T-Observers

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Abstract

We present behavioral existence and parametrization results for input observers of IO (input/output) behaviors and for pseudo state observers of Rosenbrock equations, i.e., of systems given by polynomial matrix descriptions. Our results significantly extend those of Wolovich from 1974. Valcher and Willems started the behavioral theory of observers in 1999 and Fuhrmann treated all aspects of observers in a recent comprehensive paper. We use the (behavioral) observers and associated error behaviors of these authors, but in contrast to them require the observers to be IO behaviors which are proper, but not necessarily consistent. Our results are also applicable to their more general behaviors and, conversely, their theorems are applicable to our situations. More recently Bisiacco, Valcher and Willems also considered non-consistent dead-beat observers.We discuss the relation of our work to that of our predecessors in some detail. The T in the title refers to a multiplicatively closed set of ordinary differential or shift operators in the standard cases, gives rise to T-autonomy, T-stability and T-observers and enables the simultaneous study of tracking, asymptotic, dead-beat, exact and other observers both in the continuous and the discrete cases. We derive new algorithms for the construction of proper T-observers and apply them in an instructive example, computed with MAPLE. Our proofs rely on module-behavior duality and on linear algebra over the ring of proper and T-stable rational functions.

Keywords : observer, Rosenbrock equations, polynomial matrix descriptions, behavior, proper transfer matrix, asymptotic stability

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1 Introduction

Since Luenberger's [13] ingenious and seminal introduction of state observers for Kalman state equations these observers have played a significant part for the feedback stabilization of linear systems. The first treatment of pseudo state observers of systems described by Rosenbrock equations (= differential operator representations = polynomial matrix descriptions) [18], [23, Ch.5], [11, Ch.8], [20, Ch.2], [1, Ch.7] is due to Wolovich [23, §5.5, Ch.7] under the name *Frequency domain compensation*. Wolovich's theory is applicable to IO (input/output) *behaviors* since these are defined by special Rosenbrock equations. Notice that Wolovich's theory precedes Willems' introduction of behaviors by more than ten years. Valcher and Willems [19] started the behavioral theory of observers in 1999 and Fuhrmann [9] wrote a very comprehensive recent paper on all aspects of them. We use the (behavioral) observers and associated error behaviors of these authors, but in contrast to them require the observers to be IO behaviors which are proper, but not necessarily consistent as in [19] and [9]. In the more recent paper [2] Bisiacco et al. also consider non-consistent dead-beat observers.

We significantly extend Wolovich's theory and present existence and parametrization results for input observers [23, §5.5] of IO behaviors and pseudo state observers [23, Thm.7.3.23] of Rosenbrock equations. Our results are also applicable to the more general behaviors of [19] and [9]. Conversely, the theorems of these papers are applicable to our situations. In Remarks 3.16 and 4.8 we relate our results to those of Wolovich, Valcher/Willems and Fuhrmann in some detail.

Our approach is characterized by the following features:

1. We discuss T-observers for an arbitrary multiplicatively closed subset of the polynomial algebra $F[s]$ (F a field) which in the standard cases is the ring of ordinary differential or shift operators. This T gives rise to the notions of T-autonomy, T-stability and T-observers and enables the simultaneous study of *tracking, asymptotic, dead-beat, exact and other* observers following [19], [2] and [9] both in the continuous and the discrete cases. For this purpose we use the quotient ring $F[s]_T$ of T-*stable* rational functions and the ring $\mathcal{S}$ of the *proper* functions in $F[s]_T$. According to one of the reviewers special quotient rings $F[s]_T$ were already used by Morse in [14].

2. We constantly use the *module-behavior duality* which was derived in [15] for multidimensional systems, and especially finitely generated modules and matrices over the rings $F[s]_T$ and $\mathcal{S}$. A predecessor of our duality theory is Fuhrmann's paper [6], a more complete version of which is [8]. Matrices, but not modules, and algorithms, especially the Smith form algorithm, over special rings $\mathcal{S}$ play a prominent part in the books [21] and [20] by Vidyasagar resp. Vardulakis. Our algorithms are distinct from those in [21] and [20]. Our module theoretic proofs and algorithms are different from those in the literature and have many advantages in our opinion.

3. Essentially we only consider T-observers which are proper IO behaviors and can therefore be realized by Kalman state space equations. In our opinion such observers suffice, the observers derived in [19] and [9] are more general, however.

4. The IO structures and transfer matrices of the given equations and behaviors and of the observers play an important part in our assumptions and derivations as already in Wolovich's work [23].

5. The non-obvious algorithms for the construction of *proper* T-observers use new techniques.

Our main observer results are Theorems 3.7 and 3.12 on proper input T-observers for IO behaviors, Theorems 4.3 and 4.5 on proper T-observers of the pseudo state of Rosenbrock equations and Theorem 4.4 on proper T-observers for internally proper Rosenbrock equations. Following Wolovich [23, §5.5] we also include Theorem 3.15 on output T-controllers. An important technical ingredient is given by Theorem 2.13 on the existence and construction of left inverses of matrices over principal ideal domains which is applied in all theorems listed above to the rings $F[s]_T$ and $\mathcal{S}$ and which is implicitly also used by our predecessors in particular cases. Theorem 2.15 resp. Corollary 2.18 characterize T-autonomy and T-stability resp. T-observability. The algorithms are contained in Theorems 3.12, 4.4 and 4.9 and are demonstrated by the instructive Example 4.10, computed with MAPLE.

2 T-autonomy and T-observability

We first recall Willems' one-dimensional behavior theory [22],[17], but in the module theoretic language and with the results of [15], [16] where the corresponding multidimensional theory was developed. See also Remark 2.2 for some historical notes.

Let $\mathcal{D} := F[s]$ be the polynomial ring over a field F with its quotient field $\mathcal{K} := F(s)$ of rational functions and let $\mathcal{F}$ be an *injective cogenerator* signal module over $\mathcal{D}$ with the scalar multiplication $f \circ y,\ f \in \mathcal{D},\ y \in \mathcal{F}$ [15, p.29-30, Thm.2.54]. A module $\mathcal{F}$ over the principal ideal domain $\mathcal{D}$ is injective if and only if it is divisible. This signifies that each equation $f \circ y = u$ with nonzero f and given right side $u \in \mathcal{F}$ has a solution $y \in \mathcal{F}$. The injective module $\mathcal{F}$ is a cogenerator if $\mathrm{Hom}_{\mathcal{D}}(M, \mathcal{F})$ is nonzero whenever M is nonzero. The standard injective cogenerator signal spaces $\mathcal{F}$ are the following.

Example 2.1. 1. *Continuous case*: $F = \mathbb{R}$ or $F = \mathbb{C}$, $\mathcal{F} := \mathcal{C}^\infty(\mathbb{R}, F)$ or $\mathcal{F} := \mathcal{D}'(\mathbb{R}, F) := \{F\text{-valued distributions}\}$ with the action by differentiation, i.e., $s \circ y = dy/dt$.

2. *Discrete case*:

$$\mathcal{F} := F^{\mathbb{N}} := \{y : \mathbb{N} \to F\} = \{F\text{-valued sequences}\} \cong F[[s^{-1}]]$$
$$y = (y(0), y(1), y(2), \cdots) \leftrightarrow \sum_{t \in \mathbb{N}} y(t) s^{-t}$$

with the action by left shift, i.e., $(s^k \circ y)(t) := y(t+k)$.

Notice that the scalar multiplication $f \circ y$ of the module $\mathcal{F}$ is the action by the *operator* $f\circ : \mathcal{F} \to \mathcal{F}$.

We consider $\mathcal{F}$-behaviors which in the standard cases are solutions of linear systems of differential or difference equations with constant coefficients and thus of the form

$$\mathcal{B} := \{w \in \mathcal{F}^l;\ R \circ w = 0\},\ R \in \mathcal{D}^{k \times l}. \tag{1}$$

The matrix R gives also rise to the modules

$$U := \mathcal{D}^{1 \times k} R \subseteq \mathcal{D}^{1 \times l} \text{ and } M := \mathcal{D}^{1 \times l}/U, \tag{2}$$

the latter being furnished with its canonical system of generators

$$\overline{\delta_j} := \delta_j + U \in M,\ \delta_j = (0, \cdots, 0, \overset{j}{1}, 0, \cdots, 0) \in \mathcal{D}^{1\times l},\ j = 1, \cdots, l. \quad (3)$$

Then

$$\begin{gathered}\mathcal{B} = U^{\perp} := \{w \in \mathcal{F}^l;\ U \circ w = 0\} \text{ and} \\ \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{F}) \cong \mathcal{B},\ \varphi \leftrightarrow w = (w_1, \cdots, w_l)^{\top},\ \varphi(\overline{\delta_j}) = w_j,\end{gathered} \quad (4)$$

is a canonical isomorphism (following Malgrange). The cogenerator property of $\mathcal{F}$ implies [17, Thm.3.6.2], [15, Cor.2.48]

$$\mathcal{D}^{1\times k} R = U = \mathcal{B}^{\perp} := \{\xi \in \mathcal{D}^{1\times l};\ \xi \circ \mathcal{B} = 0\}, \quad (5)$$

i.e., all equations satisfied by the trajectories $w \in \mathcal{B}$ are linear combinations of the given equations $R \circ w = 0$. Equation (5) especially implies

$$\mathrm{ann}_{\mathcal{D}}(M) := \{f \in \mathcal{D};\ fM = 0\} = \mathrm{ann}_{\mathcal{D}}(\mathcal{B}) := \{f \in \mathcal{D};\ f \circ \mathcal{B} = 0\}. \quad (6)$$

In the standard cases the signal module $\mathcal{F}$ is even a *large* injective cogenerator. This signifies that each finitely generated module can be embedded into some finite power $\mathcal{F}^l$. In the discrete case with $\mathcal{F} = F^{\mathbb{N}} = F[[s^{-1}]]$ this was already observed and essentially used in [4]. The injective cogenerator property of $\mathcal{F}$ induces a categorical duality

$$M = \mathcal{D}^{1\times l}/U \longleftrightarrow \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{F}) \cong \mathcal{B} := U^{\perp} \quad (7)$$

between the categories of finitely generated $\mathcal{D}$-modules and that of behaviors [15, Thm.2.56]. In particular, for modules and behaviors

$$\begin{gathered}U_i \subseteq \mathcal{D}^{1\times l_i},\ M_i := \mathcal{D}^{1\times l_i}/U_i \text{ and } \mathcal{B}_i = U_i^{\perp},\ i = 1, 2, \text{ the isomorphism} \\ \mathrm{Hom}_{\mathcal{D}}(M_2, M_1) \cong \mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)\end{gathered} \quad (8)$$

holds. Therefore each *behavior morphism* from $\mathcal{B}_1$ to $\mathcal{B}_2$ has the form

$$P\circ : \mathcal{B}_1 \to \mathcal{B}_2 \text{ with } P \in \mathcal{D}^{l_2 \times l_1} \text{ and } U_2 P \subseteq U_1. \quad (9)$$

This morphism is zero if and only if $\mathcal{D}^{1\times l_2} P \subseteq U_1$.

Remark 2.2. The duality theory by means of injective cogenerators [15] has, of course, various predecessors. The paper [15] was inspired by Willems' work [22] and essentially used the fundamental principle, i.e., the injectivity of various multidimensional signal modules, proven in deep papers by Ehrenpreis, Malgrange and Palamodov in the beginning 1960s. The divisibility and therefore injectivity of the standard one-dimensional signal modules were already used by Hinrichsen/Prätzel-Wolters [10] and Blomberg/Ylinen [3]. Fuhrmann's discrete one-dimensional duality [4], [6], [8] between polynomial and rational models is also an instance of the quoted categorical duality. Behavior morphisms were already studied in [5] and again in [7].

The *torsion submodule* of M is defined as $\mathrm{t}(M) := \{x \in M;\ \exists 0 \neq t \in \mathcal{D} \text{ with } tx = 0\}$. The module M is called *torsionfree* resp. a *torsion* module if $\mathrm{t}(M) = 0$ resp. $\mathrm{t}(M) = M$. A torsionfree module over the principal ideal domain $\mathcal{D}$ is free, see Corollary 2.6 below. In particular U is free, i.e., has a

basis. Therefore it is always possible, but not necessary to assume that the rows of R are a basis of U or that $k=\dim_{\mathcal{D}}(U)$.
The matrix R is contained in $\mathcal{K}^{k\times l}=F(s)^{k\times l}$ and as such has the usual $\operatorname{rank}(R)$. Derived from this are the ranks

$$\begin{aligned} p &:= \operatorname{rank}(U) := \dim_{\mathcal{D}}(U) = \dim_{\mathcal{K}}(\mathcal{K}U) = \operatorname{rank}(R) \text{ and} \\ m &:= \operatorname{rank}(\mathcal{B}) := \operatorname{rank}(M) := \dim_{\mathcal{K}}\left(\mathcal{K}^{1\times l}/\mathcal{K}U\right) = l - \operatorname{rank}(R). \end{aligned} \tag{10}$$

The behavior is *autonomous* if it satisfies the following equivalent properties [4, Cor.3.3 and Lem.3.6], [17, §3.2],[15, Thm.2.69, p.159]:

$$\begin{gathered} M \text{ is a torsion module } \iff \exists 0 \neq t \in \mathcal{D} \text{ with } tM = 0 \text{ or } t\circ\mathcal{B} = 0 \iff \\ \operatorname{rank}(U) = \operatorname{rank}(R) = l \iff \operatorname{rank}(M) = 0. \end{gathered} \tag{11}$$

The behavior is *controllable* if and only if its module is torsionfree and thus free [17, 5.2.10], [15, Thms.7.21,7.52,7.53].
An *IO structure* of $\mathcal{B}$ consists in the choice of $p=\operatorname{rank}(R)$ linearly independent columns of R and gives rise, possibly after a permutation of the columns of R and the entries of w, to an IO (input/output) representation of $\mathcal{B}$ of the form [17, §3.3], [15, Thm.2.69]

$$\begin{gathered} \mathcal{B} := \left\{ w = \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{p+m};\ P\circ y = Q\circ u \right\} \text{ where } R = (P,-Q) \in \mathcal{D}^{k\times(p+m)}, \\ U^0 := U\begin{pmatrix} \mathrm{id}_p \\ 0 \end{pmatrix} = \mathcal{D}^{1\times k}P,\ p = \operatorname{rank}(R) = \operatorname{rank}(P) \text{ or } \mathcal{K}U^0 = \mathcal{K}^{1\times p}, \\ m = \operatorname{rank}(M),\ PH = Q,\ H \in \mathcal{K}^{p\times m}. \end{gathered} \tag{12}$$

The matrix H depends on $\mathcal{B}$ and the chosen IO structure only and is called the *transfer matrix* of the IO behavior $\mathcal{B}$. Since $\operatorname{rank}(P)=p$ the behavior

$$\mathcal{B}^0 := (U^0)^\perp = \{y \in \mathcal{F}^p;\ P\circ y = 0\} \tag{13}$$

is autonomous and called the *autonomous part* of the IO behavior. For every choice of u there is a trajectory $\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)$ in $\mathcal{B}$ and therefore u resp y are called the *input* resp. the *output* of $\mathcal{B}$.
For a general behavior the transfer matrix is replaced by the transfer space, called signal flow space in [15, p.43].

Result 2.3 (Transfer space [15, Thm.2.91])**.**

1. *For a behavior* $\mathcal{B}=\left\{w\in\mathcal{F}^l;\ R\circ w=0\right\}$, $R\in\mathcal{D}^{k\times l}$, *we define its* transfer space $\mathcal{B}_{\mathcal{K}}$ *as the solution space*

$$\mathcal{B}_{\mathcal{K}} := \left\{\tilde{w} \in \mathcal{K}^l;\ R\tilde{w} = 0\right\} \subseteq \mathcal{K}^l,\ \mathcal{K} = F(s),$$

of dimension $\dim_{\mathcal{K}}(\mathcal{B}_{\mathcal{K}}) = l - \operatorname{rank}(R) = \operatorname{rank}(\mathcal{B})$.

2. *If behaviors*

$$\begin{gathered} \mathcal{B}_i = \{w\in\mathcal{F}^{l_i};\ R_i\circ w = 0\}, i = 1,2,3, \text{ and matrices} \\ P_1 \in \mathcal{D}^{l_2\times l_1},\ P_2 \in \mathcal{D}^{l_3\times l_2} \text{ are given and if } \mathcal{B}_1 \xrightarrow{P_1\circ} \mathcal{B}_2 \xrightarrow{P_2\circ} \mathcal{B}_3 \end{gathered}$$

is well-defined and exact then so is the sequence of transfer spaces

$$\mathcal{B}_{1,\mathcal{K}} \xrightarrow{P_1\circ} \mathcal{B}_{2,\mathcal{K}} \xrightarrow{P_2\circ} \mathcal{B}_{3,\mathcal{K}}.$$

3. *If* $\mathcal{B} = \{ \binom{y}{u} \in \mathcal{F}^{p+m};\ P \circ y = Q \circ u \}$ *is an IO behavior then*

$$\mathcal{B}_{\mathcal{K}} = \left\{ \begin{pmatrix} \widetilde{y} \\ \widetilde{u} \end{pmatrix} \in \mathcal{K}^{p+m};\ P\widetilde{y} = Q\widetilde{u} \right\} \cong \mathcal{K}^m,\ \begin{pmatrix} \widetilde{y} \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} H\widetilde{u} \\ \widetilde{u} \end{pmatrix} \leftrightarrow \widetilde{u}.$$

This signifies that the transfer space of the IO behavior is the graph of the transfer matrix.

In the sequel we will repeatedly need the Smith form of a matrix with respect to a specified ring, especially with respect to the ring $\mathcal{S}$ of proper and stable rational functions (see below). We recall the basic properties.

Reminder 2.4. Let R be any principal ideal domain, $K = \mathrm{quot}(R)$ its quotient field and $M \in K^{k\times l}$ a matrix with $\mathrm{rank}(M) =: p$. The matrix M has a *Smith form* or *Smith-McMillan form*

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = UMV \quad \text{where} \quad E = \begin{pmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_p \end{pmatrix}$$

with respect to R which is defined by the following properties:

$$U \in \mathrm{Gl}_k(R), \quad V \in \mathrm{Gl}_l(R), \quad 0 \neq e_i \in K \text{ for } i = 1, \dots, p, \quad \text{and}$$
$$e_1 \underset{R}{|} e_2 \underset{R}{|} \dots \underset{R}{|} e_p, \text{ i.e., there are } r_i \in R \text{ such that } e_{i+1} = r_i e_i,\ i = 1, \dots, p-1.$$

The elements $e_1, \dots, e_p$ are called the *elementary divisors* and are unique up to association, i.e., up to units in R. The element $e_{\mathrm{rank}(M)} = e_p$ is called the highest elementary divisor of M.

Remark 2.5. 1. The Smith form of a matrix in $F(s)^{k\times l}$ with respect to $F[s]$ can be easily computed by means of any standard computer algebra system.

2. The Smith form of a matrix in $K^{k\times l}$ with respect to R is also the Smith form with respect to any principal ideal domain R' with $R \subseteq R' \subseteq K$.

Corollary 2.6. *Assume* $M \in R^{k\times l}$ *in Reminder 2.4. The Smith form induces the module isomorphisms*

$$\begin{gathered} R^{1\times l}/R^{1\times k}M \cong R/Re_1 \times \cdots \times R/Re_p \times R^{l-p} \\ \overline{\xi} := \xi + R^{1\times k}M \longleftrightarrow \overline{\eta} := (\eta_1 + Re_1, \cdots, \eta_p + Re_p, \eta_{p+1}, \cdots, \eta_l) \\ \eta = \xi V,\ \xi = \eta V^{-1} \end{gathered}$$

and

$$\mathrm{t}\left(R^{1\times l}/R^{1\times k}M\right) \cong R/Re_1 \times \cdots \times R/Re_p = \mathrm{t}\left(R/Re_1 \times \cdots \times R/Re_p \times R^{l-p}\right)$$

In particular, the module $R^{1\times l}/R^{1\times k}M$ *is torsionfree and then free if and only if all elementary divisors of* M *or, equivalently, its highest one are units in* R. *For* $R = \mathcal{D} = F[s]$ *the module is a torsion module if and only if its* F*-dimension is finite. Also*

$$\mathrm{ann}_{\mathcal{D}}(\mathrm{t}(M)) = \mathcal{D}e_1 \cap \cdots \cap \mathcal{D}e_p = \mathcal{D}e_p. \tag{14}$$

Definition and Lemma 2.7 (Universal left annihilators)**.** *Let R be a principal ideal domain, $K := \mathrm{quot}(R)$ its quotient field and M a matrix in $K^{k\times l}$. A matrix $L \in R^{m\times k}$ is called a* universal left annihilator *of M if the sequence*

$$0 \longrightarrow R^{1\times m} \xrightarrow{\circ L} R^{1\times k} \xrightarrow{\circ M} K^{1\times l}$$

is exact or, in other words, if

$$\ker\left(\circ M : R^{1\times k} \to K^{1\times l}\right) = \{\xi \in R^{1\times k}; \xi M = 0\} = R^{1\times m}L. \tag{15}$$

A universal left annihilator of a matrix $M \in R^{k\times l}$ can be computed in the following fashion: If $S = UMV$ is the Smith form of M the matrix of the last $k - \mathrm{rank}(M)$ rows of U is a universal left annihilator of M.

The injectivity of the signal module implies and is indeed equivalent to the following result.

Result 2.8 (Images of behaviors [17, Thm.6.2.6], [15, Thm.2.34])**.** *Consider a behavior*

$$\mathcal{B}_1 = \left\{w_1 \in \mathcal{F}^{l_1};\ R_1 \circ w_1 = 0\right\},\ R_1 \in \mathcal{D}^{k_1\times l_1},\ \text{ and } P \in \mathcal{D}^{l_2\times l_1}.$$

Then the image

$$P \circ \mathcal{B}_1 := \left\{w_2 \in \mathcal{F}^{l_2};\ \text{ there exists a } w_1 \in \mathcal{B}_1 \text{ such that } w_2 = P \circ w_1\right\}.$$

is also a behavior, indeed

$$P \circ \mathcal{B}_1 = \left\{w_2 \in \mathcal{F}^{l_2};\ R_2 \circ w_2 = 0\right\}$$

where $(-X, R_2)$ is a universal left annihilator of $\left(\begin{smallmatrix} R_1 \\ P \end{smallmatrix}\right)$.

Example 2.9 (Elimination of the pseudo state of a Rosenbrock system [15, Cor.2.41])**.** Consider *Rosenbrock equations* (= *differential operator representation* in [23, p.135] = *polynomial matrix description* in [20, p.55])

$$\begin{array}{c} A \circ x = B \circ u,\ y = C \circ x + D \circ u \text{ where} \\ A \in \mathcal{D}^{n\times n},\ \det(A) \neq 0,\ B \in \mathcal{D}^{n\times m},\ C \in \mathcal{D}^{p\times n},\ D \in \mathcal{D}^{p\times m}. \end{array} \tag{16}$$

They give rise to two behaviors

$$\begin{array}{c} \mathcal{B}_1 := \left\{\begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{F}^{n+m};\ A \circ x = B \circ u\right\} \text{ and} \\ \mathcal{B}_2 := \begin{pmatrix} C & D \\ 0 & \mathrm{id}_m \end{pmatrix} \circ \mathcal{B}_1 = \left\{\begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{p+m};\ \exists x \in \mathcal{F}^n \text{ with } (16)\right\}. \end{array} \tag{17}$$

In this case Result 2.8 can be simplified and $\mathcal{B}_2$ can be computed as

$$\mathcal{B}_2 = \left\{\begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{p+m};\ P \circ y = (YB + PD) \circ u\right\}$$

where $(-Y, P)$ is a universal left annihilator of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$. Moreover $\mathcal{B}_1$ resp. $\mathcal{B}_2$ are IO behaviors with transfer matrices $H_1 = A^{-1}B$ resp. $H_2 = D + CH_1 = D + CA^{-1}B$.

Result 2.10 (Unique controllable realization [4, Thm.6.1], [15, Thms.7.21,7.24])**.** *For any transfer matrix $H \in F(s)^{p\times m}$ there is a unique* controllable IO realization *of H, i.e., a controllable IO behavior*

$$\mathcal{B}_{\text{cont}} = \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{p+m};\ P_{\text{cont}} \circ y = Q_{\text{cont}} \circ u \right\},\ P_{\text{cont}} \in \mathcal{D}^{p\times p},\ P_{\text{cont}} H = Q_{\text{cont}},$$

with transfer matrix H. The matrices $(P_{\text{cont}}, -Q_{\text{cont}})$ resp. P_{cont} satisfy

$$\mathcal{D}^{1\times p}(P_{\text{cont}}, -Q_{\text{cont}}) = \ker\left(\circ \begin{pmatrix} H \\ \mathrm{id}_m \end{pmatrix} : \mathcal{D}^{1\times(p+m)} \to F(s)^{1\times m} \right) \quad \text{and}$$
$$\mathcal{D}^{1\times p} P_{\text{cont}} = \left\{ \xi \in \mathcal{D}^{1\times p};\ \xi H \in \mathcal{D}^{1\times m} \right\}.$$

Hence the matrix $(P_{\text{cont}}, -Q_{\text{cont}})$ can be computed as universal left annihilator of $\left(\begin{smallmatrix} H \\ \mathrm{id}_m \end{smallmatrix}\right)$ or of $d \cdot \left(\begin{smallmatrix} H \\ \mathrm{id}_m \end{smallmatrix}\right)$ where $d \neq 0$ is a common denominator of all entries of H, i.e., $dH \in \mathcal{D}^{p\times m}$.
If $\mathcal{B}$ is any IO behavior with transfer matrix H the behavior $\mathcal{B}_{\text{cont}}$ is the largest controllable subbehavior of $\mathcal{B}$. Its module $M_{\text{cont}} = \mathcal{D}^{1\times(p+m)}/\mathcal{B}_{\text{cont}}^{\perp}$ is canonically isomorphic to $M/\operatorname{t}(M)$ where $M = \mathcal{D}^{1\times(p+m)}/\mathcal{B}^{\perp}$ is the module of $\mathcal{B}$.

According to Kalman and, for instance, Wolovich [23, §5.4] any IO behavior $\mathcal{B} := \{\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P \circ y = Q \circ u\}$ with transfer matrix $H \in F(s)^{p\times m}$ admits a unique (up to similarity) observable Kalman state space realization, i.e., there are essentially unique matrices

$$\begin{gathered} A \in F^{n\times n},\ B \in F^{n\times m},\ C \in F^{p\times n},\ D \in F[s]^{p\times m} \text{ such that} \\ \begin{pmatrix} C & D \\ 0 & \mathrm{id}_m \end{pmatrix} \circ : \mathcal{B}_1 := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{F}^{n+m};\ (s\,\mathrm{id}_n - A) \circ x = Bu \right\} \cong \mathcal{B} : \\ \begin{pmatrix} x \\ u \end{pmatrix} \mapsto \begin{pmatrix} Cx + D \circ u \\ u \end{pmatrix} \text{ and } H = D + C(s\,\mathrm{id}_m - A)^{-1} B. \end{gathered} \tag{18}$$

The matrix D is constant too, i.e., $D \in F^{p\times m}$, if and only if the IO behavior or, equivalently by definition, its transfer matrix H are *proper*. This signifies that the entries of H belong to the ring

$$F(s)_{\text{pr}} := \left\{ r := \frac{f}{g} \in F(s);\ f, g \in F[s],\ \deg_s(r) := \deg_s(f) - \deg_s(g) \leq 0 \right\} \tag{19}$$

of proper rational functions. In the standard cases the behavior can then be technically realized as interconnection of adders, multipliers and integrators or delay elements. This is one of the important technical implications of the properness of H.

Definition and Corollary 2.11 (Characteristic variety [15, Cor.7.78], [16, Thm.2])**.** *Let $F := \mathbb{R}, \mathbb{C}$ be the real or complex field. For $f \in \mathcal{D} = F[s]$ let $V_{\mathbb{C}}(f)$ denote the set of* complex *roots of f. Let $\mathcal{B} \subseteq \mathcal{F}^l$ be any behavior, $\mathcal{B}^{\perp} = \mathcal{D}^{1\times k} R$ its module of equations with $p := \operatorname{rank}(R) = \dim_{\mathcal{D}}(\mathcal{B}^{\perp})$, $M := \mathcal{D}^{1\times l}/\mathcal{B}^{\perp}$ its module and e_p the highest elementary divisor of R. Corollary 2.6 furnishes* $\operatorname{ann}_{\mathcal{D}}(\operatorname{t}(M)) = \mathcal{D}e_p$ *. Then*

$$\operatorname{ch}(\mathcal{B}) := \operatorname{ch}(M) := V_{\mathbb{C}}(e_p) = \{\lambda \in \mathbb{C};\ \operatorname{rank}(R(\lambda)) < p = \operatorname{rank}(R)\}$$

is called the characteristic variety *of M and $\mathcal{B}$ and is finite.*
Hence $\mathcal{B}$ is controllable or M is free or $\mathrm{t}(M) = 0$ if and only if $\mathrm{ch}(\mathcal{B}) = \emptyset$ [17, Thm.5.2.5],[15, Cor.7.71].
If $\mathcal{B} = \{\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P \circ y = Q \circ u\}$ is an IO behavior with transfer matrix H and autonomous part $\mathcal{B}^0 = \{y \in \mathcal{F}^p;\ P \circ y = 0\}$ the (finite) variety $\mathrm{ch}(\mathcal{B}^0)$ contains $\mathrm{ch}(\mathcal{B})$ and the elements of the first resp. the second are the poles *resp. the* uncontrollable poles *of $\mathcal{B}$ in the usual language.*

Result 2.12 ([17, Thm.3.2.5], [16, Thm.2,(38),(62),(69)])**.** *In the situation of the preceding definition assume $\mathcal{F} := \mathcal{D}'(\mathbb{R}, \mathbb{C})$ and that $\mathcal{B} \subset \mathcal{F}^l$ is autonomous, i.e.,* $\mathrm{rank}(R) = l$. *Then*

$$\begin{gathered} \dim_F(\mathcal{B}) = \dim_F(M) < \infty \text{ and} \\ \mathcal{B} \subseteq \oplus_{\lambda \in \mathrm{ch}(\mathcal{B})} \mathbb{C}[t]^l e^{\lambda t} \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})^l \subseteq \mathcal{D}'(\mathbb{R}, \mathbb{C})^l \end{gathered} \tag{20}$$

where $\oplus_{\lambda \in \mathbb{C}} \mathbb{C}[t] e^{\lambda t} = \mathrm{t}(\mathcal{D}'(\mathbb{R}, \mathbb{C}))$ is the space of polynomial-exponential functions. For the other standard F-signal spaces over $F = \mathbb{R}, \mathbb{C}$ analogous results hold [16]. Equation (20) *describes the analytic significance of the characteristic variety.*

The following theorem will be applied whenever the existence of a left inverse matrix of a given matrix M in a specified ring has to be checked. It also provides all possible left inverses if there are any. Since full column rank of M is a necessary condition for the existence of a left inverse we will assume this in the theorem.

Theorem 2.13 (Compare [18, Thm.6.1])**.** *Let R be a principal ideal domain, $K = \mathrm{quot}(R)$ its quotient field and M a matrix in $K^{k \times l}$ with $\mathrm{rank}(M) = l$, hence M has a left inverse in $K^{l \times k}$. Let*

$$\begin{pmatrix} E \\ 0 \end{pmatrix} = UMV \text{ with } E := \begin{pmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_l \end{pmatrix}$$

be the Smith form of M with respect to R.

1. *The following statements are equivalent:*
   (a) *The matrix M has a left inverse $M' \in R^{l \times k}$ (not only in $K^{l \times k}$!), i.e., satisfying $M'M = \mathrm{id}_l$.*
   (b) *$E^{-1} \in R^{l \times l}$.*
   (c) *$e_l^{-1} \in R$.*
2. *If the conditions in (1) are satisfied the set of all left inverses of M in $R^{l \times k}$ is the affine submodule*
   $$V(E^{-1}, 0)U + R^{l \times (k-l)} U_2$$
   where U_2 consists of the last $(k - l)$ rows of U.
3. *If the entries of M and thus its highest elementary divisor e_l belong to R condition (c) of item 1. signifies that e_l and then all e_i are units of R.*

Proof. 1. It is obvious that (1b) implies (1c) since e_l^{-1} is an entry of E^{-1}.

$(1c) \Rightarrow (1b)$: The relationship $e_{i+1} = r_i e_i$ leads to

$$e_i^{-1} = e_{i+1}^{-1} r_i \quad \text{for} \quad i = 1, \ldots, l-1.$$

Hence, $e_{l-1}^{-1} = e_l^{-1} r_{l-1}$ is an element of R since e_l^{-1} is so by statement (1c), and so, inductively, $e_i^{-1} \in R$ for $i = 1, \ldots, l$. Thus the matrix

$$E^{-1} = \begin{pmatrix} e_1^{-1} & & 0 \\ & \ddots & \\ 0 & & e_l^{-1} \end{pmatrix}$$

is in $R^{l \times l}$ as well.

$(1b) \Rightarrow (1a)$: Computing

$$V(E^{-1}, 0)UM = V(E^{-1}, 0)U \cdot U^{-1} \begin{pmatrix} E \\ 0 \end{pmatrix} V^{-1} = \mathrm{id}_l$$

yields that $M' := V(E^{-1}, 0)U \in R^{l \times k}$ is one possible left inverse of M.

$(1a) \Rightarrow (1b)$: The equation $M'M = \mathrm{id}_l$ implies $M'U^{-1}UMV = V$ and hence
$V^{-1}M'U^{-1} \left(\begin{smallmatrix} E \\ 0 \end{smallmatrix} \right) = \mathrm{id}_l$. Since M' has entries in R so does

$$Z := V^{-1}M'U^{-1} =: (Z_1, Z_2) \in R^{l \times k} = R^{l \times (l + (k-l))} \text{ with}$$
$$Z_1 E = (Z_1, Z_2) \begin{pmatrix} E \\ 0 \end{pmatrix} = \mathrm{id}_l, \text{ hence } Z_1 = E^{-1} \in R^{l \times l}.$$

2. According to Definition and Lemma 2.7 the sequence

$$0 \longrightarrow R^{1 \times (k-l)} \xrightarrow{\circ U_2} R^{1 \times k} \xrightarrow{\circ M} K^{1 \times l}$$

is exact. If M' is any matrix in $R^{l \times k}$ it is a left inverse of M if and only if

$$M'M = \mathrm{id}_l = V(E^{-1}, 0)UM, \quad \text{i.e.,} \ \left(M' - V(E^{-1}, 0)U\right) M = 0.$$

By the exactness of the preceding sequence this is equivalent to the existence of an

$$N \in R^{l \times (k-l)} \text{ such that } M' - V(E^{-1}, 0)U = NU_2, \text{ i.e.,}$$
$$M' = V(E^{-1}, 0)U + NU_2 \text{ or } M' \in V(E^{-1}, 0)U + R^{l \times (k-l)}U_2.$$

□

We are now going to define T-autonomy, T-stability and T-observability for a multiplicatively closed set T of polynomials. For this purpose let $T \subseteq \mathcal{D} \setminus \{0\}$ be such a set, i.e., satisfying (i) $1 \in T$ and (ii) $(t_1, t_2 \in T \implies t_1 t_2 \in T)$, and let

$$\mathcal{D}_T = \left\{ \frac{f}{t} \in F(s);\, t \in T \right\} \subseteq \mathcal{K} = F(s)$$

denote the quotient ring with respect to T, also called the ring of *T-stable* rational functions. We may and do always assume that T is *saturated*, i.e., that

it contains all divisors of elements of T. If this is not the case we replace T by the saturated multiplicatively closed set of all these divisors with the same quotient ring. Saturation implies $T = \mathcal{D} \cap \mathrm{U}(\mathcal{D}_T)$ and $\mathrm{U}(\mathcal{D}_T) = \{t_1 t_2^{-1};\ t_1, t_2 \in T\}$ where $\mathrm{U}(\mathcal{D}_T)$ denotes the group of units or invertible elements of $\mathcal{D}_T$. The ring $\mathcal{D}_T$ is also a principal ideal domain. A representative system of its prime elements, up to units, are the monic irreducible polynomials $f \in \mathcal{D} = F[s]$ which do not belong to T.
More generally we also consider the quotient module [12, §II.3] M_T for a $\mathcal{D}$-module M:

$$\begin{gathered} M_T := \left\{\frac{x}{t};\ x \in M,\ t \in T\right\} \\ \text{with the canonical map } \operatorname{can} : M \to M_T,\ x \mapsto \frac{x}{1}, \text{ and} \\ \mathrm{t}_T(M) := \ker(\operatorname{can}) = \{x \in M;\ \exists t \in T \text{ with } tx = 0\}. \end{gathered} \tag{21}$$

The module $\mathrm{t}_T(M)$ is called the *T-torsion* submodule of M and contained in $\mathrm{t}(M)$. The module M_T is a $\mathcal{D}_T$-module in the natural fashion, and the the functor (assignment) $M \mapsto M_T$ is exact, i.e. it maps exact sequences of $\mathcal{D}$-modules onto exact sequences of $\mathcal{D}_T$-modules.
For the construction of proper observers in Sections 3 and 4 we need Smith form computations over the ring

$$\mathcal{S} := \mathcal{D}_T \cap F(s)_{\mathrm{pr}} \tag{22}$$

of *proper and T-stable* rational functions. In [21, Ch.2] and [20, Ch.5] it is shown that in many cases this ring is euclidean and therefore admits a Smith form algorithm. Instead we derive a different algorithm which is implemented in every standard computer algebra system. Assume that T contains a linear polynomial

$$\begin{gathered} s - \alpha \text{ and pose } \sigma := \frac{1}{s-\alpha}, \text{ hence } \deg_s(\sigma) = -1,\ \sigma \in \mathcal{S}, \\ F[\sigma] = F\left[\frac{1}{s-\alpha}\right] \subseteq \mathcal{S} \text{ and } \mathcal{K} = F(s) = F(\sigma). \end{gathered} \tag{23}$$

If $f(s) = \sum_{i=0}^{n} a_i (s-\alpha)^i \in F[s] = F[s-\alpha]$ is any nonzero polynomial with $\deg_s(f) = n$ the rational function

$$\begin{gathered} \widehat{f}(\sigma) := f\sigma^n = \frac{f}{(s-\alpha)^n} = a_n + a_{n-1}\frac{1}{s-\alpha} + \cdots + a_0 \frac{1}{(s-\alpha)^n} = \\ a_n + a_{n-1}\sigma + \cdots + a_0\sigma^n \text{ with } \deg_s(\widehat{f}) = \deg_s(f) - n = 0 \text{ and } f = \frac{\widehat{f}(\sigma)}{\sigma^n} \end{gathered} \tag{24}$$

is a polynomial in σ. Its σ-degree $\deg_\sigma(\widehat{f}(\sigma))$ is equal to $n = \deg_s(f)$ if and only if $f(\alpha) = a_0 \neq 0$.

Definition and Lemma 2.14. *The subset*

$$T_1 := \left\{\widehat{t} = \frac{t}{(s-\alpha)^{\deg(t)}};\ t \in T\right\} \subseteq F[\sigma]$$

is multiplicatively closed and saturated in $F[\sigma]$ and contained in the group $\mathrm{U}(\mathcal{S})$ of units of $\mathcal{S}$. Its quotient ring is $\mathcal{S} = F[\sigma]_{T_1}$.

Hence $\mathcal{S}$ is a principal ideal domain. The Smith form of a rational matrix $R \in F(s)^{k\times l} = F(\sigma)^{k\times l}$ with respect to $\mathcal{S}$ is the same as that with respect to $F[\sigma]$ and can be easily computed with any standard computer algebra system.

Proof. The equation

$$\frac{t}{(s-\alpha)^{\deg(t)}}\frac{(s-\alpha)^{\deg(t)}}{t} = 1 \text{ in } \mathcal{S} \text{ implies } T_1 \subseteq \mathrm{U}(\mathcal{S}) \text{ and } F[\sigma]_{T_1} \subseteq \mathcal{S}_{\mathrm{U}(\mathcal{S})} = \mathcal{S}.$$

Conversely, if

$$0 \neq r = ft^{-1} \in \mathcal{S} = F(s)_{\mathrm{pr}} \cap \mathcal{D}_T, \text{ i.e.,}$$
$$t \in T \text{ and } \deg(f) \leq \deg(t) \text{ then } r = ft^{-1} =$$
$$\left(\sigma^{\deg(t)-\deg(f)} f\sigma^{\deg(f)}\right)\left(t\sigma^{\deg(t)}\right)^{-1} = \sigma^{\deg(t)-\deg(f)}\widehat{f}\widehat{t}^{-1} \in F[\sigma]_{T_1}.$$

□

Theorem and Definition 2.15 (T-autonomy and T-stability)**.**

1. *The following properties are equivalent for a behavior $\mathcal{B} = \{w \in \mathcal{F}^p;\ R \circ w = 0\}$ with $R \in \mathcal{D}^{k\times l}$, $U := \mathcal{D}^{1\times k}R$ and $M := \mathcal{D}^{1\times l}/U$:*

   (a) $M = \mathrm{t}_T(M)$ or, equivalently, $M_T = 0$.

   (b) There is a $t \in T$ with $tM = 0$ or, equivalently, $t \circ \mathcal{B} = 0$.

   (c) The matrix R has a left inverse in $\mathcal{D}_T^{l\times k}$.

   (d) $\mathrm{rank}(R) = l$, i.e., $\mathcal{B}$ is autonomous, and the highest elementary divisor of R belongs to T. If R is square (without loss of generality) this also signifies that $\det(R) \in T$ since this determinant is the product of all elementary divisors of R.

   *Under these conditions $\mathcal{B}$ is called T-*autonomous*. Trajectories w that satisfy $t \circ w = 0$ for some $t \in T$ are called T-*small *or T-*negligible*. Hence a behavior is T-autonomous if all its trajectories are T-small.*

2. *An IO behavior $\mathcal{B} = \{\binom{y}{u} \in \mathcal{F}^{p+m};\ P \circ y = Q \circ u\}$ is called T-*stable *if it satisfies the following equivalent conditions:*

   (a) Its autonomous part $\mathcal{B}^0 := \{y \in \mathcal{F}^p;\ P \circ y = 0\}$ is T-autonomous or, equivalently, P has a left inverse in $\mathcal{D}_T^{p\times k}$ or its highest elementary divisor belongs to T.

   (b) (i) $H \in \mathcal{D}_T^{p\times m}$ (ii) The highest elementary divisor of $(P, -Q)$ belongs to T. This condition holds in particular if $\mathcal{B}$ is controllable, i.e., if this elementary divisor is even a nonzero constant in F.

Proof. 1. $(b) \implies (a)$: obvious.
$(a) \implies (b)$: Let $x_j := \overline{\delta_j}$ denote the canonical generators of M. By assumption there are elements $t_j \in T$ with $t_j x_j = 0$. Then $t := t_1 * \cdots * t_l \in T$ annihilates M, i.e. $tM = 0$. Moreover

$$tM = 0 \iff t\mathcal{D}^{1\times l} \subseteq U \iff \mathcal{B} = U^\perp \subseteq \left(t\mathcal{D}^{1\times l}\right)^\perp =$$
$$\{w \in \mathcal{F}^l;\ (t\mathcal{D}^{1\times l}) \circ w = \mathcal{D}^{1\times l} \circ (t \circ w) = 0\} = \{w \in \mathcal{F}^l;\ t \circ w = 0\} \iff$$
$$t \circ \mathcal{B} = 0.$$

$(a) \iff (c)$:

$$0 = M_T = \left(\mathcal{D}^{1\times l}/\mathcal{D}^{1\times k}R\right)_T = \mathcal{D}_T^{1\times l}/\mathcal{D}_T^{1\times k}R \iff$$
$$\mathcal{D}_T^{1\times l} = \mathcal{D}_T^{1\times k}R \iff \exists X \in \mathcal{D}_T^{l\times k} \text{ with } XR = \mathrm{id}_l\,.$$

$(c) \iff (d)$: According to Theorem 2.13 the matrix $R \in \mathcal{D}^{k\times l} \subseteq \mathcal{D}_T^{k\times l}$ has a left inverse in $\mathcal{D}_T^{l\times k}$ if and only if $l = \mathrm{rank}(R)$ and its highest elementary divisor is invertible in $\mathcal{D}_T$. But $T = \mathcal{D} \cap \mathrm{U}(\mathcal{D}_T)$ since T is saturated.
2. We assume $k = p$ without loss of generality.
$(a) \implies (b)$: By assumption and 1.(d) $P \in \mathrm{Gl}_p(\mathcal{D}_T)$ or, equivalently, $\det(P) \in \mathcal{D} \cap \mathrm{U}(\mathcal{D}_T) = T$. Then (i) $H = P^{-1}Q \in \mathcal{D}_T^{p\times m}$. (ii) Due to $P \in \mathrm{Gl}_p(\mathcal{D}_T)$ and $H \in \mathcal{D}_T^{p\times m}$ the matrices $(\mathrm{id}_p, -H)$ and $(P, -Q) = P(\mathrm{id}_p, -H)$ are row equivalent in $\mathcal{D}_T^{p\times(p+m)}$ and have the same elementary divisors. But those of $(\mathrm{id}_p, -H)$ are one.
$(b) \implies (a)$: Let $X(P, -Q)Y = (E, 0)$ be the Smith form of $(P, -Q)$ with respect to $\mathcal{D}$. The conditions (i) and (ii) imply

$$H \in \mathcal{D}_T^{p\times m} \text{ and } E \in \mathrm{Gl}_p(\mathcal{D}_T), \text{ hence } XP\left((\mathrm{id}_p, -H)Y\begin{pmatrix} E^{-1} \\ 0 \end{pmatrix}\right) = \mathrm{id}_p \text{ in } \mathcal{D}_T^{p\times p}.$$

This implies $P \in \mathrm{Gl}_p(\mathcal{D}_T)$. □

Example 2.16. Let $F = \mathbb{R}, \mathbb{C}$ be the real or complex field and $\mathcal{F}$ one of the standard continuous or discrete injective cogenerator signal modules.

1. $T = \{1\}$: This set is not saturated, its saturation is $\mathrm{U}(F[s]) = F \setminus \{0\}$. Only the zero behavior is T-autonomous. This set T is defined for all base fields F.

2. $T := \mathcal{D} \setminus \{0\}$, hence $\mathcal{D}_T = F(s)$: In this case each nonzero polynomial is invertible in $\mathcal{D}_T$ and hence T-autonomy and autonomy coincide. This set T is also defined for any base field F.

3. Let $\mathcal{F}$ be one of the standard injective cogenerator signal modules and let

$$\mathbb{C} := \Lambda_1 \uplus \Lambda_2 \text{ with } \Lambda_1 \subseteq \begin{cases} \{\lambda \in \mathbb{C};\ \Re(\lambda) < 0\} & \text{in the continuous case} \\ \{\lambda \in \mathbb{C};\ |\lambda| < 1\} & \text{in the discrete case} \end{cases} \tag{25}$$

be a non-trivial disjoint decomposition of the complex plane into a *stable* region Λ_1 and an *unstable* region Λ_2. We assume that Λ_1 and hence Λ_2 are symmetric with respect to the real axis, i.e., $\overline{\Lambda_1} = \Lambda_1$ with the complex conjugate $\overline{\lambda}$. Define

$$T_\Lambda := \{t \in F[s];\ V_\mathbb{C}(t) \subseteq \Lambda_1\} \text{ resp. } \mathcal{D}_{T_\Lambda} = \left\{\frac{f}{t};\ f \in F[s], t \in T_\Lambda\right\}. \tag{26}$$

The polynomials in T_Λ and rational functions in $\mathcal{D}_{T_\Lambda}$ are called stable *for the chosen decomposition* (25). These stable objects were also defined and discussed in [21, Ch.2, p.14] and [20, Ch.5]. According to Theorem 2.15 and Result 2.12 T_Λ-autonomy of a behavior $\mathcal{B} \subset \mathcal{D}'(\mathbb{R}, \mathbb{C})^l$ signifies that it is $\mathbb{C}$-finite-dimensional and contained in $\oplus_{\lambda\in\Lambda_1}\mathbb{C}[t]^l e^{\lambda t}$, and analogous properties hold for all other standard cases. In particular, all its trajectories w satisfy $\lim_{t\to\infty} w(t) = 0$. If $\binom{y_1}{u}$ and $\binom{y_2}{u}$ are two trajectories of a T_Λ-stable IO behavior with same input u the outputs y_1 and y_2 are asymptotically equal, i.e. , $\lim_{t\to\infty}(y_1(t) - y_2(t)) = 0$.

4. The set $T := \{s^k;\ k \in \mathbb{N}\}$ is not saturated, its saturation is the set $\widetilde{T} = \{\alpha s^k;\ 0 \neq \alpha \in F,\ k \in \mathbb{N}\}$. Let $\mathcal{B}$ be T-autonomous in one of the standard signal spaces. In the continuous case this signifies that all trajectories of $\mathcal{B}$ are polynomial functions. In the discrete case all trajectories $w \in \mathcal{B}$ are finally zero, ie., there is a time instant t_0 such that $w(t) = 0$ for all $t \geq t_0$. This set T is also defined for arbitrary base fields F.

Corollary 2.17. *Let $T := T_\Lambda$ from item 3. of the preceding example and $\mathcal{B}$ a behavior as in Theorem 2.15.*

1. *If $\mathcal{B}$ is autonomous it is T-autonomous if and only if $\mathrm{ch}(\mathcal{B}) \subseteq \Lambda_1$.*
2. *If $\mathcal{B}$ is an IO behavior it is T-stable if and only if $\mathrm{ch}(\mathcal{B}^0) \subseteq \Lambda_1$, i.e., if all poles of $\mathcal{B}$ belong to Λ_1, or, equivalently, if $H \in \mathcal{D}_{T_\Lambda}^{p\times m}$ and $\mathrm{ch}(\mathcal{B}) \subseteq \Lambda_1$.*

Proof. According to Definition and Corollary 2.11 the characteristic variety of $\mathcal{B}$ resp. $\mathcal{B}^0$ equals $V_{\mathbb{C}}(e)$ where e is the highest elementary divisor of R resp. P, and the inclusion $V_{\mathbb{C}}(e) \subseteq \Lambda_1$ signifies $e \in T$. The assertion thus follows from items 1.d resp. 2.a of Theorem 2.15. □

Definition and Corollary 2.18 (T-observability)**.** *Consider a behavior*

$$\mathcal{B}_1 = \{w \in \mathcal{F}^{l_1};\ R_1 \circ w = 0\},\ R_1 \in \mathcal{D}^{k_1\times l_1},\ \textit{and } P \in \mathcal{D}^{l_2\times l_1}.$$

*We call $w \in \mathcal{B}_1$ resp. $\mathcal{B}_1$ T-*observable *from $P \circ w$ resp. $P \circ \mathcal{B}_1$ if the behavior*

$$\ker\left(P\circ : \mathcal{B}_1 \to \mathcal{F}^{l_2}\right) = \left\{w \in \mathcal{F}^{l_1};\ \begin{pmatrix} R_1 \\ P \end{pmatrix} \circ w = 0\right\}$$

is T-autonomous. According to Theorem 2.15 this signifies that there is a left inverse matrix $(Y, Z) \in \mathcal{D}_T^{l_1\times(k_1+l_2)}$ of $\binom{R_1}{P}$ or that for $w_1, \tilde{w}_1 \in \mathcal{B}_1$ with equal image $P \circ w_1 = P \circ \tilde{w}_1$ the difference $w_1 - \tilde{w}_1$ is T-small.
For the Rosenbrock equations from Example 2.9 the behavior $\mathcal{B}_1$ is T-observable from $\mathcal{B}_2$ if and only if $\binom{A}{C}$ has a left inverse in $\mathcal{D}_T^{n\times(n+p)}$ since

$$\ker\begin{pmatrix} A \\ C \end{pmatrix}\circ \cong \ker\begin{pmatrix} A & -B \\ C & D \\ 0 & \mathrm{id}_m \end{pmatrix}\circ,\ x \longleftrightarrow \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Example 2.19. In the situation of the Definition 2.18 and Example 2.16 different choices of T furnish the following special cases of T-observability and T-observers (see the following sections):

1. $T = \{1\}$: In this case the morphism $P\circ$ is injective on $\mathcal{B}_1$ and $\mathcal{B}_1$ is called *observable* from $P \circ \mathcal{B}_1$ [15, Def.7.62, Thm.7.63], [17, Def.5.3.2], [19, Def.3.1], [9, Defs.3.1,4.1]. If $(Y, Z) \in \mathcal{D}^{l_1\times(k_1+l_2)}$ is a left inverse of $\binom{R_1}{P}$, i.e., $YR_1 + ZP = \mathrm{id}_{l_1}$ and if $w \in \mathcal{B}_1$ then $R_1 \circ w = 0$ and therefore $w = \mathrm{id}_{l_1} \circ w = Z \circ (P \circ w)$. Thus w can be computed from $P \circ w$, but

only by means of the operator $Z\circ$, i.e., in general and the standard cases, by higher derivatives or shifts which are unsuitable from the engineering point of view. Proper asymptotic observers [13] were introduced to avoid these higher derivatives.

2. $T = \{s^k;\ k \in \mathbb{N}\}$: $\mathcal{B}_1$ is *reconstructible* from $P \circ \mathcal{B}_1$ [9, Defs.3.1,4.1]. In the discrete case a T-observer is called a *dead-beat observer* [2].

3. $T = T_\Lambda$: $\mathcal{B}_1$ is Λ-*detectable* from $P \circ \mathcal{B}_1$ [17, Def.5.3.16], [19, Def.3.1]. The associated observers are called Λ-*asymptotic* [9, Defs.3.1,4.1].

4. $T = \mathcal{D} \setminus \{0\}$: $\mathcal{B}_1$ is *trackable* from $P \circ \mathcal{B}_1$ [9, Def.3.1].

In the following sections we will define T-observers of a desired component of a trajectory as suitable IO behaviors and discuss their existence and construction.

3 Input T-observers and output T-controllers

This section can be considered as a behavioral extension of Wolovich's work in [23, Sect.5.5] where he treated input function observability and output function controllability. Concerning the behavioral definitions, especially of observers and their error behaviors, we follow Valcher/Willems [19] and Fuhrmann [9]. Also in our theory Result 2.8 on image behaviors and elimination plays an important part, compare [9, Rem. on p.104].

Definition 3.1. In this section we start with two IO behaviors

$$\begin{aligned}
\mathcal{B}_1 &:= \left\{ \begin{pmatrix} y_1 \\ u \end{pmatrix} \in \mathcal{F}^{p+m};\ P_1 \circ y_1 = Q_1 \circ u \right\},\ P_1 \in \mathcal{D}^{p\times p},\ \det(P_1) \neq 0, \\
\mathcal{B}_2 &:= \left\{ \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \in \mathcal{F}^{m+p};\ P_2 \circ y_2 = Q_2 \circ y_1 \right\},\ P_2 \in \mathcal{D}^{m\times m},\ \det(P_2) \neq 0
\end{aligned}$$

with transfer matrices $H_1 = P_1^{-1}Q_1$ resp. $H_2 = P_2^{-1}Q_2$ which can be connected in series. We define the *serial interconnection* behavior $\mathcal{B}$ and the *error* behavior $\mathcal{B}_{\mathrm{err}}$ as

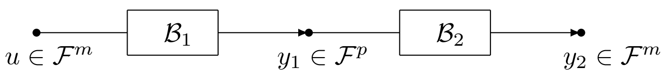
$$\begin{aligned}
\mathcal{B} &:= \left\{ \begin{pmatrix} y_2 \\ y_1 \\ u \end{pmatrix} \in \mathcal{F}^{m+p+m};\ \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \in \mathcal{B}_2 \text{ and } \begin{pmatrix} y_1 \\ u \end{pmatrix} \in \mathcal{B}_1 \right\} \\
&= \left\{ \begin{pmatrix} y_2 \\ y_1 \\ u \end{pmatrix} \in \mathcal{F}^{m+p+m};\ \begin{pmatrix} P_2 & -Q_2 & 0 \\ 0 & P_1 & -Q_1 \end{pmatrix} \circ \begin{pmatrix} y_2 \\ y_1 \\ u \end{pmatrix} = 0 \right\} \quad \text{and} \\
\mathcal{B}_{\mathrm{err}} &:= \left\{ y_2 - u \in \mathcal{F}^m;\ \exists \begin{pmatrix} y_2 \\ y_1 \\ u \end{pmatrix} \in \mathcal{B} \right\} \\
&= (\mathrm{id}_m, 0, -\mathrm{id}_m) \circ \mathcal{B} =: \{ e \in \mathcal{F}^m;\ P_{\mathrm{err}} \circ e = 0 \}
\end{aligned}$$

where P_{err} exists according to Result 2.8. Then $\mathcal{B}_2$ is called an *input T-observer* of $\mathcal{B}_1$ or $\mathcal{B}_1$ an *output T-controller* of $\mathcal{B}_2$ if $\mathcal{B}_{\mathrm{err}}$ is T-autonomous (compare [23, p.164]).

Remark 3.2. 1. The serial behavior $\mathcal{B}$ in Definition 3.1 has the representation

$$\mathcal{B} = \left\{ \begin{pmatrix} y_2 \\ y_1 \\ u \end{pmatrix} \in \mathcal{F}^{m+p+m};\ \begin{pmatrix} P_2 & -Q_2 \\ 0 & P_1 \end{pmatrix} \circ \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ Q_1 \end{pmatrix} \circ u \right\}$$

which shows that it is an IO behavior with input u and output $\left(\begin{smallmatrix} y_2 \\ y_1 \end{smallmatrix}\right)$. The following picture is a visualization of $\mathcal{B}$:



2. In the standard cases of $T := T_\Lambda$ from Example 2.16,(3), the T-autonomy of $\mathcal{B}_{\mathrm{err}}$ implies that all trajectories in $\mathcal{B}_{\mathrm{err}}$ tend to zero for $t \to \infty$, i.e., that all trajectories $\left(\begin{smallmatrix} y_2 \\ y_1 \\ u \end{smallmatrix}\right) \in \mathcal{B}$ satisfy $\lim_{t\to\infty}(y_2(t) - u(t)) = 0$. If an input T-observer $\mathcal{B}_2$ of a given plant $\mathcal{B}_1$ is used in serial connection the output of the interconnected system is asymptotically equal to the input u of the plant $\mathcal{B}_1$. Therefore an input T-observer is called an *asymptotic* input observer in this case.
Likewise, if an output T-controller $\mathcal{B}_1$ of a given plant $\mathcal{B}_2$ is used in serial connection the output y_2 of the interconnected system y_2 is asymptotically equal to a desired output u which is taken as input of the controller.
3. According to Theorem 2.15 $\mathcal{B}$ is T-stable if and only if $\det(P) = \det(P_1)\det(P_2)$ belongs to T. Since T is saturated this signifies that both $\mathcal{B}_1$ and $\mathcal{B}_2$ are T-stable.

For the proof of the theorems in this section we need some lemmas.

Lemma 3.3. *For two IO systems $\mathcal{B}_1$ and $\mathcal{B}_2$ as in Definition 3.1 let $(X, Y) \in \mathcal{D}^{k\times(m+p)}$ be a universal left annihilator of $\begin{pmatrix} P_2 & -Q_2 \\ -Q_1 & P_1 \end{pmatrix}$. Then the matrix P_{err} of equations of $\mathcal{B}_{\mathrm{err}}$ is $P_{\mathrm{err}} = XP_2$.*

Proof. It follows from the basic Result 2.8 on image behaviors that P_{err} can be computed by means of a universal left annihilator $(X, Y, -P_{\mathrm{err}})$ of the matrix

$$\begin{pmatrix} P_2 & -Q_2 & 0 \\ 0 & P_1 & -Q_1 \\ \mathrm{id}_m & 0 & -\mathrm{id}_m \end{pmatrix}.$$

This annihilator satisfies the three conditions

$$XP_2 = P_{\mathrm{err}},\ XQ_2 = YP_1 \text{ and } YQ_1 = P_{\mathrm{err}} \text{ or}$$
$$XP_2 = YQ_1,\ XQ_2 = YP_1 \text{ and } P_{\mathrm{err}} = XP_2$$

which, in turn, are equivalent to

$$(X, Y)\begin{pmatrix} P_2 & -Q_2 \\ -Q_1 & P_1 \end{pmatrix} = 0 \text{ and } P_{\mathrm{err}} = XP_2.$$

We conclude that (X, Y) is a universal left annihilator of $\begin{pmatrix} P_2 & -Q_2 \\ -Q_1 & P_1 \end{pmatrix}$ and $P_{\mathrm{err}} = XP_2$. □

Lemma 3.4. *In the situation of Lemma 3.3 assume that the transfer matrix H_2 is a left inverse of H_1, i.e. $H_2H_1 = \mathrm{id}_m$. Then:*

1. $\mathrm{rank}(X, Y) = m$. *Therefore we can and do assume that $(X, Y) \in \mathcal{D}^{m\times(m+p)}$.*

2. $P_{\mathrm{err}} = XP_2 \in \mathcal{D}^{m\times m}$ *has* $\mathrm{rank}(P_{\mathrm{err}}) = m$, *i.e., $\mathcal{B}_{\mathrm{err}}$ is autonomous, and*

$$(X, Y) = P_{\mathrm{err}}(P_2^{-1}, H_2P_1^{-1}).$$

Proof. 1.

$$\begin{aligned} \begin{pmatrix} P_2 & -Q_2 \\ -Q_1 & P_1 \end{pmatrix} &= \begin{pmatrix} P_2 & 0 \\ 0 & P_1 \end{pmatrix} \begin{pmatrix} \mathrm{id}_m & -H_2 \\ -H_1 & \mathrm{id}_p \end{pmatrix} \\ &= \begin{pmatrix} P_2 & 0 \\ 0 & P_1 \end{pmatrix} \begin{pmatrix} \mathrm{id}_m & -H_2 \\ 0 & \mathrm{id}_p \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -H_1 & \mathrm{id}_p \end{pmatrix}, \end{aligned}$$

the last equality following from $H_2H_1 = \mathrm{id}_m$. The first matrix in the last product has rank $p + m$, the second one as well, and the third one has rank p. This implies $\mathrm{rank} \begin{pmatrix} P_2 & -Q_2 \\ -Q_1 & P_1 \end{pmatrix} = p$. Since (X, Y) is a universal left annihilator of this matrix we infer

$$\mathrm{rank}(X, Y) = (m + p) - \mathrm{rank} \begin{pmatrix} P_2 & -Q_2 \\ -Q_1 & P_1 \end{pmatrix} = (p + m) - p = m.$$

2. We define the matrices X^1 and Y^1:

$$\begin{aligned} 0 &= (X, Y) \begin{pmatrix} P_2 & -Q_2 \\ -Q_1 & P_1 \end{pmatrix} \\ &= \underbrace{(X, Y) \begin{pmatrix} P_2 & 0 \\ 0 & P_1 \end{pmatrix} \begin{pmatrix} \mathrm{id}_m & -H_2 \\ 0 & \mathrm{id}_p \end{pmatrix}}_{=:(X^1,Y^1)} \begin{pmatrix} 0 & 0 \\ -H_1 & \mathrm{id}_p \end{pmatrix} \end{aligned}$$

with $X^1 = XP_2 = P_{\mathrm{err}}$ and $\mathrm{rank}(X^1, Y^1) = \mathrm{rank}(X, Y) = m$. The equation

$$0 = (X^1, Y^1) \begin{pmatrix} 0 & 0 \\ -H_1 & \mathrm{id}_p \end{pmatrix} = (-Y^1H_1, Y^1)$$

implies $Y^1 = 0$ and thus $\mathrm{rank}(X^1) = m$, hence $\mathrm{rank}(P_{\mathrm{err}}) = \mathrm{rank}(X^1) = m$. We finally rewrite the matrix (X, Y) as

$$\begin{aligned} (X, Y) &= (X^1, Y^1) \begin{pmatrix} \mathrm{id}_m & H_2 \\ 0 & \mathrm{id}_p \end{pmatrix} \begin{pmatrix} P_2^{-1} & 0 \\ 0 & P_1^{-1} \end{pmatrix} \\ &= (P_{\mathrm{err}}, 0) \begin{pmatrix} P_2^{-1} & H_2P_1^{-1} \\ 0 & P_1^{-1} \end{pmatrix} \\ &= P_{\mathrm{err}}(P_2^{-1}, H_2P_1^{-1}). \end{aligned}$$

□

Lemma 3.5. *If in the situation of Lemma 3.3 H_2 is a left inverse of H_1, i.e., $H_2H_1 = \mathrm{id}_m$, and if, in addition, $P_2^{-1} \in \mathcal{D}_T^{m\times m}$ and $H_2P_1^{-1} \in \mathcal{D}_T^{m\times p}$ then the behavior $\mathcal{B}_{\mathrm{err}}$ is T-autonomous.*

Proof. The equation

$$(P_2^{-1}, H_2P_1^{-1})\begin{pmatrix} P_2 & -Q_2 \\ -Q_1 & P_1 \end{pmatrix} = (\mathrm{id}_m - H_2H_1, -H_2 + H_2) = 0$$

shows that

$$(P_2^{-1}, H_2P_1^{-1}) \in \mathcal{D}_T^{m\times(m+p)} \text{ is a left annihilator of } \begin{pmatrix} P_2 & -Q_2 \\ -Q_1 & P_1 \end{pmatrix}.$$

According to Lemma 3.3 $(X, Y) \in \mathcal{D}^{m\times(m+p)}$ is a *universal* left annihilator of this matrix with respect to $\mathcal{D}$ and hence also with respect to $\mathcal{D}_T$ because the functor (assignment) $M \mapsto M_T$ preserves exactness. By definition of a universal annihilator there exists a matrix $X^2 \in \mathcal{D}_T^{m\times m}$ such that

$$(P_2^{-1}, H_2P_1^{-1}) = X^2(X, Y).$$

On the other hand Lemma 3.4 yields

$$(X, Y) = P_{\mathrm{err}}(P_2^{-1}, H_2P_1^{-1}).$$

Combining these two equations implies

$$\begin{aligned} &(X, Y) = P_{\mathrm{err}}X^2(X, Y), \text{ hence } P_{\mathrm{err}}X^2 = \mathrm{id}_m \text{ since } \operatorname{rank}(X, Y) = m \text{ and} \\ &1 = \det(P_{\mathrm{err}})\det(X^2), \det(X^2) \in \mathcal{D}_T, \text{ thus } \det(P_{\mathrm{err}}) \in \mathcal{D} \cap \mathrm{U}(\mathcal{D}_T) = T. \end{aligned}$$

According to Theorem 2.15 this signifies that $\mathcal{B}_{\mathrm{err}}$ is T-autonomous. □

Lemma 3.6. *Let $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}$ and $\mathcal{B}_{\mathrm{err}}$ be the behaviors from Definition 3.1. If $\mathcal{B}_{\mathrm{err}}$ is T-autonomous then*

1. $H_2H_1 = \mathrm{id}_m$ *and*
2. $\mathcal{B}_2$ *is T-stable.*

Proof. 1. For $\tilde{u} \in \mathcal{K}^m$, $\mathcal{K} := \mathrm{quot}(\mathcal{D}) = F(s)$, we obtain

$$\begin{pmatrix} H_1\tilde{u} \\ \tilde{u} \end{pmatrix} \in \mathcal{B}_{1\mathcal{K}}, \begin{pmatrix} H_2H_1\tilde{u} \\ H_1\tilde{u} \end{pmatrix} \in \mathcal{B}_{2\mathcal{K}}, \begin{pmatrix} H_2H_1\widetilde{u} \\ H_1\widetilde{u} \\ \widetilde{u} \end{pmatrix} \in \mathcal{B}_{\mathcal{K}} \text{ and}$$
$$H_2H_1\tilde{u} - \tilde{u} \in \mathcal{B}_{\mathrm{err}\mathcal{K}}$$

where $\mathcal{B}_{1\mathcal{K}}$ etc. denote the transfer spaces according to Result 2.3 and where we have also used the exactnes of the functor (assignment) $\mathcal{B} \mapsto \mathcal{B}_{\mathcal{K}}$.

But $\mathcal{B}_{\mathrm{err}\mathcal{K}} = 0$ because $\mathcal{B}_{\mathrm{err}}$ is T-autonomous and thus autonomous. Hence

$$H_2H_1\tilde{u} = \tilde{u} \text{ for all } \tilde{u} \in \mathcal{K}^m, \quad \text{i.e.} \quad H_2H_1 = \mathrm{id}_m.$$

2. The equation $H_2H_1 = \mathrm{id}_m$ and Lemma 3.4 imply $P_{\mathrm{err}} = XP_2 \in \mathcal{D}^{m\times m}$. By assumption we have $\det(P_{\mathrm{err}}) = \det(X)\det(P_2) \in T$. From the saturation of T we conclude that also $\det(P_2) \in T$. This signifies that $\mathcal{B}_2$ is a T-stable IO behavior.

□

We finally prove our main result on input T-observers.

Theorem 3.7 (Input T-observers). *Let*

$$\mathcal{B}_1 := \left\{ \begin{pmatrix} y_1 \\ u \end{pmatrix} \in \mathcal{F}^{p+m};\ P_1 \circ y_1 = Q_1 \circ u \right\},\ P_1 \in \mathcal{D}^{p\times p},\ \det(P_1) \neq 0,$$

be an IO behavior with transfer matrix $H_1 = P_1^{-1}Q_1$.
A. Existence of a T-observer $\mathcal{B}_2$ of $\mathcal{B}_1$*: The following three statements are equivalent:*

1. *There exists an input T-observer $\mathcal{B}_2$ of $\mathcal{B}_1$.*
2. *There is a matrix $H_2 \in \mathcal{D}_T^{m\times p}$ such that*
   (a) *$H_2H_1 = \mathrm{id}_m$ and*
   (b) *$H_2P_1^{-1} \in \mathcal{D}_T^{m\times p}$.*
3. *The matrix Q_1 has a left inverse $Z \in \mathcal{D}_T^{m\times p}$ or, equivalently according to Definition and Corollary 2.18, u is T-observable from y_1, and then $H_2 = ZP_1$ satisfies (A2).*

If these conditions are satisfied each input T-observer $\mathcal{B}_2$ is automatically T-stable. The unique controllable realization $\mathcal{B}_2$ of H_2 is the unique controllable T-input observer of $\mathcal{B}_1$ with transfer matrix H_2 and, of course, also T-stable. Condition (A3) can be constructively checked by means of Theorem 2.13.
B. Parametrization*: If the equivalent conditions of item A. hold all input T-observers of $\mathcal{B}_1$ are obtained by the following algorithmic steps:*

1. *Compute one and then all left inverses Z of Q_1 in $\mathcal{D}_T^{m\times p}$ according to Theorem 2.13. Define $H_2 := ZP_1$ for a chosen Z.*
2. *Compute the unique controllable realization of $H_2 = ZP_1$ by means of Result 2.10:*

$$\mathcal{B}_{\mathrm{cont}} = \left\{ \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \in \mathcal{F}^{m+p};\ P_{\mathrm{cont}} \circ y_2 = Q_{\mathrm{cont}} \circ y_1 \right\}.$$

3. *Choose a matrix $\widetilde{P}_2 \in \mathcal{D}^{m\times m}$ with $\det(\widetilde{P}_2) \in T$ and define*

$$(P_2, -Q_2) := \widetilde{P}_2(P_{\mathrm{cont}}, -Q_{\mathrm{cont}}).$$

The IO behaviors $\mathcal{B}_2 := \{\left(\begin{smallmatrix} y_2 \\ y_1 \end{smallmatrix}\right) \in \mathcal{F}^{m+p}; P_2 \circ y_2 = Q_2 \circ y_1\}$ are exactly the input T-observers of $\mathcal{B}_1$. In other words: The inverses $Z \in \mathcal{D}_T^{m\times p}$ of Q_1 parametrize all controllable input T-observers and the pairs $(Z, \widetilde{P}_2)$ all input T-observers. According to Lemma 3.3 the error behavior of this observer is

$$\mathcal{B}_{\mathrm{err}} = \{e \in \mathcal{F}^m;\ XP_2 \circ e = 0\} \text{ with a universal left annihilator } (X, Y) \text{ of}$$
$$\begin{pmatrix} P_2 & -Q_2 \\ -Q_1 & P_1 \end{pmatrix} \text{ or } X\widetilde{P_2}(P_{\mathrm{cont}}, -Q_{\mathrm{cont}}) = X(P_2, -Q_2) = Y(Q_1, -P_1).$$

C. Properness*:*

1. *If $(Z, \widetilde{P}_2)$ is chosen as in item B. the associated observer $\mathcal{B}_2$ and especially its transfer matrix $H_2 = ZP_1$ are T-stable, hence $H_2 \in \mathcal{D}_T^{m\times p}$. If H_2 is also proper the input T-observer is proper by definition. The existence and construction of such Z and H_2 will be discussed in Theorem 3.12.*

2. *If $\mathcal{B}_1$ is T-stable, i.e., $\det(P_1) \in T$, and if H_2 is any left inverse of H_1 in $\mathcal{S}^{m\times p}$ then $Z = H_2P_1^{-1} \in \mathcal{D}_T^{m\times p}$ and thus all conditions in A. are satisfied. The corresponding input T-observer according to B. is proper, and all proper input T-observers are obtained in this fashion.*
   Theorem 2.13 again enables to check the existence of such a left inverse H_2 of H_1 and to construct all of them if there is one.

Proof. A. 1. $\Rightarrow$ 2.: Let $\mathcal{B}$, $\mathcal{B}_{\mathrm{err}}$, X, and Y denote the same behaviors resp. matrices as in Definition 3.1 and Lemma 3.3. The assumption that $\mathcal{B}_2$ is an input T-observer of $\mathcal{B}_1$ implies by definition that $\mathcal{B}_{\mathrm{err}}$ is T-autonomous. By means of Lemma 3.6 we conclude

$$P_2 \in \mathrm{Gl}_m(\mathcal{D}_T), \text{ hence } H_2 = P_2^{-1}Q_2 \in \mathcal{D}_T^{m\times p}, \text{ and } H_2H_1 = \mathrm{id}_m\,.$$

Now we apply Lemma 3.4 and use the equation

$$(X, Y) = P_{\mathrm{err}}(P_2^{-1}, H_2P_1^{-1}).$$

The T-autonomy of $\mathcal{B}_{\mathrm{err}}$ implies $P_{\mathrm{err}} \in \mathrm{Gl}_m(\mathcal{D}_T)$ and then

$$H_2P_1^{-1} = P_{\mathrm{err}}^{-1}Y \in \mathcal{D}_T^{m\times p}.$$

2. $\Rightarrow$ 1.: Let $\mathcal{B}_2$ be the unique controllable realization of H_2 according to Result 2.10:

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \in \mathcal{F}^{m+p};\ P_2 \circ y_2 = Q_2 \circ y_1 \right\},\ P_2 \in \mathcal{D}^{m\times m},\ \det(P_2) \neq 0.$$

Since $\mathcal{B}_2$ is controllable the elementary divisors of $(P_2, -Q_2)$ are units in $\mathcal{D}$ and therefore in $\mathcal{D}_T$. With $H_2 \in \mathcal{D}_T^{m\times p}$ according to condition 2. we conclude from Theorem 2.15, 2., that $\mathcal{B}_2$ is a T-stable IO behavior and thus $P_2 \in \mathrm{Gl}_m(\mathcal{D}_T)$ and $P_2^{-1} \in \mathcal{D}_T^{m\times m}$.

The conditions 2. and Lemma 3.5 finally imply that $\mathcal{B}_{\mathrm{err}}$ is T-autonomous, i.e., that $\mathcal{B}_2$ is really a (T-stable) input T-observer of $\mathcal{B}_1$.
2. $\Rightarrow$ 3.: Define $Z := H_2P_1^{-1}$. Then 2. implies

$$Z \in \mathcal{D}_T^{m\times p} \quad \text{and} \quad ZQ_1 = H_2P_1^{-1}Q_1 = H_2H_1 = \mathrm{id}_m\,.$$

3. $\Rightarrow$ 2.: Define $H_2 := ZP_1 \in \mathcal{D}_T^{m\times p}$ since $Z \in \mathcal{D}_T^{m\times p}$ by condition 3. Then

$$H_2H_1 = ZP_1H_1 = ZQ_1 \underset{3.}{=} \mathrm{id}_m \text{ and } H_2P_1^{-1} = Z \in \mathcal{D}_T^{m\times p}.$$

B. *All input T-observers are obtained by these three steps*: If $\mathcal{B}_2$ is such an observer with transfer matrix H_2 the conditions from part A. are satisfied. Therefore H_2 and $Z = H_2P_1$ are obtained as in step 1. If $\mathcal{B}_{\mathrm{cont}}$ is the unique controllable realization of H_2 according to step 2. Result 2.10 implies the inclusion $\mathcal{B}_{\mathrm{cont}} \subseteq \mathcal{B}_2$ and thus the existence of some $\widetilde{P}_2 \in \mathcal{D}^{m\times m}$ with

$$(P_2, -Q_2) = \widetilde{P}_2(P_{\mathrm{cont}}, -Q_{\mathrm{cont}}), \text{ hence}$$
$$P_2 = \widetilde{P}_2P_{\mathrm{cont}} \text{ and } \det(P_2) = \det(\widetilde{P}_2)\det(P_{\mathrm{cont}}) \in T.$$

Since T is saturated this implies $\det(\widetilde{P_2}) \in T$ as in step 3.
All constructed $\mathcal{B}_2$ are indeed input T-observers: The assumptions imply

$$ZQ_1 = \mathrm{id}_m,\ Z \in \mathcal{D}_T^{m\times p},\ H_2 := ZP_1, \text{ hence } H_2,\ H_2P_1^{-1} \in \mathcal{D}_T^{m\times p}.$$

From part A. of the proof we know that $\mathcal{B}_{\underbrace{\text{cont}}}$ is a T-stable input T-observer, hence $P_{\text{cont}} \in \mathrm{Gl}_m(\mathcal{D}_T)$ and then also $P_2 = \widetilde{P_2}P_{\text{cont}} \in \mathrm{Gl}_m(\mathcal{D}_T)$ by the choice of $\widetilde{P_2}$. This signifies that $\mathcal{B}_2$ is T-stable. Again like in part A. of the proof we conclude from Lemma 3.5 that $\mathcal{B}_{\text{err}}$ is T-stable and that $\mathcal{B}_2$ is an input T-observer too. □

Example 3.8. We consider the case $p = m = 1$, i.e., P_1 and Q_1 are polynomials, $P_1 \neq 0$, and $H_1 = \frac{Q_1}{P_1} = \frac{Q_{1,\text{cont}}}{P_{1,\text{cont}}}$ with $\gcd(P_{1,\text{cont}}, Q_{1,\text{cont}}) = 1$ is a rational function. We want to check whether

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} y_1 \\ u \end{pmatrix} \in \mathcal{F}^{1+1};\ P_1 \circ y_1 = Q_1 \circ u \right\}$$

admits a proper input T-observer $\mathcal{B}_2$. By Theorem 3.7 this is equivalent to the existence of $Z \in \mathcal{D}_T$ with

$$ZQ_1 = 1 \text{ or } Q_1 \in T \quad \text{and}$$
$$H_2 = ZP_1 = \frac{P_1}{Q_1} = \frac{P_{1,\text{cont}}}{Q_{1,\text{cont}}} \in F(s)_{\text{pr}},\ \text{i.e., } \deg(P_1) \leq \deg(Q_1).$$

If this is the case all input T-observers have the form

$$\mathcal{B}_2 := \left\{ \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \in \mathcal{F}^{1+1};\ \widetilde{P}_2Q_{1,\text{cont}} \circ y_2 = \widetilde{P}_2P_{1,\text{cont}} \circ y_1 \right\},\ \widetilde{P}_2 \in T.$$

Notice that for $\widetilde{P}_2 = 1$ the matrix

$$(Q_{1,\text{cont}}, -P_{1,\text{cont}}) = \frac{1}{f}(Q_1, -P_1) \text{ with } f := \gcd(P_1, Q_1)$$

defines the unique controllable T-input observer $\mathcal{B}_2$ of $\mathcal{B}_1$ where $\frac{1}{f}$ is a rational function, but not a polynomial in general, compare [19, Thm.3.4] and [9, Thm.4.1].

We are now going to discuss the construction of *proper* input T-observers in general, i.e., without the assumption of T-stability of $\mathcal{B}_1$ as in Theorem 3.7, item (C2). We assume that an input T-observer exists, and that $Z^0 \in \mathcal{D}_T^{m\times p}$ is one left inverse of $Q_1 \in \mathcal{D}^{p\times m}$ according to Theorem 3.7, part A, hence $\mathrm{rank}(Q_1) = m$. Let $L \in \mathcal{D}^{(p-m)\times p}$ be a universal left annihilator of Q_1. Then

$$\begin{aligned} \mathrm{rank}(L) = p - m,\ \mathcal{D}^{1\times(p-m)}L = \left\{\xi \in \mathcal{D}^{1\times p};\ \xi Q_1 = 0\right\}, \\ \mathcal{D}_T^{1\times(p-m)}L = \left\{\xi \in \mathcal{D}_T^{1\times p};\ \xi Q_1 = 0\right\}. \end{aligned} \tag{27}$$

According to Theorem 2.13 all left inverses Z of Q_1 in $\mathcal{D}_T^{m\times p}$ are of the form

$$Z = Z^0 + X^0L,\ X^0 \in \mathcal{D}_T^{m\times(p-m)}. \tag{28}$$

We study when $H_2 := ZP_1$ is proper. From $\mathrm{rank}(P_1) = p$ we infer $\mathrm{rank}(LP_1) = p - m$. We need the Smith form of LP_1 with respect to $\mathcal{S}$, compare Reminder 2.4, i.e.,

$$\begin{gathered}U^1(LP_1)V^1 = (E,0) \in F(s)^{(p-m)\times p},\ E = \mathrm{diag}(e_1,\cdots,e_{p-m}) \in \mathrm{Gl}_{p-m}(F(s)),\\ U^1 \in \mathrm{Gl}_{p-m}(\mathcal{S}),\ V^1 \in \mathrm{Gl}_p(\mathcal{S}),\ \text{hence } H = Y + X(E,0) \text{ with}\\ H := ZP_1V^1 \in \mathcal{D}_T^{m\times p},\ Y := Z^0P_1V^1 \in \mathcal{D}_T^{m\times p} \text{ and}\\ X := X^0(U^1)^{-1} \in \mathcal{D}_T^{m\times(p-m)}.\end{gathered} \tag{29}$$

Notice that E, H, Y and X are rational matrices. But

$$H = Y + X(E,0) \iff \\ H_{ij} = \begin{cases} Y_{ij} + X_{ij}e_j & \text{for } 1 \le i \le m,\ 1 \le j \le p-m \\ Y_{ij} & \text{for } 1 \le i \le m,\ p-m+1 \le j \le p. \end{cases} \tag{30}$$

The inclusions $U^1, V^1 \in \mathrm{Gl}_\bullet(\mathcal{S}) \subseteq \mathrm{Gl}_\bullet(\mathcal{D}_T)$ imply the equivalences

$$\begin{gathered}H = ZP_1V^1 \in \mathcal{S}^{m\times p} \iff H_2 = ZP_1 \in \mathcal{S}^{m\times p},\\ X \in \mathcal{D}_T^{m\times(p-m)} \iff X^0 = XU^1 \in \mathcal{D}_T^{m\times(p-m)}.\end{gathered}$$

Corollary 3.9. *Assume that the equivalent conditions A. of Theorem 3.7 are satisfied, that $Z^0 \in \mathcal{D}_T^{m\times p}$ is one left inverse of Q_1 and the additional data from equations* (27)-(30). *The T-stability of $\mathcal{B}_1$ is not assumed. Then $\mathcal{B}_1$ admits a* proper *input T-observer if and only if*

$$Y_{ij} \in \begin{cases} \mathcal{D}_Te_j + \mathcal{S} & \textit{for } 1 \le i \le m,\ 1 \le j \le p-m \\ \mathcal{S} & \textit{for } 1 \le i \le m,\ p-m+1 \le j \le p. \end{cases} \tag{31}$$

If this is the case and if

$$\begin{gathered}Y_{ij} = -X_{ij}e_j + H_{ij} \in \mathcal{D}_Te_j + \mathcal{S} \textit{ for } 1 \le i \le m,\ 1 \le j \le p-m,\\ X_{ij} \in \mathcal{D}_T,\ H_{ij} \in \mathcal{S},\ X := (X_{ij})_{ij} \in \mathcal{D}_T^{m\times(p-m)},\end{gathered} \tag{32}$$

the matrix $Z := Z^0 + XU^1L \in \mathcal{D}_T^{m\times p}$ is a left inverse of Q_1 with proper $H_2 = ZP_1$ and gives rise to a proper input T-observer of $\mathcal{B}_1$ according to Theorem 3.7, part B.

We have yet to compute $\mathcal{D}_Te_j + \mathcal{S}$ to make the preceding corollary constructive.

Lemma 3.10. *Let $\mathcal{S}$ be any principal ideal domain with quotient field $\mathcal{K}$, σ a prime element of $\mathcal{S}$ and $f, g \in \mathcal{S}$ two nonzero coprime elements. Consider the quotient ring*

$$\mathcal{S}_\sigma := \left\{\frac{h}{\sigma^k} \in \mathcal{K};\ h \in \mathcal{S},\ k \ge 0\right\} = \bigcup_{k=0}^{\infty} \mathcal{S}\sigma^{-k}. \textit{ Then}$$

$$S_\sigma = \mathcal{S}_\sigma f + \mathcal{S}g.$$

Proof. The coprimeness of f and g implies $\mathcal{S} = \mathcal{S}f + \mathcal{S}g$ and $\mathcal{S}_\sigma = \mathcal{S}_\sigma f + \mathcal{S}_\sigma g$. The only $\mathcal{S}$-submodules of

$$\mathcal{S}_\sigma/\mathcal{S} = \bigcup_{k=0}^{\infty} \mathcal{S}\overline{\sigma^{-k}},\ \mathcal{S}\overline{\sigma^{-k}} = \mathcal{S}\sigma^{-k}/\mathcal{S} \cong \mathcal{S}/\mathcal{S}\sigma^k,$$

are $\mathcal{S}_\sigma/\mathcal{S}$ and the $\mathcal{S}\sigma^{-k}/\mathcal{S},\ k \geq 0$, as is easily seen. This is indeed a standard result, and $\mathcal{S}_\sigma/\mathcal{S}$ is the unique minimal injective cogenerator over the local ring $\mathcal{S}_{\mathcal{S}\setminus\mathcal{S}\sigma}$. Therefore the only $\mathcal{S}$-submodules of $\mathcal{S}_\sigma$ containing $\mathcal{S}$ are $\mathcal{S}_\sigma$ and the $\mathcal{S}\sigma^{-k},\ k \geq 0$. But $\mathcal{S}_\sigma f + \mathcal{S}g$ is such a submodule. Assume

$$\mathcal{S}_\sigma f + \mathcal{S}g = \mathcal{S}\sigma^{-k} \implies \forall \ell:\ \sigma^{-\ell} f \in \mathcal{S}\sigma^{-k} \implies f \in \bigcap_{k=0}^{\infty} \mathcal{S}\sigma^k = 0.$$

This is a contradiction to $f \neq 0$, and therefore $\mathcal{S}_\sigma f + \mathcal{S}g = \mathcal{S}_\sigma$.

□

The preceding lemma is applicable to the ring $\mathcal{S}$ of proper T-stable rational functions (compare Definition and Lemma 2.14)

$$\mathcal{S} = F[\sigma]_{T_1},\ T_1 := \left\{ \frac{t}{(s-\alpha)^{\deg(t)}};\ t \in T \right\} \text{ with } \sigma := \frac{1}{s-\alpha}.$$

The indeterminate σ is a prime element of $F[\sigma]$ and not contained in T_1 and therefore also a prime element of the quotient ring $\mathcal{S}$. Moreover

Lemma 3.11. $\mathcal{D}_T = \mathcal{S}_\sigma = \bigcup_{k=0}^{\infty} \mathcal{S}\sigma^{-k} = \bigcup_{k=0}^{\infty} \mathcal{S}(s-\alpha)^k$.

Proof. $\supseteq$: Both $\mathcal{S}$ and $s-\alpha$ are contained in $\mathcal{D}_T$.
$\subseteq$: $r = ft^{-1} \in \mathcal{D}_T \implies r = f\left(t(s-\alpha)^{\deg(f)}\right)^{-1}(s-\alpha)^{\deg(f)} \in \mathcal{S}(s-\alpha)^{\deg(f)}$.

□

Theorem 3.12 (Proper input T-observer)**.** *Assume that the equivalent conditions 1. of Theorem 3.7 are satisfied, i.e., that $\mathcal{B}_1$ admits an input T-observer and that $Z^0 \in \mathcal{D}_T^{m\times p}$ is one left inverse of Q_1. Consider the additional data from equations* (27)-(30)*. The T-stability of $\mathcal{B}_1$ is not assumed. Let*

$$e_j = \frac{f_j(\sigma)}{g_j(\sigma)},\ f_j, g_j \in F[\sigma],\ \gcd(f_j, g_j) = 1,\ j = 1, \cdots, p-m$$

be the reduced representations of the elementary divisors e_j. Then $\mathcal{B}_1$ admits a proper *input T-observer if and only if*

$$Y_{ij} \in \begin{cases} \mathcal{D}_T g_j^{-1} & \text{for } 1 \leq i \leq m,\ 1 \leq j \leq p-m \\ \mathcal{S} & \text{for } 1 \leq i \leq m,\ p-m+1 \leq j \leq p. \end{cases} \tag{33}$$

All *proper input T-observers are then constructed according to Theorem 3.7 and Corollary 3.9.*

Proof. Since the polynomials f_j, g_j are nonzero and coprime in $F[\sigma]$ they have the same property with respect to the overring $\mathcal{S} \supset F[\sigma]$. By means of Lemmas 3.10 and 3.11 we infer

$$\mathcal{D}_T e_j + \mathcal{S} = \mathcal{S}_\sigma \frac{f_j}{g_j} + \mathcal{S} = \frac{1}{g_j}\left(\mathcal{S}_\sigma f_j + \mathcal{S} g_j\right) = \frac{1}{g_j}\mathcal{S}_\sigma = \frac{1}{g_j}\mathcal{D}_T.$$

The theorem now follows from Corollary 3.9.

□

There is still the need to find explicit representations in (32).

Algorithm 3.13. *Let $f(\sigma), g(\sigma) \in F[\sigma]$ be coprime polynomials and let $f = \sigma^\ell f_1(\sigma)$ be the decomposition with $f_1(0) \neq 0$ or $\gcd(\sigma, f_1) = 1$. Compute a representation $1 = a_1(\sigma)f(\sigma) + a_2(\sigma)g(\sigma)$ by means of the euclidean algorithm. Let $r := h(s)t(s)^{-1}$ be any element in $\mathcal{D}_T$. The following algorithm computes a representation*

$$r = af + bg,\ a \in \mathcal{D}_T,\ b \in \mathcal{S} \tag{34}$$

as needed in Corollary 3.9. As in Definition and Lemma 2.14 write r in the form

$$r = (s-\alpha)^{\deg(h)-\deg(t)} \left(h(s-\alpha)^{-\deg(h)}\right) \left(t(s-\alpha)^{-\deg(t)}\right)^{-1} =$$
$$\sigma^n \widehat{h}(\sigma)\widehat{t}(\sigma)^{-1},\ n := \deg(t) - \deg(h),\ \widehat{t} \in T_1 \subset F[\sigma].$$

1.case $\deg(h) \leq \deg(t)$ or $n \geq 0$*: Then r belongs to $\mathcal{S} = F(s)_{\mathrm{pr}} \cap \mathcal{D}_T$ and the desired representation (34) is given by $r = r1 = (ra_1)f + (ra_2)g$.*

2.case $n < 0$*: Since σ^{-n} and f_1 are coprime the euclidean algorithm for $F[\sigma]$ furnishes a representation*

$$\widehat{h} = a_3 f_1 + a_4\sigma^{-n} \implies \sigma^n\widehat{h} = a_3\sigma^n f_1 + a_4 = a_3\sigma^{n-\ell} f + a_4 a_1 f + a_4 a_2 g =$$
$$\left(a_3\sigma^{n-\ell} + a_4 a_1\right) f + a_4 a_2 g \in F[\sigma, \sigma^{-1}]f + F[\sigma]g \implies$$
$$r = \sigma^n\widehat{h}\widehat{t}^{-1} = \left(\left(a_3\sigma^{n-\ell} + a_4a_1\right)\widehat{t}^{-1}\right) f + \left(a_4a_2\widehat{t}^{-1}\right) g \in \mathcal{S}_\sigma f + \mathcal{S}g = \mathcal{D}_T f + \mathcal{S}g.$$

Corollary 3.14. *Assume the situation of the preceding algorithm, and let $(a^0, b^0) \in \mathcal{D}_T \times \mathcal{S}$ satisfy*

$$r = a^0 f + b^0 g.$$

Then all other pairs (a, b) satifying (34) can be constructed as

$$\begin{aligned} a &= a^0 - c\sigma^{-\ell} g, \quad \textit{and} \\ b &= b^0 + c\sigma^{-\ell} f \end{aligned}$$

where c is an arbitrary element of $\mathcal{S}$ and f is again decomposed as $f(\sigma) = \sigma^\ell f_1(\sigma),\ f_1(0) \neq 0$.
Algorithm 3.13 with the choices of the present corollary furnishes all proper input T-observers.

Proof. Equation (34) implies that

$$\overline{r} = \overline{b}\,\overline{g} \quad \text{in } \mathcal{S}_\sigma/\mathcal{S}_\sigma f,\ \overline{g} := g + \mathcal{S}_\sigma f,\ \mathcal{D}_T = \mathcal{S}_\sigma.$$

Since $\gcd(f, g) = 1$ the element $\overline{g}$ is invertible in $\mathcal{S}_\sigma/\mathcal{S}_\sigma f$ and hence

$$\overline{b} = \overline{r}\,(\overline{g})^{-1} \quad \text{in } \mathcal{S}_\sigma/\mathcal{S}_\sigma f,$$

i.e., b is uniquely determined modulo f. In other words, each b satisfying (34) is of the form

$$b = b^0 + \tilde{c}f \quad \text{for some } \tilde{c} \in \mathcal{S}_\sigma.$$

By (34) b has to be an element of $\mathcal{S}$ and therefore $\tilde{c} \in \mathcal{S}_\sigma$ must be choosen such that $\tilde{c}f \in \mathcal{S}$. Again $\tilde{c} \in \mathcal{S}_\sigma$ can be written as $\tilde{c} = c_1\sigma^{-k}$ for some $c_1 \in \mathcal{S}$ with

$c_1(0) \neq 0$ and $k \in \mathbb{N}$. Remember that $f(\sigma) = \sigma^\ell f_1(\sigma)$, $f_1(0) \neq 0$. With this notation, $\tilde{c} f \in \mathcal{S}$ is equivalent to

$$c_1\sigma^{-k} \cdot \sigma^\ell f_1 \in \mathcal{S} \text{ or } k \leq \ell \text{ or } \widetilde{c} = c_1\sigma^{-k} \in \mathcal{S}\sigma^{-\ell}.$$

Therefore any admissible $\widetilde{c}$ has the form

$$\tilde{c} = c\sigma^{-\ell} \quad \text{for some arbitrary} \quad c \in \mathcal{S}.$$

The asserted form of b follows now directly and (34) implies the equation for a. $\square$

The next theorem furnishes conditions for the existence of output T-controllers for T-stable IO behaviors.

Theorem 3.15. *Let*

$$\mathcal{B}_2 := \left\{ \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \in \mathcal{F}^{m+p};\ P_2 \circ y_2 = Q_2 \circ y_1 \right\},\ P_2 \in \mathcal{D}^{m\times m},\ \det(P_2) \neq 0,$$

be an IO behavior with transfer matrix $H_2 = P_2^{-1}Q_2$. Then the following statements are equivalent:

1. *There exists a T-stable resp. proper T-stable output T-controller $\mathcal{B}_1$ of $\mathcal{B}_2$.*
2. *$\mathcal{B}_2$ is T-stable and its transfer matrix H_2 has a right inverse H_1 in $\mathcal{D}_T^{p\times m}$ resp. $\mathcal{S}^{p\times m}$.*

If H_1 is a right inverse of H_2 in $\mathcal{D}_T^{p\times m}$ resp. $\mathcal{S}^{p\times m}$ the controllable realization $\mathcal{B}_1$ of H_1 is one T-stable resp. proper T-stable output T-controller of $\mathcal{B}_2$. Condition 2. of this theorem can be constructively checked by means of Theorem 2.13, applied to $H_2^\top$, and Theorem 2.15.

Proof. Let $\mathcal{B}$ and $\mathcal{B}_{\text{err}}$ be the behaviors introduced in Definition 3.1. We show the equivalence of the statements for proper output T-controllers $\mathcal{B}_1$, for non-proper ones the proof is analogous.

1. $\Rightarrow$ 2.: By assumption the behavior $\mathcal{B}_{\text{err}}$ is T-autonomous. Hence Lemma 3.6 yields that $\mathcal{B}_2$ is T-stable and $H_2H_1 = \mathrm{id}_m$. Since $\mathcal{B}_1$ is assumed to be proper and T-stable its transfer matrix $H_1 = P_1^{-1}Q_1$ belongs to $\mathcal{S}^{p\times m}$.

2. $\Rightarrow$ 1.: Let $H_1 \in \mathcal{S}^{p\times m}$ be a right inverse of H_2 and

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} y_1 \\ u \end{pmatrix} \in \mathcal{F}^{p+m};\ P_1 \circ y_1 = Q_1 \circ u \right\}$$

the controllable realization of H_1. The condition $H_1 \in \mathcal{D}_T^{p\times m}$, the controllability of $\mathcal{B}_1$ and Theorem 2.15,(2), yield that $\mathcal{B}_1$ is T-stable, i.e., $P_1 \in \mathrm{Gl}_p(\mathcal{D}_T)$. This and the T-stability of $\mathcal{B}_2$ imply

$$P_2^{-1} \in \mathcal{D}_T^{m\times m} \text{ and } H_2P_1^{-1} \in \mathcal{D}_T^{m\times p}.$$

With Lemma 3.5 we conclude that $\mathcal{B}_{\text{err}}$ is T-autonomous and that $\mathcal{B}_1$ is a proper T-stable output T-controller of $\mathcal{B}_2$. $\square$

Remark 3.16 (Relation to [23], [19], [9]).

1. For $T = F[s] \setminus \{0\}$ autonomy and T-autonomy coincide. In this case the equivalence A, 1. $\Leftrightarrow$ 2., of Theorem 3.7 and the corresponding equivalence in Theorem 3.15 were already proven by Wolovich in [23, Thm.5.5.7], but in a different language and under additional conditions [23, (5.5.6)]. It is implicitly used that a behavior is autonomous if and only if its trajectories are determined by the initial conditions. The input T-observer $\mathcal{B}_2$ is called a *left inverse system* to $\mathcal{B}_1$ since H_2 is a *left inverse* of H_1 whereas the output controller is called a *right inverse system*. In the Remarks 1-4 in [23, pp.171-174] controllability and stability properties of the left and right inverse systems are also discussed.

2. Consider the special case $T = T_\Lambda$, $\Lambda_1 := \{z \in \mathbb{C};\ \Re(z) < 0\}$. Valcher and Willems discuss proper T-observers in [19, §IV]. Here we apply their results to our situation with our notations, their admissible plants are more general. The translation of their notations into ours is given by

$$\begin{array}{c||c|c|c|c|c|c|c|}VW & R_1 & R_2 & Q & P & w_1 & w_2 & \widehat{W} \\ \hline BO & P_1 & Q_1 & P_2 & Q_2 & y_1 & u & H_2\end{array}.$$

The condition $\mathrm{rank}(H_1) = \mathrm{rank}(Q_1) = \mathrm{rank}(R_2) = m$ is assumed. After elementary row transformations we assume their normal form [19, (5),(6)]

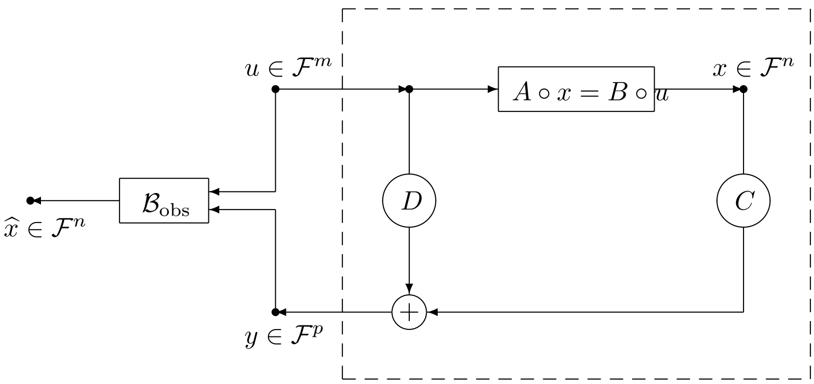
$$(P_1, -Q_1) = \begin{pmatrix} N_1 & -D_2 \\ D_1 & 0 \end{pmatrix} \in \mathcal{D}^{(m+(p-m))\times(p+m)},\ \det(D_2) \neq 0. \tag{35}$$

Properness refers to the transfer matrix $\widehat{W} = H_2$, hence their observer is also an IO behavior in this case. The transfer matrix H_1 does not appear in [19] because their general plant is not assumed IO. The equation $(P_2, -Q_2) = T(Q_1, -P_1)$ [19, Thm.3.4, p.2302] shows that they consider *consistent* IO observers only which are only part of those parametrized in Theorem 3.7. According to Definition 2.18 condition (A3) from Theorem 3.7 signifies that u is T-observable or detectable from y_1 or that $\det(D_2) \in T$, hence $(A1) \Leftrightarrow (A3)$ for this special T follows from [19, Prop.2.2,3.2]. Theorem 4.3 of [19] characterizes the existence of a proper input T-observer and constructs *one* if it exists. The proof and algorithm are quite distinct from our existence and parametrization Theorems 3.7 and 3.12. The proper pseudo state T-observers of Section 4 are applicable to the general plants of [19] and [9], see Remark 4.8.
Fuhrmann discusses consistent proper observers in [9, Prop.4.4]. In the normal form (35) he chooses an IO structure for $D_1 \circ y_1 = 0$ and from that and $D_2 \circ u = N_1 \circ y_1$ derives a transfer matrix whose properness is *sufficient* for the existence of a proper input T-observer.

4 Pseudo state T-observers

In this section we treat the existence and construction of proper pseudo state T-observers of Rosenbrock systems with the data from Example 2.9. We always assume that the two transfer matrices $H_1 = A^{-1}B$ and $H_2 = D + CH_1$ are proper. The significance of Rosenbrock systems and their observers is, for instance, discussed in [23, Ch.5, Ch.7], [11, Ch.8] and [1, Ch.7, Part 2]. In Remark 4.8 we relate our results to those of our predecessors.

A T-observer $\mathcal{B}_{\text{obs}}$ of the pseudo state x is an IO system with input y and u and output $\widehat{x}$ such that $(\widehat{x}-x)$ is T-small, i.e., that x and $\widehat{x}$ are asymptotically equal in the standard cases. With such an observer one can thus estimate the pseudo state. Principally we discuss proper T-observers since these can be realized by standard Kalman state equations. The situation is visualized in the following picture:



Definition 4.1. For the given data from Example 2.9 and an IO behavior $\mathcal{B}_{\text{obs}}$ of the form

$$\mathcal{B}_{\text{obs}} := \left\{ \begin{pmatrix} \widehat{x} \\ y \\ u \end{pmatrix} \in \mathcal{F}^{n+p+m};\ P_{\text{obs}} \circ \widehat{x} = Q_{\text{obs}} \circ \begin{pmatrix} y \\ u \end{pmatrix} \right\},\ Q_{\text{obs}} = (Q_y, Q_u),$$
$$P_{\text{obs}} \in \mathcal{D}^{n\times n}, \det(P_{\text{obs}}) \neq 0,\ Q_y \in \mathcal{D}^{n\times p},\ Q_u \in \mathcal{D}^{n\times m},$$

we define the derived behaviors

$$\begin{aligned}
\mathcal{B} &:= \left\{ \begin{pmatrix} \widehat{x} \\ x \\ y \\ u \end{pmatrix} \in \mathcal{F}^{n+n+p+m};\ \begin{array}{rcl} A \circ x &=& B \circ u, \\ y &=& C \circ x + D \circ u, \\ P_{\text{obs}} \circ \widehat{x} &=& Q_y \circ y + Q_u \circ u \end{array} \right\}, \\
\mathcal{B}_3 &:= \left\{ \begin{pmatrix} \widehat{x}-x \\ u \end{pmatrix} \in \mathcal{F}^{n+m};\ \exists \begin{pmatrix} \widehat{x} \\ x \\ y \\ u \end{pmatrix} \in \mathcal{B} \right\} \\
&= \begin{pmatrix} \mathrm{id}_n & -\mathrm{id}_n & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_m \end{pmatrix} \circ \mathcal{B}, \quad \text{and} \\
\mathcal{B}_{\text{err}} &:= \left\{ \widehat{x}-x \in \mathcal{F}^n;\ \exists \begin{pmatrix} \widehat{x} \\ x \\ y \\ u \end{pmatrix} \in \mathcal{B} \right\} \\
&= \begin{pmatrix} \mathrm{id}_n & 0 \end{pmatrix} \circ \mathcal{B}_3.
\end{aligned}$$

Then $\mathcal{B}_{\text{obs}}$ is called a *T-observer of the pseudo state x* if $\mathcal{B}_{\text{err}}$ is T-autonomous.

Remark 4.2. The behavior $\mathcal{B}$ describes the entire system shown in the picture above. Later we will see that $\mathcal{B}$ is an IO system with input u and output $\widehat{x}$, x, and y and that $\mathcal{B}_3$ is an IO system with input u and output $\widehat{x}-x$. In the standard cases $\lim_{t\to\infty}(\widehat{x}(t)-x(t))=0$ this suggests to call $\mathcal{B}_{\mathrm{obs}}$ an *asymptotic* observer.

Recall from Definition and Corollary 2.18 that the Rosenbrock equations are T-observable if and only if $\left(\begin{smallmatrix}A\\C\end{smallmatrix}\right)$ has a left inverse with entries in $\mathcal{D}_T$.

Theorem 4.3. *We consider the Rosenbrock system and derived data from above. Let $(X,Y)\in\mathcal{D}_T^{n\times n}\times\mathcal{S}^{n\times p}$ be a left inverse of $\left(\begin{smallmatrix}A\\C\end{smallmatrix}\right)$, i.e., $XA+YC=\mathrm{id}_n$. Let*

$$\mathcal{B}_{\mathrm{obs}}:=\left\{\begin{pmatrix}\widehat{x}\\y\\u\end{pmatrix}\in\mathcal{F}^{n+p+m};\ P_{\mathrm{obs}}\circ\widehat{x}=Q_{\mathrm{obs}}\circ\begin{pmatrix}y\\u\end{pmatrix}\right\},$$
$$P_{\mathrm{obs}}\in\mathcal{D}^{n\times n},\ \det(P_{\mathrm{obs}})\neq 0$$

be the unique controllable realization of the transfer matrix $(Y,XB-YD)\in\mathcal{S}^{n\times p}\times\mathcal{D}_T^{n\times m}$. Then $\mathcal{B}_{\mathrm{obs}}$ is a proper, controllable and T-stable T-observer of the given Rosenbrock system.

Proof. 1. Let $(P_{\mathrm{cont}},-Q_{\mathrm{cont}})\in\mathcal{D}^{n\times(n+(n+p))}$ denote the matrix of the unique controllable realization of the transfer matrix $(X,Y)\in\mathcal{D}_T^{n\times n}\times\mathcal{S}^{n\times p}$, i.e.,

$$\mathcal{D}^{1\times n}P_{\mathrm{cont}}=\left\{\xi\in\mathcal{D}^{1\times n};\ \xi(X,Y)\in\mathcal{D}^{n\times(n+p)}\right\},P_{\mathrm{cont}}(X,Y)=Q_{\mathrm{cont}}.$$

Since (X,Y) is T-stable this IO realization is also T-stable according to Theorem 2.15,(2b), hence $\det(P_{\mathrm{cont}})\in T$. We infer

$$P_{\mathrm{cont}}(Y,XB-YD)\in\mathcal{D}^{n\times(p+m)}.\ \text{Since}$$
$$\mathcal{D}^{1\times n}P_{\mathrm{obs}}=\left\{\xi\in\mathcal{D}^{1\times n};\ \xi(Y,XB-YD)\in\mathcal{D}^{1\times(p+m)}\right\}$$

according to Result 2.10 there is a matrix

$$P\in\mathcal{D}^{n\times n}\ \text{with}\ PP_{\mathrm{obs}}=P_{\mathrm{cont}},\ \text{hence}$$
$$\det(P)\det(P_{\mathrm{obs}})=\det(P_{\mathrm{cont}})\in T\ \text{and}\ \det(P_{\mathrm{obs}})\in T.$$

This signifies that $\mathcal{B}_{\mathrm{obs}}$ is T-stable.

2. We are going to show that

$$\mathcal{B}_{\mathrm{err}}\subseteq\{w\in\mathcal{F}^n;\ P_{\mathrm{cont}}\circ w=0\}$$

and hence that $\mathcal{B}_{\mathrm{err}}$ is T-autonomous: Let

$$\widehat{x}-x\in\mathcal{B}_{\mathrm{err}},\ y\in\mathcal{F}^m,\ u\in\mathcal{F}^p\ \text{such that}\ \begin{pmatrix}\hat{x}\\x\\y\\u\end{pmatrix}\in\mathcal{B}.$$

Then by definition of $\mathcal{B}$ the following equations are satisfied:

$$\begin{aligned}A\circ x&=B\circ u,\\ y&=C\circ x+D\circ u,\\ P_{\mathrm{obs}}\circ\widehat{x}&=P_{\mathrm{obs}}Y\circ y+P_{\mathrm{obs}}(XB-YD)\circ u\end{aligned}$$

Multiplication of the last line with P and substitution of the first two lines into the third one leads to

$$P_{\rm cont} \circ \widehat{x} = P_{\rm cont} YC \circ x + P_{\rm cont} YD \circ u + P_{\rm cont} XA \circ x - P_{\rm cont} YD \circ u =$$
$$= P_{\rm cont}(XA + YC) \circ x = P_{\rm cont} \circ x \text{ since } (X, Y)\begin{pmatrix} A \\ C \end{pmatrix} = {\rm id}_n .$$

Notice here that $P_{\rm cont} = PP_{\rm obs}$ and

$$P_{\rm obs}(XB - YD) \in \mathcal{D}^{n\times m}, \text{ but } P_{\rm cont} X \in \mathcal{D}^{n\times n} \text{ and } P_{\rm cont} Y \in \mathcal{D}^{n\times p}.$$

We infer

$$P_{\rm cont} \circ (\widehat{x} - x) = 0 \text{ for } \widehat{x} - x \in \mathcal{B}_{\rm err} \text{ and } \mathcal{B}_{\rm err} \subseteq \{w \in \mathcal{F}^n;\ P_{\rm cont} \circ w = 0\}$$

as asserted.

3. We finally show that the transfer matrix $(Y, XB - YD)$ of $\mathcal{B}_{\rm obs}$ and thus $\mathcal{B}_{\rm obs}$ itself are proper. For Y this holds by assumption. Moreover

$$XA + YC = {\rm id}_n \implies H_1 + YD = {\rm id}_n H_1 + YD =$$
$$(XA + YC)A^{-1}B + YD = XB + Y\left(CA^{-1}B + D\right) =$$
$$XB + YH_2 \implies XB - YD = H_1 - YH_2.$$

Since H_1, H_2, Y are proper so is $XB - YD$.

□

Theorem 4.4 (T-observers of internally proper Rosenbrock equations, compare [23, Thm.7.3.23], [9, Prop.3.3])**.** *Assume that the Rosenbrock equations are* internally proper *[20, Ch.4.5], i.e., not only $H_1 = A^{-1}B$ and $H_2 = D + CH_1$, but also A^{-1} and CA^{-1} are proper. This holds, for instance, for Kalman state equations. Let*

$$U\begin{pmatrix} A \\ C \end{pmatrix}V = \begin{pmatrix} E \\ 0 \end{pmatrix},\ E = {\rm diag}(e_1, \cdots, e_n),\ U \in {\rm Gl}_{n+p}(\mathcal{S}),\ V \in {\rm Gl}_n(\mathcal{S}),$$

be the Smith form of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ with respect to $\mathcal{S}$ and $L \in \mathcal{S}^{p\times(n+p)}$ the matrix of the last p rows of U which is a universal left annihilator of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$. Then the following statements are equivalent:

1. *The Rosenbrock equations are T-observable, i.e., by Definition and Corollary 2.18, the matrix $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ has a left inverse with entries in $\mathcal{D}_T$.*

2. *The matrix $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ has a left inverse with entries in $\mathcal{S}$.*

3. *$e_n^{-1} \in \mathcal{S}$.*

If these equivalent conditions are satisfied all left inverses of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ with entries in $\mathcal{S}$ are given as $V(E^{-1}, 0)U + ZL,\ Z \in \mathcal{S}^{n\times p}$. Each such left inverse (X, Y) gives rise to a unique proper controllable T-observer with transfer matrix $(Y, XB - YD)$ which is also T-stable.
For Kalman state equations and special T the equivalence 1. $\Leftrightarrow$ 2. is essentially the same as [9, Prop.3.3, (a) $\Leftrightarrow$ (b)].

Proof. The equivalence follows easily from Theorem 2.13.
Since $(A^{-1},0)\in F(s)_{\mathrm{pr}}^{n\times(n+p)}$ is a left inverse of $\left(\begin{smallmatrix}A\\C\end{smallmatrix}\right)$ that theorem implies

$$e_n^{-1}\in F(s)_{\mathrm{pr}},\ \text{hence } e_n^{-1}\in\mathcal{D}_T \iff e_n^{-1}\in\mathcal{S}=F(s)_{\mathrm{pr}}\cap\mathcal{D}_T.$$

The equivalence and the construction of all inverses is now also a special case of that theorem. □

The next result is the converse of Theorem 4.3 for Rosenbrock equations as in Example 2.9.

Theorem 4.5. *1. Assume that*

$$\mathcal{B}_{\mathrm{obs}}=\left\{\begin{pmatrix}\widehat{x}\\y\\u\end{pmatrix}\in\mathcal{F}^{n+p+m};\ P_{\mathrm{obs}}\circ\widehat{x}=Q_{\mathrm{obs}}\circ\begin{pmatrix}y\\u\end{pmatrix}\right\},$$
$$P_{\mathrm{obs}}\in\mathcal{D}^{n\times n},\ \det(P_{\mathrm{obs}})\neq 0,\ Q_{\mathrm{obs}}=(Q_y,Q_u)\in\mathcal{D}^{n\times(p+m)},$$

is any controllable and proper T-observer of the pseudo state of the given Rosenbrock system with transfer matrix

$$H_{\mathrm{obs}}:=(H_y,H_u):=P_{\mathrm{obs}}^{-1}(Q_y,Q_u)\in F(s)_{\mathrm{pr}}^{n\times(p+m)}.$$

Since the observer is controllable it is the unique controllable realization of its transfer matrix.
Then $\mathcal{B}_{\mathrm{obs}}$ is T-stable, especially $H_{\mathrm{obs}}=(H_y,H_u)\in\mathcal{S}^{n\times(p+m)}$, and there exists a matrix $X\in\mathcal{D}_T^{n\times n}$ such that

$$XA+H_yC=(X,H_y)\begin{pmatrix}A\\C\end{pmatrix}=\mathrm{id}_n\ \text{ and } H_u=XB-H_yD. \tag{36}$$

In other words, with $Y:=H_y$, the pair (X,Y) satisfies the conditions of Theorem 4.3 and $\mathcal{B}_{\mathrm{obs}}$ is the unique controllable realization of $(Y,XB-YD)=(H_y,H_u)=H_{\mathrm{obs}}$, i.e., $\mathcal{B}_{\mathrm{obs}}$ is constructed from (X,Y) as in Theorem 4.3.
2. Parametrization*: According to item 1. the left inverses $(X,Y)\in\mathcal{D}_T^{n\times n}\times\mathcal{S}^{n\times p}$ of $\left(\begin{smallmatrix}A\\C\end{smallmatrix}\right)$ parametrize the set of all controllable proper T-observers of the Rosenbrock system. Two such inverses (X_i,Y_i), $i=1,2$, give rise to the same observer if and only if*

$$Y_1=Y_2\ \text{ and } (X_1-X_2)B=0.$$

Proof. 1. *The transfer matrix H of $\mathcal{B}$*: We show that the behavior $\mathcal{B}$ from Definition 4.1 is an IO behavior and compute its transfer matrix:

$$\begin{aligned}\mathcal{B}&=\left\{\begin{pmatrix}\widehat{x}\\x\\y\\u\end{pmatrix}\in\mathcal{F}^{n+n+p+m};\ \begin{array}{rcl}A\circ x&=&B\circ u,\\ y&=&C\circ x+D\circ u,\\ P_{\mathrm{obs}}\circ\widehat{x}&=&Q_y\circ y+Q_u\circ u\end{array}\right\}\\
&=\left\{\begin{pmatrix}\widehat{x}\\x\\y\\u\end{pmatrix}\in\mathcal{F}^{n+n+p+m};\ \underbrace{\begin{pmatrix}P_{\mathrm{obs}}&0&-Q_y\\0&A&0\\0&-C&\mathrm{id}_p\end{pmatrix}}_{=:P}\circ\begin{pmatrix}\widehat{x}\\x\\y\end{pmatrix}=\underbrace{\begin{pmatrix}Q_u\\B\\D\end{pmatrix}}_{=:Q}\circ u\right\}\\
&=\left\{\begin{pmatrix}w\\u\end{pmatrix}\in\mathcal{F}^{(n+n+p)+m};\ P\circ w=Q\circ u\right\}.\end{aligned}$$

Elementary transformations show that P is really invertible and lead to

$$H = P^{-1}Q = \begin{pmatrix} H_u + H_y H_2 \\ H_1 \\ H_2 \end{pmatrix}.$$

2. *The transfer matrix H_3 of $\mathcal{B}_3$*:

$$\begin{aligned}
\mathcal{B}_3 &= \begin{pmatrix} \mathrm{id}_n & -\mathrm{id}_n & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_m \end{pmatrix} \circ \mathcal{B} \\
&= \begin{pmatrix} C_3 & 0 \\ 0 & \mathrm{id}_m \end{pmatrix} \circ \mathcal{B} \qquad \text{with } C_3 := \begin{pmatrix} \mathrm{id}_n & -\mathrm{id}_n & 0 \end{pmatrix} \\
&=: \left\{ \begin{pmatrix} e \\ u \end{pmatrix} \in \mathcal{F}^{n+m};\ P_3 \circ e = Q_3 \circ u \right\}.
\end{aligned}$$

Since the behavior $\mathcal{B}_3$ is derived from the Rosenbrock equations

$$P \circ w = Q \circ u,\ e = C_3 \circ w, \tag{37}$$

as in Example 2.9 it is itself an IO behavior with $P_3 \in \mathcal{D}^{n\times n}$, $\det(P_3) \neq 0$ and transfer matrix

$$H_3 = C_3 H = H_u + H_y H_2 - H_1. \tag{38}$$

Since $\mathcal{B}_{\mathrm{err}}$ is T-autonomous it has the form

$$\mathcal{B}_{\mathrm{err}} = \begin{pmatrix} \mathrm{id}_n & 0 \end{pmatrix} \circ \mathcal{B}_3 =: \{e \in \mathcal{F}^n;\ P_{\mathrm{err}} \circ e = 0\},\ P_{\mathrm{err}} \in \mathcal{D}^{n\times n},\ \det(P_{\mathrm{err}}) \in T.$$

In particular, a vector e will belong to $\mathcal{B}_{\mathrm{err}}$ whenever $\left(\begin{smallmatrix} e \\ u \end{smallmatrix}\right) \in \mathcal{B}_3$ for some $u \in \mathcal{F}^m$. In other words, $P_3 \circ e = Q_3 \circ u$ implies $P_{\mathrm{err}} \circ e = 0$. Rewriting this relation as

$$(P_3, -Q_3) \circ \begin{pmatrix} e \\ u \end{pmatrix} = 0 \Longrightarrow (P_{\mathrm{err}}, 0) \circ \begin{pmatrix} e \\ u \end{pmatrix} = 0,$$

we infer that there is an

$$\begin{gathered}
\tilde{X} \in \mathcal{D}^{n\times n} \text{ such that } (P_{\mathrm{err}}, 0) = \tilde{X}(P_3, -Q_3), \text{ i.e.,} \\
P_{\mathrm{err}} = \tilde{X} P_3 \text{ and } \tilde{X} Q_3 = 0 \implies \\
\det(P_{\mathrm{err}}) = \det(\tilde{X}) \det(P_3) \in T \implies \det(\tilde{X}) \neq 0 \implies \\
Q_3 = 0 \implies H_u + H_y H_2 - H_1 \underset{(38)}{=} H_3 = P_3^{-1} Q_3 = 0.
\end{gathered} \tag{39}$$

Since T is saturated we also get $\det(P_3) \in T$.
3. *The exact equations of $\mathcal{B}_3$*: Let $(-K, L) \in \mathcal{D}^{n\times((2n+p)+n)}$ be a universal left annihilator of $\left(\begin{smallmatrix} P \\ C_3 \end{smallmatrix}\right)$, hence $KP = LC_3$. According to Example 2.9 the Rosenbrock equations

$$\begin{gathered}
P \circ w = Q \circ u,\ e = C_3 \circ w,\ C_3 := (\mathrm{id}_n, -\mathrm{id}_n, 0) \text{ imply} \\
\mathcal{B}_3 = \left\{ \begin{pmatrix} e \\ u \end{pmatrix} \in \mathcal{F}^{n+m};\ L \circ e = KQ \circ u \right\},\ \text{hence} \\
P_3 = L \text{ and } 0 = Q_3 = KQ.
\end{gathered}$$

Here we used $Q_3 = 0$ from part 2. of the proof. With $K = (K^1, K^2, K^3) \in \mathcal{D}^{n\times(n+n+p)}$ and $C_3 = (\mathrm{id}_n, -\mathrm{id}_n, 0)$ the equation $KP = LC_3$ yields

$$\begin{pmatrix} K^1, & K^2, & K^3 \end{pmatrix} \begin{pmatrix} P_{\mathrm{obs}} & 0 & -Q_y \\ 0 & A & 0 \\ 0 & -C & \mathrm{id}_p \end{pmatrix} = \begin{pmatrix} L, & -L, & 0 \end{pmatrix}$$

or, equivalently,

$$K^1 P_{\mathrm{obs}} = L, \quad -K^2 A + K^3 C = L \quad \text{and} \quad K^3 = K^1 Q_y.$$

With $L = P_3$ and $\det(P_3) \in T$ according to part 2. of the proof this gives

$$P_3 = L = K^1 P_{\mathrm{obs}} = -K^2 A + K^1 Q_y C. \tag{40}$$

4. *Equations* (36): With $P_3 \in \mathrm{Gl}_n(\mathcal{D}_T)$ equation (40) furnishes

$$\mathrm{id}_n = -P_3^{-1} K^2 A + P_3^{-1} K^1 Q_y C = (-P_3^{-1} K^2) A + (P_{\mathrm{obs}}^{-1} Q_y) C = XA + YC$$
$$\text{with} \quad X := -P_3^{-1} K^2 \in \mathcal{D}_T^{n\times n} \text{ and } Y := H_y := P_{\mathrm{obs}}^{-1} Q_y \in \mathcal{D}_T^{n\times p}.$$

Moreover

$$XA + YC = \mathrm{id}_n \implies H_1 = A^{-1}B = (XA + YC)A^{-1}B =$$
$$XB + YCH_1 = XB - YD + Y(D + CA^{-1}B) = XB - YD + H_y H_2 \implies$$
$$XB - YD = H_1 - H_y H_2 \underset{(39)}{=} H_u.$$

5. Since $\mathcal{B}_{\mathrm{obs}}$ is controllable it is the unique controllable realization of its transfer matrix $(H_y, H_u) = (Y, XB - YD)$ and therefore constructed from (X, Y) according to Theorem 4.3. This signifies that the left inverses $(X, Y) \in \mathcal{D}_T^{n\times n} \times \mathcal{S}^{n\times p}$ of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ indeed parametrize the set of proper controllable T-observers of the given Rosenbrock system. Also recall from Result 2.10 that two controllable T-observers $\mathcal{B}_i$ coincide if and only if their transfer matrices $(Y_i, X_i B - Y_i D)$ do. $\square$

Corollary 4.6 (Non-proper and non-controllable T-observers)**.**

1. *Theorems 4.3 and 4.5 remain true if the properness of $H_1 = A^{-1}B$, $H_2 = D + CH_1$ and $H_{\mathrm{obs}} = (H_y, H_u)$ is dropped. In this case the (X, Y) are left inverses of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ in $\mathcal{D}_T^{n\times(n+p)}$ and parametrize all controllable T-observers of x.*

2. *Any, not necessarily controllable, T-observer $\widetilde{\mathcal{B}}$ of x is T-stable. Its controllable part $\mathcal{B}_{\mathrm{obs}} := \widetilde{\mathcal{B}}_{\mathrm{cont}}$ is also a T-observer and thus satisfies the conditions of Theorem 4.5. With the notations of this theorem $\widetilde{\mathcal{B}}$ has the form*

$$\widetilde{\mathcal{B}} = \left\{ \begin{pmatrix} \widetilde{x} \\ y \\ u \end{pmatrix} ;\ PP_{\mathrm{obs}} \circ \widetilde{x} = P(Q_y, Q_u) \circ \begin{pmatrix} y \\ u \end{pmatrix} \right\},\ P \in \mathcal{D}^{n\times n},\ \det(P) \in T. \tag{41}$$

In other words, the tripels (X, Y, P) with

$$(X, Y) \in \mathcal{D}_T^{n\times(n+p)},\ XA + YC = \mathrm{id}_n,\ P \in \mathcal{D}^{n\times n},\ \det(P) \in T,$$

parametrize all T-observers of x. If $H_1 = A^{-1}B$ and $H_2 = D + CH_1$ are proper the tripels with proper Y parametrize the proper T-observers.

Proof. The proofs of Theorems 4.3 and 4.5 remain valid, in particular, $\widetilde{\mathcal{B}}$ is T-stable. Any IO behavior and its controllable subbehavior are related by an equation (41) with some matrix $P \in \mathcal{D}^{n\times n}$ and $\det(P) \neq 0$. The T-stability of $\widetilde{\mathcal{B}}$ implies $\det(P) \in T$. □

Example 4.7. We consider any IO behavior

$$\mathcal{B}_1 := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{F}^{n+m};\ A \circ x = B \circ u \right\},\ A \in \mathcal{D}^{n\times n},\ \det(A) \neq 0, \tag{42}$$

with transfer matrix $H_1 = A^{-1}B$. For application of Theorems 4.3 and 4.5 we define special Rosenbrock equations with trivial C and D, viz.

$$A \circ x = B \circ u,\ A \in \mathcal{D}^{n\times n},\ B \in \mathcal{D}^{n\times m},\ p := 0,\ C := \emptyset,\ D := \emptyset, \tag{43}$$

and assume that these are T-observable, i.e., that A is (left) invertible in $\mathcal{D}_T^{n\times n}$ or $\det(A) \in T$. Let $(P_1, -Q_1)$ denote the matrix of the unique controllable realization of H_1, hence $\mathcal{B}_{1,\mathrm{cont}} = \{ \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right);\ P_1 \circ x = Q_1 \circ u \}$. Since $(Y, XB - YD) = (\emptyset, A^{-1}B) = H_1$ Theorem 4.3 shows that $\mathcal{B}_{1,\mathrm{cont}}$ is the unique controllable T-observer of x. All other such observers have the form

$$\mathcal{B}_{\mathrm{obs}} := \left\{ \begin{pmatrix} \widehat{x} \\ u \end{pmatrix} \in \mathcal{F}^{n+m};\ PP_1 \circ \widehat{x} = PQ_1 \circ u \right\},\ P \in \mathcal{D}^{n\times n},\ \det(P) \in T.$$

These observers are proper if and only if $\mathcal{B}_1$ is proper.
For $p > 0$, $C := 0 \in \mathcal{D}^{p\times m}$ and $D := 0$ all matrices (A^{-1}, Y), $Y \in \mathcal{D}_T^{p\times(n+p)}$, are left inverses of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$, but the observers are essentially the same as those for $p = 0$ since the output $y = 0$ is superfluous for the estimation.

Remark 4.8 (Connection with [23], [21], [9] and [19])**.** We relate our results to those of our predecessors and also discuss various remarks of colleagues after the talk of the first author at the recent MTNS conference 2008 where some of our results were presented. We do not yet comment unpublished work of Trumpf and Willems of which the first author learned at the same conference.

1. Theorem 4.4 generalizes Theorem 7.3.23 in [23] with the following translation of our notation into that of [23]:

Wolovich	P	id_n	R	0	K	H	Q	F	$Q^{-1}K$	$Q^{-1}H$
BO	A	B	C	D	Q_u	Q_y	P_{obs}	id_n	H_u	H_y

   In contrast to [23] we consider the case $F = \mathrm{id}_n$ only, i.e., observers of x and not of $F \circ x$. In comparison to the simple derivation of Theorem 4.4 the proof of [23, 7.3.23] is rather involved and therefore omitted in [1, p.611].

2. In [21, §5.6] asymptotic observers appear in context with two parameter compensators. The IO systems are, however, given by their transfer matrix and correspond to controllable IO behaviors according to Result 2.10. The autonomous part of the behavior is not discussed.

3. In the special case of Kalman state systems

$$(s\,\mathrm{id}_n - F) \circ x = Gu,\ y = Hx + Ju \tag{44}$$

the equivalence 1. $\Leftrightarrow$ 2. of Theorem 4.4 is derived in [9, Prop.3.3, $(a) \Leftrightarrow (b)$], but differently. Notice the proof economy of our results due to the use of an arbitrary multiplicatively closed set T and the simplicity of the proof of Theorem 4.4.
It is not possible to reduce our theorems to the Kalman case. To see this consider the general situation of our theorems. Then there are Kalman equations (44) (compare (18)) such that

$$\begin{pmatrix} H & J \\ 0 & \mathrm{id}_m \end{pmatrix} \circ : \left\{ \begin{pmatrix} x' \\ u \end{pmatrix};\ s \circ x' = Fx' + Gu \right\} \cong \mathcal{B}_1 := \left\{ \begin{pmatrix} x \\ u \end{pmatrix};\ A \circ x = B \circ u \right\}$$

is a behavior isomorphism. The corresponding Rosenbrock equations for x', u, y are

$$(s\,\mathrm{id}_{n'} - F) \circ x' = Gu,\ y = C \circ (Hx' + Ju) + D \circ u = (CH) \circ x' + (CJ + D) \circ u.$$

The matrices CH and $CJ + D$ are not constant, so these equations are not of the type (44).

4. Valcher/Willems [19] and Fuhrmann [9] consider special sets T and behaviors
$$\mathcal{B}_{\mathrm{sys}} = \left\{ \begin{pmatrix} w_2 \\ w_1 \end{pmatrix} \in \mathcal{F}^{n+m};\ R_2 \circ w_2 = R_1 \circ w_1 \right\} \tag{45}$$
with the estimated resp. measured components w_2 resp. w_1 and observers or estimators
$$\mathcal{B}_{\mathrm{est}} = \left\{ \begin{pmatrix} \widehat{w_2} \\ w_1 \end{pmatrix};\ Q \circ \widehat{w}_2 = P \circ w_1 \right\}. \tag{46}$$
In [19, Prop.3.2] and [9, Prop.4.2,(1)] they show that the existence of an observer implies that R_2 has full column rank n and that $\mathcal{B}_{\mathrm{sys}}$ has a normal form (compare (35))
$$\begin{gathered} \mathcal{B}_{\mathrm{sys}} = \left\{ \begin{pmatrix} w_2 \\ w_1 \end{pmatrix};\ D_2 \circ w_2 = N_1 \circ w_1,\ D_1 \circ w_1 = 0 \right\} \subseteq \\ \mathcal{B}_1 := \left\{ \begin{pmatrix} w_2 \\ w_1 \end{pmatrix} \in \mathcal{F}^{n+m};\ D_2 \circ w_2 = N_1 \circ w_1 \right\},\ D_2 \in \mathcal{D}^{n \times n},\ \det(D_2) \neq 0, \end{gathered} \tag{47}$$
which they use for their further considerations. The behavior
$$\mathcal{B}_1 = \left\{ \begin{pmatrix} w_2 \\ w_1 \end{pmatrix} \in \mathcal{F}^{n+m};\ D_2 \circ w_2 = N_1 \circ w_1 \right\}$$
is an IO behavior as in Example 4.7 and has the T-observers derived there. Their T-observers are consistent and described by equations [19, Thm.3.4], [9, Thm.4.1]
$$\begin{gathered} Q \circ \widehat{w}_2 = P \circ w_1 \text{ with } (Q, -P) = (Y, X) \begin{pmatrix} D_2 & -N_1 \\ 0 & -D_1 \end{pmatrix} \text{ or} \\ Y D_2 \circ \widehat{w}_2 = (Y N_1 + X D_1) \circ w_1,\ Y \in \mathcal{D}^{n \times n},\ \det(Y) \in T. \end{gathered} \tag{48}$$
For $\left(\begin{smallmatrix} w_2 \\ w_1 \end{smallmatrix}\right) \in \mathcal{B}_{\mathrm{sys}}$ with $D_1 \circ w_1 = 0$ this furnishes the observer equations $Y D_2 \circ \widehat{w}_2 = Y N_1 \circ w_1$ whereas the, possibly not consistent, T-observer

equations of $\mathcal{B}_1$ according to Example 4.7 are $PP_1 \circ \widehat{w}_2 = PQ_1 \circ w_1$ where $(P_1, -Q_1)$ defines the controllable realization of $D_2^{-1}N_1$ and where $P \in \mathcal{D}^{n\times n}$ with $\det(P) \in T$ is arbitrary. This implies the existence of $Z \in \mathcal{D}^{n\times n}$ with $(D_2, -N_1) = Z(P_1, -Q_1)$ and thus that Example 4.7 furnishes more effective observers than those considered in [19] and [9]. The component XD_1 of P in (48) is practically uneffective and therefore superfluous.

5. Conversely, the general Rosenbrock equations of this section can be written in the form (45), viz.

$$\mathcal{B}_{\rm sys} := \left\{ \begin{pmatrix} A \\ C \end{pmatrix} \circ x = \begin{pmatrix} 0 & B \\ \mathrm{id}_p & -D \end{pmatrix} \circ \begin{pmatrix} y \\ u \end{pmatrix} \right\},\ w_2 := x,\ w_1 := \begin{pmatrix} y \\ u \end{pmatrix}$$

where $R_2 = \left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ and $\mathrm{rank}(R_2) = \mathrm{rank}(A) = n$, and therefore the results of [19] and [9] can be applied to our situation. However, transformation of these equations into their normal form (47) changes the IO structures and their associated transfer matrices which are the basis of our approach.

We finally describe an algorithm for the computation of all (X, Y) from Theorem 4.3. By means of Theorem 2.13 we check whether $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ has a left inverse in $\mathcal{D}_T^{n\times(n+p)}$ and compute a special such left inverse (X^0, Y^0) if there is one. The goal is to then find all such left inverses (X^1, Y^1) with $Y^1 \in \mathcal{S}^{n\times p}$. The algorithm is similar to that of Theorem 3.12. Since $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right) \in \mathcal{D}^{(n+p)\times n}$ and $\mathrm{rank}\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right) = n$ we can compute a universal left annihilator of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ of the form

$$L = (L^X, L^Y) \in \mathcal{D}^{p\times(n+p)},\ \text{hence also } \mathcal{D}_T^{1\times p}L = \left\{ \xi \in \mathcal{D}_T^{1\times(n+p)};\ \xi \begin{pmatrix} A \\ C \end{pmatrix} = 0 \right\}. \tag{49}$$

According to Theorem 2.13 each left inverse of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ with entries in $\mathcal{D}_T$ has the form

$$\begin{gathered} (X^1, Y^1) = (X^0, Y^0) + Z^0(L^X, L^Y),\ Z^0 \in \mathcal{D}_T^{n\times p}, \\ X^1 = X^0 + Z^0 L^X \in \mathcal{D}_T^{n\times n},\ Y^1 = Y^0 + Z^0 L^Y \in \mathcal{D}_T^{n\times p}. \end{gathered} \tag{50}$$

We have to check when Y^1 is proper. For this purpose we compute the Smith form of L^Y with respect to $F[\sigma] = F[\frac{1}{s-\alpha}]$ and thus *with respect to* $\mathcal{S}$:

$$\begin{gathered} UL^Y V = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix},\ U, V \in \mathrm{Gl}_p(\mathcal{S}),\ E := \mathrm{diag}(e_1, \cdots, e_r),\ r := \mathrm{rank}(L^Y),\ \text{hence} \\ Y^1 V = Y + Z \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \text{ with } Y := Y^0 V \in \mathcal{D}_T^{n\times p},\ Z := Z^0 U^{-1} \in \mathcal{D}_T^{n\times p} \text{ or} \\ (Y^1 V)_{ij} = \begin{cases} Y_{ij} + Z_{ij} e_j & \text{if } 1 \le j \le r \\ Y_{ij} & \text{if } r+1 \le j \le p \end{cases},\ 1 \le i \le n. \end{gathered} \tag{51}$$

Theorem 4.9 (Algorithm for Theorem 4.5). *For the given Rosenbrock system from Example 2.9 assume that* $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ *has a left inverse* $(X^0, Y^0) \in \mathcal{D}_T^{n\times(n+p)}$ *which has been constructed via Theorem 2.13. Consider the derived data from equations* (49)-(51)*. Let*

$$e_j = \frac{f_j}{g_j},\ f_j, g_j \in F[\sigma],\ \gcd(f_j, g_j) = 1,\ j = 1, \cdots, r,$$

be the reduced representations of the e_j as rational functions in $F(\sigma) = F(s)$, hence

$$\mathcal{D}_T e_j + \mathcal{S} = g_j^{-1}(\mathcal{D}_T f_j + \mathcal{S} g_j) = g_j^{-1}\mathcal{D}_T \text{ for } j = 1, \cdots, r \tag{52}$$

as in Theorem 3.12.
Then $\binom{A}{C}$ has a left inverse $(X^1, Y^1) \in \mathcal{D}_T^{n\times n} \times \mathcal{S}^{n\times p}$ as needed in Theorem 4.3 if and only if

$$Y_{ij} \in \begin{cases} \mathcal{D}_T g_j^{-1} & \text{if } 1 \le j \le r \\ \mathcal{S} & \text{if } r+1 \le j \le p \end{cases}, \ 1 \le i \le n. \tag{53}$$

If these conditions are satisfied and if

$$Y_{ij} = g_j^{-1}(-Z_{ij} f_j + s_{ij} g_j) = -Z_{ij} e_j + s_{ij},\ Z_{ij} \in \mathcal{D}_T,\ s_{ij} \in \mathcal{S},\ 1 \le j \le r, \tag{54}$$

also choose arbitrary $Z_{ij} \in \mathcal{D}_T$ for $1 \le i \le n$, $r+1 \le j \le p$, hence $Z \in \mathcal{D}_T^{n\times p}$. Then the matrix

$$(X^1, Y^1) := (X^0, Y^0) + ZUL \in \mathcal{D}_T^{n\times n} \times \mathcal{S}^{n\times p} \tag{55}$$

is a left inverse of $\binom{A}{C}$ as needed in Theorem 4.3, and all such inverses are obtained by this construction.

Proof. With the data from above (X^1, Y^1) is a left inverse of $\binom{A}{C}$ with entries in $\mathcal{D}_T$. Moreover the following equivalences hold:

$$Y^1 \in \mathcal{S}^{n\times p} \iff Y^1 V = Y^0 V + Z^0 U^{-1}\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = Y + Z\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}^{n\times p}$$

$$\underset{(51)}{\iff} \begin{cases} Y_{ij} + Z_{ij} e_j \in \mathcal{S} & \text{if } 1 \le j \le r \\ Y_{ij} \in \mathcal{S} & \text{if } r+1 \le j \le p \end{cases}, \ 1 \le i \le n \iff$$

$$\exists s_{ij} \in \mathcal{S} \text{ with } Y_{ij} = \begin{cases} -Z_{ij} e_j + s_{ij} \in \mathcal{D}_T e_j + \mathcal{S} & \text{if } 1 \le j \le r \\ s_{ij} \in \mathcal{S} & \text{if } r+1 \le j \le p \end{cases}, \ 1 \le i \le n.$$

These equivalences finally imply the asserted equivalence

$$\exists (X^1, Y^1) \in \mathcal{D}_T^{n\times n} \times \mathcal{S}^{n\times p} \text{ with } (X^1, Y^1)\begin{pmatrix} A \\ C \end{pmatrix} = \mathrm{id}_n \iff$$

$$Y_{ij} \in \begin{cases} \mathcal{D}_T e_j + \mathcal{S} & \text{if } 1 \le j \le r \\ \mathcal{S} & \text{if } r+1 \le j \le p \end{cases}, \ 1 \le i \le n.$$

The remaining assertions follow by reading backwards the equations (49)-(54) □

Example 4.10. As an example for the described algorithms we consider the complex continuous case, the stable region $\Lambda_1 := \{\lambda \in \mathbb{C};\ \Re(\lambda) < 0\}$, $\sigma := \frac{1}{s+1}$ and the Rosenbrock system given by the following matrices:

$$A := \begin{pmatrix} 1 - s^2 & 4\,s^2 + 6\,s + 2 \\ -s^3 - s^2 & 2\,s^3 + 3\,s^2 + 3\,s + 2 \end{pmatrix}, \quad B := \begin{pmatrix} s+1 & 1 \\ s & s^2 - 4 \end{pmatrix},$$

$$C := \begin{pmatrix} -s^2 - s & 2\,s^2 + 2\,s \end{pmatrix}, \quad \text{and} \qquad D := \begin{pmatrix} 1 & -s \end{pmatrix}.$$

Computation of the transfer matrices $H_1 = A^{-1}B$ and $H_2 = D + CH_1$ shows that they are proper as required. As a next step we determine whether the matrix $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ has a left inverse in $\mathcal{D}_T$: The Smith form of this matrix (with respect to $\mathcal{D}_T$) is

$$\begin{pmatrix} s+1 & 0 \\ 0 & 2+3\,s+s^2 \\ 0 & 0 \end{pmatrix},$$

the inverse of the greatest elementary divisor $2 + 3\,s + s^2 = (s+1)(s+2)$ is contained in $\mathcal{D}_T$. Hence, by Theorem 2.13, the matrix does indeed permit left inverse matrices with entries in $\mathcal{D}_T$. Applying part (2) of that theorem yields that any such left inverse (X^1, Y^1) is of the form

$$(X^1, Y^1) = (X^0, Y^0) + Z^0(L^X, L^Y) \tag{56}$$

for some $Z^0 \in \mathcal{D}_T^{n\times p}$ where (X^0, Y^0) is one particular left inverse and (L^X, L^Y) a universal left annihilator of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$. In our case we get that

$$\begin{aligned} X^0 &= \begin{pmatrix} 1 & -\frac{2\,(s^2+3\,s+1)}{2+3\,s+s^2} \\ 0 & \frac{1}{2+3\,s+s^2} \end{pmatrix}, \quad Y^0 = \begin{pmatrix} \frac{s^2\,(2\,s+5)}{2+3\,s+s^2} \\ -\frac{s}{2+3\,s+s^2} \end{pmatrix}, \\ L^X &= \quad \begin{pmatrix} s & -2\,s \end{pmatrix}, \quad \text{and} \quad L^Y = \begin{pmatrix} 2\,s^2 - s + 1 \end{pmatrix}. \end{aligned}$$

Here the matrix $L^Y \in \mathcal{D}^{1\times 1}$ is equal to its Smith form $\left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right) = UL^YV$ (i.e., $U = V = 1 \in \mathcal{D}^{1\times 1}$). Consequently, $E = \operatorname{diag}(e_1, \ldots, e_{\operatorname{rank}(L^Y)}) = e_1 = L^Y$, $Y = Y^0$ and $Z = Z^0$ in the notation of (51). Considering e_1 as a rational function in $\sigma = \frac{1}{s+1}$ yields

$$e_1 = 2\left(\frac{1-\sigma}{\sigma}\right)^2 - \frac{1-\sigma}{\sigma} + 1 = \frac{4\sigma^2 - 5\sigma + 2}{\sigma^2} =: \frac{f_1}{g_1}.$$

By Theorem 4.9 the matrix $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ has a left inverse $(X^1, Y^1) \in \mathcal{D}_T^{2\times 2} \times \mathcal{S}^{2\times 1}$ if and only if

$$\begin{aligned} Y_{11} = \frac{s^2\,(2\,s+5)}{2+3\,s+s^2} \quad &\in \quad \mathcal{D}_T(s+1)^2 = \mathcal{D}_T g_1^{-1} \quad \text{and} \\ Y_{21} = -\frac{s}{2+3\,s+s^2} \quad &\in \quad \mathcal{D}_T(s+1)^2 = \mathcal{D}_T g_1^{-1}. \end{aligned}$$

This condition is obviously fulfilled. Now we have to find representations

$$Y_{i1} = -Z_{i1}e_1 + s_{i1} = g_1^{-1}(-Z_{i1}f_1 + s_{i1}g_1) \in g_1^{-1}(\mathcal{D}_T f_1 + \mathcal{S} g_1) = g_1^{-1}\mathcal{D}_T.$$

One possible choice computed with Algorithm 3.13 is

$$Z = \begin{pmatrix} -\frac{s^2\,(2\,s+5)\,(2\,s+7)}{4\,(s+1)^3\,(2+3\,s+s^2)} \\ \frac{s\,(2\,s+7)}{4\,(s+1)^3\,(2+3\,s+s^2)} \end{pmatrix}.$$

By Corollary 3.14 any other possible choice of Z could be obtained by adding $c_i(s+1)^0 g_1 = c_i(s+1)^{-2}$, $c_i \in \mathcal{S}$ arbitrary, to Z_{i1} for $i = 1, 2$.

By means of the constructed matrix Z we can now compute a left inverse $(X^1, Y^1) \in \mathcal{D}_T^{2\times 2} \times \mathcal{S}^{2\times 1}$ of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ by (56) (note that $Z = Z^0$ in our case). With our data we get

$$X^1 = \begin{pmatrix} \dfrac{21\,s^3+64\,s^2+36\,s+8}{4\,(s+1)^3\,(2+3\,s+s^2)} & -\dfrac{17\,s^3+52\,s^2+24\,s+4}{2\,(s+1)^3\,(2+3\,s+s^2)} \\ \dfrac{s^2\,(2\,s+7)}{4\,(s+1)^3\,(2+3\,s+s^2)} & -\dfrac{s^2-6\,s-2}{2\,(s+1)^3\,(2+3\,s+s^2)} \end{pmatrix},$$

$$Y^1 = \begin{pmatrix} \dfrac{s^2\,(2\,s+5)\,(17\,s-3)}{4\,(s+1)^3\,(2+3\,s+s^2)} \\ -\dfrac{s\,(17\,s-3)}{4\,(s+1)^3\,(2+3\,s+s^2)} \end{pmatrix}.$$

Checking the properties of these matrices shows that they are really contained in $\mathcal{D}_T^{2\times 2}$ resp. $\mathcal{S}^{2\times 1}$ and that $(X^1, Y^1)\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right) = \mathrm{id}_2$ is indeed fulfilled.

At last we compute the transfer matrix $H_{\mathrm{obs}} := (Y^1, X^1B - Y^1D)$ and its controllable realization $\mathcal{B}_{\mathrm{obs}} = \left\{ \left(\begin{smallmatrix} \hat{x} \\ y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{2+1+2};\ P_{\mathrm{obs}} \circ \widehat{x} = (Q_y, Q_u) \circ \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \right\}$ (using Result 2.10) and get that

$$P_{\mathrm{obs}} = \begin{pmatrix} -\frac{17}{4}\,(s+1) & -\frac{17}{4}\,(s+1)\,s\,(2\,s+5) \\ 0 & \frac{4}{17}\,(s^5+6\,s^4+14\,s^3+16\,s^2+9\,s+2) \end{pmatrix},$$

$$Q_y = \begin{pmatrix} 0 \\ \frac{1}{17}\,(-17\,s^2+3\,s) \end{pmatrix}, \quad \text{and}$$

$$Q_u = \begin{pmatrix} -\frac{17}{4}\,(s^2+s+1) & \frac{17}{4}\,(s^2-s-5) \\ \frac{1}{17}\,(2\,s^4+7\,s^3+36\,s^2+s) & \frac{1}{17}\,(-2\,s^4-3\,s^3+22\,s^2-48\,s-16) \end{pmatrix}.$$

Checking properness of H_{obs}, T-stability of $\mathcal{B}_{\mathrm{obs}}$ and T-autonomy of $\mathcal{B}_{\mathrm{err}}$ (after computing $\mathcal{B}_{\mathrm{err}}$) yields that $\mathcal{B}_{\mathrm{obs}}$ really is a proper T-stable T-observer of the pseudo state of the given Rosenbrock system.

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