2\begin{pmatrix}\widehat{N}_1\\ \widehat{D}_1\end{pmatrix} = \widehat{R}'_2\begin{pmatrix}\widehat{N}_1\\ \widehat{D}_1\end{pmatrix} = \mathrm{id}_m$, and consequently $\widehat{R}_2 = \widehat{R}'_2$ and $X = X'$. The row equivalence of A and A' follows from $\mathscr{D}^{1\times m}AR_{2,\text{cont}} = \mathscr{D}^{1\times m}R_2 = \mathscr{D}^{1\times m}R'_2 = \mathscr{D}^{1\times m}A'R_{2,\text{cont}}$.

2. The controllabiliy of $\mathscr{B}_{1,T}$ is a necessary condition for T-stabilizability by Theorem 3.2 and sufficient – even for the existence of *properly* T-stabilizing compensators – due to the construction in 1. Recall that almost all $\widehat{P}_2$ in Lemma 3.10 have non-zero determinant and can be chosen in the construction in 1.

□

Remark 3.13. Computer calculations of the data in the preceding theorem require the following possibilities only:

1. The Smith form algorithm over the polynomial algebra $\mathscr{D} = F[s]$, hence also over $\widehat{\mathscr{D}} = F[\sigma] = F[\frac{1}{s-\alpha}]$, $\alpha \in F$.
2. A decision method for the inclusion $t \in T$.

For $F = \mathbb{Q}$ as in all practical examples the computations of 1 are exact, no numerical approximation is required. Note however that the Smith form transformation matrices that are also required usually get very complicated.

Theorem 3.14 (Computation of the transfer matrix of $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$)**.** *Assume that $\mathscr{B}_1$ is T-stabilizable, the data from* (6) - (8)*, and a compensator $\mathscr{B}_2$ constructed according to Theorem 3.12. Let $\begin{pmatrix}\widehat{D}_2\\ \widehat{N}_2\end{pmatrix} = \begin{pmatrix}\widehat{D}_2^0\\ \widehat{N}_2^0\end{pmatrix} - \begin{pmatrix}\widehat{N}_1\\ \widehat{D}_1\end{pmatrix}X \in \mathscr{S}^{(p+m)\times p}$ denote the right inverse of $\widehat{R}_1$ corresponding to $\widehat{R}_2 = \left(-\widehat{Q}_2, \widehat{P}_2\right) = \widehat{R}_2^0 + X\widehat{R}_1$. Let P, Q, $\widehat{P}$, $\widehat{Q}$ denote the matrices from* (9)*. Then*

$$H := P^{-1}Q = \widehat{P}^{-1}\widehat{Q} = \begin{pmatrix}\widehat{N}_1\widehat{Q}_2 & \widehat{D}_2\widehat{Q}_1\\ \widehat{D}_1\widehat{Q}_2 & \widehat{N}_2\widehat{Q}_1\end{pmatrix} =: \begin{pmatrix}H_{y_1,u_2} & H_{y_1,u_1}\\ H_{y_2,u_2} & H_{y_2,u_1}\end{pmatrix}.$$

Moreover,

$$H+\mathrm{id}_{p+m}=\begin{pmatrix}\widehat{D}_2\widehat{P}_1 & \widehat{N}_1\widehat{P}_2\\ \widehat{N}_2\widehat{P}_1 & \widehat{D}_1\widehat{P}_2\end{pmatrix}.$$

Proof. The definitions of the involved matrices in (6) - (8) and Lemma 3.10 imply that

$$\begin{pmatrix}\widehat{P}_1 & -\widehat{Q}_1\\ -\widehat{Q}_2 & \widehat{P}_2\end{pmatrix}\begin{pmatrix}\widehat{D}_2 & \widehat{N}_1\\ \widehat{N}_2 & \widehat{D}_1\end{pmatrix}=\mathrm{id}_{p+m},\quad \text{i.e.,}\quad \begin{pmatrix}\widehat{D}_2 & \widehat{N}_1\\ \widehat{N}_2 & \widehat{D}_1\end{pmatrix}=\widehat{P}^{-1}.$$

The assertion on H follows directly by computing $H=\widehat{P}^{-1}\widehat{Q}$. The claimed form of $H+\mathrm{id}_{p+m}$ is a consequence of the equation

$$\begin{pmatrix}\mathrm{id}_p & 0\\ 0 & \mathrm{id}_m\end{pmatrix}=\widehat{P}^{-1}\widehat{P}=\begin{pmatrix}\widehat{D}_2\widehat{P}_1-\widehat{N}_1\widehat{Q}_2 & -\widehat{D}_2\widehat{Q}_1+\widehat{N}_1\widehat{P}_2\\ \widehat{N}_2\widehat{P}_1-\widehat{D}_1\widehat{Q}_2 & -\widehat{N}_2\widehat{Q}_1+\widehat{D}_1\widehat{P}_2\end{pmatrix}=\begin{pmatrix}\widehat{D}_2\widehat{P}_1 & \widehat{N}_1\widehat{P}_2\\ \widehat{N}_2\widehat{P}_1 & \widehat{D}_1\widehat{P}_2\end{pmatrix}-H.$$

□

Next we discuss the question of pole placement or spectral assignability. Consider the following data:

$$\begin{gathered}R_1=(P_1,-Q_1)\in\mathscr{D}^{p\times(p+m)},\quad \det(P_1)\neq 0,\quad H_1:=P_1^{-1}Q_1,\\ U_1:=\mathscr{D}^{1\times p}R_1,\quad M_1:=\mathscr{D}^{1\times(p+m)}/U_1,\\ \mathscr{B}_1:=U_1^{\perp}=\left\{\begin{pmatrix}y_1\\ u_1\end{pmatrix}\in\mathscr{F}^{p+m};\, P_1\circ y_1=Q_1\circ u_1\right\},\\ \text{and the Smith form}\quad X_1R_1Y_1=(E,0),\quad X_1\in\mathrm{Gl}_p(\mathscr{D}),\quad Y_1\in\mathrm{Gl}_{p+m}(\mathscr{D}),\\ E=\begin{pmatrix}e_1 & & 0\\ & \ddots & \\ 0 & & e_p\end{pmatrix}\in\mathscr{D}^{p\times p}\text{ with } e_1|\dots|e_p\neq 0.\text{ Then}\\ \mathscr{D}e_p=\mathrm{ann}_{\mathscr{D}}(\mathrm{t}(M_1))=\{d\in\mathscr{D};\, d\cdot\mathrm{t}(M_1)=0\}\text{ where}\\ \mathrm{t}(M_1)=\left\{\xi\in M_1;\,\exists d\in\mathscr{D}\setminus\{0\}\text{ with } d\xi=0\right\}\end{gathered}\tag{12}$$

is the torsion module of M_1. Let $\mathscr{P}$ denote the representative system of primes in $\mathscr{D}=F[s]$ containing all monic irreducible polynomials, $\alpha\in F$, and define

$$\begin{gathered}t_1:=(s-\alpha)e_p,\quad \mathscr{P}_1:=\{s-\alpha\}\cup\left\{q\in\mathscr{P};\, q|e_p\right\}=\{q\in\mathscr{P};\, q|t_1\},\quad\text{and}\\ T_1:=\left\{\beta\prod_{q\in\mathscr{P}_1}q^{\mu(q)};\,\beta\in F\setminus\{0\},\,\mu(q)\geq 0\right\}=\left\{t\in\mathscr{D}\setminus\{0\};\,\exists\mu: t|t_1^{\mu}\right\}\end{gathered}\tag{13}$$

which is the saturated monoid generated by t_1. For all $\mathscr{D}$-modules M the quotient modules M_{t_1} and M_{T_1} coincide, especially

$$\mathscr{D}_{t_1}=\left\{\frac{d}{t_1^{\mu}}\in F(s);\, d\in\mathscr{D},\mu\geq 0\right\}=\mathscr{D}_{T_1}\subset F(s).$$

By construction t_1 and thus its divisors e_p and e_i, $1\leq i\leq p$, are invertible in $\mathscr{D}_{T_1}$, hence $E\in\mathrm{Gl}_p(\mathscr{D}_{T_1})$. This implies that

$$\begin{gathered}\mathrm{id}_p=(E,0)\left(\begin{smallmatrix}E^{-1}\\ 0\end{smallmatrix}\right)=X_1R_1Y_1\left(\begin{smallmatrix}E^{-1}\\ 0\end{smallmatrix}\right)\quad\text{and thus}\\ \mathrm{id}_p=R_1Y_1\left(\begin{smallmatrix}E^{-1}\\ 0\end{smallmatrix}\right)X_1,\quad Y_1\left(\begin{smallmatrix}E^{-1}\\ 0\end{smallmatrix}\right)X_1\in\mathscr{D}_{T_1}^{(p+m)\times p}.\end{gathered}$$

Therefore R_1 is right invertible over $\mathscr{D}_{T_1}$.

Theorem 3.15 (Pole placement). *Consider the behavior $\mathscr{B}_1$ and the accompanying data from* (12) *and the saturated monoid T_1 from* (13). *Let $T \subset \mathscr{D}$ be any other saturated monoid with $(s-\alpha) \in T$.*

1. *By definition the monoid T_1 is the least saturated one which contains $(s-\alpha)$ and for which R_1 is right invertible over $\mathscr{D}_{T_1}$ or, in other words, for which $\mathscr{B}_1$ is T_1-stabilizable.*
2. *The behavior $\mathscr{B}_1$ is T-stabilizable if and only if the monoid T_1 from* (13) *is contained in T or, in other words, if the finitely many irreducible factors of e_p belong to T.*
3. *If, in particular, $t_2 \in F[s]$ is any multiple of $t_1 = (s-\alpha)e_p$ and T_2 is the saturated monoid of all divisors of powers of t_2, i.e.,*

$$T_2 := \left\{ \beta \prod_q q^{m(q)};\ \beta \in F \setminus \{0\},\ m(q) \geq 0,\ q \text{ irreducible factor of } t_2 \right\}$$

   then this T_2 contains T_1, $\mathscr{B}_1$ is T_2-stabilizable and all properly T_2-stabilizing compensators can be constructed according to Thm. 3.12 applied to T_2.
4. *If in item 3 $F = \mathbb{R}$ and $\mathscr{D} = \mathbb{R}[s]$, then the T_2-stabilizability of $\mathscr{B}_1$ resp. the T_2-stability of the feedback behavior $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ signify that the uncontrollable poles of $\mathscr{B}_1$ resp. the poles of the feedback behavior $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ are zeros of t_2.*
   This is the generalization of the standard pole placement result via state feedback for stabilizable state space systems.

Proof. Assertions 1, 3, and 4 are clear.

2. The T-stabilizability of $\mathscr{B}_1$ is equivalent to controllablity of $\mathscr{B}_{1,T}$, i.e., to the existence of a right inverse of R_1 with entries in $\mathscr{D}_T$. This signifies that the greatest elementary divisor e_p of R_1 (w.r.t. $\mathscr{D}$) is invertible in $\mathscr{D}_T$, i.e., contained in T. Since T is saturated and $(s-\alpha) \in T$ by assumption, this is equivalent to $t_1 = (s-\alpha)e_p \in T$ or $T_1 \subseteq T$.

□

Remark 3.16. Computer calculations of the T_2-stabilizing compensators in Theorem 3.15.3 require the Smith form algorithm over $F[s]$ (and $F[\sigma]$) only. A non-zero $t \in F[s]$ belongs to T_2 if and only if $t \mid t_2^{\deg(t)}$ and this is trivially checked. For $F = \mathbb{Q}$ these computations can be executed exactly with all computer algebra systems.

Our next aim is the study of T-stabilizing compensators such that both $\mathscr{B}_2$ and $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ are proper. We start with further results regarding the rings $F(s)_{\mathrm{pr}}$ and $\mathscr{S}$.

Lemma 3.17 (The map ν_q). *Let R be a principal ideal domain with quotient field K and q a prime of R. Then q induces the saturated monoid $T(q) := R \setminus Rq$, the discrete valuation ring $R_{T(q)} = \left\{ \frac{f}{g} \in K; f, g \in R,\ q \not| \, g \right\}$, i.e., a principal ideal domain with the unique prime q (up to association), and the residue field $k(q) := R/Rq$ with the canonical map* $\mathrm{can} : R \longrightarrow k(q),\ f \longmapsto f + Rq$.
The canonical map can be uniquely extended to the ring epimorphism

$$\nu_q :\ R_{T(q)} \longrightarrow k(q),\ r = \frac{f}{g} \longmapsto \nu_q(r) := \mathrm{can}(g)^{-1}\,\mathrm{can}(f) = \mathrm{can}(g'f)$$
$$\text{where} \quad g'g \equiv 1 \mod q.$$

Then $\ker(\nu_q) = R_{T(q)}q$ *and hence* $R_{T(q)}/R_{T(q)}q \cong k(q) = R/Rq$.

Proof. Obviously $\nu_q(q) = 0$ and hence $R_{T(q)}q \subseteq \ker\left(\nu_q : R_{T(q)} \longrightarrow k(q)\right)$. If, conversely, $\nu_q(r) = \mathrm{can}(g)^{-1}\mathrm{can}(f) = 0$ in $k(q)$, then $\mathrm{can}(f) = 0$ and $f \in Rq$, hence $r = \frac{f}{g} \in R_{T(q)}q$. □

Application of the preceding lemma to the ring $R = \widehat{\mathscr{D}} = F[\sigma]$ and the prime σ yields

$$\begin{gathered}\nu_\sigma : \widehat{\mathscr{D}}_{T(\sigma)} \longrightarrow \widehat{\mathscr{D}}/\widehat{\mathscr{D}}\sigma = F,\ f = \frac{\widehat{f}}{\widehat{g}} \longmapsto \widehat{g}(0)^{-1}\widehat{f}(0)\\ \text{where}\quad T(\sigma) := \widehat{\mathscr{D}} \setminus \widehat{\mathscr{D}}\sigma = \left\{\widehat{g} \in \widehat{\mathscr{D}};\, \widehat{g}(0) \neq 0\right\}.\end{gathered}$$

Lemma 3.18 (The ring $F(s)_{\mathrm{pr}}$ as quotient ring of $\widehat{\mathscr{D}}$)**.**

$$F(s)_{\mathrm{pr}} = \widehat{\mathscr{D}}_{T(\sigma)} \subseteq F(s) = F(\sigma), \quad \sigma = \frac{1}{s-\alpha}.$$

In particular, $F(s)_{\mathrm{pr}}$ *is a discrete valuation ring with the unique prime* σ*, up to association. The prime factor decomposition of a non-zero rational function* $r = \frac{f}{g}$*,* $f,g \in \mathscr{D}$*, has the form* $r = u\sigma^{v(r)}$ *where* $v(r) := -\deg(r) = \deg(g) - \deg(f)$ *is the standard valuation of* r *and where* u *is a unit in* $F(s)_{\mathrm{pr}}$*, i.e., with* $v(u) = 0$*.*

Proof. Let $0 \neq r = \frac{\widehat{f}}{\widehat{g}} \in F(s) = F(\sigma)$, $\widehat{f}, \widehat{g} \in F[\sigma]$, $\gcd(\widehat{f}, \widehat{g}) = 1$, $\widehat{f} = a_m\sigma^m + \ldots + a_0$, $a_m \neq 0$, $\widehat{g} = b_n\sigma^n + \ldots + b_0$, $b_n \neq 0$, $\widehat{g}(0) = b_0$. Then, for any number $N \geq m,n$,

$$\begin{aligned}\widehat{f}\sigma^{-N} &= a_m(\sigma^{-1})^{N-m} + \ldots + a_0(\sigma^{-1})^N = a_m(s-\alpha)^{N-m} + \ldots + a_0(s-\alpha)^N,\\ \widehat{g}\sigma^{-N} &= b_n(\sigma^{-1})^{N-n} + \ldots + b_0(\sigma^{-1})^N = b_n(s-\alpha)^{N-n} + \ldots + b_0(s-\alpha)^N,\\ r &= \frac{\widehat{f}}{\widehat{g}} = \frac{\widehat{f}\sigma^{-N}}{\widehat{g}\sigma^{-N}} = \frac{a_0(s-\alpha)^N + \ldots + a_m(s-\alpha)^{N-m}}{b_0(s-\alpha)^N + \ldots + b_n(s-\alpha)^{N-n}}.\end{aligned}$$

If $\widehat{g} \in T(\sigma)$, i.e., $\sigma \not| \widehat{g}$ or $b_0 = \widehat{g}(0) \neq 0$, then $\deg_s(r) \leq N - N = 0$, hence $r \in F(s)_{\mathrm{pr}}$. Assume, conversely, $b_0 = \widehat{g}(0) = 0$. Then $a_0 = \widehat{f}(0) \neq 0$ since $\gcd(\widehat{f}, \widehat{g}) = 1$ and hence $\deg_s(r) \geq N - (N-1) = 1$ and consequently $r \notin F(s)_{\mathrm{pr}}$. □

Remark 3.19. In systems theory, compare [26, Ch.3], [27, Ch.2], and also in one-dimensional projective algebraic geometry, $\widehat{\mathscr{D}}\sigma$ is called the place or prime at infinity and the following notation is used:

$$\begin{gathered}r(\infty) := \nu_\sigma(r) = \widehat{g}(0)^{-1}\widehat{f}(0)\\ \text{for}\quad r = \widehat{g}^{-1}\widehat{f} \in F(s)_{\mathrm{pr}} = \widehat{\mathscr{D}}_{T(\sigma)}, \quad \text{i.e.,}\quad \widehat{f}, \widehat{g} \in \widehat{\mathscr{D}},\ \widehat{g}(0) \neq 0.\end{gathered}$$

If $F = \mathbb{R}$ or $F = \mathbb{C}$ and $r = g^{-1}f$ where $f, g \in \mathscr{D} = F[s]$ and $g \neq 0$, this implies

$$r(\infty) = \lim_{t\to\infty} r(t) = \lim_{t\to\infty} \frac{f(t)}{g(t)}.$$

Corollary 3.20 (Direct sum decomposition of $\widehat{\mathscr{D}}$, $\mathscr{S}$, and $F(s)_{\mathrm{pr}}$)**.**

1.
$$F \cong \widehat{\mathscr{D}}/\widehat{\mathscr{D}}\sigma \cong \mathscr{S}/\mathscr{S}\sigma \cong \widehat{\mathscr{D}}_{T(\sigma)}/\widehat{\mathscr{D}}_{T(\sigma)}\sigma = F(s)_{\mathrm{pr}}/F(s)_{\mathrm{pr}}\sigma.$$

Therefore all these residue fields will be identified, especially $r(\infty) = \nu_\sigma(r)$ for $r \in F(s)_{\mathrm{pr}}$, $\nu_\sigma(\widehat{f}) = \widehat{f}(0)$ for $\widehat{f} \in \widehat{\mathscr{D}}$.

2.
$$\widehat{\mathscr{D}} = F \oplus \widehat{\mathscr{D}}\sigma \subseteq \mathscr{S} = F \oplus \mathscr{S}\sigma \subseteq F(s)_{\mathrm{pr}} = F \oplus F(s)_{\mathrm{pr}}\sigma \ni r = \nu_\sigma(r) + (r - \nu_\sigma(r)).$$

Proof. 1. The canonical maps $F \cong \widehat{\mathscr{D}}/\widehat{\mathscr{D}}\sigma \longrightarrow \mathscr{S}/\mathscr{S}\sigma \longrightarrow \widehat{\mathscr{D}}_{T(\sigma)}/\widehat{\mathscr{D}}_{T(\sigma)}\sigma$ are field homomorphisms and thus injective. The isomorphism $\widehat{\mathscr{D}}/\widehat{\mathscr{D}}\sigma \cong \widehat{\mathscr{D}}_{T(\sigma)}/\widehat{\mathscr{D}}_{T(\sigma)}\sigma$ from Lemma 3.17 yields the assertion.

2. The injection $F \longrightarrow \widehat{\mathscr{D}}_{T(\sigma)}$, $k \longmapsto k$, is a right inverse of

$$\nu_\sigma : \widehat{\mathscr{D}}_{T(\sigma)} \longrightarrow \widehat{\mathscr{D}}_{T(\sigma)}/\widehat{\mathscr{D}}_{T(\sigma)}\sigma = F, \quad \text{hence}$$
$$\widehat{\mathscr{D}}_{T(\sigma)} = F \oplus \ker(\nu_\sigma) = F \oplus \widehat{\mathscr{D}}_{T(\sigma)}\sigma \ni r = \nu_\sigma(r) + (r - \nu_\sigma(r)).$$

The same argument applies to the other rings.

□

Remark 3.21. The ideal $F(s)_{\mathrm{spr}} := F(s)_{\mathrm{pr}}\sigma$ of $F(s)_{\mathrm{pr}}$ is the ideal of *strictly proper* rational functions and $F(s)_{\mathrm{pr}} = F \oplus F(s)_{\mathrm{pr}}\sigma$ is a subdecomposition of the standard decomposition
$$F(s) = F[s] \oplus F(s)_{\mathrm{pr}}\sigma \ni r = r_{\mathrm{pol}} + r_{\mathrm{spr}}$$
of a rational function into its polynomial and strictly proper part.

Corollary 3.22. *For all $k, \ell \in \mathbb{N}_{>0}$ we get the decompositions*
$$\widehat{\mathscr{D}}^{k\times\ell} = F^{k\times\ell} \oplus \widehat{\mathscr{D}}^{k\times\ell}\sigma \subseteq \mathscr{S}^{k\times\ell} = F^{k\times\ell} \oplus \mathscr{S}^{k\times\ell}\sigma \subseteq F(s)_{\mathrm{pr}}^{k\times\ell} = F^{k\times\ell} \oplus F(s)_{\mathrm{pr}}^{k\times\ell}\sigma$$
$$\ni X = \nu_\sigma(X) + (X - \nu_\sigma(X)), \quad \nu_\sigma(X) := (\nu_\sigma(X_{ij}))_{i,j}, \ \nu_\sigma(Y) = Y(0) \text{ if } Y \in \widehat{\mathscr{D}}^{k\times\ell}.$$

Corollary 3.23 (Invertible elements of $F(s)_{\mathrm{pr}}$)**.** *1. Since $F(s)_{\mathrm{pr}} = \widehat{\mathscr{D}}_{T(\sigma)}$ is a discrete valuation ring with the unique prime σ (up to association) an element $r \in F(s)_{\mathrm{pr}}$ is invertible in $F(s)_{\mathrm{pr}}$, i.e., $r \in \mathrm{U}\left(F(s)_{\mathrm{pr}}\right)$ if and only if σ does not divide r in $F(s)_{\mathrm{pr}}$ or $\nu_\sigma(r) \neq 0$. For $\widehat{f} \in \widehat{\mathscr{D}}$ this implies $\widehat{f} \in \mathrm{U}(F(s)_{\mathrm{pr}}) \Leftrightarrow \nu_\sigma(\widehat{f}) = \widehat{f}(0) \neq 0$.*

2. Let $P \in F(s)_{\mathrm{pr}}^{p\times p}$. Then P is invertible in $F(s)_{\mathrm{pr}}^{p\times p}$, i.e., $P \in \mathrm{U}(F(s)_{\mathrm{pr}}^{p\times p}) = \mathrm{Gl}_p(F(s)_{\mathrm{pr}})$ if and only if $\nu_\sigma(P) = \left(\nu_\sigma(P_{ij})\right)_{i,j} \in \mathrm{Gl}_p(F)$. Especially, $\widehat{P} \in \widehat{\mathscr{D}}^{p\times p}$ is contained in $\mathrm{Gl}_p(F(s)_{\mathrm{pr}})$ if and only if $\widehat{P}(0) \in \mathrm{Gl}_p(F)$.

Proof. We only have to prove the second assertion. But $P \in \mathrm{Gl}_p(F(s)_{\mathrm{pr}})$ if and only if $\det(P) \in \mathrm{U}(F(s)_{\mathrm{pr}})$, and this is the case if and only if $\nu_\sigma(\det(P)) = \det(\nu_\sigma(P)) \neq 0$, i.e., if $\nu_\sigma(P) \in \mathrm{Gl}_p(F)$. □

Lemma 3.24. *Assume a T-stabilizable behavior $\mathscr{B}_1$ and the usual data from* (6) - (8)*. The split exact sequences*

$$\begin{array}{c} 0 \longrightarrow \widehat{\mathscr{D}}^{1\times p} \xrightarrow{\circ\left(\widehat{P}_1,-\widehat{Q}_1\right)} \widehat{\mathscr{D}}^{1\times(p+m)} \xrightarrow{\circ\begin{pmatrix}\widehat{N}_1\\ \widehat{D}_1\end{pmatrix}} \widehat{\mathscr{D}}^{1\times m} \longrightarrow 0, \\ 0 \longleftarrow \widehat{\mathscr{D}}^{1\times p} \xleftarrow{\circ\begin{pmatrix}\widehat{D}_2^0\\ \widehat{N}_2^0\end{pmatrix}} \widehat{\mathscr{D}}^{1\times(p+m)} \xleftarrow{\circ\left(-\widehat{Q}_2^0,\widehat{P}_2^0\right)} \widehat{\mathscr{D}}^{1\times m} \longleftarrow 0 \end{array} \tag{14}$$

from (7), (8) *with* $\left(-\widehat{Q}_2^0, \widehat{P}_2^0\right)\begin{pmatrix}\widehat{N}_1\\ \widehat{D}_1\end{pmatrix} = \mathrm{id}_m$ *and* $\left(\widehat{P}_1, -\widehat{Q}_1\right)\begin{pmatrix}\widehat{D}_2^0\\ \widehat{N}_2^0\end{pmatrix} = \mathrm{id}_p$ *induce the split exact sequences*

$$\begin{array}{c} 0 \longrightarrow F^{1\times p} \xrightarrow{\circ\left(\widehat{P}_1(0),-\widehat{Q}_1(0)\right)} F^{1\times(p+m)} \xrightarrow{\circ\begin{pmatrix}\widehat{N}_1(0)\\ \widehat{D}_1(0)\end{pmatrix}} F^{1\times m} \longrightarrow 0 \\ 0 \longleftarrow F^{1\times p} \xleftarrow{\circ\begin{pmatrix}\widehat{D}_2^0(0)\\ \widehat{N}_2^0(0)\end{pmatrix}} F^{1\times(p+m)} \xleftarrow{\circ\left(-\widehat{Q}_2^0(0),\widehat{P}_2^0(0)\right)} F^{1\times m} \longleftarrow 0 \end{array} \tag{15}$$

with $\left(-\widehat{Q}_2^0(0), \widehat{P}_2^0(0)\right)\begin{pmatrix}\widehat{N}_1(0)\\ \widehat{D}_1(0)\end{pmatrix} = \mathrm{id}_m$ *and* $\left(\widehat{P}_1(0), -\widehat{Q}_1(0)\right)\begin{pmatrix}\widehat{D}_2^0(0)\\ \widehat{N}_2^0(0)\end{pmatrix} = \mathrm{id}_p$*. In particular,* $\operatorname{rank}\left(\widehat{P}_1(0), -\widehat{Q}_1(0)\right) = p$*.*

Proof. Application of the functor $(-)\otimes_{\widehat{\mathscr{D}}} \widehat{\mathscr{D}}/\widehat{\mathscr{D}}\sigma = (-)\otimes_{\widehat{\mathscr{D}}} F$ to (14) furnishes (15) which is again exact since additive functors preserve *split* exact sequences. Recall that the tensor product is only right exact in general. $\square$

Lemma 3.25 (Characterization of compensators with proper transfer matrix)**.** *Assume that $\mathscr{B}_{1,T}$ is controllable and the ensuing data from* (6) - (8)*. Let* $\left(-\widehat{Q}_2, \widehat{P}_2\right) := \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) + X\left(\widehat{P}_1, -\widehat{Q}_1\right) \in \mathscr{S}^{m\times(p+m)}$ *where $X \in \mathscr{S}^{m\times p}$. According to Thm. 3.12* $\left(-\widehat{Q}_2, \widehat{P}_2\right)$ *gives rise to a properly T-stabilizing compensator if and only if* $\det(\widehat{P}_2) \neq 0$*. The following assertions are equivalent:*

1. $\det(\widehat{P}_2) \neq 0$ *and* $H_2 := \widehat{P}_2^{-1}\widehat{Q}_2$ *is proper.*
2. $\nu_\sigma(\widehat{P}_2) \in \mathrm{Gl}_m(F)$*, i.e.,* $\det(\nu_\sigma(\widehat{P}_2)) \neq 0$ *or, equivalently,* $\widehat{P}_2 \in \mathrm{Gl}_m(F(s)_{\mathrm{pr}})$*.*

Recall that for $\widehat{f}, \widehat{g} \in \widehat{\mathscr{D}}$ and $\widehat{g}(0) \neq 0$ the rational function $\widehat{g}^{-1}\widehat{f}$ is proper, $\nu_\sigma(\widehat{g}^{-1}\widehat{f}) = \widehat{g}(0)^{-1}\widehat{f}(0)$ *and* $\nu_\sigma(\widehat{P}_2) = \left(\nu_\sigma\left((\widehat{P}_2)_{ij}\right)\right)_{i,j}$*.*
The condition 2 thus characterizes the situation where both the compensator and the feedback behavior are proper.

Proof. It is obvious that the second statement implies the first one. For the other implication, apply the functor $(-)\otimes_{\mathscr{S}} \mathscr{S}/\mathscr{S}\sigma = (-)\otimes_{\mathscr{S}} F$ to the split exact sequences from (11). This furnishes the split exact sequences

$$\begin{array}{c} 0 \longrightarrow F^{1\times p} \xrightarrow{\circ\left(\widehat{P}_1(0),-\widehat{Q}_1(0)\right)} F^{1\times \ell} \xrightarrow{\circ\begin{pmatrix}\widehat{N}_1(0)\\ \widehat{D}_1(0)\end{pmatrix}} F^{1\times m} \longrightarrow 0 \quad \text{and} \\ 0 \longleftarrow F^{1\times p} \xleftarrow{\circ\begin{pmatrix}\nu_\sigma(\widehat{D}_2)\\ \nu_\sigma(\widehat{N}_2)\end{pmatrix}} F^{1\times \ell} \xleftarrow{\circ\left(-\nu_\sigma(\widehat{Q}_2),\nu_\sigma(\widehat{P}_2)\right)} F^{1\times m} \longleftarrow 0 \end{array}$$

and especially rank $\left(-\nu_\sigma(\widehat{Q}_2),\, \nu_\sigma(\widehat{P}_2)\right) = m$.

Since H_2 is proper by assumption $\nu_\sigma(H_2) \in F^{p\times m}$ is well-defined, and $\widehat{P}_2 H_2 = \widehat{Q}_2$ implies $\nu_\sigma(\widehat{P}_2)\nu_\sigma(H_2) = \nu_\sigma(\widehat{P}_2 H_2) = \nu_\sigma(\widehat{Q}_2)$, and consequently $\left(-\nu_\sigma(\widehat{Q}_2),\, \nu_\sigma(\widehat{P}_2)\right) = \nu_\sigma(\widehat{P}_2)\left(-\nu_\sigma(H_2), \mathrm{id}_m\right)$. From the fact that rank $\left(-\nu_\sigma(\widehat{Q}_2),\, \nu_\sigma(\widehat{P}_2)\right) = m$, we deduce that rank $\left(\nu_\sigma(\widehat{P})\right) = m$, i.e., that $\det\left(\nu_\sigma(\widehat{P}_2)\right) \neq 0$ or $\widehat{P}_2 \in \mathrm{Gl}_m(F(s)_{\mathrm{pr}})$.

□

Lemma 3.26. *Let $\mathscr{B}_{1,T}$ be controllable and assume the usual data from* (6) - (8). *Then the polynomial*

$$g(\Xi) := \det\left(\widehat{P}_2^0(0) - \Xi\widehat{Q}_1(0)\right) \in F[\Xi]$$

where $\Xi = (\Xi_{ij})_{1\le i\le m, 1\le j\le p}$ is non-zero. Note that, since F is an infinite *field, this signifies that $g(X_0) \neq 0$ for almost all $X_0 \in F^{m\times p}$.*

Proof. 1. Recall from Lemma 2.3 and Assumption 3.6 that

$$\widehat{P}^0 := \begin{pmatrix} \widehat{P}_1 & -\widehat{Q}_1 \\ -\widehat{Q}_2^0 & \widehat{P}_2^0 \end{pmatrix} \in \mathrm{Gl}_{p+m}(\widehat{\mathscr{D}}) \subseteq \mathrm{Gl}_{p+m}(\mathscr{S}) \subseteq \mathrm{Gl}_{p+m}(F(s)_{\mathrm{pr}}) \text{ and hence}$$

$$\widehat{P}^0(0) = \begin{pmatrix} \widehat{P}_1(0) & -\widehat{Q}_1(0) \\ -\widehat{Q}_2^0(0) & \widehat{P}_2^0(0) \end{pmatrix} \in \mathrm{Gl}_{p+m}(F) \text{ and}$$

$$\mathrm{rank}\begin{pmatrix} -\widehat{Q}_1(0) \\ \widehat{P}_2^0(0) \end{pmatrix} = m \text{ or } F^{1\times(p+m)}\begin{pmatrix} -\widehat{Q}_1(0) \\ \widehat{P}_2^0(0) \end{pmatrix} = F^{1\times m}.$$

2. Consider, more generally, any matrix $\binom{B}{C} \in F^{(p+m)\times m}$ with rank $\binom{B}{C} = m$. By induction on $\mathrm{rank}(C) =: r$ we transform $\binom{B}{C}$ into $\binom{B}{C+AB}$ with $\det(C+AB) \neq 0$ by at most $m - \mathrm{rank}(C)$ elementary row operations, the case $r = m$ being obvious. For $r < m$ we assume without loss of generality that the last row of C is linearly dependent of the preceding rows, i.e.,

$$F^{1\times m}C = \sum_{j=1}^{m-1} FC_{j-} \subsetneq F^{1\times(p+m)}\begin{pmatrix} B \\ C \end{pmatrix} = F^{1\times m}.$$

Then there exists $i \le p$ such that B_{i-} is linearly independent of all rows of C, i.e.,

$$F^{1\times m}C \subsetneq F^{1\times m}C \oplus FB_{i-} = \sum_{j=1}^{m-1} FC_{j-} \oplus F(C_{m-} + B_{i-}) = F^{1\times m}\begin{pmatrix} C_{1-} \\ \vdots \\ C_{(m-1)-} \\ C_{m-}+B_{i-} \end{pmatrix}.$$

Obviously the last matrix has rank $r+1$, and it can be obtained from C as $C+AB$ where A is the matrix with $A_{m,i} = 1$ and 0 at all other entries.

□

Theorem 3.27 (Constructive parametrization of proper and properly T-stabilizing compensators)**.** *Assume that $\mathscr{B}_1 = \left\{\binom{y_1}{u_1} \in \mathscr{F}^{p+m};\, P_1 \circ y_1 = Q_1 \circ u_1\right\}$ is T-stabilizable and the derived data from* (6) - (8). *Then*

1. *The determinant* $\det\left(\widehat{P}_2^0(0) - X_0\widehat{Q}_1(0)\right)$ *is non-zero for almost all $X_0 \in F^{m\times p}$.*

2. *The triples* $(X_0, Y, A) \in F^{m\times p} \times \mathscr{S}^{m\times p} \times \mathscr{D}^{m\times m}$ *with* $\det\left(\widehat{P}_2^0(0) - X_0\widehat{Q}_1(0)\right) \neq 0$ *and* $\det(A) \in T$ *parametrize the set of all proper and properly* T*-stabilizing compensators* $\mathscr{B}_2 = \{\binom{u_2}{y_2} \in \mathscr{F}^{p+m};\, P_2 \circ y_2 = Q_2 \circ u_2\}$ *of* $\mathscr{B}_1$ *by the following construction:*

   (a) $X := X_0 + \sigma Y \in \mathscr{S}^{m\times p}$, $\left(-\widehat{Q}_2, \widehat{P}_2\right) := \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) + X\left(\widehat{P}_1, -\widehat{Q}_1\right) \in \mathscr{S}^{m\times(p+m)}$. *Then* $\det(\widehat{P}_2) \neq 0$ *and* $H_2 := \widehat{P}_2^{-1}\widehat{Q}_2 \in F(s)_{\mathrm{pr}}^{m\times p}$.

   (b) *Let* $H_2 = P_{2,\mathrm{cont}}^{-1} Q_{2,\mathrm{cont}}$ *be a left coprime factorization of* H_2 *over* $\mathscr{D}$ *and define* $(-Q_2, P_2) := A\,(-Q_{2,\mathrm{cont}}, P_{2,\mathrm{cont}})$.

   Two such triples (X_0, Y, A) *and* (X_0', Y', A') *give rise to the same compensator if and only if* $X_0 = X_0'$, $Y = Y'$, *and* A *and* A' *are row equivalent over* $\mathscr{D}$.

3. *The transfer matrix* H_1 *is proper if and only if* $\widehat{P}_1(0) \in \mathrm{Gl}_p(F)$. *In this case* $X_0 := \widehat{Q}_2^0(0)\widehat{P}_1(0)^{-1}$ *satisfies the inequality of item 1 and gives rise to exactly those compensators with strictly proper* H_2 *(compare [27, Lemma 5.2.25]).*

4. *If the transfer matrix* H_1 *is strictly proper and hence* $\widehat{Q}_1(0) = \widehat{P}_1(0)v_\sigma(H_1) = 0$ *then* $X_0 \in F^{m\times p}$ *can be chosen arbitrarily and hence all properly* T*-stabilizing compensators are proper (compare [27, Cor. 5.2.20])*

Proof. 1. This is a consequence of the previous lemma.
2. By the results derived above, any $X \in \mathscr{S}^{m\times p}$ with $\det\left(\widehat{P}_2^0(0) - v_\sigma(X)\widehat{Q}_1(0)\right) \neq 0$ gives rise to a transfer matrix H_2 of a proper and properly T-stabilizing compensator $\mathscr{B}_2$ of $\mathscr{B}_1$, and all such transfer matrices are obtained in this fashion. Recall the decomposition $\mathscr{S}^{m\times p} = F^{m\times p} \oplus \mathscr{S}^{m\times p}\sigma \ni X = v_\sigma(X) + (X - v_\sigma(X)) =: X_0 + \sigma Y$ from Corollary 3.22.
3. The equivalence of the properness of H_1 with $\widehat{P}_1(0) \in \mathrm{Gl}_p(F)$ follows as in Lemma 3.25 where $\det(\widehat{P}_1) \neq 0$ by assumption. The inclusion

$$\mathrm{Gl}_{p+m}(F) \underset{\text{Lemma 3.26}}{\ni} \begin{pmatrix} \widehat{P}_1(0) & -\widehat{Q}_1(0) \\ -\widehat{Q}_2^0(0) & \widehat{P}_2^0(0) \end{pmatrix} =$$

$$\begin{pmatrix} \widehat{P}_1(0) & 0 \\ 0 & \mathrm{id}_m \end{pmatrix} \begin{pmatrix} \mathrm{id}_p & 0 \\ -\widehat{Q}_2^0(0) & \mathrm{id}_m \end{pmatrix} \begin{pmatrix} \mathrm{id}_p & -\widehat{P}_1(0)^{-1}\widehat{Q}_1(0) \\ 0 & \widehat{P}_2^0(0) - \widehat{Q}_2^0(0)\widehat{P}_1(0)^{-1}\widehat{Q}_1(0) \end{pmatrix} \text{ implies}$$

$$\det\left(\widehat{P}_2^0(0) - X_0\widehat{Q}_1(0)\right) = \det\left(\widehat{P}_2^0(0) - \widehat{Q}_2^0(0)\widehat{P}_1(0)^{-1}\widehat{Q}_1(0)\right) \neq 0$$

and hence the condition of item 1. The equations

$$\widehat{P}_2 H_2 = \widehat{Q}_2, \text{ hence } v_\sigma\left(\widehat{P}_2\right) v_\sigma(H_2) = v_\sigma\left(\widehat{Q}_2\right) = \widehat{Q}_2^0(0) - X_0\widehat{P}_1(0) \text{ and } v_\sigma(\widehat{P}_2) \in \mathrm{Gl}_m(F)$$

show that H_2 is strictly proper, i.e., $v_\sigma(H_2) = 0$, if and only if

$$v_\sigma\left(\widehat{Q}_2\right) = \widehat{Q}_2^0(0) - X_0\widehat{P}_1(0) = 0, \text{ i.e., } X_0 = \widehat{Q}_2^0(0)\widehat{P}_1(0)^{-1}.$$

4. By construction the determinant $\det\left(\widehat{P}_2^0(0) - X_0\widehat{Q}_1(0)\right) = \det\left(\widehat{P}_2^0(0)\right)$ is not zero for all X_0 and otherwise does not depend on the parameter X_0, hence is non-zero for all X_0. □

Remark 3.28 (Pole placement)**.** The results on pole placement in Thm. 3.15 also hold mutatis mutandis for the T-stabilizing compensators of Thm. 3.27.

Remark 3.29 (State space realizations)**.** Let the plant $\mathscr{B}_1$ be proper and let $\mathscr{B}_2$ be a compensator constructed according to Theorem 3.27. Assume that state space (Kalman) realizations

$$s\circ x_i = A_i x_i + B_i u_i,\ y_i = C_i x_i + D_i u_i,\ x_i \in \mathscr{F}^{n_i},\ y_1, u_2 \in \mathscr{F}^p,\ y_2, u_1 \in \mathscr{F}^m$$

of $\mathscr{B}_i$, $i = 1,2$, are given or constructed. Recall that every input/output behavior admits an essentially unique observable state space representation [29, Ch.5.2]. On the other hand every state space system gives rise to a unique IO behavior by elimination of the state (compare, for example, [20, Ch.6], [17, Cor. and Def.2.41, p.27]).

In order that the assumptions of Theorem 3.27 are satisfied we need that the plant Kalman equations are T-stabilizable and T-observable (=T-detectable), i.e., that for $\mathscr{B}_1^s = \{\left(\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}\right) \in \mathscr{F}^{n_1+m};\ s\circ x_1 = A_1 x_1 + B_1 u_1\}$ the quotient behavior $\mathscr{B}_{1,T}^s$ is controllable and that $\left(\begin{smallmatrix} C_1 & D_1 \\ 0 & \mathrm{id}_m \end{smallmatrix}\right) : \mathscr{B}_{1,T}^s \cong \mathscr{B}_{1,T}$ is an isomorphism. We may and do assume that the Kalman equations of the compensator $\mathscr{B}_2$ are the essentially unique observable ones. The constant matrix D_i is the constant part of the proper transfer matrix H_i. Then the proper and T-stable IO behavior $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ has the T-stable and T-observable Kalman realization (compare [28, §10.5])

$$s\circ x = Ax + Bu,\ y = Cx + Du \quad \text{with}$$

$$x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathscr{F}^{n_1+n_2},\ u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathscr{F}^{m+p},\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathscr{F}^{p+m},$$

$$D_0 := \mathrm{id}_m - D_2 D_1 \in \mathrm{Gl}_m(F),\ \begin{pmatrix} \mathrm{id}_p & -D_1 \\ -D_2 & \mathrm{id}_m \end{pmatrix}^{-1} = \begin{pmatrix} \mathrm{id}_p + D_1 D_0^{-1} D_2 & D_1 D_0^{-1} \\ D_0^{-1} D_2 & D_0^{-1} \end{pmatrix},$$

$$A := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id}_p & -D_1 \\ -D_2 & \mathrm{id}_m \end{pmatrix}^{-1} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix},$$

$$B := \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} + \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id}_p & -D_1 \\ -D_2 & \mathrm{id}_m \end{pmatrix}^{-1} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

$$C := \begin{pmatrix} \mathrm{id}_p & -D_1 \\ -D_2 & \mathrm{id}_m \end{pmatrix}^{-1} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix},\quad D := \begin{pmatrix} \mathrm{id}_p & -D_1 \\ -D_2 & \mathrm{id}_m \end{pmatrix}^{-1} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

In particular, if $\mathscr{B}_1$ is strictly proper and hence $D_1 = 0$, this yields

$$A = \begin{pmatrix} A_1 + B_1 D_2 C_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & B_1 D_2 \\ 0 & B_2 \end{pmatrix},$$

$$C = \begin{pmatrix} C_1 & 0 \\ D_2 C_1 & C_2 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & D_2 \end{pmatrix}.$$

Corollary 3.30 (State space realizations)**.** *Data of Remark 3.29. Assume a T-observable (=T-detectable) state space realization of the proper and T-stabilizable plant $\mathscr{B}_1$. Then every observable T-stabilizing compensator in state space form is the observable state space realization of a compensator $\mathscr{B}_2$ of $\mathscr{B}_1$ constructed according to Theorem 3.27. The equations of the compensator are contained in the preceding Remark 3.28. The condition for T-stability of the feedback behavior is* $\det(s\,\mathrm{id}_{n_1+n_2} - A) \in T$. *The same arguments apply to the later compensators for various control tasks. Notice that the theorems in [9, §7.5] and [28, §10.5], for instance, construct* some*, but not* all *stabilizing compensators in state space form.*

4 Tracking and disturbance rejection

Assumption 4.1. In the remainder of this paper we always consider IO behaviors

$$\mathscr{B}_1 = \left\{ \begin{pmatrix} y_1 \\ u_1 \end{pmatrix} \in \mathscr{F}^{p+m};\, P_1 \circ y_1 = Q_1 \circ u_1 \right\}, \quad (P_1, -Q_1) \in \mathscr{D}^{p\times(p+m)},\ \det(P_1) \neq 0,$$
$$\mathscr{B}_2 = \left\{ \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \in \mathscr{F}^{p+m};\, P_2 \circ y_2 = Q_2 \circ u_2 \right\}, \quad (-Q_2, P_2) \in \mathscr{D}^{m\times(p+m)},\ \det(P_2) \neq 0$$

such that $\mathscr{B}_2$ is a proper T-stabilizing compensator of the plant $\mathscr{B}_1$ for which the feedback behavior $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ is proper and, of course, T-stable. These $\mathscr{B}_2$ have been parametrized in Thm. 3.27. We use the data from Assumption 3.6, Lemma 3.10 and Thm. 3.27. Furthermore, we assume a *signal generator* behavior

$$\mathscr{B}_3 := \{ w \in \mathscr{F}^p;\, R \circ w = 0 \}, \quad R \in \mathscr{D}^{k\times p}.$$

The trajectories of $\mathscr{B}_3$ are the reference signals that shall be tracked resp. the disturbances that shall be rejected in the following.

Definition 4.2 (T-tracking and T-rejecting compensators)**.** The compensator $\mathscr{B}_2$ is called a *T-tracking compensator* resp. *T-(disturbance) rejecting compensator* of $\mathscr{B}_1$ for signals u_2 in $\mathscr{B}_3$ if $u_2 \in \mathscr{B}_3$, $u_1 = 0$, and $\begin{pmatrix} y_1 \\ y_2 \\ u_2 \\ 0 \end{pmatrix} \in \mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ imply that $e_2 := y_1 + u_2$ resp. y_1 is T-small, compare Figure 2 for an interconnection diagram. This signifies that for zero input $u_1 = 0$ the output y_1 T-tracks any signal $-u_2 \in \mathscr{B}_3$ resp. that any disturbance input $u_2 \in \mathscr{B}_3$ has no significant effect on the output y_1.

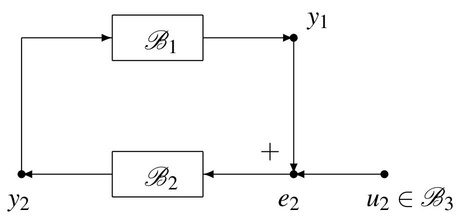

Figure 2: Tracking resp. disturbance rejection interconnection.

Corollary 4.3. *Assume that $\mathscr{B}_2$ is a T-disturbance rejecting compensator of $\mathscr{B}_1$, let $u_2 \in \mathscr{B}_3$ be a disturbance signal and let $u_1 \in \mathscr{F}^m$ be arbitrary. Furthermore, assume*

$$P \circ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Q \circ \begin{pmatrix} 0 \\ u_1 \end{pmatrix} \quad \text{and} \quad P \circ \begin{pmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \end{pmatrix} = Q \circ \begin{pmatrix} u_2 \\ u_1 \end{pmatrix},$$

i.e., $\binom{y_1}{y_2}$ is an output of $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ *to the input $\binom{0}{u_1}$, and $\begin{pmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \end{pmatrix}$ is an output to the* disturbed *input $\binom{u_2}{u_1}$. Then $P \circ \begin{pmatrix} \widetilde{y}_1 - y_1 \\ \widetilde{y}_2 - y_2 \end{pmatrix} = Q \circ \begin{pmatrix} u_2 \\ 0 \end{pmatrix}$, and hence $\widetilde{y}_1 - y_1$ is T-small. Consequently, the difference between the disturbed output $\widetilde{y}_1$ and the undisturbed output y_1 is T-negligible.*

Example 4.4. A standard choice for the behavior $\mathscr{B}_3$, in particular for its use via the internal model principle, is $R = \phi\,\mathrm{id}_p$ where $\phi \in \mathscr{D} = F[s]$ is a non-zero polynomial whose roots determine the frequencies and growth of the tracking resp. disturbance signals.

In the following considerations we derive necessary and sufficient conditions for the existence of T-tracking resp. T-rejecting compensators of a given T-stabilizable IO behavior $\mathscr{B}_1$ and parametrize all such compensators.

Theorem 4.5 (Characterization of T-tracking and T-rejecting compensators)**.** *Assumption 4.1 is in force.*

1. *The behavior $\mathscr{B}_2$ is a T-tracking compensator of $\mathscr{B}_1$ for signals $u_2 \in \mathscr{B}_3$ if and only if there is a matrix $Z_t \in \mathscr{D}_T^{p\times k}$ such that*
$$\widehat{N}_1\widehat{Q}_2 + \mathrm{id}_p = \widehat{N}_1(\widehat{Q}_2^0 - X\widehat{P}_1) + \mathrm{id}_p = Z_t R.$$
In this case $\mathrm{rank}(R) = p$ *and $\mathscr{B}_3$ is thus autonomous.*

2. *The behavior $\mathscr{B}_2$ is a T-rejecting compensator of $\mathscr{B}_1$ for signals $u_2 \in \mathscr{B}_3$ if and only if there is a matrix $Z_d \in \mathscr{D}_T^{p\times k}$ such that*
$$\widehat{N}_1\widehat{Q}_2 = \widehat{N}_1(\widehat{Q}_2^0 - X\widehat{P}_1) = Z_d R.$$

Proof. We prove 1, the proof of 2 is analogous. Recall from Theorem 3.14 that $H_{y_1,u_2} = \widehat{N}_1\widehat{Q}_2$ and $H_{y_1,u_2} + \mathrm{id}_p = H_{e_2,u_2} = \widehat{D}_2\widehat{P}_1$.

1. The feedback behavior is proper and T-stable by definition and hence especially $P \in \mathrm{Gl}_{p+m}(\mathscr{D}_T)$ and $H = P^{-1}Q \in \mathscr{D}_T^{(p+m)\times(p+m)}$. Recall that $\mathscr{D}_T$ acts on $\mathscr{F}_T$. The following equivalences hold: $\mathscr{B}_2$ is a T-tracking compensator of $\mathscr{B}_1$ for $u_2 \in \mathscr{B}_3 \Leftrightarrow$

$$\Leftrightarrow \quad \left\{ y_1 + u_2 \in \mathscr{F}^p;\, y_1, u_2 \in \mathscr{F}^p, R \circ u_2 = 0, \exists y_2 \in \mathscr{F}^m : P \circ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Q \circ \begin{pmatrix} u_2 \\ 0 \end{pmatrix} \right\}$$
is T-autonomous
$$\Leftrightarrow \quad \left\{ y_1 + u_2 \in \mathscr{F}_T^p;\, y_1, u_2 \in \mathscr{F}_T^p, R \circ u_2 = 0, \exists y_2 \in \mathscr{F}_T^m : P \circ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Q \circ \begin{pmatrix} u_2 \\ 0 \end{pmatrix} \right\}$$
$= 0$
$$\Leftrightarrow \quad \left\{ y_1 + u_2 \in \mathscr{F}_T^p;\, u_2 \in \mathscr{F}_T^p, R \circ u_2 = 0, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = H \circ \begin{pmatrix} u_2 \\ 0 \end{pmatrix} \right\} = 0$$
$$\Leftrightarrow \quad \left\{ y_1 + u_2 \in \mathscr{F}_T^p;\, u_2 \in \mathscr{F}_T^p, R \circ u_2 = 0, y_1 = \underbrace{H_{y_1,u_2}}_{=\widehat{N}_1\widehat{Q}_2} \circ u_2 \right\} = 0$$
$$\Leftrightarrow \quad \left\{ (\widehat{N}_1\widehat{Q}_2 + \mathrm{id}_p) \circ u_2 \in \mathscr{F}_T^p;\, u_2 \in \mathscr{F}_T^p, R \circ u_2 = 0 \right\} = 0$$
$$\Leftrightarrow \quad \left\{ u_2 \in \mathscr{F}_T^p;\, R \circ u_2 = 0 \right\} \subseteq \left\{ u_2 \in \mathscr{F}_T^p;\, (\widehat{N}_1\widehat{Q}_2 + \mathrm{id}_p) \circ u_2 = 0 \right\}$$
$$\Leftrightarrow \quad \exists\, Z_t \in \mathscr{D}_T^{p\times k} \quad \text{such that} \quad \widehat{N}_1\widehat{Q}_2 + \mathrm{id}_p = Z_t R.$$

The second equivalence holds by [4, Thm.1.9] and since $(-)_T$ is an exact functor on behaviors, cf. [4, Cor.1.10], the third one since $P \in \mathrm{Gl}_{p+m}(\mathscr{D}_T)$ and $H = P^{-1}Q \in \mathscr{D}_T^{(p+m)\times(p+m)}$. The last equivalence is true since $\mathscr{F}_T$ is a cogenerator over $\mathscr{D}_T$, compare [4, Thm.1.6].

2. Since $\widehat{N}_1\widehat{Q}_2+\mathrm{id}_p = H_{y_1,u_2}+\mathrm{id}_p = \widehat{D}_2\widehat{P}_1$ is non-singular, the equation

$$\widehat{D}_2\widehat{P}_1 = \widehat{N}_1\widehat{Q}_2+\mathrm{id}_p = Z_t R$$

implies that $\mathrm{id}_p = \widehat{P}_1^{-1}\widehat{D}_2^{-1}Z_t R$, hence $\mathrm{rank}(R)=p$. But this signifies that $\mathscr{B}_3$ is autonomous.

□

Corollary 4.6 (Dimension relations for tracking)**.** *Let $\mathscr{B}_1$ be a plant and consider the behavior $\mathscr{B}_3=\{w\in\mathscr{F}^p;\, R\circ w=0\}$, $R\in\mathscr{D}^{k\times p}$, containing tracking signals as above. Let $a_1\mid\ldots\mid a_s$ denote the invariant factors of the module $\mathscr{D}_T^{1\times p}/\mathscr{D}_T^{1\times k}R$, i.e., the elementary divisors of R w.r.t. $\mathscr{D}$ which are non-units of $\mathscr{D}_T$.*
1. If a T-tracking compensator $\mathscr{B}_2$ of $\mathscr{B}_1$ for signals $u_2\in\mathscr{B}_3$ exists then $s\le m$.
2. For $R=\phi\,\mathrm{id}_p$ where $\phi\in\mathscr{D}\setminus T$ is non-zero (compare Example 4.4) this signifies $p\le m$ [8, Thm.31 on p.201], [26, Cor.7.6].

Proof. From Thm. 4.5 we infer $\mathrm{rank}(R)=p$. If $s=0$ the behavior $\mathscr{B}_3$ is T-autonomous and the assertion is trivial, hence assume $s>0$. Let $URV=\begin{pmatrix}E\\0\end{pmatrix}$ be the Smith form of R w.r.t. $\mathscr{D}_T$ with $E=\mathrm{diag}(1,\ldots,1,a_1,\ldots,a_s)$. Let q be a prime of $\mathscr{D}_T$ which divides a_1 and hence all a_i with its associated canonical map (compare Lemma 3.17)

$$\nu_q:\mathscr{D}_T \xrightarrow{\mathrm{can}} \mathscr{D}_T/\mathscr{D}_T q =: k(q).$$

By Thm. 4.5.1 the equation $\widehat{N}_1\widehat{Q}_2+\mathrm{id}_p=Z_tR$ holds for some $Z_t\in\mathscr{D}_T^{p\times k}$, hence

$$\begin{aligned} p=\mathrm{rank}(\mathrm{id}_p)=\mathrm{rank}(\nu_q(\mathrm{id}_p))&=\mathrm{rank}(\nu_q(Z_t)\nu_q(R)-\nu_q(\widehat{N}_1)\nu_q(\widehat{Q}_2))\\ &\le \mathrm{rank}(\nu_q(R))+\mathrm{rank}(\nu_q(\widehat{N}_1))\le \mathrm{rank}(\nu_q(R))+m \end{aligned}\tag{16}$$

because $\widehat{N}_1\in\widehat{\mathscr{D}}^{p\times m}$. The Smith form of R implies that $\nu_q(U)\nu_q(R)\nu_q(V)=\begin{pmatrix}\nu_q(E)\\0\end{pmatrix}$, $\nu_q(E)=\mathrm{diag}(1,\ldots,1,0,\ldots,0)$ since $q|a_i$, $i=1,\ldots,s$. We deduce that $\mathrm{rank}(\nu_q(R))=\mathrm{rank}(\nu_q(E))=\mathrm{rank}\begin{pmatrix}\mathrm{diag}(1,\ldots,1,0,\ldots,0)\\0\end{pmatrix}=p-s$ since $\nu_q(U)$ and $\nu_q(V)$ are invertible over $k(q)$. Substituting this in (16), we get $p\le(p-s)+m$, i.e., $s\le m$ as asserted. □

Remark 4.7 (Signals with left bounded support, compare [8, Def.35 on p.113])**.** Consider the complex continuous standard situation with the signal module $\mathscr{F}=\mathscr{D}'(\mathbb{R},\mathbb{C})$, and denote by $Y:\mathbb{R}\longrightarrow\mathbb{C}$ the Heaviside function. Let $\mathscr{B}_2$ be a T-tracking resp. T-rejecting compensator of $\mathscr{B}_1$ for signals $u_2\in\mathscr{B}_3$. Assume that the input to the feedback behavior $\mathrm{fb}(\mathscr{B}_1,\mathscr{B}_2)$ is of the form $\binom{u_2}{u_1}=Y\begin{pmatrix}\widetilde{u}_2\\0\end{pmatrix}$ where $\widetilde{u}_2\in\mathscr{B}_3\cap\mathscr{C}^\infty(\mathbb{R},\mathbb{C})^p$, and let $\binom{y_1}{y_2}\in\mathscr{F}^{p+m}$ be the uniquely determined output with left bounded support corresponding to this input. Then it can be shown that y_1+u_2 resp. y_1 is of the form $Y\widetilde{v}$ for some T-small signal $\widetilde{v}$ in the case of tracking resp. disturbance rejection. In other words, the errors occuring by tracking resp. disturbance rejection are "truncated" T-small signals. This signifies that any T-tracking resp. T-rejecting compensator does also track resp. reject signals of the form $Y\widetilde{u}_2$ where $\widetilde{u}_2\in\mathscr{C}^\infty(\mathbb{R},\mathbb{C})^p$ is a signal in $\mathscr{B}_3$. Properness of the feedback behavior $\mathrm{fb}(\mathscr{B}_1,\mathscr{B}_2)$ (or, more precisely, of the submatrix H_{y_1,u_2} of the transfer matrix H of $\mathrm{fb}(\mathscr{B}_1,\mathscr{B}_2)$) is essential for this result.

In the following we assume that the given IO behavior $\mathscr{B}_1$ is T-stabilizable. We use the same notations as in Section 3, (6) - (8) in Assumption 3.6. We first treat the

existence of a T-tracking resp. a T-rejecting compensator $\mathscr{B}_2$ for signals $u_2 \in \mathscr{B}_3$ and then *parametrize* all such compensators.
By Theorem 4.5 and Theorem 3.27 there exists a T-tracking resp. T-rejecting compensator if and only if the equation

$$M = \widehat{N}_1 X \widehat{P}_1 + ZR \tag{17}$$

$$\text{where} \quad M := \begin{cases} \widehat{N}_1 \widehat{Q}_2^0 + \mathrm{id}_p & \text{in the case of tracking} \\ \widehat{N}_1 \widehat{Q}_2^0 & \text{in the case of disturbance rejection} \end{cases}$$

has a solution $(X, Z) \in \mathscr{S}^{m\times p} \times \mathscr{D}_T^{p\times k}$ such that

$$\left(-\widehat{Q}_2, \widehat{P}_2\right) := \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) + X\left(\widehat{P}_1, -\widehat{Q}_1\right)$$

has the correct IO structure and proper transfer matrix, i.e.,

$$\det\left(\nu_\sigma(\widehat{P}_2)\right) = \det\left(\widehat{P}_2^0(0) - \nu_\sigma(X)\widehat{Q}_1(0)\right) \neq 0.$$

This condition can be checked algorithmically due to the following considerations:
Assume that (17) has a solution $(X^0, Z^0) \in \mathscr{S}^{m\times p} \times \mathscr{D}_T^{p\times k}$. Note that (17) is an inhomogeneous linear equation in the entries of X and Z, i.e., it can be rewritten as an equation of the form $(\underline{x}, \underline{z})\left(\frac{A}{B}\right) = \underline{m}$ where $\underline{x} \in \mathscr{S}^{1\times(mp)}$ resp. $\underline{z} \in \mathscr{D}_T^{1\times(pk)}$ contains the entries of $X \in \mathscr{S}^{m\times p}$ resp. of $Z \in \mathscr{D}_T^{p\times k}$ etc. Consequently, Algorithm 7.1 and Algorithm 7.2 allow to check solvability of (17) first in $\mathscr{D}_T^{m\times p} \times \mathscr{D}_T^{p\times k}$, then in $\mathscr{S}^{m\times p} \times \mathscr{D}_T^{p\times k}$, and to compute such a matrix (X^0, Z^0).

Now consider the associated homogeneous equation

$$\widehat{N}_1 X \widehat{P}_1 + ZR = 0 \tag{18}$$

and its solution module over $\widehat{\mathscr{D}}$

$$\left\{(X, Z) \in \widehat{\mathscr{D}}^{m\times p} \times \widehat{\mathscr{D}}^{p\times k}; (18)\right\} = \sum_{i=1}^{\mu} \widehat{\mathscr{D}}(X^{h,i}, Z^{h,i}). \tag{19}$$

Again, since (18) is a linear equation in the entries of X and Z, Algorithm 7.1 (over the ring $R = \widehat{\mathscr{D}}$) can be applied in order to compute the matrices $(X^{h,i}, Z^{h,i}) \in \widehat{\mathscr{D}}^{m\times p} \times \widehat{\mathscr{D}}^{p\times k}$ appearing in (19).

Equation (19) implies that

$$\mathscr{X}^h := \left\{X \in \widehat{\mathscr{D}}^{m\times p}; \exists Z \in \widehat{\mathscr{D}}^{p\times k} : (18)\right\} = \sum_{i=1}^{\mu} \widehat{\mathscr{D}} X^{h,i}$$

and, since localization preserves exactness, that

$$\mathscr{D}_T \mathscr{X}^h = \left\{X \in \mathscr{D}_T^{m\times p}; \exists Z \in \mathscr{D}_T^{p\times k} : (18)\right\} = \sum_{i=1}^{\mu} \mathscr{D}_T X^{h,i}. \tag{20}$$

By means of Algorithm 7.4, again after arranging the entries of the matrices $X^{h,i} \in \widehat{\mathscr{D}}^{m\times p}$ as rows $\underline{x}^{h,i} \in \widehat{\mathscr{D}}^{1\times(mp)}$, we determine $B^{(1)}, \ldots, B^{(\nu)} \in \widehat{\mathscr{D}}^{m\times p}$ such that

$$\widehat{\mathscr{D}}_\sigma \mathscr{X}^h \cap \widehat{\mathscr{D}}^{m\times p} = \bigoplus_{j=1}^{\nu} \widehat{\mathscr{D}} B^{(j)}. \tag{21}$$

Note that the $B^{(j)} \in \widehat{\mathscr{D}}^{m\times p}$ are computed by means of $\widehat{N}_1$, $\widehat{P}_1$, and R, i.e., the $B^{(j)}$ depend on $\mathscr{B}_1$ and $\mathscr{B}_3$, but not on T.

Equation (20) and application of $(-)_{\widehat{T}}$ (where $\widehat{\mathscr{D}}_{\widehat{T}} = \mathscr{S}$ and $(\widehat{\mathscr{D}}_\sigma)_{\widehat{T}} = \mathscr{D}_T$) to (21) imply

$$\left\{X \in \mathscr{S}^{m\times p};\, \exists Z \in \mathscr{D}_T^{p\times k}:\ (18)\right\} = \mathscr{D}_T\mathscr{X}^h \cap \mathscr{S}^{m\times p} = \bigoplus_{j=1}^{\nu} \mathscr{S}B^{(j)}. \tag{22}$$

Consequently, considering again the inhomogeneous equation (17) with its solution $(X^0, Z^0) \in \mathscr{S}^{m\times p} \times \mathscr{D}_T^{p\times k}$, we get the result

$$\left\{X \in \mathscr{S}^{m\times p};\, \exists Z \in \mathscr{D}_T^{p\times k}:\ (17)\right\} = X^0 + \left(\mathscr{D}_T\mathscr{X}^h \cap \mathscr{S}^{m\times p}\right) = X^0 + \bigoplus_{j=1}^{\nu} \mathscr{S}B^{(j)}. \tag{23}$$

Let now $X := X^0 + \sum_{j=1}^{\nu} \eta_j B^{(j)}$, $\eta = (\eta_1, \dots, \eta_\nu) \in \mathscr{S}^\nu$ arbitrarily, be any element of $X^0 + (\mathscr{D}_T\mathscr{X}^h \cap \mathscr{S}^{m\times p})$. Then the matrix $\left(-\widehat{Q}_2, \widehat{P}_2\right) := \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) + X\left(\widehat{P}_1, -\widehat{Q}_1\right)$ defines a T-tracking resp. T-rejecting compensator if and only if $\det\left(\nu_\sigma(\widehat{P}_2)\right) \neq 0$, i.e.,

$$\begin{aligned}\det\left(\nu_\sigma(\widehat{P}_2)\right) &= \det\left(\widehat{P}_2^0(0) - \nu_\sigma(X)\widehat{Q}_1(0)\right)\\ &= \det\left(\widehat{P}_2^0(0) - \left[\nu_\sigma(X^0) + \sum_{j=1}^{\nu} \nu_\sigma(\eta_j)B^{(j)}(0)\right]\widehat{Q}_1(0)\right) \neq 0.\end{aligned}$$

Remember that $\nu_\sigma(\widehat{f}) = \widehat{f}(0)$ for $\widehat{f} \in \widehat{\mathscr{D}}$, and $\nu_\sigma(r) = \nu_\sigma\left(\frac{\widehat{f}}{\widehat{g}}\right) = \frac{\widehat{f}(0)}{\widehat{g}(0)}$ for $r \in F(s)_{\mathrm{pr}}$, $\widehat{f}, \widehat{g} \in \widehat{\mathscr{D}}$, $\widehat{g}(0) \neq 0$. Define the polynomial

$$g(\Xi) := \det\left(\widehat{P}_2^0(0) - \left[\nu_\sigma(X^0) + \sum_{j=1}^{\nu} \Xi_j B^{(j)}(0)\right]\widehat{Q}_1(0)\right) \in F[\Xi] \tag{24}$$

in the indeterminates $\Xi = (\Xi_1, \dots, \Xi_\nu)$. Notice that the polynomial g depends on X^0 and the $B^{(j)}$ which in turn depend on $\mathscr{B}_1$ and $\mathscr{B}_3$, but not on T. This will be important for the discussion of spectral assignability.

We summarize the preceding considerations in the following theorems:

Theorem 4.8 (Existence of T-tracking and T-rejecting compensators)**.** *For a plant $\mathscr{B}_1$ and a signal generator $\mathscr{B}_3$ with the notations from above the following conditions are equivalent:*

1. There exists a T-tracking resp. T-rejecting compensator $\mathscr{B}_2$ of $\mathscr{B}_1$ for signals in $\mathscr{B}_3$.

2. (a) $\mathscr{B}_1$ is T-stabilizable, i.e., $(P_1, -Q_1)$ has a right inverse matrix in $\mathscr{D}_T^{(p+m)\times m}$,

(b) the equation (17)

$$M = \widehat{N}_1 X \widehat{P}_1 + ZR \tag{25}$$

$$\textit{where}\quad M := \begin{cases} \widehat{N}_1\widehat{Q}_2^0 + \mathrm{id}_p & \textit{in the case of tracking} \\ \widehat{N}_1\widehat{Q}_2^0 & \textit{in the case of disturbance rejection}\end{cases}$$

has a solution $(X^0, Z^0) \in \mathscr{S}^{m\times p} \times \mathscr{D}_T^{p\times k}$, and *(c) the polynomial g from* (24) *is non-zero.*

Proof. The assertion follows directly from Section 3, Theorem 4.5, and the considerations from above. $\square$

Theorem 4.9 (Constructive parametrization of T-tracking and T-rejecting compensators)**.** *Assume that the conditions of the previous theorem are satisfied. Then all T-tracking resp. T-rejecting compensators are obtained in the following fashion:*

1. *Let $\xi = (\xi_1, \ldots, \xi_\nu) \in F^\nu$ be a non-zero of the polynomial g, i.e., $g(\xi_1, \ldots, \xi_\nu) \neq 0$. Note that, since $g \neq 0$ and F is an infinite field, almost all $\xi \in F^\nu$ satisfy this condition.*

2. *Choose arbitrary $\zeta_1, \ldots, \zeta_\nu \in \mathscr{S}$ and define*
$$\eta_j := \xi_j + \sigma\zeta_j, \quad j = 1, \ldots, \nu, \quad X := X^0 + \sum_{j=1}^{\nu} \eta_j B^{(j)}, \quad \textit{and}$$
$$\left(-\widehat{Q}_2, \widehat{P}_2\right) := \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) + X\left(\widehat{P}_1, -\widehat{Q}_1\right).$$
Then $\det\left(\nu_\sigma(\widehat{P}_2)\right) = g(\xi_1, \ldots, \xi_\nu) \neq 0$*, hence* $\widehat{P}_2 \in \mathrm{Gl}_m(F(s)_{\mathrm{pr}})$ *and*
$$H_2 := \widehat{P}_2^{-1}\widehat{Q}_2 \in F(s)_{\mathrm{pr}}^{m \times p}.$$

3. *Let $H_2 = P_{2,\mathrm{cont}}^{-1} Q_{2,\mathrm{cont}}$ be a left coprime factorization of H_2 over $\mathscr{D}$, choose $A \in \mathscr{D}^{m \times m}$ with $\det(A) \in T$, and define $(-Q_2, P_2) := A(-Q_{2,\mathrm{cont}}, P_{2,\mathrm{cont}})$. Then $\mathscr{B}_2 := \{\binom{u_2}{y_2} \in \mathscr{F}^{p+m}; P_2 \circ y_2 = Q_2 \circ u_2\}$ is a T-tracking resp. T-rejecting compensator of $\mathscr{B}_1$ for $u_2 \in \mathscr{B}_3$, and all such compensators can be obtained in this fashion.*

In other terms: The T-tracking resp. T-rejecting compensators are parametrized by the triples $(\xi, \zeta, A) \in F^\nu \times \mathscr{S}^\nu \times \mathscr{D}^{m \times m}$ with $g(\xi) \neq 0$ and $\det(A) \in T$.

Proof. Follows directly from the above considerations. $\square$

Corollary 4.10. *Theorem 3.27.4 implies that the requirement that g is non-zero resp. that ξ is a non-zero of g in Theorem 4.8 resp. 4.9 is automatically satisfied if the plant $\mathscr{B}_1$ is* strictly proper.

In the following we treat the problem of pole placement or spectral assignability for tracking and disturbance rejection.

Theorem 4.11 (Pole placement for tracking and disturbance rejection)**.** *Consider a plant $\mathscr{B}_1$ and the signal generator $\mathscr{B}_3$ with the notations from above. Choose $\alpha \in F$ and define $\sigma := (s-\alpha)^{-1}$ and $\widehat{\mathscr{D}} := F[\sigma]$ as always. Let $e_p \in \mathscr{D} = F[s]$ be the greatest elementary divisor of $R_1 = (P_1, -Q_1)$ w.r.t. $\mathscr{D}$ as in* (13).

1. *Assume that* (17) *has a solution in $F(s)_{\mathrm{pr}}^{m \times p} \times F(s)^{p \times k}$; this can be checked by means of Algorithms 7.1 and 7.2 with $T = F[s] \setminus \{0\}$. Then Algorithm 7.1.2 yields a "minimal" polynomial $t_2 \in \mathscr{D} = F[s]$ such that* (17) *has a solution in $\mathscr{D}_{t_2}^{m \times p} \times \mathscr{D}_{t_2}^{p \times k}$. Let $t_{\min} := (s-\alpha)e_p t_2$, $T_{\min}$ the saturated monoid of all divisors of powers of t_{min}, i.e.*
$$T_{\min} = \left\{\beta \prod_q q^{m(q)} \in F[s];\ 0 \neq \beta \in F,\ m(q) \geq 0,\ q \textit{ an irreducible factor of } t_{\min}\right\},$$

and $\mathscr{S}_{\min} := \mathscr{D}_{T_{\min}} \cap F(s)_{\mathrm{pr}}$*. Algorithm 7.2 furnishes a solution* $(X^0, Z^0) \in \mathscr{S}_{\min}^{m\times p} \times \mathscr{D}_{T_{\min}}^{p\times k}$ *of* (17)*. Define the polynomial* g *from* (24) *with this* X^0*. Then* $\mathscr{B}_1$ *admits a* $T_{\min}$*-tracking resp.* $T_{\min}$*-rejecting compensator if and only if* $g \neq 0$.

2. *If* $T \subseteq \mathscr{D} \setminus \{0\}$ *is any saturated monoid containing* $(s-\alpha)$ *then* $\mathscr{B}_1$ *admits a* T*-tracking resp.* T*-rejecting compensator if and only if* (17) *has a solution in* $F(s)_{\mathrm{pr}}^{m\times p} \times F(s)^{p\times k}$*, the polynomial* g *from item 1 is non-zero, and* $T_{\min} \subseteq T$*. In particular,* $T_{\min}$ *is the least saturated monoid* T *containing* $(s-\alpha)$ *with a* T*-tracking resp. rejecting compensator for* $\mathscr{B}_1$*. All such compensators can be constructed via Thm. 4.9 with* X^0 *and* g *from item 1.*
3. *If* (17) *has a solution in* $F(s)_{\mathrm{pr}}^{m\times p} \times F(s)^{p\times k}$*, the polynomial* g *from item 1 is non-zero,* $t_3 \in F[s]$ *is any multiple of* $(s-\alpha)e_p t_2$*, and hence the saturated monoid* T_3 *of all divisors of powers of* t_3 *contains* $T_{\min}$*, then there are* T_3*-tracking resp.* T_3*-rejecting compensators of* $\mathscr{B}_1$ *and all such compensators can be constructed via Thm. 4.9 applied to* T_3 *with the data from item 1.*
4. *For* $F = \mathbb{R}$ *and* $\mathscr{D} = \mathbb{R}[s]$ *in item 3 the poles of all feedback behaviors with the compensators from item 3 are contained in the finite set of complex numbers*

$$\mathrm{V}_{\mathbb{C}}(t_3) \supseteq \mathrm{V}_{\mathbb{C}}(t_{\min}) = \{\alpha\} \cup \mathrm{V}_{\mathbb{C}}(e_p) \cup \mathrm{V}_{\mathbb{C}}(t_2) = \{\alpha\} \cup \mathrm{ch}(\mathscr{B}_1) \cup \mathrm{V}_{\mathbb{C}}(t_2).$$

Proof. 1. follows directly from Theorem 4.8 applied to $\mathscr{B}_1$, $\mathscr{B}_3$, and $T_{\min}$.

2. $\Rightarrow$: Assume that a T-tracking resp. rejecting compensator of $\mathscr{B}_1$ exists. In particular, equation (17) has a solution in $(\mathscr{D}_T \cap F(s)_{\mathrm{pr}})^{m\times p} \times \mathscr{D}_T^{p\times k} \subseteq F(s)_{\mathrm{pr}}^{m\times p} \times F(s)^{p\times k}$. This implies that the data $T_{\min}$, $(X^0, Z^0) \in (\mathscr{D}_{T_{\min}} \cap F(s)_{\mathrm{pr}})^{m\times p} \times \mathscr{D}_{T_{\min}}^{p\times k}$, and g from item 1 can be constructed. Since the monoid $T_{\min}$ is the least saturated one such that $(s-\alpha) \in T_{\min}$, $\mathscr{B}_1$ is $T_{\min}$-stabilizable, and (17) has a solution in $\mathscr{D}_{T_{\min}}^{m\times p} \times \mathscr{D}_{T_{\min}}^{p\times k}$ we conclude that $T_{\min} \subseteq T$. Also $(X^0, Z^0) \in (\mathscr{D}_T \cap F(s)_{\mathrm{pr}})^{m\times p} \times \mathscr{D}_T^{p\times k}$. Hence the data (X^0, Z^0) and g can be used both for $T_{\min}$ and for T in Theorem 4.8. The existence of a T-tracking resp. rejecting compensator and Theorem 4.8 then imply $g \neq 0$.

$\Leftarrow$: According to item 1 there is a $T_{\min}$-tracking resp. rejecting compensator. Because of $T_{\min} \subseteq T$ this is also a T-tracking resp. rejecting compensator.

3. The assertions in item 3 and 4 follow directly from item 2.

□

Finally, we study the problem of *simultaneously* T-tracking signals in one behavior $\mathscr{B}_t$ and T-rejecting signals in another behavior $\mathscr{B}_d$. We also admit disturbances at the input u_1 of the plant.

Corollary 4.12 (Simultaneous tracking and disturbance rejection)**.** *Assume that* $\mathscr{B}_1$ *is* T*-stabilizable and three behaviors*

$$\begin{aligned}
\mathscr{B}_{1,d} &= \left\{u_1 \in \mathscr{F}^m;\, R_{1,d} \circ u_1 = 0\right\},\\
\mathscr{B}_{2,d} &= \left\{u_2 \in \mathscr{F}^p;\, R_{2,d} \circ u_2 = 0\right\}, \quad \textit{and}\\
\mathscr{B}_{2,t} &= \left\{u_2 \in \mathscr{F}^p;\, R_{2,t} \circ u_2 = 0\right\}.
\end{aligned}$$

Then there is a T-stabilizing compensator which simultaneously rejects disturbances $u_1 \in \mathscr{B}_{1,d}$ at the input and $u_2 \in \mathscr{B}_{2,d}$ at the output and tracks signals $u_2 \in \mathscr{B}_{2,t}$ if and only if the inhomogeneous linear system

$$\begin{aligned} \widehat{D}_2^0 \widehat{Q}_1 &= \widehat{N}_1 X \widehat{Q}_1 + Z_{1,d} R_{1,d}, \\ \widehat{N}_1 \widehat{Q}_2^0 &= \widehat{N}_1 X \widehat{P}_1 + Z_{2,d} R_{2,d}, \\ \widehat{D}_2^0 \widehat{P}_1 &= \widehat{N}_1 X \widehat{P}_1 + Z_{2,t} R_{2,t} \end{aligned}$$

is solvable such that $X \in \mathscr{S}^{m\times p}$ and the Z_k have entries in $\mathscr{D}_T$ and the ensuing polynomial corresponding to g from (24) *is non-zero. All preceding results and proofs of this section are applicable to this more general situation.*

5 Model matching

Consider three IO behaviors

$$\begin{aligned} \mathscr{B}_1 &= \left\{ \begin{pmatrix} y_1 \\ u_1 \end{pmatrix} \in \mathscr{F}^{p+m};\ P_1 \circ y_1 = Q_1 \circ u_1 \right\}, \\ \mathscr{B}_2 &= \left\{ \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \in \mathscr{F}^{p+m};\ P_2 \circ y_2 = Q_2 \circ u_2 \right\}, \quad \text{and} \\ \mathscr{B}_m &= \left\{ \begin{pmatrix} y_m \\ u_2 \end{pmatrix} \in \mathscr{F}^{p+p};\ P_m \circ y_m = Q_m \circ u_2 \right\},\ H_m := P_m^{-1} Q_m. \end{aligned}$$

Definition 5.1 (Model matching T-compensators)**.** Under Assumption 4.1 we call $\mathscr{B}_2$ a *model matching T-compensator* of $\mathscr{B}_1$ for the model behavior $\mathscr{B}_m$ if $y_1 - y_m$ is T-small whenever $\begin{pmatrix} y_1 \\ y_2 \\ u_2 \\ 0 \end{pmatrix} \in \operatorname{fb}(\mathscr{B}_1, \mathscr{B}_2)$ and $\binom{y_m}{u_2} \in \mathscr{B}_m$.

Theorem 5.2 (Characterization of model matching T-compensators)**.** *Under assumption 4.1 the compensator $\mathscr{B}_2$ is a model matching T-compensator of $\mathscr{B}_1$ for $\mathscr{B}_m$ if and only if $\mathscr{B}_m$ is T-stable and $H_m = H_{y_1,u_2}$.*

Proof. 1. Assume that $\mathscr{B}_2$ is such a compensator. For any $y_m \in \mathscr{B}_m^0$ the equations $P \circ \binom{0}{0} = Q \circ \binom{0}{0}$ and $P_m \circ y_m = Q_m \circ 0$ imply that $y_m = y_m - 0$ is T-small. But this signifies that $\mathscr{B}_m$ is T-stable, i.e., $P_m \in \mathrm{Gl}_p(\mathscr{D}_T)$ and $H_m = P_m^{-1} Q_m \in \mathscr{D}_T^{p\times p}$.
2. We now show the equivalence of the two conditions under the assumption that $\mathscr{B}_m$ is T-stable, i.e., $P_m \in \mathrm{Gl}_p(\mathscr{D}_T)$. By definition $\mathscr{B}_2$ is a model matching T-compensator of $\mathscr{B}_1$ for $\mathscr{B}_m$ if and only if

$$\Big\{ y_1 - y_m \in \mathscr{F}^p;\ y_1, y_m \in \mathscr{F}^p,\ \exists u_2 \in \mathscr{F}^p\ \exists y_2 \in \mathscr{F}^m \text{ with } P \circ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Q \circ \begin{pmatrix} u_2 \\ 0 \end{pmatrix},\ P_m \circ y_m = Q_m \circ u_2 \Big\}$$

is T-autonomous. This is equivalent to

$$\Big\{ y_1 - y_m \in \mathscr{F}_T^p;\ y_1, y_m \in \mathscr{F}_T^p,\ \exists u_2 \in \mathscr{F}_T^p\ \exists y_2 \in \mathscr{F}_T^m \text{ with } P \circ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Q \circ \begin{pmatrix} u_2 \\ 0 \end{pmatrix},\ P_m \circ y_m = Q_m \circ u_2 \Big\} = 0.$$

Since $P \in \mathrm{Gl}_{p+m}(\mathscr{D}_T)$ and $P_m \in \mathrm{Gl}_p(\mathscr{D}_T)$, we can rewrite this as

$$\left\{y_1 - y_m \in \mathscr{F}_T^p; \exists u_2 \in \mathscr{F}_T^p : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = H \circ \begin{pmatrix} u_2 \\ 0 \end{pmatrix}, y_m = H_m \circ u_2 \right\} = 0,$$

or, equivalently, as

$$\left\{y_1 - y_m \in \mathscr{F}_T^p; \exists u_2 \in \mathscr{F}_T^p : y_1 = H_{y_1,u_2} \circ u_2, y_m = H_m \circ u_2 \right\} = 0, \quad \text{i.e.,}$$
$$(H_{y_1,u_2} - H_m) \circ u_2 = 0 \quad \text{for} \quad u_2 \in \mathscr{F}_T^p.$$

Since $\mathscr{F}_T$ is an injective cogenerator over $\mathscr{D}_T$ this is equivalent to $H_{y_1,u_2} = H_m$. □

Theorem 5.3 (Existence of model matching T-compensators). *For given IO behaviors $\mathscr{B}_1$ and $\mathscr{B}_m$ the following two conditions are equivalent:*

1. *There exists a model matching T-compensator $\mathscr{B}_2$ of $\mathscr{B}_1$ for the model behavior $\mathscr{B}_m$.*
2. (a) *The matrix $(P_1, -Q_1)$ has a right inverse matrix in $\mathscr{D}_T^{(p+m)\times p}$, i.e., $\mathscr{B}_1$ is T-stabilizable,*

   (b) *$\mathscr{B}_m$ is T-stable, i.e., $P_m \in \mathrm{Gl}_p(\mathscr{D}_T)$,*

   (c) *with the notations from Assumption 3.6 the equation*

   $$\widehat{N}_1 \widehat{Q}_2^0 - H_m = \widehat{N}_1 X \widehat{P}_1 \tag{26}$$

   has a solution $X^0 \in \mathscr{S}^{m\times p}$, and

   (d) *the polynomial*

   $$g(\Xi) := \det\left(\widehat{P}_2^0(0) - \left[\nu_\sigma(X^0) + U(0)\Xi\right] \widehat{Q}_1(0)\right) \in F[\Xi] \tag{27}$$

   in the indeterminates $\Xi = (\Xi_{ij})_{1\le i\le m-r, 1\le j\le p}$ is non-zero where $r := \mathrm{rank}(\widehat{N}_1)$ and $U \in \widehat{\mathscr{D}}^{m\times(m-r)}$ denotes a universal right annihilator of $\widehat{N}_1$ (cf. [5, Def. and Lem.2.7]).

Proof. By Thm. 5.2 we assume the necessary conditions 2a and 2b and have to look for compensators $\mathscr{B}_2$ with $H_{y_1,u_2} = H_m$ among those parametrized in Theorem 3.27, i.e., with

$$\left(-\widehat{Q}_2, \widehat{P}_2\right) := \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) + X\left(\widehat{P}_1, -\widehat{Q}_1\right), \; X \in \mathscr{S}^{m\times p},$$
$$H_m = H_{y_1,u_2} = \widehat{N}_1\widehat{Q}_2 = \widehat{N}_1\widehat{Q}_2^0 - \widehat{N}_1 X \widehat{P}_1, \; \det\left(\nu_\sigma(\widehat{P}_2)\right) = \det\left(\widehat{P}_2^0(0) - \nu_\sigma(X)\widehat{Q}_1(0)\right) \neq 0.$$

These equations and inequalities are solvable if and only if conditions 2c and 2d are satisfied. Since $\det(\widehat{P}_1) \neq 0$ all solutions of (26) are of the form

$$X = X^0 + US, \; S \in \mathscr{S}^{(m-r)\times p}, \text{ hence } \nu_\sigma(X) = \nu_\sigma(X^0) + U(0)\nu_\sigma(S).$$

□

Theorem 5.4 (Constructive parametrization of model matching T-compensators). *Assume that the conditions of Thm. 5.3 are satisfied and use X^0 and g from that theorem. Then all model matching T-compensators are obtained in the following fashion:*

1. *Let $\xi = (\xi_{ij})_{1\leq i\leq m-r, 1\leq j\leq p} \in F^{(m-r)\times p}$ be a non-zero of the polynomial g from* (27). *Note that, since $g \neq 0$ and F is an infinite field, almost all $\xi \in F^{(m-r)\times p}$ satisfy this condition.*

2. *Choose an arbitrary matrix $\zeta \in \mathscr{S}^{(m-r)\times p}$ and define*

$$S := \xi + \sigma\zeta \in \mathscr{S}^{(m-r)\times p},\ X := X^0 + US \text{ and}$$
$$\left(-\widehat{Q}_2, \widehat{P}_2\right) := \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) + X\left(\widehat{P}_1, -\widehat{Q}_1\right). \text{ Then}$$
$$\det\left(\nu_\sigma(\widehat{P}_2)\right) \neq 0,\ \widehat{P}_2 \in \mathrm{Gl}_m(F(s)_{\mathrm{pr}}),\ \text{and } H_2 := \widehat{P}_2^{-1}\widehat{Q}_2 \in F(s)_{\mathrm{pr}}^{m\times p}.$$

3. *Let $R_{2,\mathrm{cont}} := (-Q_{2,\mathrm{cont}}, P_{2,\mathrm{cont}})$ be the controllable realization of H_2 over $\mathscr{D}$, choose an arbitrary matrix $A \in \mathscr{D}^{m\times m}$ with $\det(A) \in T$, and define $R_2 := (-Q_2, P_2) := AR_{2,\mathrm{cont}}$. Then $\mathscr{B}_2 := \{\binom{u_2}{y_2} \in \mathscr{F}^{p+m}; P_2 \circ y_2 = Q_2 \circ u_2\}$ is a model matching T-compensator of $\mathscr{B}_1$ for the model behavior $\mathscr{B}_m$, and all such compensators can be obtained in this fashion.*

In other terms: The model matching T-compensators are parametrized by the triples $(\xi, \zeta, A) \in F^{(m-r)\times p} \times \mathscr{S}^{(m-r)\times p} \times \mathscr{D}^{m\times m}$ with $g(\xi) \neq 0$ and $\det(A) \in T$.

Proof. The assertions follow directly from Thm. 5.3. □

Remark 5.5. The two previous theorems are constructive: The conditions 2a and 2c in Theorem 5.3 can be checked by means of Algorithm 7.1 (by transposing the occuring equations, in the case of (26) after multiplying with $\widehat{P}_1^{-1}$ from the right), the universal right annihilator U can be computed as described in [5, Def. and Lem.2.7] (again by transposing).

Finally, we treat the question of pole placement or spectral assignability for model matching. The following result is an analogue of Theorem 4.11.

Theorem 5.6 (Pole placement for model matching)**.** *Consider the plant and model*

$$\mathscr{B}_1 = \left\{\binom{y_1}{u_1} \in \mathscr{F}^{p+m}; P_1 \circ y_1 = Q_1 \circ u_1\right\} \text{ and } \mathscr{B}_m = \left\{\binom{y_m}{u_2} \in \mathscr{F}^{p+p}; P_m \circ y_m = Q_m \circ u_2\right\}.$$

Choose $\alpha \in F$ and define $\sigma := (s-\alpha)^{-1}$ and $\widehat{\mathscr{D}} := F[\sigma]$ as always. Let e_p^1 be the greatest elementary divisor of $R_1 = (P_1, -Q_1)$ w.r.t. $\mathscr{D}$ as in (13) *and $e_p^m \in \mathscr{D}$ the greatest elementary divisor of P_m, i.e., minimally such that $\mathscr{B}_m$ is $\langle e_p^m\rangle$-stable.*

1. *Assume that* (26) *has a solution in $F(s)_{\mathrm{pr}}^{m\times p}$. Then Algorithm 7.1.2 yields a "minimal" polynomial $t_2 \in \mathscr{D} = F[s]$ such that* (26) *has a solution in $\mathscr{D}_{t_2}^{m\times p}$. Define $t_{\min} := (s-\alpha)e_p^1 e_p^m t_2$, $T_{\min}$ as the saturated monoid of all divisors of powers of $t_{\min}$ and $\mathscr{S}_{\min} := \mathscr{D}_{T_{\min}} \cap F(s)_{\mathrm{pr}}$. Then Algorithms 7.1 and 7.2 furnish a solution $X^0 \in \mathscr{S}_{\min}^{m\times p}$ of* (26) *which gives rise to the polynomial g from* (27).

2. *Let T be any saturated monoid containing $(s-\alpha)$. Then $\mathscr{B}_1$ admits a model matching T-compensator if and only if* (26) *has a solution in $F(s)_{\mathrm{pr}}^{m\times p}$, the polynomial g from item 1 is non-zero, and $T_{\min} \subseteq T$. In particular, $T_{\min}$ is the least saturated monoid T which contains $(s-\alpha)$ with a model matching T-compensator.*

3. *Assume that* (26) *has a solution in* $F(s)_{\mathrm{pr}}^{m\times p}$*, the polynomial g from item 1 is nonzero,* t_3 *is any multiple of* $t_{\min}$ *and* T_3 *is the saturated monoid of all divisors of powers of* t_3 *and hence* $T_3 \supseteq T_{\min}$*. Then there are* T_3*-model matching compensators of* $\mathscr{B}_1$ *for the model* $\mathscr{B}_m$ *and all these can be constructed via Thm. 5.4 applied to* T_3 *with* X^0 *and g from item 1.*

4. *If in item 3* $F=\mathbb{R}$ *and* $\mathscr{D}=\mathbb{R}[s]$*, then all poles of feedback behaviors constructed with the* T_3*-model matching compensators are zeros of* t_3*.*

Proof. The proof proceeds parallel to that of Theorem 4.11. □

6 Decoupling

We consider IO behaviors $\mathscr{B}_1$ and $\mathscr{B}_2$ as in Assumption 4.1.

Definition 6.1 (Decoupling T-compensators)**.** The compensator $\mathscr{B}_2$ is called a *decoupling T-compensator* of $\mathscr{B}_1$ if H_{y_1,u_2} is diagonal. Recall $H_{y_1,u_2}=\widehat{N}_1\widehat{Q}_2$ from Theorem 3.14.

Obviously, there exists a decoupling T-compensator $\mathscr{B}_2$ of a given T-stabilizable behavior $\mathscr{B}_1$ if and only if there is a matrix $X\in\mathscr{S}^{m\times p}$ such that $\widehat{N}_1\widehat{Q}_2^0-\widehat{N}_1X\widehat{P}_1$ is a diagonal matrix and $\left(-\widehat{Q}_2,\widehat{P}_2\right):=\left(-\widehat{Q}_2^0,\widehat{P}_2^0\right)+X\left(\widehat{P}_1,-\widehat{Q}_1\right)$ has the correct IO structure and proper transfer matrix, i.e., $\det\left(\nu_\sigma(\widehat{P}_2)\right)\neq 0$. Diagonality of $\widehat{N}_1\widehat{Q}_2^0-\widehat{N}_1X\widehat{P}_1$ signifies that $\widehat{N}_1\widehat{Q}_2^0-\widehat{N}_1X\widehat{P}_1=\mathrm{diag}(Z_1,\dots,Z_p)$ for some $(Z_1,\dots,Z_p)\in\mathscr{D}_T^{1\times p}$.

The problem of decoupling can hence be treated completely along the lines of the theory on tracking and disturbance rejection displayed in Section 4: Substitute Equation (17) by

$$\widehat{N}_1\widehat{Q}_2^0=\widehat{N}_1X\widehat{P}_1+\mathrm{diag}(Z) \tag{28}$$

where $X\in\mathscr{S}^{m\times p}$ and $Z=(Z_1,\dots,Z_p)\in\mathscr{D}_T^{1\times p}$. Since (28) is again an inhomogeneous linear equation in the entries of X and $Z_1,\dots,Z_p$, the existence of solutions $(X,Z)\in\mathscr{S}^{m\times p}\times\mathscr{D}_T^{1\times p}$ of (28) can be checked and one such solution (X^0,Z^0) can be computed by means of Algorithm 7.1 and Algorithm 7.2. Analogously to the derivations in Section 4, a parametrization of all $X\in\mathscr{S}^{m\times p}$ satisfying (28) for some $\mathrm{diag}(Z_1,\dots,Z_p)\in\mathscr{D}_T^{p\times p}$ can be obtained, leading to the appropriate definition of a polynomial $g(\Xi)\in F[\Xi]$, $\Xi=(\Xi_1,\dots,\Xi_\nu)$.

The characterization of the existence of T-tracking resp. T-rejecting compensators in Theorem 4.8 and the constructive parametrization of all such compensators in Theorem 4.9 hold mutatis mutandis for the case of decoupling T-compensators. Also the results on pole placement remain valid.

7 Algorithms

Algorithm 7.1. *1. We cite from [4, Alg.3.1] (cf. also [27, p.152, Lem.4]): Let R be a principal ideal domain with* quot$(R)=:K$ *and let* $A\in K^{a\times d}$*,* $M\in K^{c\times d}$*. The following algorithm determines whether there exists a matrix* $X\in R^{c\times a}$ *such that* $XA=M$ *and, if this is the case, parametrizes all such matrices. Let*

$$\begin{pmatrix}E&0\\0&0\end{pmatrix}=UAV,\quad E=\begin{pmatrix}e_1&&0\\&\ddots&\\0&&e_r\end{pmatrix},\quad r=\mathrm{rank}(A),$$

be the Smith form of A with respect to R. Then the following equivalences hold:

$$
\begin{aligned}
& \exists\, X \in R^{c\times a}: \quad XA = M \\
\Leftrightarrow\quad & \exists\, X \in R^{c\times a}: \quad \underbrace{XU^{-1}}_{=:\widetilde{X}}\,\underbrace{UAV}_{\left(\begin{smallmatrix} E & 0 \\ 0 & 0\end{smallmatrix}\right)} = \underbrace{MV}_{=:\widetilde{M}} \\
\Leftrightarrow\quad & \exists\, \widetilde{X} \in R^{c\times a}: \quad \widetilde{X}\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = \widetilde{M} \\
\Leftrightarrow\quad & \exists\, \widetilde{X} \in R^{c\times a}: \quad \widetilde{M}_{ij} = \begin{cases} \widetilde{X}_{ij}e_j & \text{for} \quad 1 \le j \le r, \\ 0 & \text{for} \quad r < j \le d, \end{cases} \quad 1 \le\; i \le c. \\
\Leftrightarrow\quad & \forall\, i \in \{1,\dots,c\}\; \forall\, j \in \{1,\dots,r\}: \quad \widetilde{M}_{ij}e_j^{-1} \in R, \\
& \forall\, i \in \{1,\dots,c\}\; \forall\, j \in \{r+1,\dots,d\}: \quad \widetilde{M}_{ij} = 0.
\end{aligned}
$$

If this is the case, define $\widetilde{X}^1 \in R^{c\times a}$ by

$$\widetilde{X}^1_{ij} := \begin{cases} \widetilde{M}_{ij}e_j^{-1} & \text{for} \quad 1 \le j \le r, \\ 0 & \text{for} \quad r < j \le a, \end{cases} \quad 1 \le i \le c.$$

Then $X^1 := \widetilde{X}^1 U \in R^{c\times a}$ satisfies

$$X^1 A = M \text{ and } \left\{X \in R^{c\times a};\, XA = M\right\} = X^1 + R^{c\times(a-r)}U_2$$

where $U =: \binom{U_1}{U_2} \in R^{(r+(a-r))\times a}$, i.e., U_2 is a universal left annihilator of A.

2. *In part 1 consider $\mathscr{D} = F[s] \subset \mathscr{K} = F(s)$ and the Smith form of A w.r.t. $\mathscr{D}$. Assume that $XA = M$ has a solution in $\mathscr{K}^{c\times a}$, i.e.,*

$$\widetilde{M}_{ij} = 0 \text{ for } 1 \le i \le c, r < j \le d. \text{ For } 1 \le i \le c,\ 1 \le j \le r \text{ write}$$

$$\widetilde{M}_{ij}e_j^{-1} = \frac{f_{ij}}{g_{ij}} \in F(s),\ f_{ij}, g_{ij} \in F[s],\ \gcd(f_{ij}, g_{ij}) = 1, \text{ and define } t_2 := \mathrm{lcm}_{i,j}\, g_{ij}.$$

Then there is a solution of $XA = M$ in $\mathscr{D}_{t_2}^{c\times a}$, and the saturated monoid T_2 of all divisors of powers of $(s-\alpha)t_2$ (compare (13)) is the least saturated monoid T containing $s-\alpha$ for which there is a solution $X \in \mathscr{D}_T^{c\times a}$ of $XA = M$.

Proof. Compare Algorithm 3.1 in [4]. □

Algorithm 7.2. *Consider the ring $\mathscr{D}_T$ for a multiplicatively closed saturated set $T \subseteq \mathscr{D}\setminus\{0\}$. Assume that T contains an element of the form $(s-\alpha)$ for some $\alpha \in F$, define $\sigma := (s-\alpha)^{-1}$ and $\widehat{\mathscr{D}} := F[\sigma]$. Let $A \in F(s)^{a\times d}$, $B \in F(s)^{b\times d}$, $M \in F(s)^{c\times d}$ and assume that the equation*

$$XA + ZB = M \tag{29}$$

has a solution $(X^1, Z^1) \in \mathscr{D}_T^{c\times(a+b)}$. Then, by Algorithm 7.1, the set of all solutions $(X, Z) \in \mathscr{D}_T^{c\times(a+b)}$ of (29) is given by $(X^1, Z^1) + \mathscr{D}_T^{c\times s}(C, D)$ where $s := a + b - \operatorname{rank}\binom{A}{B}$ and $(C, D) \in \widehat{\mathscr{D}}^{s\times(a+b)}$ denotes a universal left annihilator of $\binom{A}{B}$ w.r.t. $\widehat{\mathscr{D}}$ and hence also w.r.t. $\mathscr{D}_T$. Hence,

$$\left\{X \in \mathscr{D}_T^{c\times a};\, \exists Z \in \mathscr{D}_T^{c\times b} : (29)\right\} = X^1 + \mathscr{D}_T^{c\times s}C.$$

1. *The existence of* $Y \in \mathscr{D}_T^{c\times s}$ *such that* $X^1 + YC$ *is proper, i.e., contained in* $\mathscr{S}^{c\times a} = (\mathscr{D}_T \cap F(s)_{\mathrm{pr}})^{c\times a}$*, can be checked as follows (compare [5, Cor.3.9 - Cor.3.14], [4, Alg.3.2]): Let*

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = UCV, \quad E = \begin{pmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_r \end{pmatrix}, \quad r = \mathrm{rank}(C),$$

be the Smith form of C *with respect to* $\widehat{\mathscr{D}}$*. From*

$$U \in \mathrm{Gl}_s(\widehat{\mathscr{D}}) \subseteq \mathrm{Gl}_s(\mathscr{D}_T) \text{ and } V \in \mathrm{Gl}_a(\widehat{\mathscr{D}}) \subseteq \mathrm{Gl}_a(\mathscr{S})$$

we conclude the following equivalences:

$$\begin{gathered} \exists Y \in \mathscr{D}_T^{c\times s} \text{ with } X^1 + YC \in \mathscr{S}^{c\times a} \;\Leftrightarrow \\ \exists \widetilde{Y} := YU^{-1} \in \mathscr{D}_T^{c\times s} \text{ with } X^1V + YCV = X^1V + \widetilde{Y}\left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathscr{S}^{c\times a} \;\Leftrightarrow \\ (X^1V)_{ij} \in \begin{cases} \mathscr{S} + \mathscr{D}_T e_j & \text{for} \quad 1 \le j \le r \\ \mathscr{S} & \text{for} \quad r < j \le a \end{cases}, \; 1 \le i \le c \;\Leftrightarrow \\ (X^1V)_{ij} \in \mathscr{S} \quad \text{for} \quad 1 \le i \le c,\; r < j \le a \end{gathered}$$

where the last equivalence holds by Lemma 3.10 in [5] or Algorithm 7.3 below. Note that in the next to last line of item 1 in Algorithm 3.2 in [4] it is incorrectly asserted that $(X^1V)_{ij} = 0$ *instead of* $(X^1V)_{ij} \in \mathscr{S}$ *for* $r < j$*.*

If this condition is satisfied Algorithm 7.3 below yields representations

$$(X^1V)_{ij} = \widetilde{X}^0_{ij} - \widetilde{Y}_{ij}e_j \in \mathscr{S} + \mathscr{D}_T e_j \text{ for } 1 \le i \le c,\; 1 \le j \le r.$$

Enlarge the set of entries $\widetilde{X}^0_{ij}$ *and* $\widetilde{Y}_{ij}$ *to matrices* $\widetilde{X}^0 \in \mathscr{S}^{c\times a}$ *and* $\widetilde{Y} \in \mathscr{D}_T^{c\times s}$ *by*

$$\begin{gathered} (\widetilde{X}^0)_{ij} := (X^1V)_{ij} \text{ and } \widetilde{Y}_{ij} := 0 \text{ for } 1 \le i \le c,\; r < j. \text{ Then} \\ \widetilde{X}^0 = X^1V + \widetilde{Y}\left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right) \text{ and } X^0 := \widetilde{X}^0 V^{-1} = X^1 + YC \in \mathscr{S}^{c\times a} \text{ with } Y = \widetilde{Y}U \in \mathscr{D}_T^{c\times s}. \end{gathered}$$

2. *Assume that* (29) *has a solution in* $\mathscr{K}^{c\times(a+b)}$*, let* T_2 *be constructed as in Algorithm 7.1.2, and let* (X^1, Z^1) *be a solution of* (29) *in* $\mathscr{D}_{T_2}^{c\times(a+b)}$*. Then* (29) *has also a solution* $(X, Z) \in (\mathscr{D}_{T_2} \cap F(s)_{\mathrm{pr}})^{c\times a} \times \mathscr{D}_{T_2}^{c\times b}$ *if and only if there exists a matrix* Y *in* $F(s)^{c\times s}$ *such that* $X^1 + YC$ *is proper, i.e., that in part 1* $(X^1V)_{ij}$ *is proper for* $1 \le i \le c$, $r < j \le a$*. Moreover* T_2 *is then also the least saturated monoid* T *containing* $s - \alpha$ *such that* (29) *has a solution* $(X,Z) \in \left(\mathscr{D}_T \cap F(s)_{\mathrm{pr}}\right)^{c\times a} \times \mathscr{D}_T^{c\times b}$*.*

Algorithm 7.3. *Let* $r \in \mathscr{D}_T$ *and* $0 \ne e \in \widehat{\mathscr{D}} = F[\sigma]$*. The following algorithm yields* $x \in \mathscr{S}$ *and* $y \in \mathscr{D}_T$ *such that* $r = x + ye \in \mathscr{S} + \mathscr{D}_T e$*.*

Find representations $r = \widehat{h}\widehat{t}^{-1}\sigma^{-n} \in \mathscr{D}_T = \mathscr{S}_\sigma = (\widehat{\mathscr{D}}_{\widehat{T}})_\sigma$*,* $\widehat{h} \in \widehat{\mathscr{D}}$*,* $\widehat{t} \in \widehat{T} \subseteq \widehat{\mathscr{D}}$*,* $\widehat{t}(0) \ne 0$*,* $n \in \mathbb{N}$*, and* $e = e_1\sigma^\ell \in \widehat{\mathscr{D}}$*,* $e_1 \in \widehat{\mathscr{D}}$*,* $e_1(0) \ne 0$*,* $\ell \in \mathbb{N}$*. Since* $\gcd(\sigma^n, e_1) = 1$ *by construction, there exist* $a, b \in \widehat{\mathscr{D}}$ *such that* $1 = a\sigma^n + be_1$*. Hence*

$$r = 1 \cdot \widehat{h}\widehat{t}^{-1}\sigma^{-n} = a\widehat{h}\widehat{t}^{-1}\sigma^{-n+n} + b\widehat{h}\widehat{t}^{-1}\sigma^{-n}e_1 = a\widehat{h}\widehat{t}^{-1} + b\widehat{h}\widehat{t}^{-1}\sigma^{-n-\ell}e \in \mathscr{S} + \mathscr{D}_T e.$$

Defining $x := a\widehat{h}\widehat{t}^{-1} \in \widehat{\mathscr{D}}_{\widehat{T}} = \mathscr{S}$ *and* $y := b\widehat{h}\widehat{t}^{-1}\sigma^{-n-\ell} \in (\widehat{\mathscr{D}}_{\widehat{T}})_\sigma = \mathscr{D}_T$ *yields the asserted representation of* r*.*

Algorithm 7.4. *Assume a matrix $R \in \widehat{\mathscr{D}}^{k\times\ell}$. The following algorithm determines a matrix $R' \in \widehat{\mathscr{D}}^{r\times\ell}$ such that $\widehat{\mathscr{D}}_{\sigma}^{1\times k}R \cap \widehat{\mathscr{D}}^{1\times\ell} = \widehat{\mathscr{D}}^{1\times r}R'$ and $\operatorname{rank}(R') = r$. Localization w.r.t. $\widehat{T}$ (with $\widehat{\mathscr{D}}_{\widehat{T}} = \mathscr{S}$, compare (5)) then yields*

$$\mathscr{D}_T^{1\times k}R \cap \mathscr{S}^{1\times\ell} = \mathscr{S}^{1\times r}R'.$$

Let

$$\begin{aligned}
&\left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right) = XRY \quad \text{be the Smith form of } R \text{ w.r.t. } \widehat{\mathscr{D}},\\
&E = \begin{pmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_r \end{pmatrix}, \quad r = \operatorname{rank}(R), \quad e_i \in \widehat{\mathscr{D}}, \quad e_1 | \dots | e_r,\\
&e_i =: e'_i \cdot \sigma^{a_i}, \quad a_i \in \mathbb{N},\ e'_i \in \widehat{\mathscr{D}} \setminus \widehat{\mathscr{D}}\sigma,\\
&E' := \begin{pmatrix} e'_1 & & 0 \\ & \ddots & \\ 0 & & e'_r \end{pmatrix}, \qquad R' := (E', 0)Y^{-1}.
\end{aligned}$$

Then $R' \in \widehat{\mathscr{D}}^{r\times\ell}$ and $\widehat{\mathscr{D}}_{\sigma}^{1\times k}R \cap \widehat{\mathscr{D}}^{1\times\ell} = \widehat{\mathscr{D}}^{1\times r}R'$.

Proof. We study the modules $\widehat{\mathscr{D}}_{\sigma}^{1\times k}R$ and $\widehat{\mathscr{D}}^{1\times r}R'$:

$$\begin{aligned}
\widehat{\mathscr{D}}_{\sigma}^{1\times k}R &= \widehat{\mathscr{D}}_{\sigma}^{1\times k}\left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right)Y^{-1} = \bigoplus_{i=1}^{r} \widehat{\mathscr{D}}_{\sigma} e_i (Y^{-1})_{i-}\\
&= \bigoplus_{i=1}^{r} \widehat{\mathscr{D}}_{\sigma} \sigma^{a_i} e'_i (Y^{-1})_{i-} = \bigoplus_{i=1}^{r} \widehat{\mathscr{D}}_{\sigma} e'_i (Y^{-1})_{i-},\\
\widehat{\mathscr{D}}^{1\times r}R' &= \widehat{\mathscr{D}}^{1\times r}(E', 0)Y^{-1} = \bigoplus_{i=1}^{r} \widehat{\mathscr{D}} e'_i (Y^{-1})_{i-}.
\end{aligned}$$

It is obvious that $\widehat{\mathscr{D}}^{1\times r}R' \subseteq \widehat{\mathscr{D}}_{\sigma}^{1\times k}R \cap \widehat{\mathscr{D}}^{1\times\ell}$.
On the other hand, let $\xi = \sum_{i=1}^{r} \eta_i e'_i (Y^{-1})_{i-} \in \widehat{\mathscr{D}}_{\sigma}^{1\times k}R \cap \widehat{\mathscr{D}}^{1\times\ell}$, $\eta_i \in \widehat{\mathscr{D}}_{\sigma}$. Since $\xi \in \widehat{\mathscr{D}}^{1\times\ell}$ and $Y \in \mathrm{Gl}_{\ell}(\widehat{\mathscr{D}})$, we deduce that $\xi Y \in \widehat{\mathscr{D}}^{1\times\ell}$, $\xi Y = \sum_{i=1}^{r} \eta_i e'_i \delta_i$ where $\delta_i \in F^{1\times\ell}$, $(\delta_i)_j := \delta_{ij}$. Consequently $\eta_i e'_i \in \widehat{\mathscr{D}}$ for $1 \le i \le r$, and hence $\eta_i \in \widehat{\mathscr{D}}_{\sigma} \cap \widehat{\mathscr{D}}\frac{1}{e'_i} \subseteq \widehat{\mathscr{D}}_{\sigma} \cap \widehat{\mathscr{D}}_{\widehat{\mathscr{D}}\setminus\widehat{\mathscr{D}}\sigma} = \widehat{\mathscr{D}}$ since $e'_i \in \widehat{\mathscr{D}} \setminus \widehat{\mathscr{D}}\sigma$ by construction. It follows that $\xi \in \widehat{\mathscr{D}}^{1\times k}R'$. □

8 Example

We conclude this paper with an example illustrating the application of the techniques described in the preceding sections.

Example 8.1. We consider the continuous standard case, i.e., $F = \mathbb{R}$, $\mathscr{D} = \mathbb{R}[s]$, $\mathscr{F} = \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R})$ with $T := \{t \in \mathbb{R}[s];\ \mathrm{V}_{\mathbb{C}}(t) \subseteq \Lambda\}$ where $\Lambda := \{\lambda \in \mathbb{C};\ \Re(\lambda) < 0\}$, i.e., a signal in $\mathscr{F}$ is T-small if and only if it is polynomial exponential and converges to zero for t tending to infinity. Assume the behaviors $\mathscr{B}_1 := \left\{\left(\begin{smallmatrix} y_1 \\ u_1 \end{smallmatrix}\right) \in \mathscr{F}^{1+2};\ P_1 \circ y_1 = Q_1 \circ u_1\right\}$, $\mathscr{B}_t := \left\{w \in \mathscr{F}^1;\ R_t \circ w = 0\right\}$, and $\mathscr{B}_d := \left\{w \in \mathscr{F}^1;\ R_d \circ w = 0\right\}$ where

$$P_1 := \left(s^2 - 4\right), \quad Q_1 := \left(-(s+2)^2, \quad -s^2(s+2)\right), \quad R_t := \left(s\right), \quad R_d := \left(s^2+1\right).$$

Our goal is to construct a T-stabilizing compensator that simultaneously T-tracks signals $u_2 \in \mathscr{B}_t$ and T-rejects signals $u_2 \in \mathscr{B}_d$. This signifies that any constant signal is admissible as tracking signal whereas any signal of the form $u_2(t) = a\sin(t) + b\cos(t)$ (for arbitrary constants a and b) is considered as disturbance and shall not have any significant influence onto the output y_1 of the full feedback behavior. Of course we require both the feedback behavior and the compensator to be proper.

Note that $\mathscr{B}_1$ is not T-stable since the greatest elementary divisor (s^2-4) of P_1 has zeroes 2 and -2 and is consequently not contained in T. $\mathscr{B}_1$ is also not controllable, but T-stabilizable since the greatest elementary divisor of $R_1 = (P_1, -Q_1)$ is $(s+2)$ with zero $-2 \in \Lambda$. Note moreover that the transfer matrix

$$H_1 = P_1^{-1}Q_1 = \left(-\tfrac{s+2}{s-2}, -\tfrac{s^2}{s-2} \right)$$

is *not* proper. It is convenient to choose $\alpha := -2$ for the definition of $\sigma := (s-\alpha)^{-1}$ and $\widehat{\mathscr{D}} := \mathbb{R}[\sigma]$. Then

$$H_1 = \left(\tfrac{1}{4\sigma-1}, \tfrac{(2\sigma-1)^2}{\sigma(4\sigma-1)} \right).$$

Now we can construct the matrices introduced in Section 3, Assumption 3.6: a left coprime factorization $H_1 = \widehat{P}_1^{-1}\widehat{Q}_1$ of H_1 over $\widehat{\mathscr{D}}$ can be obtained via the controllable realization $\left(\widehat{P}_1, -\widehat{Q}_1\right)$ of H_1 over $\widehat{\mathscr{D}}$, i.e., via the computation of a universal left annihilator of $\left(\begin{smallmatrix} H_1 \\ \mathrm{id}_2 \end{smallmatrix}\right)$, compare e.g. [5, Res.2.10, Def. and Lem.2.7]. A right coprime factorization $H_1 = \widehat{N}_1\widehat{D}_1^{-1}$ over $\widehat{\mathscr{D}}$ can be computed in a similar fashion. In our case we get:

$$\widehat{P}_1 = \begin{pmatrix} -16\sigma^2 + 4\sigma \end{pmatrix}, \qquad \widehat{Q}_1 = \begin{pmatrix} -4\sigma, & -16\sigma^2 + 16\sigma - 4 \end{pmatrix},$$
$$\widehat{N}_1 = \begin{pmatrix} 0, & -1 \end{pmatrix}, \qquad \widehat{D}_1 = \begin{pmatrix} -4\sigma^2 + 4\sigma - 1 & -4\sigma + 1 \\ \sigma & 0 \end{pmatrix}.$$

A left inverse $\widehat{R}_2^0 = \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) \in \widehat{\mathscr{D}}^{2\times(1+2)}$ of $\left(\begin{smallmatrix} \widehat{N}_1 \\ \widehat{D}_1 \end{smallmatrix}\right)$ computed by Algorithm 7.1 is

$$\widehat{P}_2^0 = \begin{pmatrix} -1 & -4\sigma + 4 \\ 0 & 0 \end{pmatrix}, \quad \widehat{Q}_2^0 = \begin{pmatrix} -4\sigma + 1 \\ 1 \end{pmatrix}.$$

According to Corollary 4.12 we have to solve the equation

$$\begin{pmatrix} \widehat{N}_1\widehat{Q}_2^0 + \mathrm{id}_p \\ \widehat{N}_1\widehat{Q}_2^0 \end{pmatrix} = \begin{pmatrix} \widehat{N}_1 \\ \widehat{N}_1 \end{pmatrix} X\widehat{P}_1 + \begin{pmatrix} Z_t & 0 \\ 0 & Z_d \end{pmatrix} \begin{pmatrix} R_t \\ R_d \end{pmatrix}. \tag{30}$$

This can be achieved by means of Algorithm 7.1 after rearranging the entries of the occuring matrices, compare the considerations after Remark 4.7. Here we obtain

$$\left\{ X \in \mathscr{D}_T^{2\times 1};\, \exists Z_t \in \mathscr{D}_T^{1\times 1}, \exists Z_d \in \mathscr{D}_T^{1\times 1} : (30) \right\} = X^0 + \sum_{i=1}^{\mu} \mathscr{D}_T X^{h,i}$$
$$\text{with} \quad X^0 = \begin{pmatrix} 0 \\ -\frac{s(278s-29)}{20(s+2)^4} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\sigma^2}{20}(2\sigma-1)(585\sigma-278) \end{pmatrix} \in \mathscr{S}^{2\times 1},$$
$$\mu = 2, \quad X^{h,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X^{h,2} = \begin{pmatrix} 0 \\ \frac{s(s^2+1)}{(s+2)^3} \end{pmatrix} = \begin{pmatrix} 0 \\ -(2\sigma-1)(5\sigma^2 - 4\sigma + 1) \end{pmatrix}.$$

with the notations from after Remark 4.7. The smallest saturated monoid T_2 such that (30) has solutions (X, Z_t, Z_d) with entries in $\mathscr{D}_{T_2}$ according to Algorithm 7.1.2 is the saturated monoid

$$T_2 := \left\{\beta(s+2)^k;\, \beta \in F \setminus \{0\},\, k \geq 0\right\}$$

generated by $t_2 := (s+2)$.

It is easily seen that Algorithm 7.4 yields $B^{(j)} := X^{h,j} \in \widehat{\mathscr{D}}^{2\times 1}$ for $j = 1,2$. Hence, we get

$$\left\{X \in \mathscr{S}^{2\times 1};\, \exists Z_t \in \mathscr{D}_T^{1\times 1},\, \exists Z_d \in \mathscr{D}_T^{1\times 1} : (30)\right\} = X^0 + \sum_{j=1}^{2} \mathscr{S} B^{(j)}.$$

For $X = X^0 + \sum_{j=1}^{2} \eta_j B^{(j)}$ with $\eta_j \in \mathscr{S}$, the matrix

$$\left(-\widehat{Q}_2, \widehat{P}_2\right) = \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) + X\left(\widehat{P}_1, -\widehat{Q}_1\right)$$

gives rise to a proper T-stabilizing compensator if and only if it has the right IO structure and proper transfer matrix, i.e., $\det(\nu_\sigma(\widehat{P}_2)) \neq 0$. Writing $\eta_j =: \xi_j + \sigma\zeta_j$, $\xi_j \in F$, $\zeta_j \in \mathscr{S}$, and defining the polynomial $g(\Xi)$ from (24),

$$g(\Xi) := \det\left(\widehat{P}_2^0(0) - \left[\nu_\sigma(X^0) + \sum_{j=1}^{2} \Xi_j B^{(j)}(0)\right]\widehat{Q}_1(0)\right) = -4\Xi_2 \in F[\Xi_1, \Xi_2],$$

this is the case iff $g(\xi_1, \xi_2) \neq 0$. This signifies that $\xi_2 \neq 0$. We could just choose $\eta_1 := 0$, $\eta_2 := 1$. However, the resulting compensator gets simpler if we make an ansatz for η_1, η_2 as polynomials in $F[\sigma] = \widehat{\mathscr{D}} \subseteq \mathscr{S}$ with indetermined coefficients and assign the coefficients such that the degrees (w.r.t. σ) of the entries of $\left(-\widehat{Q}_2, \widehat{P}_2\right)$ get as low as possible. One possible solution is the following:

$$\eta_1 := 0, \quad \eta_2 := -\tfrac{19}{10} - \tfrac{117}{20}\sigma.$$

This yields

$$X = X^0 + \sum_{j=1}^{2} \eta_j B^{(j)} = \begin{pmatrix} 0 \\ -\frac{1}{20}(2\sigma - 1)(35\sigma - 38) \end{pmatrix}$$

and, by $\left(-\widehat{Q}_2, \widehat{P}_2\right) := \left(-\widehat{Q}_2^0, \widehat{P}_2^0\right) + X\left(\widehat{P}_1, -\widehat{Q}_1\right)$,

$$\widehat{P}_2 = \begin{pmatrix} -1 & -4\sigma+4 \\ -\frac{\sigma}{5}(2\sigma-1)(35\sigma-38) & -\frac{1}{5}(2\sigma-1)^3(35\sigma-38) \end{pmatrix}, \quad \widehat{Q}_2 = \begin{pmatrix} -4\sigma+1 \\ -\frac{1}{5}(56\sigma^2-58\sigma-5)(5\sigma^2-4\sigma+1) \end{pmatrix}.$$

It can easily be checked by means of Algorithm 7.1 that there really exist Z_t resp. Z_d in $\mathscr{D}_T^{1\times 1}$ satisfying $\widehat{N}_1\widehat{Q}_2 + \mathrm{id}_1 = Z_t R_t$ resp. $\widehat{N}_1\widehat{Q}_2 = Z_d R_d$.

Now we compute the transfer matrix $H_2 = \widehat{P}_2^{-1}\widehat{Q}_2$:

$$H_2 = \begin{pmatrix} \frac{280\sigma^3-514\sigma^2+283\sigma-58}{(2\sigma-1)(35\sigma-38)} \\ -\frac{5}{(2\sigma-1)(35\sigma-38)} \end{pmatrix} = \begin{pmatrix} -\frac{58s^3+65s^2+78s+80}{s(s+2)(38s+41)} \\ -\frac{5(s+2)^2}{s(38s+41)} \end{pmatrix}.$$

Note that H_2 is by construction proper. A left coprime factorization $H_2 = P_2^{-1}Q_2$ over $\mathscr{D} = F[s]$ is given by

$$P_2 = \begin{pmatrix} -s-2 & 4s+4 \\ -38s^2-79s-6 & 12 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} s-2 \\ 58s^2+7s+74 \end{pmatrix}.$$

The matrices obtained from the algorithm described above are somewhat more complicated, some elementary row operations yield the (row equivalent) matrices stated here.

The behavior $\mathscr{B}_2 = \left\{ \binom{u_2}{y_2} \in \mathscr{F}^{1+2}; P_2 \circ y_2 = Q_2 \circ u_2 \right\}$ is by construction a T-stabilizing compensator of $\mathscr{B}_1$ that T-tracks signals $u_2 \in \mathscr{B}_t$ and T-rejects signals $u_2 \in \mathscr{B}_d$. Moreover, both $\mathscr{B}_2$ and $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ are proper, i.e., have proper transfer matrices.

We can check these properties by computing the feedback behavior $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ and the error behaviors

$$\mathscr{B}_{\mathrm{err},t} := \left\{ y_1 + u_2 \in \mathscr{F}^p; y_1 \in \mathscr{F}^p, u_2 \in \mathscr{B}_t, \exists y_2 \in \mathscr{F}^m : \begin{pmatrix} y_1 \\ y_2 \\ u_2 \\ 0 \end{pmatrix} \in \mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2) \right\}$$
$$\text{resp.} \quad \mathscr{B}_{\mathrm{err},d} := \left\{ y_1 \in \mathscr{F}^p; \exists u_2 \in \mathscr{B}_d \, \exists y_2 \in \mathscr{F}^m : \begin{pmatrix} y_1 \\ y_2 \\ u_2 \\ 0 \end{pmatrix} \in \mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2) \right\}$$

describing the deviation of $-y_1$ from the tracking signal $u_2 \in \mathscr{B}_t$ resp. the error y_1 caused by a disturbance input $u_2 \in \mathscr{B}_d$. These computations are not carried out in detail here, but we find that $\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)$ is really proper and T-stable and $\mathscr{B}_{\mathrm{err},t}$ and $\mathscr{B}_{\mathrm{err},d}$ are really T-autonomous. More precisely:

$$\mathrm{ch}(\mathrm{fb}(\mathscr{B}_1, \mathscr{B}_2)^0) = \{-2\}, \quad \mathrm{ch}(\mathscr{B}_{\mathrm{err},t}) = \{-2\}, \quad \text{and} \quad \mathrm{ch}(\mathscr{B}_{\mathrm{err},d}) = \{-2\}$$

in accordance with Theorem 4.11.

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